ON THE APPLICATION OF OPTIMAL CONTROL IN A MILITARY ENVIRONMENT

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Abstract

In this paper, we formulated a mathematical model using the concept of optimal control, where the state of the system is governed by a differential equation controlled by two combatants (combatant A and combatant B) refer to as the evader and pursuer respectively, with conflicting motives. A control *u(.)* defines a notion where the combatant A tries to escape capture from the combatant B defined by control *v(.)*. The game terminates if capture occurs prior to a pre-assigned time *t* or at a time *t*. In a situation where capture has not occurred prior to *t*, the control u(.) sole-aim would be to escape capture, while combatant B sole-aim is to ensure that combatant A is captured.

Key words: Optimal control, terminal miss, dynamics, perturbation and differential games

1.0 Introduction

The existence of optimal control theory can be traced back to the days of Newton, Lagrange and Cauchy, (Emilo, 2012). One of the first instances of optimization theory concerns the finding of a geometric curve of given length which will together with a straight line, enclose the largest possible area, (Francis, 2014). The fundamental problem of optimal control problems is concerned with the choosing of a function that minimizes certain functional, (Jack,2014).

The theory of optimization involves the maximization or minimization of a function (sometimes unknown), which represent the performance of a system, (Jery, 2012). This is carried out by finding values for those variables (which are both quantifiable and controllable) which cause the function to yield an optimal value.

The Optimal control problem can be stated, (Rao ,1984), (Robert,2015) as:

Find the control vector;

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad (1.1)$$

which minimizes the functional called the *Performance* index;

$$J = \int_0^T f_0(x,u,t) \quad (1.2)$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (1.3)$$

Equation (1.3) is called the state vector,

t = time parameter

T = terminal time

f_0 = is a function of x, u, t

In this paper, we formulate a mathematical model by extending the notion of equation (1.1) to a two control problem defined by control *u(.)* and control *v(.)*.

2.0 Model Formulation

We considered a dynamic system, (Shehu, (2004), given by equation (2.1)

$$\dot{x} = \mathbf{f}(x, u, v, t), \qquad x(t_0) = x_0 \qquad (2.1)$$

Where;

control *u(.)* = combatant A

control *v(.)* = combatant B

Equation (2.1) is a state a system that is governed by a differential equation controlled by two combatants (combatant A and combatant B) refers to as the evader and pursuer respectively, with conflicting motives.

A control *u(.)* defines a notion where the combatant A tries to escape capture from combatant B defined by control *v(.)*.

We define a terminal constraints;

$$\psi(x(t_f), t_f \qquad (2.2)$$

and the performance criterion;

$$P = \phi(x(t_f), t_f) + \int_0^{t_f} f_0(x, u, v, t)dt \qquad (2.3)$$

Where;

t_f = Terminal Time

u = Control for Combatant A

v = Control for Combatant B

A strategy Γ for combatant A is a sequence of instructions given by Γ_n

A strategy ∇ for combatant A is a sequence of instructions given by ∇_n

We define the following saddle point property for both controls;

$$p(u^0, v^0) = \min_u p(u, v^0) \tag{2.4}$$

Where,

$$v^0 = v(t : x_0, t_0) \tag{2.5}$$

Similarly;

$$e\,(u^0, v^0) = \max_v e(u^0, v\,) \tag{2.6}$$

$$u^0 = u(t : x_0, t_0) \tag{2.7}$$

Where,

u, v are controls for Combatant A and B respectively

and

u^0, v^0 are optimal controls for Combatant A and B respectively.

The equation of motion for the two controls is defined as;

$$\dot{v} = ap - ae \qquad v(t_0) = v_0 \tag{2.8}$$

$$\dot{y} = v \qquad y(t_0) = 0 \tag{2.9}$$

Where,

$v(t)$ = Relative Velocity to the Initial Line of Site

$y(t)$ = Relative Displacement to the initial Line of Site

Combatant B whishes to minimize the terminal miss ($y(t_f)$),to make capture possible whereas combatant A wishes to maximize the terminal miss to make capture invariably impossible.

We define the performance index as;

$$J = \frac{1}{2}\left(y(t_f)\right)^2 \tag{2/10}$$

We now form the Hamiltonian system for the two combatants as

$$H=\lambda_v(a_p-a_e)+\lambda_y v \tag{2.11}$$

Where;

λ = Control parameter for both the velocity and the displacement

a_p = Acceleration of the Pursuer (combatant B)

a_e = Acceleration of the Evader (combatant A)

The adjoint equations are;

$$\dot{\lambda}_v=-\lambda_y, \qquad \lambda_v(t_f)=0 \tag{2.12}$$

$$\dot{\lambda}_y=0, \qquad \lambda_y(t_f)=y(t_f) \tag{2.13}$$

The optimality conditions are;

$$a_p(t)=-a_{pm}\operatorname{sgn}\lambda_v \tag{2.14}$$

$$a_e(t)=-a_{em}\operatorname{sgn}\lambda_v \tag{2.15}$$

Integrating the adjoint equations we have;

$$\lambda_v(t)=(t_f-t)\ y(t_f) \tag{2.16}$$

Where;

$$\lambda_y(t_f)=y(t_f) \tag{2.17}$$

$$\lambda_y(t\ )=y(t\ )=constatnt \tag{2.18}$$

We note that

$$\operatorname{sgn}\lambda_v(t)=\operatorname{sgn}y(t_f)=constatnt \tag{2.19}$$

Substituting (2.19) into equations (2.14) and (2.15) we have;

$$a_p(t)=-a_{pm}\ \operatorname{sgn}y(t_f) \tag{2.20}$$

$$a_e(t)=-a_{em}\ \operatorname{sgn}y(t_f) \tag{2.21}$$

Substituting equations (2.20) and (2.21) into equations (2.8) and (2.9) yields;

$$\dot{v}=(-a_{pm}+a_{em})\operatorname{sgn}\ y(t_f) \tag{2.22}$$

$$\dot{v}=(a_{pm}-a_{em})\operatorname{sgn}y(t_f) \tag{2.23}$$

$$v(t_f - t_0) = (t_f - t_0)(a_{pm} - a_{em}) \operatorname{sgn} y(t_f) \tag{2.24}$$

$$v(t_f) - v(t_0) = (t_f - t_0)(a_{pm} - a_{em}) \operatorname{sgn} y(t_f) \tag{2.25}$$

$$v(t_f) = v_0 - (t_f - t_0)(a_{pm} - a_{em}) \operatorname{sgn} y(t_f); \quad v(t_0) = v_0 \tag{2.26}$$

$$\dot{y} = v_0 - (t_f - t_0)(a_{pm} - a_{em}) \operatorname{sgn} y(t_f); \qquad \dot{y} = v \tag{2.27}$$

Whose solution may be written as;

$$y(t_f) = v_0(t_f - t_0) - 1/2(a_{pm} - a_{em})(t_f - t_0)^2 \operatorname{sgn} y(t_f); \; y(t_0) = 0 \tag{2.28}$$

We then have;

$$y(t_f) = \begin{bmatrix} 1/2(t_f - t_0)^2[2v_0/(t_f - t_0) - (a_{pm} - a_{em})] \\ \\ -1/2(t_f - t_0)^2[-2v_0/(t_f - t_0) - (a_{pm} - a_{em})] \end{bmatrix} \tag{2.29}$$

3.0 Discussion of Results

It is possible for Combatant B to bring the terminal miss to zero, by making $y(t_f) = 0$ in equation (2.29) This can be achieved by choosing;

$$a_p(t) = a_e(t) + 2v(t)/(t_f - t) \tag{2.30}$$

Substituting equation (2.30) into equation (2.29) $y(t_f)$ becomes;

$$y(t_f) = 1/2(t_f - t_0)^2[2v(t)/(t_f - t) - (a_e(t) + 2v(t)/(t_f - t) - a_e(t))] \tag{2.31}$$

The sole aim of the Combatant B (the pursuer) is to make capture possible by minimizing the terminal miss as minimum as possible, and Combatant B (the evader) sole aim is to maximize the terminal miss to make capture invariably impossible.

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