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DIRECT SOLUTION OF SECOND ORDER INITIAL VALUE PROBLEMS BY HYBRID BACKWARD DIFFERENTIATION FORMULAS

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ABSTRACT

In this paper, we propose a family of Hybrid Backward Differentiation Formulas (HBDF) for direct solution of general second order Initial Value Problems (IVPs) of the form $y'' = f(x, y, y')$. The method is derived by the interpolation and collocation of the assumed approximate solution and its second derivative at $x = x_{n+j}$, $j = 1,2,.....k-1$ and $x = x_{n+k}$ respectively, where k is the step number of the methods. The interpolation and collocation procedures lead to a system of (k+1) equations, which are solved to determine the unknown coefficients. The resulting coefficients are used to construct the approximate continuous solution from which the Multiple Finite Difference Methods (MFDMs) are obtained and simultaneously applied to provide the direct solution to IVPs. A specific methods for k=4 is used to illustrate the process. The methods is shown to be zero stable, consistence and hence convergence. Numerical examples are given to show the efficiency of the method.

Keywords: Hybrid method, Backward differentiation Formulas, Collocation, Interpolation, Second order.

INTRODUCTION

In recent times, the integration of Ordinary Differential Equations (ODEs) are investigated using some kind of block methods. This paper discusses the family of implicit Linear Multistep Method (LMM) for numerical integration of general second order ODEs which arise frequently in the area of science and engineering especially mechanical system, control theory and celestial mechanics and are generally written as:

$$y'' = f(x, y', y),\ \ y(a) = y_0\ ,\ y'(a) = \eta_0 \quad (1)$$

In practice the problems are reduced to systems of first order equations and any method for first order equations is used to solve them see Awoyemi (1999). It has been extensively discussed that due to the dimension of the problem after it has been reduced to a system of first order equations also, more often the reduced systems of ordinary differential equations (ODEs) is not well posed, unlike the given problem. The approach waste a lot of Computer time and human efforts, hence there is a need to develop algorithms to handle these classes of problems directly without any reduction to system of first order ODEs.

Development of LMM for solving ODE can be generated using methods such as taylor's series, numerical interpolation, numerical integration and collocation method, which are restricted by an assumed order of convergence. This paper considers the contribution of multi step collocation technique introduced by Onumayi *et al* (1994) by deriving our new method. Some researchers have attempted the solution of directly using linear multistep methods without reduction to system of first order ordinary differential equations these include Mohammed et al (2011), Yusuph and Onumayi (2002) and Onumayi *et al* (1999).

Block methods for solving ODEs have initially been proposed by Milne (1953). The Milne's idea of proceeding in blocks was developed by Rosser (1967) for Runge-Kutta method. Also block Backward Differentiation Formulas (BDF) methods are discussed and developed by many researchers (Ibrahim *et al*., 2007; Majid and Suleiman, 2007; Yahaya and Mohammed, 2009; Mohammed and Yahaya, 2010; Yahaya and Mohammed, 2010; Mohammed, 2011; Akinfenwa et al., 2011; Akinfenwa *et al*., 2013; Semenov *et al*.,2013). The method of collocation and interpolation of the power series approximate solution to generate continuous LMM has been adopted by many researchers among them are (Houwen et al.(1991), Fatunla (1991), Jiaxiang (1995). In this paper we are suggested a construction of four step Hybrid Backward Differentiation Formulas (HBDF) method, it is self-starting and can be applied for the numerical solution of IVPs (Cauchy problem) for second-order ODEs.

Development of the Method

We seek an approximation of the form

$$Y(x)=\sum_{j=0}^{r+s-1}\ell_j x^j \qquad (2)$$

Where ℓ_j are unknown coefficients to be determined and $1 \le r < k$ and $s > 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions

$$Y(x)=y_{n+j},\ j=0,1,2,\ldots,.k-1 \qquad (3)$$

$$Y''(x_{n+k})=f_{n+k} \qquad (4)$$

We note that $y_{n+\mu}$ is the numerical approximation to the analytical solution $y(x_{n+\mu}), f_{n+\mu}=f(x_{n+\mu},y_{n+\mu},y'_{n+\mu})$.

Equations (3) and (4) lead to a system of (k+1) equations which is solved by Cramer's rule to obtain ℓ_j. Our continuous approximation is constructed by substituting the values ℓ_j into equation (2). After some manipulation, the continuous method is expressed as

$$Y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+\alpha_\mu(x)y_{n+\mu}+h^2\beta_k(x)f_{n+k} \qquad (5)$$

Where $\alpha_j(x),\beta_k(x)$ and $\alpha_\mu(x)$ are continuous coefficients. We note that since the general second order ordinary differential equation involves the first derivative, the first derivative formula

$$Y'(x)=\frac{1}{h}\left(\sum_{j=0}^{k-1}\alpha'_j(x)y_{n+j}+\alpha'_\mu(x)y_{n+\mu}+h^2\beta'_k(x)f_{n+k}\right) \qquad (6)$$

$$Y'(x) = \delta(x) \tag{7}$$

$$Y'(a) = \delta_0 \tag{8}$$

Specification of Methods

Four Step Methods with One-off-step Point at Interpolation.

To derive this methods, we use Eq.(5) to obtained a continuous 4-step HBDF method with the following specification : $r = 5, s = 1, k = 4$ We also express $\alpha_j(x), \alpha_\mu(x)$ and $\beta_k(x)$ as functions of t, where $t = \dfrac{x - x_n}{h}$ to obtain the continuous form as follows:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_{\frac{7}{2}} y_{n+\frac{7}{2}} + h^2 \beta_4 f_{n+4} \tag{9}$$

where

$$\alpha_0(t) = 1 - \frac{4397}{1890}t + \frac{5566}{2835}t^2 - \frac{4357}{5670}t^3 + \frac{401}{2835}t^4 - \frac{4}{405}t^5$$

$$\alpha_1(t) = \frac{238}{45}t - \frac{946}{135}t^2 + \frac{907}{270}t^3 - \frac{187}{270}t^4 + \frac{7}{135}t^5$$

$$\alpha_2(t) = -\frac{1589}{270}t + \frac{4342}{405}t^2 - \frac{4969}{810}t^3 + \frac{572}{405}t^4 - \frac{46}{405}t^5$$

$$\alpha_3(t) = \frac{742}{135}t - \frac{4402}{405}t^2 + \frac{5659}{810}t^3 - \frac{1429}{810}t^4 + \frac{61}{401}t^5$$

$$\alpha_{\frac{7}{2}}(t) = -\frac{2432}{945}t + \frac{2944}{567}t^2 - \frac{1952}{567}t^3 + \frac{512}{567}t^4 - \frac{32}{405}t^5$$

$$\beta_4(t) = \frac{1}{270}\left(42t - 89t^2 + 64t^3 - 19t^4 + 2t^5\right)$$

Evaluating (9) at $x = x_{n+4}$ yields Hybrid Four step implicit method

$$y_{n+4} = \frac{23}{945}y_n - \frac{8}{45}y_{n+1} + \frac{86}{135}y_{n+2} - \frac{296}{135}y_{n+3} + \frac{512}{189}y_{n+\frac{7}{2}} + \frac{4}{45}h^2 f_{n+4} \qquad (10)$$

Taking the second derivative of equation (9), thereafter, evaluating the resulting continuous polynomial solution at $x = x_{n+2}, x = x_{n+3}, x = x_{n+\frac{7}{2}}$ we generate three additional methods

$$y_{n+2} = -\frac{11}{329}y_n + \frac{273}{517}y_{n+1} + \frac{307}{517}y_{n+3} - \frac{320}{3619}y_{n+\frac{7}{2}} - \frac{405}{1034}h^2 f_{n+2} - \frac{3}{1034}h^2 f_{n+4} \qquad (11)$$

$$y_{n+3} = \frac{107}{14693}y_n - \frac{141}{2099}y_{n+1} + \frac{899}{2099}y_{n+2} + \frac{9280}{14693}y_{n+\frac{7}{2}} - \frac{405}{2099}h^2 f_{n+3} + \frac{3}{2099}h^2 f_{n+4} \qquad (12)$$

$$y_{n+\frac{7}{2}} = -\frac{641}{17440}y_n + \frac{4641}{17440}y_{n+1} - \frac{16079}{17440}y_{n+2} + \frac{29519}{17440}y_{n+3} + \frac{567}{1744}h^2 f_{n+\frac{7}{2}} - \frac{231}{2180}h^2 f_{n+4} \qquad (13)$$

Since our method is design to simultaneously provide the values of $y_{n+1}, y_{n+2}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4}$ at a block point $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+\frac{7}{2}}, x_{n+4}$, the three equations

(10)-(13) are not sufficient to provide the solution for three unknown $y_{n+1}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}$.

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Thus, we obtain an additional method from (8), given by

$$1890h\delta_0 + 4397y_0 - 9996y_1 + 11123y_2 - 10388y_{n+3} + 4864y_{\frac{7}{2}} = 294h^2 f_4 \qquad (14)$$

The derivatives are obtained from (7) by imposing that $\delta(x_{n+\mu}) = \delta_{n+\mu}, \mu = \{j, v\}, j = 0,\ldots 4,$ thus, we have

$$h\delta_{n+1} = -\frac{107}{567}y_n - \frac{313}{270}y_{n+1} + \frac{181}{81}y_{n+2} - \frac{257}{162}y_{n+3} + \frac{1984}{2835}y_{n+\frac{7}{2}} - \frac{1}{27}h^2 f_{n+4}$$

$$h\delta_{n+2} = \frac{79}{1890} y_n - \frac{4}{9} y_{n+1} - \frac{137}{270} y_{n+2} + \frac{196}{135} y_{n+3} - \frac{512}{945} y_{n+\frac{7}{2}} + \frac{1}{45} h^2 f_{n+4}$$

$$h\delta_{n+3} = -\frac{17}{945} y_n + \frac{13}{90} y_{n+1} - \frac{89}{135} y_{n+2} - \frac{167}{270} y_{n+3} + \frac{1088}{945} y_{n+\frac{7}{2}} - \frac{1}{45} h^2 f_{n+4}$$

$$h\delta_{n+\frac{7}{2}} = \frac{37}{1512} y_n - \frac{133}{720} y_{n+1} + \frac{77}{108} y_{n+2} - \frac{1463}{432} y_{n+3} + \frac{2678}{945} y_{n+\frac{5}{2}} + \frac{7}{144} h^2 f_{n+4}$$

$$h\delta_{n+4} = \frac{361}{5670} y_n - \frac{62}{135} y_{n+1} + \frac{1297}{810} y_{n+2} - \frac{2006}{405} y_{n+3} + \frac{10624}{2835} y_{n+\frac{5}{2}} + \frac{49}{135} h^2 f_{n+4}$$

Error Analysis and Zero Stability

Following (Fatunla, 1991) and (Lambert, 1973) we define the local truncation error associated with the conventional form of (5) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k} \{\alpha_j y(x+jh)\} + \alpha_v y(x+vh) + h^2 \beta_v y''(x+jh) \tag{15}$$

Assuming that y(x) is sufficiently differentiable, we can expand the terms in (15) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + \ldots, + C_q h^q y^q(x) + \ldots, \tag{16}$$

where the constant coefficients $C_q, \quad q = 0,1,\ldots$ are given as follows: $C_q, \quad q = 0,1,\ldots$

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

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$$.C_q = \left[\frac{1}{q!} \sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} j^{q-2} \beta_j \right].$$

According to (Henrici, 1962), method (5) has order p if

$$C_0 = C_1 = \ldots = C_P = C_{P+1} = 0, \quad C_{P+2} \neq 0$$

In order to analyze the methods for zero stability, we normalize the HBDF schemes and write them as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det\left(RA^{(0)} - A^{(1)}\right) = R^k(R-1) \qquad (17)$$

Where $A^{(0)}$ is the identity matrix of dimension k+1, $A^{(1)}$ is the matrix of dimension k+1

Case k=4. It is easily shown that (19)-(23) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det\left(RA^{(0)} - A^{(1)}\right) = R^4(R-1)$$

Where $A^{(0)}$ is an identity matrix of dimension five and $A^{(1)}$ is a matrix of dimension five given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Following (Fatunla, 1991) the block method by combining k+1 HBDF is zero-stable, since from (17), $\rho(R) = 0$ satisfy $|R_j| \le 1 \; j = 1\ldots,k$ and for those roots with $|R_j|$ =1, the multiplicity does not exceed 2. The block method by combining k+1 HBDF is consistent since HBDF have order $P > 1$. According to (Henrici, 1962), we can safely ascertain the convergence of HBDF method.

Table 1: Order and Error Constants for the HBDF methods.

Step number	Method	Order	Error constant
4	(10)	4	$-\dfrac{49}{4050}$
	(11)	4	$-\dfrac{4123}{120}$
	(12)	4	$-\dfrac{7301}{240}$
	(13)	4	$-\dfrac{308567}{960}$
	(14)	4	$-\dfrac{15631}{60}$

We notice from the table above that our proposed methods has uniform order and small error constant which make it a suitable candidate to hand second order differential equations.

Numerical Example

The HBDF methods are implemented as simultaneous numerical integration for IVPs without requiring starting values and predictors. We proceed by explicitly obtaining initial conditions at x_{n+k}, n=0,k,….,N-k using the computed values $Y(x_{n+k}) = y_{n+k}$ and $\delta(x_{n\_k}) = \delta_{n+k}$ over sub-intervals $[x_0, x_k],\ldots,[x_{N-K}, x_N]$ which do not overlap. We give examples to illustrate the efficiency of the methods.

We report here a numerical example taken from the literature (Jator and Li, 2007 and Mohammed, 2011).

Problems 1

$y'' = 2y^3,\ y(1) = 1, y'(1) = -1, h = 0.1$

Exact Solution $y(x) = \dfrac{1}{x}$

Problems 2

$y'' + y = 0,\ y(0) = 1, y'(0) = 1, h = 0.1$

Exact Solution $y(x) = \cos(x) + \sin(x)$

Problems 3

$y'' + 1001y' + 1000y = 0,\ y(0) = 1, y'(0) = -1, h = 0.1$

Exact Solution $y(x) = e^{-x}$

Source: Jator (2007)

Problems 4

$y'' - y' = 0,\ y(0) = 0, y'(0) = -1, h = 0.1$

Exact Solution $y(x) = 1 - e^{x}$

Source: Mohammed (2011)

Table 2: Showing exact solutions and the computed results from the proposed methods for problem **1**

X	Exact Solution	K=4	Error in K=4
1	1	1	0.000000000E+00
1.1	0.9090909091	0.9089159003	1.750088000E-04
1.2	0.8333333333	0.8328649894	4.683439000E-04
1.3	0.7692307692	0.7684626550	7.681142000E-04
1.4	0.7142857143	0.7132183146	1.067399700E-03
1.5	0.6666666667	0.6655746535	1.092013200E-03
1.6	0.6250000000	0.6238664477	1.133552300E-03
1.7	0.5882352941	0.5870592595	1.176034600E-03
1.8	0.5555555556	0.5543371121	1.218443500E-03
1.9	0.5263157895	0.5250919992	1.223790300E-03
2.0	0.5	0.4987671511	1.232848900E-03

Table 3: Showing exact solutions and the computed results from the proposed methods for problem 2

X	Exact Solution	Proposed Method	Error in Proposed Method
0	1	1	0.000000000E+00
0.1	1.094837582	1.094838951	1.369000000E-06
0.2	1.178735909	1.178739619	3.710000000E-06
0.3	1.250856696	1.250862769	6.073000000E-06
0.4	1.310479336	1.310487706	8.370000000E-06
0.5	1.357008100	1.357018004	9.904000000E-06
0.6	1.389978088	1.389990556	1.246800000E-05
0.7	1.409059874	1.409074849	1.497500000E-05
0.8	1.414062800	1.414080127	1.732700000E-05
0.9	1.404936878	1.404955651	1.877300000E-05
1.0	1.38177329	1.381794415	2.112500000E-05

Table 4: Showing exact solutions and the computed results from the proposed methods for problem 3

X	Exact Solution	K=4	Error in K=4	Error in K=2 (BDF) Jator(2007)	Error in K=3(BDF) Jator(2007)
0	1	1	0.000000000E+00	0.000000000E+00	0.000000000E+00
0.1	0.9048374180	0.9048374018	1.619999990E-08	2.940180000E-04	1.111124000E-05
0.2	0.8187307531	0.8187307576	4.500000039E-09	5.571550000E-04	5.749050000E-05
0.3	0.7408182207	0.7408182184	2.300000079E-09	7.512790000E-04	9.210130000E-05
0.4	0.6703200460	0.6703200484	2.400000088E-09	9.202740000E-04	4.078390000E-05
0.5	0.6065306597	0.6065306479	1.179999998E-08	10.29514000E-04	2.530190000E-05
0.6	0.5488116361	0.5488116388	2.699999890E-09	11.26415000E-04	4.725860000E-05
0.7	0.4965853038	0.4965853031	7.000000024E-10	11.80252000E-04	1.893470000E-05
0.8	0.4493289641	0.4493289646	4.999999859E-10	12.27376000E-04	4.288120008E-05
0.9	0.4065696597	0.4065696509	8.799999951E-09	12.42326000E-04	7.966800000E-05
1.0	0.36787944	.3678794421	2.100000007E-09	12.54553000E-04	2.941190000E-05

Table 5: Showing exact solutions and the computed results from the proposed methods for problem 4

x	Exact Solution	K=4	Error in K=4	Error in K=5(BDF) Mohammed(2011)
0	0	0	0.000000000E+00	0.000000000E+00
0.1	-0.105170918	-0.1051694677	1.450300000E-06	2.198000000E-05
0.2	-0.221402758	-0.2213988084	3.949600000E-06	6.070400000E-06
0.3	-0.349858808	-0.3498522957	6.512300000E-06	1.005100000E-05
0.4	-0.491824698	-0.4918156294	9.068600000E-06	1.402530000E-05
0.5	-0.648721271	-0.6487100379	1.123310000E-05	1.799340000E-05
0.6	-0.822118800	-0.8221038376	1.496240000E-05	2.161620000E-05
0.7	-1.013752707	-1.013733921	1.878600000E-05	2.799300000E-05
0.8	-1.225540928	-1.225518326	2.260200000E-05	3.456100000E-05
0.9	-1.459603111	-1.459577282	2.582900000E-05	4.111400000E-05
1.0	-1.718281828	-1.718250438	3.139000000E-05	4.765600000E-05

Table 5: Showing exact solutions and the computed results from the proposed methods for problem 4

x	Exact Solution	K=4	Error in K=4	Error in K=5(BDF) Mohammed(2011)
0	0	0	0.000000000E+00	0.000000000E+00
0.1	-0.105170918	-0.1051694677	1.450300000E-06	2.198000000E-05
0.2	-0.221402758	-0.2213988084	3.949600000E-06	6.070400000E-06
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1.0	-1.718281828	-1.718250438	3.139000000E-05	4.765600000E-05

CONCLUSION

A Collocation technique which yields a method with Continuous Coefficients has been presented for the approximate Solution of second Order ODEs with initial conditions. Four test examples have been solved to demonstrate the efficiency of the proposed method and the results compare favorably with the exact Solution, a desirable feature of good numerical methods. Interestingly, all the discrete schemes used in the Block formulation were from a single continuous formulation (CF).

REFERENCES

Akinfenwa, O., Jator, S., Yoa, N. (2011). An eighth order Backward Differentiation Formula with Continuous Coefficients for Stiff Ordinary Differential Equations. *International Journal of Mathematical and Computer Sciences*. 7(4): 171-176.

Akinfenwa, O. A., Jator, S. N., Yao, N. M. (2013). Continuous Block Backward Differentiation Formula for Solving Stiff Ordinary Differential Equations. *Computers and Mathematics with Applications,* 65: 996–1005.

Awoyemi, D. O. (1999). A Class of Continuous Linear Multistep Method for General Second Order Initial Value Problem in Ordinary Differential Equation. *Intern. J. Comp. Math*., 72: 29–37.

Fatunla, S. O. (1991). Block Method for Second Order Differential Equation. *International Journal Computer Mathematics*, 41: 55-63.

Henrici, P. (1962). *Discrete Variable Methods in Ordinary Differential Equations*. New York: John Wiley & Sons. Pp 16-30

Ibrahim, Z. B., Othman, K. I., Suleiman, M. (2007). Implicit R-point Block Backward Differentiation Formula for Solving First-order Stiff ODEs. *Applied Mathematics and Computation*, 186: 558–565.

Jator, S. N. (2007). Solving Stiff Second Order Initial Value Problem Directly by Backward Differentiation Formulas, *Proceeding of the 2007 Int. conference on computational and Mathematical Methods in Science and Engineering, Illinois, Chicago.* Pp 223-232.

Jiaxiang, X., Cameron, I. T. (1995). Numerical Solution of DAE Systems using Block BDF Methods. *Journal of Computation and Applied Mathematics*, 62: 255-266.

Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations.* New York: John Wiley and Sons. Pp 1- 275

Majid, Z. A., Suleiman, M. B. (2007). Implementation of Four-point fully Implicit Block Method for Solving Ordinary Differential Equations. *Applied Mathematics and Computation*. 184(2): 514–522.

Milne, W. E. (1953). *Numerical Solution of Differential Equations*. New York: John Wiley & Sons. Pp 1-300

Mohammad, U., Yahaya, Y. A. (2010). Fully Implicit four Point Block Backward Differentiation Formulae for Solution of First Order Initial Value Problems *Leonardo. Journal of Sciences,* 16: 21-30.

Mohammed, U. (2011). A Class of Implicit Five Step Block Method for General Second Order Ordinary Differential Equations. *Journal of Nigerian Mathematical Society*, 30: 25-39

Onumanyi, P., Sirisena, U. W. and Jator, S. N. (1999). Continous Finite Difference Approximation for Solving Differential Equations, *Inter. J. Comp Maths*, 72(1): 15-27.

Onumanyi, P., Awoyemi, D. O., Jator, S. N., Sirisena, U. W. (1994). New Linear Multistep with Continuous Coefficients for First Order Initial Value Problems. *J. Nig. Math. Soc*. 13:37-51.

Rosser, J. B. (1967). A Runge-Kutta for all Seasons. *Siam Rev*., 9(3): 417–452.

Semenov, D. E., Mohammed U., Semenov, M. E. (2013). Continuous Multistep Methods for Solving First Order Ordinary Differential Equations. *Proceeding of Third Postgraduate Consortium International Workshop Innovations in Information and Communication Science and Technology IICST 2013*. Tomsk: Russia. Pp 165-170.

Yahaya, Y. A., Mohammed, U. (2009). A Reformulation of Implicit Five Step Backward Differentiation Formulae in Continuous Form for Solution of First Order Initial Value Problem. *Journal of General Studies*, 1(2): 134-144.

Yahaya, Y. A., Mohammed, U. (2010). Fully Implicit Three Point Backward Differentiation Formulae for Solution of First Order Initial Value Problems. *International Journal of Numerical Mathematics*, 5(3): 384-398.

Yusuph, Y., Onumayi, P. (2002). New Multiple FDMS through Multi step Collocation for $y'' = f(x, y)$. *Abacus*, 29(2): 92-100.