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A Linear K-Step Method for Solving Ordinary Differential Equations

Ndanusa, A and Adeboye, K.R
Department of Mathematics/Computer Science,
Federal University of Technology, Minna, Nigeria

Abstract
The field of differential equations, no doubt, plays a vital role in the applications of mathematics to scientific and engineering problems. A considerable number of the important physical laws of the universe, more often than not, is expressed in differential equation form. Therefore, the solution of a differential equation implies the solution of the physical problem it represents. Although a multitude of families of approximate numerical methods for solving differential equations exists, for acceptability a numerical method must exhibit convergence; more so, for it to be effective, it must converge rapidly. In this paper, we construct a numerical method of optimal order from the family of linear k-step methods. Numerical tests verifying the efficiency and accuracy of the method are also presented.
Keywords: Linear k-step method, linear multistep method, Ordinary differential equation, Taylor series, Convergence, initial value problem

Introduction

Given a first order differential equation

$$y' = f(x,y) \qquad (1)$$

We say that the differential equation (1), together with an initial condition

$$y(x_0) = y_0 \qquad (2)$$

form an *initial value problem (ivp).*

Most problems in practical applications are often modeled, not with a single differential equation, but with a system of n simultaneous first-order differential equations in n dependent variables $y_1, y_2, \dots, y_n$. This system, which may be written as,

$$\left.\begin{aligned} y_1' &= f_1(x_1, y_1, y_2, \dots, y_n) \\ y_2' &= f_2(x_1, y_1, y_2, \dots, y_n) \\ &\cdot \\ &\cdot \\ &\cdot \\ y_n' &= f_1(x_1, y_1, y_2, \dots, y_n) \end{aligned}\right\} \qquad (3)$$

together with a corresponding set of initial conditions,

$$y_1(x_0) = y_{10}, y_2(x_0) = y_{20}, \dots, y_n(x_0) = y_{n0} \qquad (4)$$

is called an *initial value problem for a first-order system.*
We may write the initial value problem (3) and (4) in the form

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}), \;\; \boldsymbol{y}(x_0) = \boldsymbol{y}_0 \qquad (5)$$

where,

$$\boldsymbol{y} = [y_1, y_2, \dots, y_n]^T, \qquad \boldsymbol{f} = [f_1, f_2, \dots, f_n]^T = \boldsymbol{f}(x, \boldsymbol{y}),$$
$$\boldsymbol{y}_0 = [y_{10}, y_{20}, \dots, y_{n0}]^T$$

An essential property of the majority of computational methods for the solution of (1) or (5) is that of discretization; that is, we seek an approximate solution, not on the continuous interval, $a \le x \le b$, but on the discrete point set $\{x_n | n = 0, 1, \dots, (b-a)/h\}$.
Let y_n be an approximation to the theoretical solution at x_n, that is, to $y(x_n)$, and let $f_n \equiv f(x_n, y_n)$. Lambert (1973) defined a *linear multistep method* or a *linear k-step*

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method to be a computational method for determining the sequence y_n which takes the form of a linear relationship between y_{n+j}, f_{n+j}, $j = 0, 1, \dots, k$. He further defined the general linear k −step method as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (6)$$

where α_j and β_j are constants; we assume that $\alpha_k \neq 0$ and that not both α_0 and β_0 are zeros.

Methodology

In deriving an optimal linear multistep method with step number 4 (i.e., a 4-step method of order $k + 2$, where $k = 4$ in this case), Lambert (1973) used the method of Taylor expansions. Ndanusa (2007) used the same method to derive a 6-step linear multistep method (lmm) of order 8. A further analysis on this earlier work resulted in the derivation of another 6-step lmm by Ndanusa and Adeboye (2008).

In this paper, we derive yet another 6-step lmm of optimal order. Considering the linear difference operator, ℓ, defined by

$$\ell\,[y(t); h] = \sum_{j=0}^{k} [\alpha_j y(t + jh) - h\beta_j y(t + jh)] \qquad (7)$$

Suppose we choose to expand $y(t + h)$ and $y'(t + jh)$ about $t + rh$; where r need not necessarily be an integer. We obtain

$$\ell\left[y(t); h\right] = D_0 y(t + rh) + D_1 h y'(t + rh) + \dots + D_q h^q y^q(t + rh) \qquad (8)$$

The formulae for the constants D_q expressed in terms of α_j, β_j are

$$\left.\begin{aligned}
D_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \dots \alpha_k \\
D_1 &= -r\alpha_0 + (1 - r)\alpha_1 + (2 - r)\alpha_2 + \dots + (k - r)\alpha_k \\
&\quad - (\beta_0 + \beta_1 + \beta_2 + \dots \beta_k) \\
&\ \ \vdots \\
D_q &= \tfrac{1}{q!}\left[(-r)^q \alpha_0 + (1 - r)^q \alpha_1 + (2 - r)^q \alpha_2 + \dots . (k - r)^q \alpha_k\right] \\
&\quad - \tfrac{1}{(q-1)!}\left[(-r)^{q-1} \beta_0 + (1 - r)^{q-1} \beta_1 + \dots + (k - r)^{q-1} \beta_k\right], \\
&\quad q = 2, 3, \dots
\end{aligned}\right\} \qquad (9)$$

In order to derive the method of our choice, i.e., a 6-step method of order 8, we require all the roots of the first characteristic polynomial $\rho(\xi)$ to lie on the unit circle. Since $\rho(\xi)$ is a polynomial of degree 6, consistency demands that it has one real root at +1 and another real root at -1. The four remaining roots must be complex. Hence we have

$$\xi_1 = +1, \quad \xi_2 = -1, \quad \xi_3 = e^{i\theta_1}, \quad \xi_4 = e^{-i\theta_1}, \quad \xi_5 = e^{i\theta_2}, \quad \xi_6 = e^{-i\theta_2},$$

Hence

$$\left.\begin{aligned}
&\alpha_6 = +1, \quad \alpha_5 = -2(a + b), \quad \alpha_4 = (4ab + 1), \quad \alpha_3 = 0, \quad \alpha_2 = -(4ab + 1), \\
&\alpha_1 = 2(a + b), \quad \alpha_0 = -1
\end{aligned}\right\} \qquad (10)$$

We now state the order requirement in terms of the coefficients D_q.

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$$

$$D_1 = -r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + (3-r)\alpha_3 + (4-r)\alpha_4 + (5-r)\alpha_5 + (6-r)\alpha_6 - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)$$

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$$D_8 = \tfrac{1}{8!}\left[(-r)^8\alpha_0 + (1-r)^8\alpha_1 + (2-r)^8\alpha_2 + (3-r)^8\alpha_3 + (4-r)^8\alpha_4 + (5-r)^8\alpha_5 + (6-r)^8\alpha_6\right] - \tfrac{1}{7!}\left[(-r)^7\beta_0 + (1-r)^7\beta_1 + (2-r)^7\beta_2 + (3-r)^7\beta_3 + (4-r)^7\beta_4 + (5-r)^7\beta_5 + (6-r)^7\beta_6\right]$$

$$D_9 = \tfrac{1}{9!}\left[(-r)^9\alpha_0 + (1-r)^9\alpha_1 + (2-r)^9\alpha_2 + (3-r)^9\alpha_3 + (4-r)^9\alpha_4 + (5-r)^9\alpha_5 + (6-r)^9\alpha_6\right] - \tfrac{1}{8!}\left[(-r)^8\beta_0 + (1-r)^8\beta_1 + (2-r)^8\beta_2 + (3-r)^8\beta_3 + (4-r)^8\beta_4 + (5-r)^8\beta_5 + (6-r)^8\beta_6\right]$$

Setting $r = 3$ and $D_q = 0,\ q = 2, 3, 4, 5, 6, 7, 8$ we have,

$$D_2 = \tfrac{1}{2!}\left[3^2\alpha_0 + 2^2\alpha_1 + \alpha_2 + \alpha_4 + 2^2\alpha_5 + 3^2\alpha_6\right] - \left[-3\beta_0 - 2\beta_1 - \beta_2 + \beta_4 + 2\beta_5 + 3\beta_6\right] = 0$$

$$D_3 = \tfrac{1}{3!}\left[-3^3\alpha_0 - 2^3\alpha_1 - \alpha_2 + \alpha_4 + 2^3\alpha_5 + 3^3\alpha_6\right] - \tfrac{1}{2!}\left[3^2\beta_0 + 2^2\beta_1 + \beta_2 + \beta_4 + 2^2\beta_5 + 3^2\beta_6\right] = 0$$

$$D_4 = \tfrac{1}{4!}\left[3^4\alpha_0 + 2^4\alpha_1 + \alpha_2 + 2^4\alpha_5 + 3^4\alpha_6\right] - \tfrac{1}{3!}\left[3^3\beta_0 + 2^3\beta_1 - \beta_2 + \beta_4 + 2^3\beta_5 + 3^3\beta_6\right] = 0$$

$$D_5 = \tfrac{1}{5!}\left[-3^5\alpha_0 - 2^5\alpha_1 - \alpha_2 + \alpha_4 + 2^5\alpha_5 + 3^5\alpha_6\right] - \tfrac{1}{4!}\left[3^4\beta_0 + 2^4\beta_1 + \beta_2 + \beta_4 + 2^4\beta_5 + 3^4\beta_6\right] = 0$$

$$D_6 = \tfrac{1}{6!}\left[3^6\alpha_0 + 2^6\alpha_1 + \alpha_2 + \alpha_4 + 2^6\alpha_5 + 3^6\alpha_6\right] - \tfrac{1}{5!}\left[-3^5\beta_0 - 2^5\beta_1 - \beta_2 + \beta_4 + 2^5\beta_5 + 3^5\beta_6\right] = 0$$

$$D_7 = \tfrac{1}{7!}\left[-3^7\alpha_0 - 2^7\alpha_1 - \alpha_2 + \alpha_4 + 2^7\alpha_5 + 3^7\alpha_6\right] - \tfrac{1}{6!}\left[3^6\beta_0 + 2^6\beta_1 + \beta_2 + \beta_4 + 2^6\beta_5 + 3^6\beta_6\right] = 0$$

$$D_8 = \tfrac{1}{8!}\left[3^8\alpha_0 + 2^8\alpha_1 + \alpha_2 + \alpha_4 + 2^8\alpha_5 + 3^8\alpha_6\right] - \tfrac{1}{7!}\left[-3^7\beta_0 - 2^7\beta_1 - \beta_2 + \beta_4 + 2^7\beta_5 + 3^7\beta_6\right] = 0$$

However, on inserting the values we have obtained for the α_j into these equations we have

$$\left.\begin{aligned}
&-3\beta_0 - 2\beta_1 - \beta_2 + \beta_4 + 2\beta_5 + 3\beta_6 = 0\\
&3^2\beta_0 + 2^2\beta_1 + \beta_2 + \beta_4 + 2^2\beta_5 + 3^2\beta_6 = \tfrac{2}{3}\left[28 + 4ab - 16(a+b)\right]\\
&-3^3\beta_0 + 2^3\beta_1 - \beta_2 + \beta_4 + 2^3\beta_5 + 3^3\beta_6 = 0\\
&3^4\beta_0 + 2^4\beta_1 + \beta_2 + \beta_4 + 2^4\beta_5 + 3^4\beta_6 = \tfrac{2}{5}\left[244 + 4ab - 64(a+b)\right]\\
&-3^5\beta_0 - 2^5\beta_1 - \beta_2 + \beta_4 + 2^5\beta_5 + 3^5\beta_6 = 0\\
&3^6\beta_0 + 2^6\beta_1 + \beta_2 + \beta_4 + 2^6\beta_5 + 3^6\beta_6 = \tfrac{2}{7}\left[2188 + 4ab - 256(a+b)\right]\\
&-3^7\beta_0 - 2^7\beta_1 - \beta_2 + \beta_4 + 2^7\beta_5 + 3^7\beta_6 = 0
\end{aligned}\right\} \quad (11)$$

In order to satisfy the first, third, fifth and seventh of the above equations we let

$$\beta_2 = \beta_4, \qquad \beta_1 = \beta_5, \qquad \beta_0 = \beta_6$$

The remaining three equations give

$$3^2\beta_0 + 2^2\beta_1 + \beta_2 = \tfrac{1}{3}\left[28 + 4ab - 16(a+b)\right] \quad (12)$$

$$3^4\beta_0 + 2^4\beta_1 + \beta_2 = \tfrac{1}{5}\left[244 + 4ab - 64(a+b)\right] \quad (13)$$

$$3^6\beta_0 + 2^6\beta_1 + \beta_2 = \tfrac{1}{7}\left[2188 + 4ab - 256(a+b)\right] \quad (14)$$

The above set of equations produce the following results,

$$\beta_0 = \tfrac{1}{[illegible]}[278 + 5ab + 16(a+b)] = \beta_6$$

$$\beta_1 = \tfrac{1}{[illegible]}[160 - 8ab - 76(a+b)] = \beta_5$$
$$\beta_2 = \tfrac{1}{105}[62 + 167ab - 272(a+b)] = \beta_4$$

Finally, solving $D_I = 0$ gives

$$\beta_3 = \tfrac{1}{945}(3008 + 5688ab - 1328(a+b))$$

We solve for the error constant, D_9

$$D_9 = \tfrac{1}{9!}\left[-3^9\alpha_0 - 2^9\alpha_1 - \alpha_2 + \alpha_4 + 2^9\alpha_5 + 3^9\alpha_6\right] - \tfrac{1}{8!}\left[3^8\beta_0 + 2^8\beta_1 + \beta_2 + \beta_4 + 2^8\beta_5 + 3^8\beta_6\right]$$
$$D_9 = -\tfrac{1}{907200}\left[6016 + 736ab - 8576(a+b)\right]$$

Since $a = \cos\theta_1$, $b = \cos\theta_2$, $0 < \theta_1 < \pi$, $0 < \theta_2 < \pi$, a and b are restricted to the range $-1 < a < 1$ *and* $-1 < b < 1$.

The following values are chosen for a and b, in order to minimize the error constant as well as develop a method that makes computation easier by reducing the number of operations involved.

$$a = {}^7\!/_8, \quad b = -{}^7\!/_8$$

This causes two coefficients α_5 and α_1 to vanish. Hence the following values are obtained for the coefficients α_j, β_j.

$\alpha_6 = +1$	$\alpha_2 = {}^{33}\!/_{16}$	$\beta_0 = \frac{17547}{60480} = \beta_6$
$\alpha_5 = 0$	$\alpha_1 = 0$	$\beta_1 = \frac{443}{280} = \beta_5$
$\alpha_4 = {}^{-33}\!/_{16}$	$\alpha_0 = -1$	$\beta_2 = \frac{-281}{448} = \beta_4$
$\alpha_3 = 0$		$\beta_3 = \frac{-155}{252}$

On inserting the above values into D_9 above, we obtain the error constant to be -0.006010251323

And, finally, we obtain the following linear 6-step method.

$$y_{n+6} - {}^{33}\!/_{16}\, y_{n+4} + {}^{33}\!/_{16}\, y_{n+2} - y_n =$$
$$h\left[\tfrac{17547}{60480} f_{n+6} + \tfrac{443}{280} f_{n+5} - \tfrac{281}{448} f_{n+4} - \tfrac{155}{252} f_{n+3} - \tfrac{281}{448} f_{n+2} + \tfrac{443}{280} f_{n+1} + \tfrac{17547}{60480} f_n\right] \quad (15)$$

Convergence Test

For the above scheme to be consistent, we establish the following:

$$\rho(1) = 1 - 1 = 0 \quad (35)$$
$$\rho'(1) = 6(1) = 6 \quad (36)$$
$$\sigma(1) = 6 \quad (37)$$

We find the roots of $\rho(\xi)$:

$$\rho(\xi) = \xi^6 - 1 = 0 \quad (38)$$

And we have the following as its roots:

$$\xi_1 = +1; \quad \xi_2 = -1; \quad \xi_3 = \frac{1 + \sqrt{3}}{2} i$$
$$\xi_4 = \frac{1 - \sqrt{3}}{2} i; \quad \xi_5 = \frac{-1 + \sqrt{3}}{2} i; \quad \xi_6 = \frac{-1 - \sqrt{3}}{2} i$$

It is obvious that $|\xi_{ri}| \leq 1$, $i = 1, 2, 3, 4, 5, 6$.

Thus ζ_i, $i = 1, 2, \ldots, 6$ satisfy the zero stability condition. Hence, we conclude that scheme 3 is convergent.

Numerical Examples

The following tables show the results of some problems solved using the three schemes.

Table 1: *PROBLEM* : $F = (1 + Y)/(2 + X)$; $Y(0) = 1$; $h = 0.1$
EXACT SOLUTION : $Y(X) = 2 + X - 1$

X	EXACT	Y(X)	ERROR
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.1000000000	1.1000000000	0.0000000000E+00
0.2	1.2000000000	1.2000000000	0.0000000000E+00
0.3	1.3000000000	1.3000000000	0.0000000000E+00
0.4	1.4000000000	1.4000000000	0.0000000000E+00
0.5	1.5000000000	1.5000000000	0.0000000000E+00
0.6	1.6000000000	1.6000000000	0.0000000000E+00
0.7	1.7000000000	1.7000000000	0.0000000000E+00
0.8	1.8000000000	1.8000000000	0.0000000000E+00
0.9	1.9000000000	1.9000000000	0.0000000000E+00
1.0	2.0000000000	2.0000000000	0.0000000000E+00

TABLE 2: *PROBLEM* : $F = X^5 + 2X^4 + 3X^3$; $Y(0) = 1$; $h = 0.1$
EXACT SOLUTION : $Y(X) = (X^6/6) + (2X^5/5) + (3X^4/4) + 1$

X	EXACT	Y(X)	ERROR
0.0	**1.0000000000**	**1.0000000000**	**0.0000000000E+00**
0.1	**1.0000791667**	**1.0000793542**	**1.8749999997E-07**
0.2	**1.0013386667**	**1.0013390833**	**4.1666666672E-07**
0.3	**1.0071685000**	**1.0071691875**	**6.8750000004E-07**
0.4	**1.0239786667**	**1.0239796667**	**9.9999999992E-07**
0.5	**1.0619791667**	**1.0619805208**	**1.3541666666E-06**
0.6	**1.1360800000**	**1.1360800000**	**0.0000000000E+00**
0.7	**1.2669111667**	**1.2669111667**	**0.0000000000E+00**
0.8	**1.4819626667**	**1.4819626667**	**0.0000000000E+00**
0.9	**1.8168445000**	**1.8168445000**	**0.0000000000E+00**
1.0	**2.3166666667**	**2.3166666667**	**0.0000000000E+00**

DISCUSSION OF RESULTS

As expected, the scheme exhibits high accuracy in Table 1. This is due to the fact that the solution of the differential equation is a polynomial of degree one. This trend is also visible in Table 2. This is according to expectation as well; since the solution of the differential equation is a polynomial of degree six and the scheme is 6-step method of order 8.

Conclusion

Due to the fact that the scheme has been proved to be convergent, and the results of Tables 1 and 2, we conclude that our 6-step implicit linear multistep method of order 8 is accurate and acceptable as a numerical method for solving ordinary differential differential equations.

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