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A FAMILY OF IMPLICIT 4-STEP BLOCK HYBRID COLLOCATION METHOD FOR ACCURATE AND EFFICIENT SOLUTION OF ODES.

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ABSTRACT: *In this paper, we present a one parameter family of modified collocation method with large region of absolute stability for accurate and efficient solution of the ordinary differential equations.* ***The*** *process produce some hybrid schemes which are combined together to form the block method for parallel or sequential solution of ODEs. The suggested approach eliminates requirement for a starting value and special predictor for off- grid values in the discrete schemes .*

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1 INTRODUCTION

.Consider the initial value problems for the system/scalar Ordinary Differential Equations (ODEs):

$$y' = f(x, y) \quad y(a) = y_0, a \le x \le b \tag{1.1}$$

In the literature, conventional linear multistep methods including hybrid ones have been made continuous through the idea of Multistep Collocation (MC),[see Lie and Norsett (1989), Onumanyi etal (1994), Onumanyi et al (1999), Yusuph and Onumanyi(2002), and Yahaya (2004)]. The Continuous Multistep Method (CMM) produces piece-wise polynomial solutions over K-steps $[X_n, X_{n+k}]$ for the first order Ordinary Differential Equation (ODEs). The implicit continuous multistep method (CMM) interpolant 3.1 is not to be used directly as a numerical integrator, but the resulting block discrete multistep method which is derived from it, is self-starting and can be applied for direct solutions of the initial value problems.

This paper is part of research effort to consider the hybrid methods for efficient and accurate use, a one parameter family of modified collocation method is considered here. The standard Quade's method in discrete or continuous forms requires starting value for initial value problems. (see Yahaya and Adegboye(2007), Lambert 1973). Moreover, the continuous method does not allow for block formulation which could have eliminated the requirement for starting values. The continuous method with variant is presented in this paper using the matrix inversion technique and the method3.1 has the following advantages:

(1.) It allows block formulation and therefore is self starting and for appropriate choice of k, overlap of solution model is eliminated.

(2.) The continuous formulation and its derivative can be used at off-grid points x_{n+r}, to obtain further discrete schemes for the block hybrid method.

2. GENERAL MC LINKED TO CONTINUOUS MULTISTEP INTERPOLANT.

Let us first give a general description for the method of Multistep Collocation (MC) and its link to Continuous Multistep (CM) method for (1.1). In the equation 1.1, f is given and y is sought as

$$y = a_1\phi_1 + a_2\phi_2 + \dots\dots\dots + a_p\phi_p \tag{2.1}$$

where

$a = (a_1, a_2 \dots\dots a_p)^T, \quad \phi = (\phi_1, \phi_2 \dots\dots \phi_p)^T$

$X_n \le X \le X_{n+k}$ where $n = 0, k, \dots, N-k$, and T denotes 'Transpose of'

Equation (2.1) can be re-written as

$$y = (a_1, a_2 \dots\dots\dots a_p)^T (\phi_1, \phi_2 \dots\dots\dots \phi_p)^T \tag{2.2}$$

The unknown coefficients $a_1, a_2 \dots\dots\dots a_p$ are determined using, respectively, the $r(0<r<=k)$ interpolation conditions and the $s>0$ distinct collocation conditions, $p=r+s$ as follows:

$$\sum_{1=1}^{p} a_j\varphi_j(x_i) = y_i, \qquad (i = 1, \dots\dots\dots r)$$

$$\sum_{1=1}^{p} a_j\varphi_j'(x_i) = f_i, \qquad (i = 1, \dots\dots\dots s) \tag{2.3}$$

This is a system of p linear equations from which we can compute values for the unknown coefficients provided (2.3) is assumed non-singular. For the distinct points x_i and c_i the non-singular system is guaranteed (see proof in Yusuph and Onumanyi(2002)). We can write (2.3) as a single set of linear equations of the form

$$\underline{D}\ \underline{a} = \underline{F}$$

$$\underline{a} = \underline{\underline{D}}^{-1}\ \underline{F} \tag{2.4}$$

where $\underline{F} = (y_1, y_2, \dots, y_r, f_1, f_2, \dots, f_s)^T \qquad (2.5)$

Substituting the vector a, given by (2.4) and F by (2.5) into (2.2) gives

$$y = (y_1, y_2, \dots, y_r, f_1, f_2, \dots, f_s) C^T (\phi_1, \phi_2 \dots\dots \phi_p)^T \tag{2.6}$$

Equation (2.6) is the continuous MC Interpolant, C^T is known explicitly in the form

$$\underline{\underline{C}}^T \phi = \begin{pmatrix} C_{11} & C_{21} & C_{p1} \\ C_{12} & C_{22} & C_{p2} \\ & & \\ C_{1r} & C_{2r} & C_{pr} \\ \vdots & & \vdots \\ C_{1p} & C_{2p} & C_{pp} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \\ \phi_r \\ \vdots \\ \phi_p \end{pmatrix} \tag{2.7}$$

$$\underline{\underline{C^T}}\phi = \begin{pmatrix} \sum_{j=1}^{p} C_{j1\phi_j} \\ \sum_{j=1}^{p} C_{jr\phi_j} \\ \sum_{j=1}^{p} C_{jr+1\phi_j} \\ \sum_{j=1}^{p} C_{jt\phi_j} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_s \end{pmatrix} \tag{2.8}$$

$$F^T C^T \phi = (\alpha_1 y_1 + \alpha_2 y_2 + \ldots\ldots + \alpha_r y_r + \beta_1 f_1 + \beta_2 f_2 + \ldots\ldots + \beta_s f_s)$$

Or

$$F^T C^T \phi = \sum_{j=1}^{r} \alpha_j y_j + h_j \left(\sum_{j=1}^{s} \beta_j/h_j \, f_j \right) \tag{2.9}$$

where, from (2.8),

$$\alpha_j = \sum_{q=i}^{p} C_{qi}\phi_j, \quad j = 1,\ldots r$$

$$\beta_j/h_j = \sum_{q=i}^{p} \left[\frac{C_{qi+r}}{h_i} \right] \phi_j \quad j = 1,\ldots s \tag{2.10}$$

Therefore,

$$y = \sum_{j=1}^{r} \alpha_j y_j + h_j \left(\sum_{j=1}^{s} \beta_j/h_j \right) f_j \tag{2.11}$$

where α_j, β_j/h_j are given by (2.10). Hence, (2.11) with (2.10) is the CM interpolant with constant or variable step-size.

2. FOUR-STEP BLOCK HYBRID METHOS WITHONE OFF- STEP COLLOCATION POINT

The general form of the proposed method is expressed as:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+3} + h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+3} + \beta_3 f_{n+4} + \beta_4 f_{n+\mu}] \tag{3.1}$$

The matrix D of the proposed method expressed as:

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \\ 0 & 1 & 2x_{n+\mu} & 3x_{n+\mu}^2 & 4x_{n+\mu}^3 & 5x_{n+\mu}^4 & 6x_{n+\mu}^5 & 7x_{n+\mu}^6 \end{pmatrix}$$

after some manipulations gives rise to the following continuous form:

$$\bar{y}(x) = \Big[228(\mu-2)(x-x_n)^7 - 2(133\mu^2+1064\mu-2685)(x-x_n)^6 h + 12(266\mu^2+394\mu-1902) (x-x_n)^5 h^2 - 15(931\mu^2-572\mu-275)(x-x_n)^4 h^3 + 4(6650\mu^2-10583\mu-6684)(x-x_n)^3 h^4 - 6(3192\mu^2-6684\mu)(x-x_n)^2 h^5 + (3591\mu^2-9180\mu+3321)h^7\Big] y_n \Big/ 27h^7(133\mu^2-340\mu+12$$

$$+ \Big[-3(8\mu-21)(x-x_n)^7 + (28\mu^2+224\mu-750)(x-x_n)^6 h - 3(119\mu^2+148\mu-1077) (x-x_n)^5 h^2 + 15(931\mu^2-572\mu-275)(x-x_n)^4 h^3 - 4(3465\mu^2 6156\mu-3888)(x-x_n)^3 h^4 + 3(882\mu^2-1944\mu)(x-x_n)^2 h^5 y_{n+1} \Big/ 4h^7(133\mu^2-340\mu+123) +$$

$$\Big[-3(88\mu-41)(x-x_n)^7 + (308\mu^2+2464\mu-1230)(x-x_n)^6 h - 3(1043\mu^2+2308\mu-1353) (x-x_n)^5 h^2 + 15(700\mu^2+304\mu-328)(x-x_n)^4 h^3 - (12845\mu^2-3116\mu-1968)(x-x_n)^3 h^4 + 3(1722\mu^2-984\mu)(x-x_n)^2 h^5\Big] y_{n+3} \Big/ 108h^7(133\mu^2-340\mu+123)$$

$$+ \Big[6(35\mu^2-80\mu-19)(x-x_n)^7 - (245\mu^3+1960\mu^2-5685\mu-1404)(x-x_n)^6 h + 6(504\mu^3+690\mu^2-4056\mu+1074) (x-x_n)^5 h^2 - 3(4634\mu^3-3520\mu^2-1474\mu+4520)(x-x_n)^4 h^3 + 2(14644\mu^3-25615\mu^2-13476\mu+6417)(x-x_n)^3 h^4 - 3(9247\mu^3+2337\mu-20920\mu^2+1476)(x-x_n)^2 h^5 + 6(1596\mu^3+1476\mu-4080\mu^2)(x-x_n)h^6\Big] f_n \Big/ 72h^6(133\mu^2-340\mu$$

$$+ \Big[(102\mu^2-251\mu+11)(x-x_n)^7 - (119\mu^3+833\mu^2-2677\mu+1257)(x-x_n)^6 h + (1351\mu^3+961\mu^2-9485\mu+5025)(x-x_n)^5 h^2 - (5411\mu^3-7111\mu^2-11019\mu+8199)(x-x_n)^4 h^3 (8841\mu^3-1995\mu^2+2520\mu+4320)(x-x_n)^3 h^4 - (4662\mu^3+6480\mu-12618\mu^2)(x-x_n)^2 h^5\Big] f_{n+1} \Big/ 12h^6(133\mu^3-473\mu^2+463\mu-123)$$

$$+ \Big[3(26\mu^2-75\mu+29)(x-x_n)^7 - (91\mu^3+455\mu^2-1989\mu+819)(x-x_n)^6 h + 3(287\mu^3-125\mu^2-1773\mu (x-x_n)^5 h^2 - 3(889\mu^3-1847\mu^2-1281\mu+983)(x-x_n)^4 h^3 + (3115\mu^3-8527\mu^2+1440\mu+1152)(x-x_n - 3(406\mu^3+576\mu-1246\mu^2)(x-x_n)^2 h^5\Big] f_{n+3} \Big/ 36h^6(133\mu^3-739\mu^2+1143\mu-369)$$

$$+ \Big[-2(3\mu^2-8\mu+3)(x-x_n)^7 - (7\mu^3+28\mu^2-119\mu+48)(x-x_n)^6 h - 2(28\mu^3-14\mu^2-140\mu+66) (x-x_n)^5 h^2 + (154\mu^3-296\mu^2-186\mu+144)(x-x_n)^4 h^3 - 2(84\mu^3-213\mu^2+36\mu+27)(x-x_n)^3 h + (63\mu^3+81\mu-180\mu^2)(x-x_n)^2 h^5\Big] f_{n+4} \Big/ 24h^6(133\mu^3-872\mu^2+1483\mu-492)$$

$$+ \left[19 (x - x_n)^7 - 234 (x - x_n)^6 h + 1074 (x - x_n)^5 h^2 - 2260 (x - x_n)^4 h^3 + 2139 (x - x_n)^3 h^4 - 738 (x - x_n) h^5 \right] f_{n+\mu} \Big/ h^6 \mu (133 \mu^5 - 1476 + 6417 \mu - 9040 \mu^2 + 5370 \mu^3 - 1404 \mu^4)$$

(3.2)

Evaluating equation (3.2) at points $x = x_{n+2}, x = x_{n+4}, x = x_{n+\frac{3}{2}}, x = x_{n+\frac{7}{2}}$ and $\mu = 7/2$ yield the following four discrete hybrid methods which are used as a block integrator:

$$y_{n+2} - \frac{12067}{60723} y_n - \frac{668}{2249} y_{n+1} - \frac{30620}{60723} y_{n+3} = \frac{h}{1416870} [76625 f_n + 619248 f_{n+1} - 595840 f_{n+3} - 33285 f_{n+4} + 202752 f_{n+\frac{7}{2}}]$$

$$y_{n+4} + \frac{55}{2249} y_n - \frac{8}{2249} y_{n+1} - \frac{2296}{2249} y_{n+3} = \frac{h}{78715} [570 f_n - 2856 f_{n+1} + 9240 f_{n+3} + 12390 f_{n+4} + 55296 f_{n+\frac{7}{2}}]$$

$$y_{n+\frac{3}{2}} - \frac{9965}{71968} y_n - \frac{52731}{71968} y_{n+1} - \frac{1159}{8996} y_{n+3} = \frac{h}{40302080} [1453035 f_n + 17596278 f_{n+1} - 5532870 f_{n+3} - 388395 f_{n+4} + 2218752 f_{n+\frac{7}{2}}]$$

$$y_{n+\frac{7}{2}} - \frac{47375}{1943136} y_n - \frac{343}{71968} y_{n+1} - \frac{471625}{485784} y_{n+3} = \frac{h}{10363392} [72625 f_n + 421890 f_{n+1} + 3172750 f_{n+3} - 128625 f_{n+4} + 2499840 f_{n+\frac{7}{2}}]$$

(3.3)

In the same vein, its derivative is evaluated at point $x = x_{n+2}$ and $x = x_{n+\frac{3}{2}}$ to obtain:

$$y_{n+3} + \frac{12800}{21140} y_n - \frac{34020}{21140} y_{n+1} = \frac{h}{21140} [-3515 f_n - 20034 f_{n+1} + 20650 f_{n+3} + 1323 f_{n+4} - 7776 f_{n+\frac{7}{2}} + 38752 f_{n+\frac{3}{2}}]$$

$$y_{n+1} + \frac{11200}{69930} y_n - \frac{81130}{69930} y_{n+3} = \frac{h}{69930} [-2703 f_n - 42378 f_{n+1} - 94458 f_{n+2} - 40698 f_{n+3} - 903 f_{n+4} + 7680 f_{n+\frac{7}{2}}]$$

(3.4)

4 IMPLIMENTATION STARATEGIES

To start the IVP on the subinterval $[X_0, X_4]$ we combine (3.3) and (3.4) when n=0.

The discrete hybrid block method 3.3 and 3.4 with one off-step collocation point has uniform order 7 and with error constant $C_8 = \left[\frac{68920}{47229}, -\frac{143033}{1889160}, -\frac{10496331}{515866624}, \frac{88549321}{221085696}, 13200, \frac{27513}{3}\right]^T$

It is thus convergent and simultaneously provides values for $y_1, y_{3/2}, y_2, y_3, y_{7/2}$ and y_4

without looking for any other method to provide y_1, y_2 *and* y_3 or special predictor to obtain $y_{3/2}$ *and* $y_{7/2}$ for computation of which was the case before (see Lambert 1973). Hence this is an improvement. The first option is to proceed in the conventional approach with the single equation on the sub-intervals $[X_1, X_4]$, $[X_2, X_5]$.....which do overlap. The second option is to use (3.3) and (3.4) and a single finite difference equation is obtained which simultaneously provides values for $y_1, y_{3/2}, y_2, y_3, \dots\dots\dots. y_{N-1}, y_N$ over sub-intervals which do not overlap.

Using the matlab package, we were able to plot the stability region of the proposed block method. This is done by reformulating the block method as general linear method to obtain the values of the matrices A,B,U,V which are then substituted into stability matrix and stability function. Then the utilized maple package yield the stability polynomial of the block method. Using a matlab program we plot the absolute stability region of proposed four step block hybrid multistep method. From figure 4.1 below the proposed four point block method (3.3) and (3.4) is A −stable.

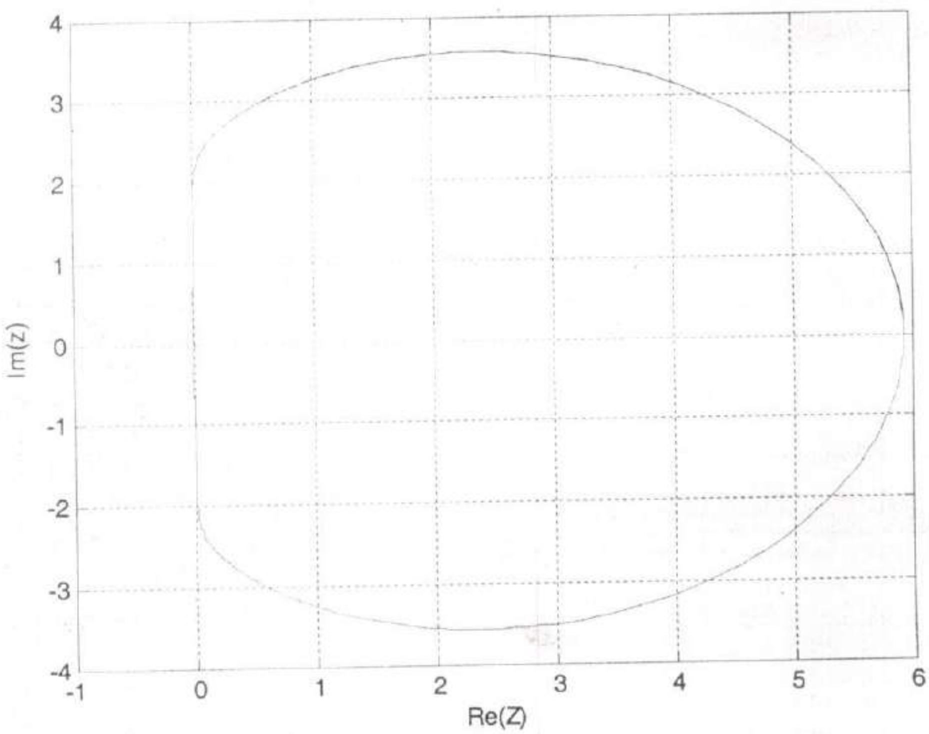

FIGURE:4.1: region of absolute stability of four step block hybrid method

5. NUMERICAL EXPERIMENTS

In this section we shall apply the derived discrete schemes in block form for solution of initial value problems

Example 5.1

Consider the initial value problem

$y' = -y,\ y(0) = 1,\ 0 \le x \le 2$ *with exact solution* $y(x) = e^{-x}$
and Taking $h = 0.1$

Example 5.2

Consider the initial value problem

$y' = -9y,\ y(0) = 1,\ 0 \le x \le 2$ *with exact solution* $y(x) = e^{1-9x}$
and Taking $h = 0.1$

Firstly we transform the schemes by substitution to get a recurrence relation. Substituting n=0,4,8,........,20 for k=4 and solving simultaneously at each step, we obtain values of y(x) and the results are summarized in Table 1and Table 2

Table 1: Example 5.1 for k=4 with one off-grid point at Collocation

N	X	Numerical Solution	Exact Solution	Absolute Error
0	0	1.000000000	1.000000000	0
1	0.1	0.9048374175	0.9048374180	4.9999993035E-10
2	0.2	0.8187307527	0.8187307531	4.0000003310E-10
3	0.3	0.7408182203	0.7408182207	4.0000003310E-10
4	0.4	0.6703200454	0.6703200460	5.9999993862E-10
5	0.5	0.6065306583	0.6065306597	1.4000000048E-09
6	0.6	0.5488116350	0.5488116361	1.1000000910E-09
7	0.7	0.4965853025	0.4965853038	1.2999999965E-09
8	0.8	0.4493289632	0.4493289641	9.0000001896E-10
9	0.9	0.4065696585	0.4065696597	1.1999999883E-09
10	1.0	0.3678794401	0.3678794412	1.0999999800E-09
11	1.1	0.3328710825	0.3328710837	1.1999999883E-09
12	1.2	0.3011942112	0.3011942119	7.0000000241E-10
13	1.3	0.2725317922	0.2725317930	8.0000001068E-10
14	1.4	0.2465969633	0.2465969639	5.9999999413E-10
15	1.5	0.2231301595	0.2231301601	5.9999999413E-10
16	1.6	0.2018965174	0.2018965180	5.9999999413E-10
17	1.7	0.1826835232	0.1826835231	1.0000000827E-10
18	1.8	0.1652988874	0.1652988882	7.9999998293E-10
19	1.9	0.1495686185	0.1495686192	7.0000000241E-10
20	2.0	0.1353352825	0.1353352832	7.0000000241E-10

Table2: Example 5.2 for k=4 with one off grid at Collocation

N	x	Numerical Solution	Exact solution	Exact error
0	0	2.71828E+00	2.71828E+00	0
1	0.1	1.10501E+00	1.10517E+00	1.64935E-04
2	0.2	4.49262E-01	4.49329E-01	6.70433E-05
3	0.3	1.82678E-01	1.82684E-01	5.32050E-06
4	0.4	7.42874E-02	7.42736E-02	1.38386E-05
5	0.5	3.01985E-02	3.01974E-02	1.11920E-06
6	0.6	1.22778E-02	1.22773E-02	4.55150E-07
7	0.7	4.99238E-03	4.99159E-03	7.84514E-07
8	0.8	2.03019E-03	2.02943E-03	7.55938E-07
9	0.9	8.25289E-04	8.25105E-04	1.84229E-07
10	1.0	3.35538E-04	3.33546E-04	1.99127E-06
11	1.1	1.36436E-04	1.36389E-04	4.67803E-08
12	1.2	5.54826E-05	5.54516E-05	3.09998E-08
13	1.3	2.25542E-05	2.25449E-05	9.27519E-09
14	1.4	9.16984E-06	9.16609E-06	3.75362E-09
15	1.5	3.72863E-06	3.72665E-06	1.97796E-09
16	1.6	1.51627E-06	1.51514E-06	1.13361E-09
17	1.7	6.16379E-07	6.16012E-07	3.66843E-10
18	1.8	2.50601E-07	2.50452E-07	1.48945E-10
19	1.9	1.01899E-07	1.01826E-07	7.29774E-11
20	2.0	4.14380E-08	4.13994E-08	3.85583E-11

Example 5.3: Consider the System of Stiff Equation

$y_1' = 198y_1 + 199y_2 \quad , y_2' = -398y_1 - 399y_2 \quad \text{with } y_1(0) = 1 \text{ and } y_2(0) = -1$

Exact Solution : $y_1(x) = e^{-x} \quad \text{and} \quad y_2(x) = -e^{-x}$

Table3: Example 5.3 for k=4 with one off grid at Collocation

	Exact Solution		Numerical Solution		Error	
	Y1	Y2	Y1	Y2	Y1	Y2
[illegible]	0.904837418	−0.9048374180	0.904837376	−0.9048373740	4.24000000E-08	4.40000000E-08
[illegible]	0.818730753	−0.8187307531	0.818730697	−0.8187307007	5.59000000E-08	5.24000000E-08
[illegible]	0.740818221	−0.7408182207	0.740818157	−0.7408181540	6.41000000E-08	6.67000001E-08
[illegible]	0.670320046	−0.6703200460	0.670319976	−0.6703199741	6.97000000E-08	7.19000000E-08
[illegible]	0.60653066	−0.6065306597	0.606530588	−0.6065305868	7.17999999E-08	7.28999999E-08
[illegible]	0.548811636	−0.5488116361	0.548811561	−0.5488115628	7.50000001E-08	7.33000001E-08
[illegible]	0.496585304	−0.4965853038	0.496585236	−0.4965852339	6.80000000E-08	6.99000000E-08
[illegible]	0.449328964	−0.4493289641	0.449328895	−0.4493288934	6.89000000E-08	7.07000000E-08
[illegible]	0.40656966	−0.4065696597	0.42708292	−0.4270312384	2.05132601E-02	2.04615787E-0
[illegible]	0.367879441	−0.3678794412	0.388303975	−0.3876573340	2.04245342E-02	1.97778928E-0

CONCLUSION

A Collocation technique which yields a method with continuous coefficients has been presented for the approximate solution of first order ODEs with initial conditions. Three test examples have been solved to demonstrate the efficiency of the proposed methods and the results compare favorably with the exact solution, a desirable feature of good numerical methods. Interestingly, all the discrete schemes used in the Block formulation were from a single Continuous Formulation (CF).

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