A01: Stability Analysis of Heat Equation using Finite Difference Method

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Abstract
In this research work, the Taylor's series expansion of two variables was used to develop a scheme of the second order heat equation using the finite difference method. The stability analysis carefully analysed proved the scheme.

Keywords: Finite difference method, Heat equation, Stability, Taylor's series expansion

1.0 Introduction

Partial Differential Equations (PDE) offer a convenient tool for modeling heat equation mathematically (Shehu, *et al.*, 2014). In Mathematics, a PDE is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand or used to create a computer model. A PDE for the function $u(x_1, x_2, \dots, x_n)$ is modeled by an equation of the form

$$f\left(x_1, x_2, \dots, x_n, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_2} \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}; \dots\right) = 0 \qquad (1)$$

Consider a Partial Differential Equation of one dimensional heat equation:

$$u_t = \tau u_{xx}; \qquad 0 \le x \le L, \qquad t > 0 \qquad (2)$$

where $u = u(x, t)$, is the temperature at the time (t) and at position (x) along a thin rod, and τ is a positive constant of the thermal diffusivity, the symbol u_t signifies the partial derivatives of the function u with respect to the time variable (t). Similarly, u_{xx} is the second partial derivative with respect to space/position variable (x).

Equation (2) describes the variation of temperature in a given region over time. It can be used to express the heat flow with diffusion τu_{xx} along the rod, where the coefficient τ is the thermal diffusivity and L is the length of the rod.

The solution of a partial differential equation thus provides a solution to the physical problem it represents. However, it offers a much more challenging problem compared to the solution of ordinary differential equations (Ndanusa, 2002).

2. Literature Review

In Mathematics, the finite difference methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives. The growth in computing power has transformed the use of realistic mathematical models in science and engineering and thereby intelligent numerical analysis is required to implement the details of the world model. Numerical methods have become a vital tool used for solving ordinary differential equations (ODEs) because of the complex nature of the problem which cannot be solved by analytical method. Over the years, there had been discovery and derivation of methods in respect to numerical solution of differential equations.

In Richmond (2006), the author develop the analytical solutions of non-trivial examples of a well-known class of initial-boundary value problems which, by the choice of parameters, can be reduced to regular or singular Sturm-Liouville problems.

In (Sweilam *et al.*, 2012) the authors present the Crank-Nicolson-Finite Difference Method to solve the linear time fractional diffusion equation. They claimed that the Crank-Nicolson-Finite Difference Method is applicable, simple and efficient for solving problems.

The author (Juan, 2006) studied the Spectral methods for solving the one dimensional parabolic heat equation. In (Hikmet *et. al.,* 2010), the authors claimed that the Adomian Decomposition Method (ADM) is more accurate. In (Subir *et. al.*, 2012), the optimal

Homotopy Analysis Method (HAM) is used to obtain approximate analytic solutions of the time-fractional nonlinear diffusion equation in the presence if and external force and an absorbent term. The fractional derivatives are considered in the Caputo sense to avoid nonzero derivative of constants. Unlike usual HAM, this method contains at the most three convergence control parameters which determine the fast convergence of the solution through different choices of convergence control parameters. Effects of proper choice of parameters on the convergence of the approximate series solution by minimizing the averaged residual error for different particular cases are depicted through tables and graphs.

According to Hirt (2009), numerical solution schemes are often referred to as being explicit or implicit. When a direct computation of the dependent variables can be made in terms of known quantities, the computation is said to be explicit. When the dependent variables are defined but coupled sets of equations and either a matrix or an iterative technique is needed to obtain the solution, the numerical method is said to be implicit.

3. Methodology

Derivation of Finite Difference Method from Taylor's Polynomial

First, assuming the function whose derivatives are to be approximated is properly-behaved, by Taylor's theorem we can create a Taylor's series expansion

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \cdots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n(x) \qquad (3)$$

Where $n!$ denotes the factorial of n and R_n(x) is a remainder term, denoting the difference between the Taylor polynomial of degree n and the original function. We will derive an approximation for the first derivative of the function "f" by first truncating the Taylor polynomial:

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + R_1(x) \tag{4}$$

Dividing across byΔx:

$$\frac{f(x+\Delta x)}{\Delta x} = \frac{f(x)}{\Delta x} + \frac{f^{'}(x)}{\Delta x}\Delta x + \frac{R_1(x)}{\Delta x} \tag{5}$$

Solving for $f^{'}(x)$:

$$f^{'}(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{R_1(x)}{\Delta x} \tag{6}$$

Assuming that $R_1(x)$ is sufficiently small, the approximation of the first derivation of $f^{'}(x)$ is:

$$f^{'}(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{7}$$

The Finite Difference Method (FDM)

Finite difference methods (FDMs) are numerical methods for approximating the solution to differential equations using finite difference equations to approximate derivatives (Wazwaz, 2002*)*. FDMs are thus discretization methods. The reduction of the differential equations makes the problem of finding the solution to a given ODE ideally suited to modern computers, hence the widespread use of FDMs in modern numerical analysis. FDMs are the dominant approach to the numerical solutions of partial differential equations.

Forward Difference in Time*:*

$$u_t \cong \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{\partial u(x,t)}{\partial t} + O(\Delta t) \tag{8}$$

Backward Difference in Time

$$u_t \cong \frac{u(x,t) - u(x,t-\Delta x)}{\Delta t} = \frac{\partial u(x,t)}{\partial t} - \Delta t \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial u(x,t)}{\partial t} - O(\Delta t) \quad (9)$$

Forward Difference in Space

$$u_{xx} \cong 2\left[\frac{u(x+\Delta x,t) - u(x,t) - \Delta x u_x}{\Delta x^2}\right] = \frac{\partial^2 u(x,t)}{\partial x^2} + O(\Delta x) \quad (10)$$

Backward Difference in Space

$$u_{xx} \cong 2\left[\frac{u(x-\Delta x,t) - u(x,t) + \Delta x u_x}{\Delta x^2}\right] = \frac{\partial^2 u(x,t)}{\partial x^2} - O(\Delta x) \quad (11)$$

Derived Solution of Heat Equation Using Finite Difference Method

Using the FDM, the heat equation is given thus:

$$u(x,t+\Delta t) = u(x,t) + \tau\left(\frac{\Delta t}{\Delta x^2}\right)\big(u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\big) \quad (12)$$

And upon Discretizing equation (12), the heat equation turns:

$$u(x_n,t_{k+1}) = u(x_n,t_k) + \tau\left(\frac{\Delta t}{\Delta x^2}\right)\big(u(x_{n+1},t_k) - 2u(x_n,t_k) + u(x_{n-1},t_k)\big) \quad (13)$$

The Stability Analysis

The stability solution to the discretized heat equation (13) is:

$$U_n^k = \phi_k e^{in\theta\Delta x} \quad (14)$$

Where ϕ_k is a function in time variable and $e^{in\theta\Delta x}$is a function in space variable.

Thus;

$$U_n^{k+1} = \phi_{k+1} e^{in\theta\Delta x}$$

(15)

$$U_{n+1}^{k} = \phi_k e^{in\theta\Delta x}.e^{i\theta\Delta x}$$

(16)

$$U_{n-1}^{k} = \phi_k e^{in\theta\Delta x}.e^{-i\theta\Delta x}$$

(17)

Substituting equations (14), (15), (16) and (17) into (13) gives:

$$\phi_{k+1} e^{in\theta\Delta x} = \phi_k e^{in\theta\Delta x} + \tau\left(\frac{\Delta t}{\Delta x^2}\right)\left(\phi_k e^{in\theta\Delta x}.e^{i\theta\Delta x} - 2\phi_k e^{in\theta\Delta x} + \phi_k e^{in\theta\Delta x}.e^{-i\theta\Delta x}\right) \quad (18)$$

$$\phi_{k+1} e^{in\theta\Delta x} = \phi_k e^{in\theta\Delta x} + \tau\left(\frac{\Delta t}{\Delta x^2}\right)\left(e^{i\theta\Delta x} - 2 + e^{-i\theta\Delta x}\right)\phi_k e^{in\theta\Delta x} \quad (19)$$

For stability purpose τ(constant) =1

$$(\phi_{k+1} - \phi_k) e^{in\theta\Delta x} = \left(\frac{\Delta t}{\Delta x^2}\right)\left(e^{i\theta\Delta x} - 2 + e^{-i\theta\Delta x}\right)\phi_k e^{in\theta\Delta x} \quad (20)$$

$$(\phi_{k+1} - \phi_k) = \left(\frac{\Delta t}{\Delta x^2}\right)\left(e^{i\theta\Delta x} - 2 + e^{-i\theta\Delta x}\right)\phi_k \quad (21)$$

$$\phi_{k+1} = \phi_k + \left(\frac{\Delta t}{\Delta x^2}\right)\left(e^{i\theta\Delta x} - 2 + e^{-i\theta\Delta x}\right)\phi_k \quad (22)$$

$$\phi_{k+1} = \left[1 + \left(\frac{\Delta t}{\Delta x^2}\right)\left(e^{i\theta\Delta x} - 2 + e^{-i\theta\Delta x}\right)\right]\phi_k \quad (23)$$

Using the Trigonometry Identity:

$$e^{\pm i\theta\Delta x} = \cos(\theta\Delta x) \pm i\sin(\theta\Delta x) \quad (24)$$

Substituting (24) into (23):

$$\phi_{k+1} = \left[1 + \left(\frac{\Delta t}{\Delta x^2}\right)(2\cos(\theta\Delta x) - 2)\right]\phi_k \quad (25)$$

But using the trigonometry identity

$$\cos(\theta\Delta x) - 1 = -2sin^2\left(\frac{\theta\Delta x}{2}\right) \tag{26}$$

Equation (25) becomes

$$\phi_{k+1} = \left[1 - 4\frac{\Delta t}{\Delta x^2}sin^2\left(\frac{\theta\Delta x}{2}\right)\right]\phi_k \tag{27}$$

For stability, it is required that:

$$|\phi_{k+1}| \leq |\phi_k|$$

So that;

$$\frac{|\phi_{k+1}|}{|\phi_k|} \leq |1|$$

(28)

$i.e$

$$|1| \leq \left|1 - 4\frac{\Delta t}{\Delta x^2}sin^2\left(\frac{\theta\Delta x}{2}\right)\right| \tag{29}$$

$$1 \leq 1 - 4\frac{\Delta t}{\Delta x^2}sin^2\left(\frac{\theta\Delta x}{2}\right) \leq -1$$

(30)

$$0 \leq -4\frac{\Delta t}{\Delta x^2}sin^2\left(\frac{\theta\Delta x}{2}\right) \leq -2 \tag{31}$$

$$\text{Since: } \sin\left(\frac{\theta\Delta x}{2}\right) \leq 1 \tag{32}$$

This condition is satisfied for all θ provided:

$$-4\frac{\Delta t}{\Delta x^2} \leq -2 \tag{33}$$

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \tag{34}$$

$\Delta t \leq \frac{\Delta x^2}{2}$; (Stability Proved)

4. Results and Discussion

In this thesis, the finite difference method was introduced for the solution of a dimensional second order heat problem with boundary and initial value problem. Several methods had been developed and used to solve diverse form of heat problem with accuracy better to the exact solution. The setback with those methods was inability to meet the higher continuity requirement for the approximate functions. One advantage of the finite difference method is that it allows the use of higher degree basis functions which meet the continuity requirements of heat problems.

5. Conclusion

One advantage of the finite difference method is that it allows the use of higher degree basis functions which meet the continuity requirements of heat problems. Therefore, it concludes that, the finite difference method gives a closer approximation to the solution of heat problem while compared with its exact solution and also, the computational technique is easier and direct in manipulating as to finite volume method and finite element method.

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