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Mathematical Analysis of Groundwater Flow in a Layered Coordinate System

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Abstract

In this paper, we formulated mathematical equations for groundwater flow in a layered coordinate system using Darcy's Equation, in a two dimensional (x, y) system flow. The transformation from Neutral (Area) coordinate λ to Cartesian coordinates (x, y) within a triangular element was considered and values for α, β and γ were computed. It was observed that groundwater flow velocity v_x and v_y is governed by the values of λ_i , λ_j and λ_k within the i, j and k nodes.

Key words: Darcy, Aquifer, Hydraulic Head, Isotropy, Anisotropy, Homogenous, Heterogeneous.

1.0 Introduction

Groundwater is water located beneath the ground surface in soilpore spaces and in the fractures of litho-logic formations[1]. A unit of rock or an unconsolidated deposit is called an aquifer when it can yield a usable quantity of water[2]. The depth at which soil pore spaces or fractures and voids in rock become completely saturated with water is called the water table. Groundwater is a vital resource in our environment. It replenishes our streams, rivers, Habitats, and also provides fresh water for irrigation, industry and communities[3].

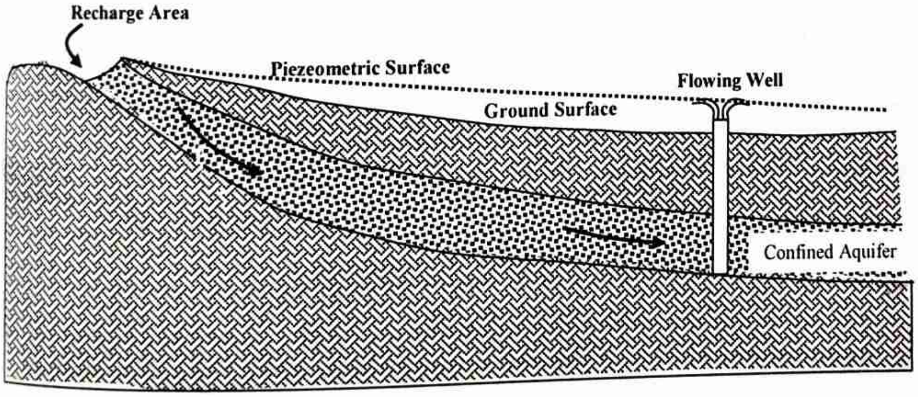

Figure 1.1: An Aquifer

According to Ashok and Bithin [4], in a three-dimensional porous medium with an isotropic hydraulic conductivity, specific discharge is given as;

$$q=q_1 i+q_2 j+q_3 k \tag{1.1}$$

where i, j, and k are unit vectors in the x, y, and z directions respectively, and q_1, q_2, and q_3 are specific discharge components in the x, y, and z directions, respectively.
Specific discharge (q) is given by Kumar [5], as shown in (1.2);

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$$q_i = -k_{ij}\frac{\partial h}{\partial x_j} \qquad (1.2)$$

Using k_{xx}, k_{yy}, and k_{zz} to represent the main diagonal terms of the Hydraulic Conductivity Tensor, equation (1.2) can be expanded to get [6]

$$q = -k_{xx}\left(\frac{\partial h}{\partial x}\right)\hat{i} - k_{yy}\left(\frac{\partial h}{\partial y}\right)\hat{j} - k_{zz}\left(\frac{\partial h}{\partial z}\right)\hat{k} \qquad (1.3)$$

Figures (1.2) and (1.3a-b)illustrates a natural hydraulic anisotropy in water – lain sediments and isotropy, anisotropy, homogenous and heterogeneous media respectively [7].

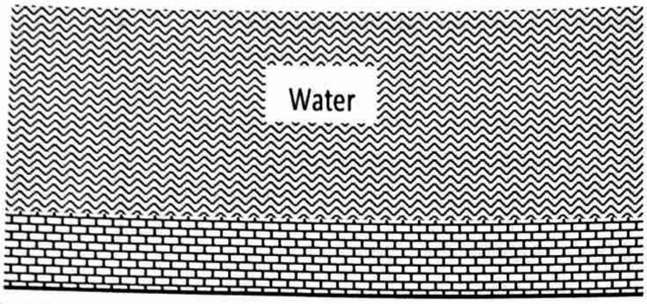

Figure 1.2: Natural Hydraulic Anisotropy

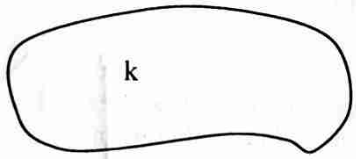

Figure 1.3a: Isotropic and Homogenous Medium

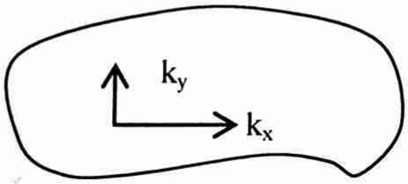

Figure 1.3b: Anistropic and Homogeneous Medium

2.0 Model Formulation

We employed the method of space $(x\ y)$ discretization of the groundwater flow, where the computational domain is divided into finite element mesh connected at nodes i, j, k

Our Governing equation is the Darcy's equation given by [2]:

$$Q_i = -k_{ij}A\left(\frac{\partial h}{\partial x_j}\right) \qquad (1.4)$$

Where;

Q	=	Flow rate
A	=	Flow Cross Section Area
h	=	Piezometric Head
k	=	Hydraulic Conductivity.

According to Durbin [1],

$$Q = Av \tag{1.5}$$

Substituting equation (1.5) into (1.4) we have,

$$Av = -k_{ij} A\left(\frac{\partial h}{\partial x_j}\right) \tag{1.6}$$

Where,
v = Flow Velocity,
Equation (1.6) gives;

$$v = -k_{ij}\left(\frac{\partial h}{\partial x_j}\right) \tag{1.7}$$

We now express equation (1.7) in a layered coordinatesystem in the directions ξ and η

$$v_\xi = -k_\xi \frac{\partial h}{\partial \xi} \tag{1.8}$$

and

$$v_\eta = -k_\eta \frac{\partial h}{\partial \eta} \tag{1.9}$$

Where;
v = Flow Velocity,
h = Piezometric Head
k = Hydraulic Conductivity.

In matrix notation, equations (1.8) and (1.9) can be expressed as:

$$\begin{bmatrix} v_\xi \\ v_\eta \end{bmatrix} = -\begin{bmatrix} k_\xi & 0 \\ 0 & k_\eta \end{bmatrix}\begin{bmatrix} \frac{\partial h}{\partial \xi} \\ \frac{\partial h}{\partial \eta} \end{bmatrix} \tag{1.10}$$

We need to perform computation within a global *x-y*coordinate system, (that is, coordinate transformation made from layer coordinates to global one); where θ is the angle between horizontal axes of the two coordinate systems (that is, axes*x* and ξ).

Equation (1.8) and (1.9) expressed in the Global $x - y$ coordinate system takes the following form:

$$v_x = -k_{xx}\frac{\partial h}{\partial x} - k_{xy}\frac{\partial h}{\partial y} \tag{1.11}$$

$$v_y = -k_{yx}\frac{\partial h}{\partial x} - k_{yy}\frac{\partial h}{\partial y} \tag{1.12}$$

where,
v_x = flow velocity in the horizontal direction
v_y = flow velocity in the vertical direction

3.0 Conclusion

Equations (1.11) and (1.12)are the flow velocities in-terms of Recharge and Discharge Directions of the Groundwater Flow at different points of the hydraulic head values.The flow velocities given by (1.11) and (1.12) shows the direction vectors ofthe groundwater flow, and this validate the fact that groundwater flows from high hydraulic heads towards the low heads.

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