Optimal Analysis and Application of Discrete Games in Decision Making Environment (A Computer Program Approach)

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Abstract

The study of game theory has gone on for ages. From most of the researches carried out, it is worth saying that the application of game theory in decision making is still in a state of flux, partly because of the difficulties in formulating the "rules of the game". This paper lays emphasis on the analysis of game theory with the use of a computer program designed in Visual Basic for solving problems posed in the form of discrete games using the method of MINMAX / MAXMIN principle.

Introduction

The mathematical theory of games was invented by John von Neumann and Oskar Morgenstern (1994). Limitations in their mathematical framework initially made the theory applicable only under special and limited conditions. This situation has gradually changed, over the past six decades, as the framework was deepened and generalized. Refinements are still being made, however, since at least the late 1970s it has been possible to say with confidence that game theory is the most important and useful tool in the analyst's kit whenever she confronts situations in which one agent's rational decision-making depends on her expectations about what one or more other agents will do, and theirs similarly depend on expectations about her. Despite the fact that game theory has been rendered mathematically and logically systematic only recently, however, game-theoretic insights can be found among philosophers and political commentators going back to ancient times. The various alternatives available for allocation of the resources are competitive in the sense that out of the various choices, only one will correspond to the optimum result. Luce Raifla; (1957) write "in a game theory problem, a decision maker has to select such strategies which will enable him to gain as much as possible or to lose as little as possible by taking into consideration the possible strategies of his competitors".

Historical Development Of Optimization Theory

The existence of optimization methods can be traced back to the days of Newton, Lagrange and Cauchy. One of the first recorded instances of optimization theory concerns the finding of a geometric curve of given length which will together with a straight line, enclose the largest possible area. Archimedes conjectured correctly that the optimal curve is a semi-circle. Some of the early results are in the form of principles which attempt to describe and explain natural phenomena.

Optimization as a tool in Decision making

One of the basic tools of solving optimization problems is the linear programming method and this technique can be used to find the best uses of an organisation's resources. It is used in Operational Research (OR) to solve particular types of problem known as allocation, transportation or assignment problems.

All organisations have to make decisions about how to allocate their resources, and there is no organisation that operates permanently with unlimited resources.

Consequently, management must continually allocate scarce resources to achieve an organisation's goals.

MINI MAX (MAXMIN) Principle

In any game problem, each player is interested in determining his own "optimal" strategy. However because of the conflicting nature of the problem, and because of the lack of information regarding the specific strategies selected by the other player(s), optimality has to be based on a conservative principle.

Definition (Minimax(Maxmin) Principle)
A situation where each player selects his strategy which guarantees a pay-off that can never be worsened by the selection of his opponents is referred to as the minimax(maxmin) principle i.e.
maxmin value $\leq$ value of the game $\leq$ minimax value -

Discrete Game Problems:

Discrete game problems are often represented in matrices form, which take the form of either n x n or m x n matrix and it involves two players with selection of strategies.

Definition (Saddle point)
The element aij of the pay-off matirx is called a *saddle point* if it is the minimum among the ith row and maximum among the jth column elements.

Example
The saddle point could be illustrated as below:
Let us consider a game for which the pay-off matrix is as shown in figure 5.0.1

Strategies for player B

		I	II		
Strategies of Player A	I	a	b	a	maxmin solution
	II	c	d	[c]	
		[c]	d		
		minimax solution			

Figure 5.0.1 The pay-off matrix

From figure 5.0.1 The saddle point element = c

The value of the game is equal to the saddle point, and the optimal strategies for the two players are given by the row that contains the saddle point for player A, and the column that contains the saddle point for player B.

Definition (Mixed Strategies)
The element aij of the pay-off matrix is of mixed a strategies if they do not have saddle point i.e the optimal pure strategy of the game can not be found readily.

Generalised method of solution for matrix game problems with mixed strategies

If the probability of player A playing I is x
Then the probability of playing II is 1-x
Similarly, if the probability of player B playing I is y
Then the probability of B playing II is 1-y

The expected value of A if B plays I through out is

$ax + c(1-x) = bx + d(1-x)$

Then

$ax - cx - bx + dx = d - c$

$$x = \frac{d - c}{a + d - (b + c)}$$

$$1 - x = \frac{a - b}{a + d - (b + c)}$$

Similarly,

$ay + b(1-y) = cy + d(1-y)$

$ay - by - cy + dy = d - b$

$$y = \frac{d - b}{a + d - (b + c)}$$

$$1 - y = \frac{a - c}{a + d - (b + c)}$$

The value of the game is

$ax + c(1-x)$

$$= \frac{ad - ac}{a + d - (b + c)} + \frac{ac - cb}{a+d - (b+c)}$$

$$= \frac{ad - bc}{a + d - (b + c)}$$

Applications

1. (Game problem with a saddle point)

Let us consider a 2x2 matrix game with pay-off matrix as shown in figure 5.0.2

Player C

Player R		C1	C2
	R1	$G_{11}=1$	$G_{12}=6$
	R2	$G_{21}=3$	$G_{22}=7$

Figure 5.0.2 A (2x2 matrix) discrete game where the order of play makes no difference.

From the above matrix, player C is maximizing, while player R is minimizing. For each pair of strategies let the corresponding pay-off be P=Gij. Player C attempts to maximize the pay-off while player R attempts to minimize it. Player C will select the column with the largest minimum while R will select the row with the smallest maximum.

The optimal choices for the game of figure 5.0.2 are R_1 and C_2 with the pay-off 6 regardless of who plays first.

i.e. we have

$$\max_{C_j} \min_{R_i} G_{ij} = 6 = \min_{R_i} \max_{C_j} G_{ij}$$

(C plays first)   (R plays first)

3.   (Game problem with mixed strategy)

Graphical solution of game problem with mixed strategy

Let us consider a matrix game with pay off as shown below:

		Player C	
		C_1	C_2
Player R	R1	$a_{11}=11$	$a_{12}=7$
	R2	$a_{21}=5$	$a_{22}=9$

Figure 5.0.2 A (2x2 matrix) discrete game with mixed strategies.

Thus if player C plays a fixed choice while R uses a random choice, the expected pay off for various probability mixes of R_1 and R_2 is as shown in figure 5.0.3 (a)

Figure 5.0.3(a) illustration of the minimax solution of figure 5.0.2

In figure 5.0.3(a) for player R to realize maximal expected pay-off R must play the probability mix half of the time R_1 and of the time R_2.

Similarly if player R plays a fixed choice while C uses a random choice, the expected pay-off for various probability mixes of C_1 and C_2 is as shown in figure.5.0.3(b)

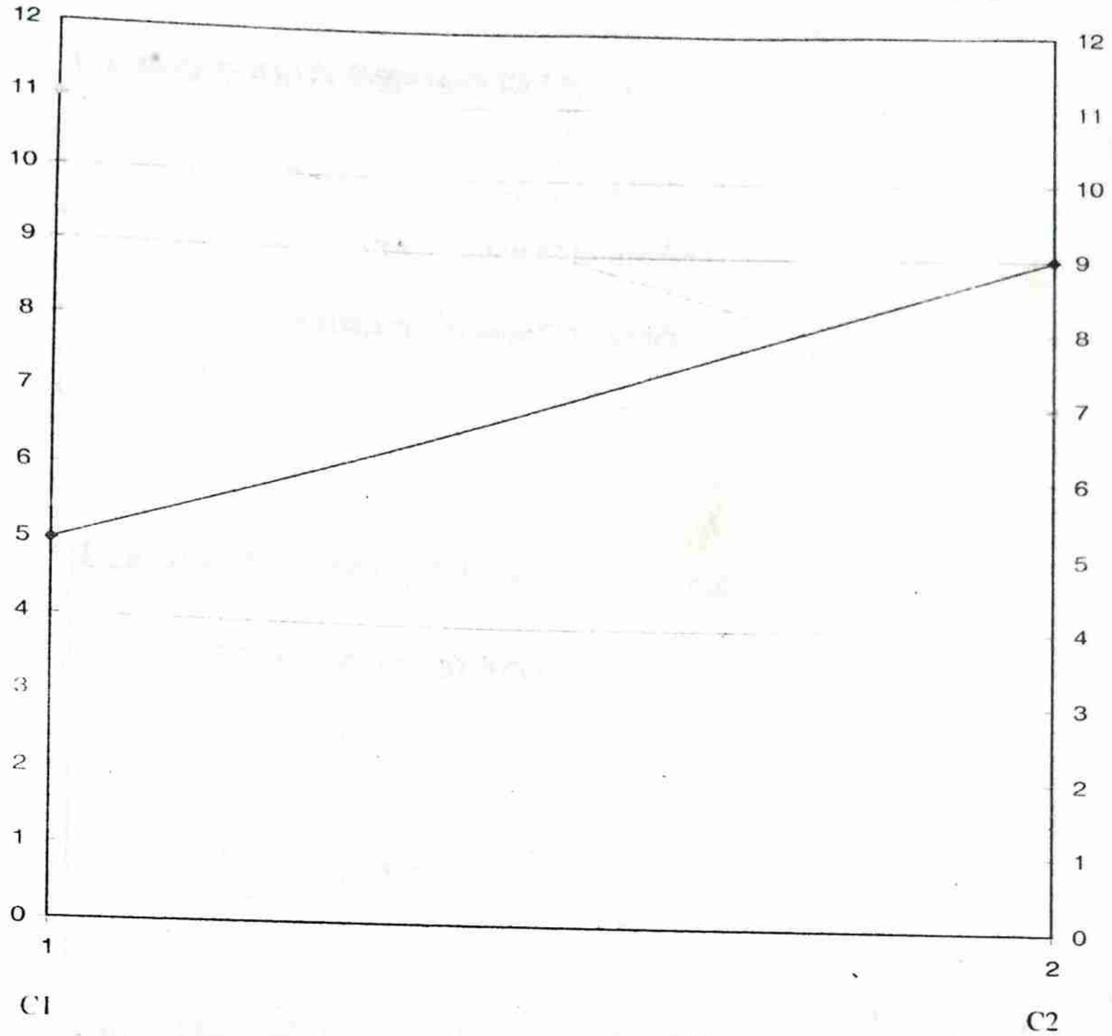


Figure 5.0.3(b) illustration of the minimax solution of figure 5.0.2

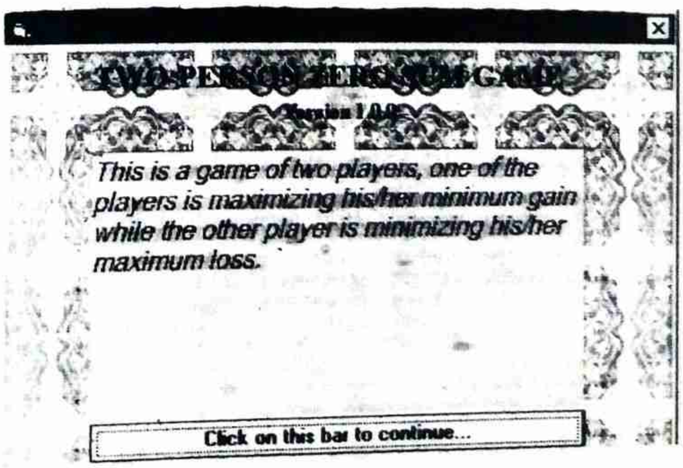
Similarly in figure 5.0.3(b) player C must play the probability mix 1/4 of the time C_1 and 3/4 of the time C_2 to realize maximal expected pay-off.

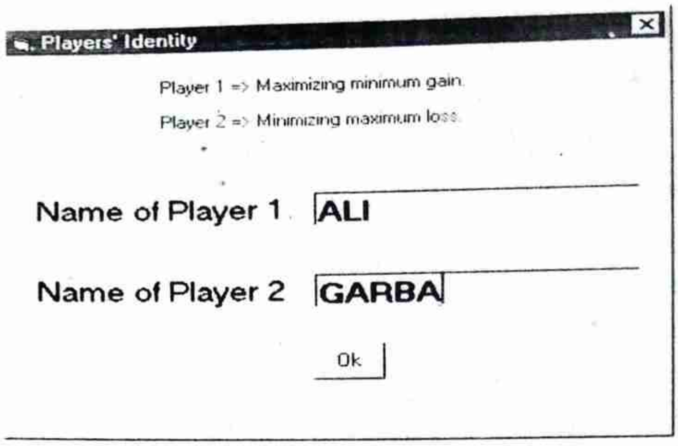
We then have that;

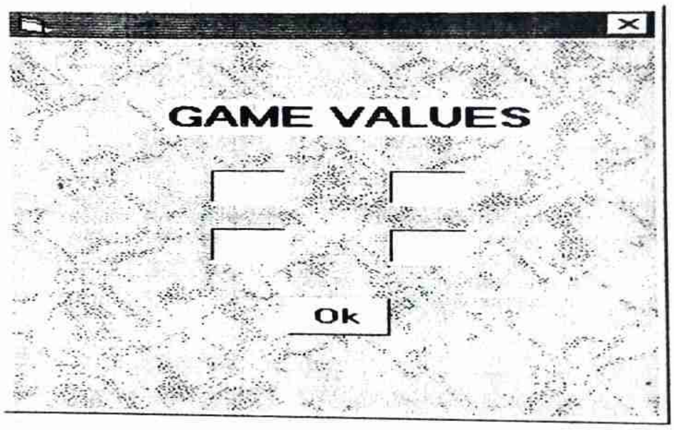
$$E[\underset{p\ q}{\text{minimax}}\ G_{ij}] = 8 = E[\underset{p\ q}{\text{maxmin}}\ G_{ij}]$$

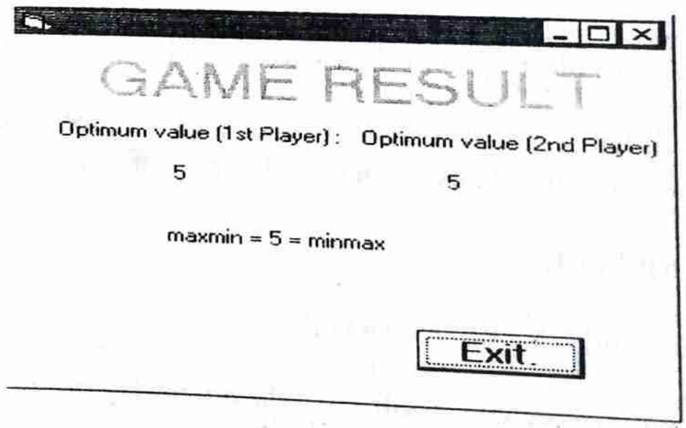
where E is the expected pay-off, p, q the probability mixes.

Computational Results (matrix game problem with saddle point)

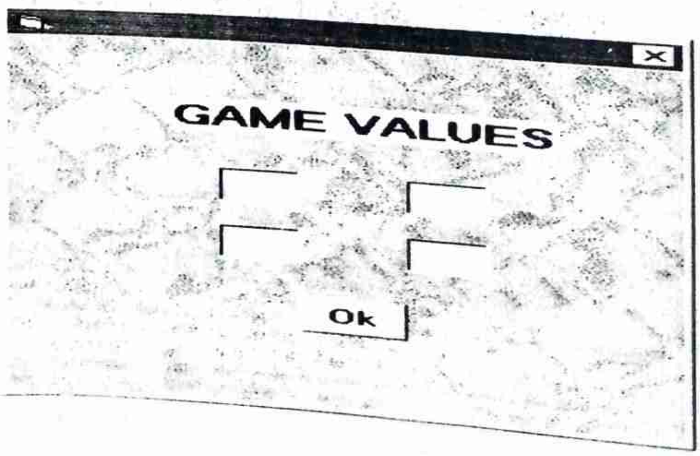


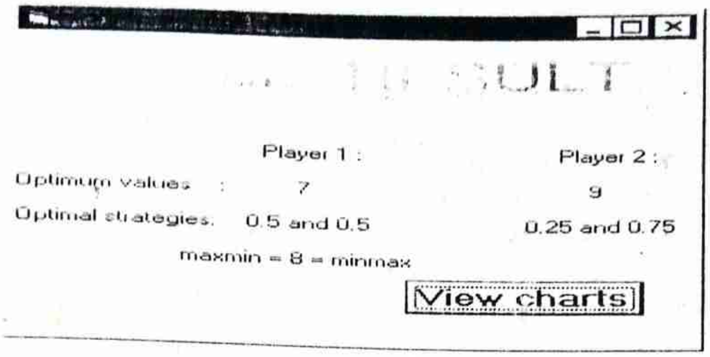


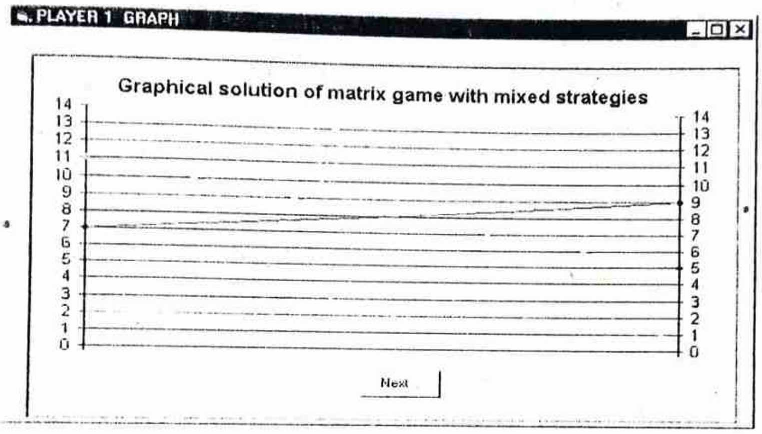


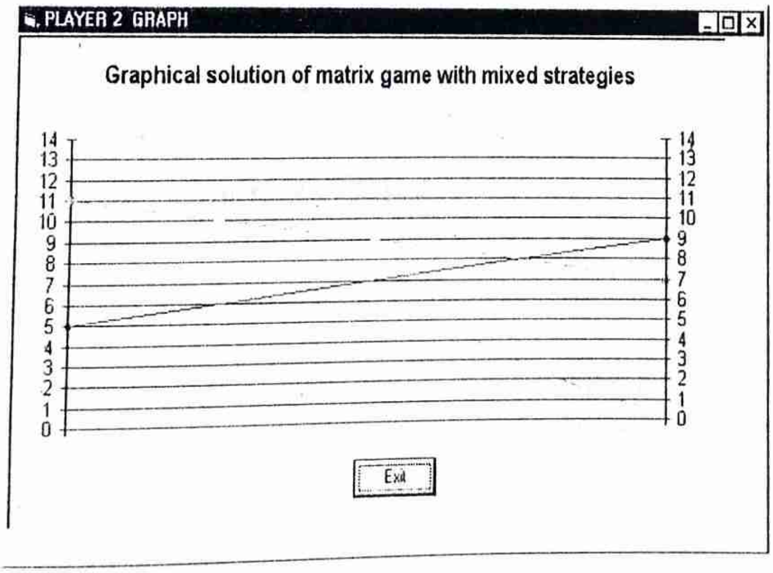


Computational Results (matrix game problem with mixed strategies)









Conclusion

The design of visual basic program for the solution of game theory makes it easy to solve game problems formulated in the form of matrixes (saddle point/mixed strategies), and this set the basis for comparison of results. We look forward to an enhancement of this program to be able to handle an m x n matrix game problem.

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