EXPONENTIALLY FITTED EXPLICIT FIFTH ORDER IMPROVED RUNGE-KUTTA METHOD FOR SOLUTION OF INITIAL VALUE PROBLEMS WITH OSCILLATORY BEHAVIOUR

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Abstract
In this article, we constructed exponentially fitted Improved Runge – Kutta (EFIRK) method for the numerical integration of initial value problems whose solutions are linear combinations of functions of the form $\{x^j e^{\omega x}, x^j e^{-\omega x}\}$, $(\omega \in \mathbb{R}$ *or* $i\mathbb{R}$, $j = 0,1,\cdots,jmax)$, *where* $0 \leq jmax \leq \lfloor s/2 - 1 \rfloor, s$ *being the number of stages of the method. The method is of order five with five stag es, wherein the coefficients depend on the frequency and step size. A heuristically chosen algorithm is adopted to determine the frequency* (ω) *in each integration interval* $[x_n, x_{n+1}]$. *The results of numerical experiments with sample initial value problems with oscillatory solutions established the superiority of the exponentially fitted method over the non – fitted method.*

Keywords: Exponential – fitting, Improved Runge – kutta method, Oscillatory solution, Initial value problem

Introduction
The Improved Runge – Kutta (IRK) methods are often employed towards solution of the initial value problem

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0 \qquad (1)$$

The IRK methods are a class of two – step methods that require lower number of stages, and by implication lower number of function evaluations per step, than the classical Runge – Kutta (RK) methods. The generalization of explicit IRK method takes the form

$$y_{n+1} = (1-\alpha)y_n + \alpha y_{n-1} + h\left(b_1 k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})\right) \qquad (2)$$

for $0 \leq \alpha \leq 1$, $1 \leq n \leq N-1$, where,

$$\left.\begin{aligned} k_1 = f(x_n, y_n),\quad k_{-1} = f(x_{n-1}, y_{n-1}) \\ k_i = f\left(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{i,j} k_j\right), \qquad 2 \leq i \leq s \\ k_{-i} = f\left(x_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{i-1} a_{i,j} k_{-j}\right), \qquad 2 \leq i \leq s \end{aligned}\right\} \qquad (3)$$

for $c_2, \cdots, c_s \in [0,1]$ and f depends on both x and y while k_i and k_{-i} depend on the values of k_j and k_{-j} for $j = 1, \cdots, i-1$ (Rabiei et al., 2013). Many authors have made considerable contributions toward improving the accuracy and efficiencey of IRK methods; these include, Wu (2004), Rabiei and Ismail (2011), Rabiei and Ismail (2012), Rabiei et al. (2012), among others

Exponential fitting is a procedure which allows the generalization of the classical algorithms and formulae. Technically speaking, it is hinged on the replacement of power functions, which are taken for reference in the classical case, by a carefully chosen mixture of power and exponential or oscillatory functions. A great many physical phenomena exhibit a pronounced oscillatory character: behaviour of pendulum-like systems, vibrations, resonances and wave propagation are all phenomena of this type in classical mechanics, while the same is true for the typical behaviour of quantum particles (Berghe et al., 2001). Of recent, the study of exponentially – fitted methods has enjoyed the patronage of some authors such as Berghe et al. (2000), Calvo, et al. (2009), Williams and Simos (2003). In the present paper, an exponentially fitted Improved Runge – Kutta method is constructed to solve initial value problems of ordinary differential equations with oscillatory solutions.

Materials and Methods

Derivation of Order Conditions for Fifth Order Exponentially Fitted IRK (EFIRK5) Methods

With $\alpha = 0$ and the introduction of γ_i into the internal stages, the general IRK method (2) becomes

$$y_{n+1} = y_n + hb_1 f(x_n, y_n) - hb_{-1} f(x_{n-1}, y_{n-1}) + h\sum_{i=2}^{s} b_i\big(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_{-i})\big) \quad (4)$$

where

$$Y_i = \gamma_i y_n + h\sum_{j=1}^{i-1} a_{ij} f(x_n + c_j h, Y_j) \quad (5)$$

$$Y_{-i} = \gamma_i y_{n-1} + h\sum_{j=1}^{i-1} a_{ij} f(x_{n-1} + c_j h, Y_{-j}) \quad (6)$$

A function whose solution is $y(x)$ is integrated exactly by an EFIRK method for all problems when

$$y_n = y(x_n) = e^{i\omega x_n} \quad (7)$$

Consequertly,

$$y_{n+1} = y(x_{n+1}) = y(x_n + h) = e^{i\omega(x_n+h)} \quad (8)$$

$$y_{n-1} = y(x_{n-1}) = y(x_n - h) = e^{i\omega(x_n-h)} \quad (9)$$

$$y'_n = i\omega e^{i\omega x_n} = f(x_n, y_n) \quad (10)$$

$$y'_{n-1} = i\omega e^{i\omega(x_n-h)} = f(x_{n-1}, y_{n-1}) \quad (11)$$

And from (3.2) we obtain

$$e^{i\omega(x_n+c_i h)} = \gamma_i e^{i\omega x_n} + h\sum_{j=1}^{i-1} a_{ij}(i\omega) e^{i\omega(x_n+c_j h)} \quad (12)$$

$$e^{i\omega x_n} \cdot e^{i\omega c_i h} = \gamma_i e^{i\omega x_n} + h\sum_{j=1}^{i-1} a_{ij}(i\omega) e^{i\omega x_n} \cdot e^{i\omega c_j h} \quad (13)$$

$$e^{i\omega(c_i h)} = \gamma_i + (i\omega)h\sum_{j=1}^{i-1} a_{ij} e^{i\omega(c_j h)} \quad (14)$$

By letting $\omega h = z$, where ω is the frequency, h is step size, (14) becomes

$$e^{ic_i z} = \gamma_i + iz\sum_{j=1}^{i-1} a_{ij} e^{ic_j z} \quad (15)$$

$$\cos(c_i z) + i\sin(c_i z) = \gamma_i + iz\sum_{j=1}^{i-1} a_{ij}[\cos(c_j z) + i\sin(c_j z)] \quad (16)$$

Thus,

$$\cos(c_i z) = \gamma_i - z\sum_{j=1}^{i-1} a_{ij}\sin(c_j z), \qquad i = 2, 3, \dots, s \quad (17)$$

$$\sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\cos(c_j z), \qquad i = 2, 3, \dots, s \quad (18)$$

Further manipulation of (4) results into

$$e^{i\omega(x_n+h)} = e^{i\omega(x_n)} + hb_1(i\omega)e^{i\omega(x_n)} - hb_{-1}(i\omega)z$$
$$+h\sum_{i=2}^{s} b_i(i\omega)[e^{i\omega(x_n+c_ih)} - e^{i\omega(x_n-h+c_ih)}] \quad (19)$$

$$e^{i\omega(x_n)}\cdot e^{i\omega(h)} = e^{i\omega(x_n)} + hb_1(i\omega)e^{i\omega(x_n)} - hb_{-1}(i\omega)e^{i\omega(x_n)}\cdot e^{-i\omega(h)}$$
$$+h\sum_{i=2}^{s} b_i(i\omega)[e^{i\omega(x_n)}\cdot e^{i\omega(c_ih)} - e^{i\omega(x_n)}\cdot e^{-i\omega(h)}\cdot e^{i\omega(c_ih)}] \quad (20)$$

$$e^{i\omega(h)} = 1 + hb_1(i\omega) - hb_{-1}(i\omega)e^{-i\omega(h)} + h(i\omega)\sum_{i=2}^{s} b_i[e^{i\omega(c_ih)}(1 - e^{-i\omega(h)})] \quad (21)$$

Again by assigning $z = \omega h$

$$e^{iz} = 1 + izb_1 - izb_{-1}e^{-iz} + iz\sum_{i=2}^{s} b_i[e^{ic_iz} - e^{iz(c_i-1)}] \quad (22)$$

$$\cos(z) + i\sin(z) = 1 + izb_1 - izb_{-1}[\cos(z) - i\sin(z)]$$
$$+iz\sum_{i=2}^{s} b_i[\cos(c_iz) + i\sin(c_iz) - (\cos(z(c_i-1)) + i\sin(z(c_i-1)))] \quad (23)$$

Thus,

$$\cos(z) = 1 - zb_{-1}\sin(z) - z\sum_{i=2}^{s} b_i\sin(c_iz) + z\sum_{i=2}^{s} b_i\sin(z(c_i-1)) \quad (24)$$

$$\sin(z) = zb_1 - zb_{-1}\cos(z) + z\sum_{i=2}^{s} b_i\cos(c_iz) - z\sum_{i=2}^{s} b_i\cos(z(c_i-1)) \quad (25)$$

Equations (17), (18), (24) and (25) are now the relations of order conditions of the proposed exponentially-fitted methods. They replace the equations of order conditions of two-step IRK methods up to order five derived by Rabiei et al. (2013) as follows.

$$\left.\begin{aligned}
&\text{order 1: } b_1 - b_{-1} = 1\\
&\text{order 2: } b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}\\
&\text{order 3: } \sum_{i=2}^{s} b_i c_i = \frac{5}{12}\\
&\text{order 4: } \sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3}\\
&\sum_{i=2,j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}\\
&\text{order 5: } \sum_{i=2}^{s} b_i c_i^3 = \frac{31}{120}\\
&\sum_{i=2,j=1}^{s} b_i c_i a_{ij} c_j = \frac{31}{240}\\
&\sum_{i=2,j=1}^{s} b_i a_{ij} c_j^2 = \frac{31}{360}\\
&\sum_{i=2,j=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{31}{720}
\end{aligned}\right\} \qquad (26)$$

These equations are solved in order to determine the coefficients c_i, γ_i, a_{ij} and b_i of an EFRK5 method by choosing values for free parameter from existing coefficients of IRK5 methodas follows:

Derivation of EFIRK5 Method wit$s = p = 5$

The Butcher tableau for the IRK5 method of order five as given Rabiei and Isma(2012) is:

Table 1 Butcher Tableau forIRK5 Method

0					
$\frac{1}{4}$	$\frac{1}{4}$				
$\frac{1}{4}$	$\frac{-1}{125}$	$\frac{259}{1000}$			
$\frac{1}{2}$	$\frac{386}{1000}$	$\frac{-531}{1000}$	$\frac{644}{1000}$		
$\frac{3}{4}$	$\frac{206}{1000}$	$\frac{-9}{10}$	$\frac{892}{1000}$	$\frac{552}{1000}$	
$\frac{1}{45}$	$\frac{46}{45}$	$\frac{1}{25}$	$\frac{-107}{1000}$	$\frac{-1}{10}$	$\frac{29}{45}$

In order to derive the fifth order five stage exponentially fitted IRK (EFIRK5) method, we substitute $s = 5, c_1 = 0, \gamma_1 = 1$ in the recursive relations (17) and (18)
for $i = 2$

$$\cos(c_2 z) - \gamma_2 = 0 \quad (27)$$
$$\sin(c_2 z) - z a_{2,1} = 0 \quad (28)$$

for $i = 3$

$$\cos(c_3 z) - \gamma_3 + z a_{32} \sin(c_2 z) = 0 \quad (29)$$
$$\sin(c_3 z) - z[a_{31} + a_{32}\cos(c_2 z)] = 0 \quad (30)$$

for $i = 4$

$$\cos(c_4 z) - \gamma_4 + z a_{42}\sin(c_2 z) + z a_{43}\sin(c_3 z) = 0 \quad (31)$$
$$\sin(c_4 z) - z[a_{41} + a_{42}\cos(c_2 z) + a_{43}\cos(c_3 z)] = 0 \quad (32)$$

for $i = 5$

$$\cos(c_5 z) - \gamma_5 + z[a_{52}\sin(c_2 z) + a_{53}\sin(c_3 z) + a_{54}\sin(c_4 z)] = 0 \quad (33)$$
$$\sin(c_5 z) - z[a_{51} + a_{52}\cos(c_2 z) + a_{53}\cos(c_3 z) + a_{54}\cos(c_4 z)] = 0 \quad (34)$$

Now, substituting $s = 5$, $c_1 = 0$ in equations (24) and (25), we have

$$\cos(z) - 1 + z b_{-1}\sin(z) + z[b_2\sin(c_2 z) + b_3\sin(c_3 z) + b_4\sin(c_4 z) + b_5\sin(c_5 z)]$$
$$-z[b_2\sin((c_2-1)z) + b_3\sin((c_3-1)z) + b_4\sin((c_4-1)z) + b_5\sin((c_5-1)z)] = 0 \quad (35)$$
$$\sin(z) - z b_1 + z b_{-1}\cos(z) - z[b_2\cos(c_2 z) + b_3\cos c_3 z + b_4\cos c_4 z + b_5\cos c_5 z]$$
$$+z[b_2\cos((c_2-1)z) + b_3\cos((c_3-1)z) + b_4\cos((c_4-1)z) + b_5\cos((c_5-1)z)] = 0 \quad (36)$$

Equations (27) – (36) are now the equations of order conditions for fifth order five stage exponentially-fitted IRK method that replaces order conditions of the original IRK5 method.

To obtain the coefficients of the method, equations (27), (29), (31), (33), (35) and (36) together with four additional equations from the order conditions (26) are solved to give ten equations in twenty unknown parameters $(b_{-1}, b_1, b_2, b_3, b_4, b_5, c_2, c_3, c_4, c_5, a_{32}, a_{42}, a_{43}, a_{52}, a_{53}, a_{54}, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, thus giving ten degrees of freedom, that is, the equations are solved in terms of ten free parameters, $(c_2, c_3, c_4, c_5, a_{32}, a_{42}, a_{43}, a_{52}, a_{53}, a_{54})$ whose values would be taken from the Butcher Table 1 for existing IRK5 method. With the aid of MAPLE Software the values of the remaining parameters in terms of the free parameters are sought. The results obtained are as follow

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{3}\frac{M_1}{M_2} \\ b_1 &= \frac{1}{3}\frac{M_3}{M_4} \\ b_2 &= -\frac{1}{1554}\frac{M_5}{M_6} \\ b_3 &= \frac{1}{1554}\frac{M_7}{M_8} \\ b_4 &= -\frac{1}{6}\frac{M_9}{M_{10}} \\ b_5 &= \frac{1}{6}\frac{M_{11}}{M_{12}} \end{aligned}\right\} \quad (37)$$

$$M_1 = 2z\cos\left(\frac{3}{4}z\right) - 2z\cos\left(\frac{1}{4}z\right) - 3\sin(z) + 3z$$
$$M_2 = \left[1 - 2\cos\left(\frac{1}{4}z\right) + 2\cos\left(\frac{3}{4}z\right) - \cos(z)\right]z$$
$$M_3 = 3z\cos(z) - 4z\cos\left(\frac{3}{4}z\right) + 4z\cos\left(\frac{1}{4}z\right) - 3\sin(z)$$
$$M_4 = \left[1 - 2\cos\left(\frac{1}{4}z\right) + 2\cos\left(\frac{3}{4}z\right) - \cos(z)\right]z$$

$$\begin{aligned}
M_5 = {} & 3387 + 2450\sin\left(\tfrac{1}{2}z\right) z\cos\left(\tfrac{3}{4}z\right) - 2450\sin\left(\tfrac{1}{2}z\right) z\cos\left(\tfrac{1}{4}z\right) - 587z\cos(z) - 6774\cos(z) \\
& + 1987z\cos(z)\sin\left(\tfrac{3}{4}z\right) + 1987z\cos(z)\sin\left(\tfrac{1}{4}z\right) + 2258z\cos\left(\tfrac{3}{4}z\right)\sin(z) \\
& - 6780z\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) - 6780z\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 2258z\cos\left(\tfrac{1}{4}z\right)\sin(z) \\
& + 6780z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) + 6780z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 1914\sin\left(\tfrac{1}{2}z\right)\sin(z) \\
& - 3387\sin(z)^2 + 3387\cos(z)^2 + 4344\sin(z)\sin\left(\tfrac{3}{4}z\right) + 4344\sin(z)\sin\left(\tfrac{1}{4}z\right) \\
& + 6774\cos\left(\tfrac{1}{4}z\right)\cos(z) - 6774\cos\left(\tfrac{3}{4}z\right)\cos(z) + 6774\cos\left(\tfrac{3}{4}z\right) - 6774\cos\left(\tfrac{1}{4}z\right) \\
& + 3387z\sin(z) - 6331z\sin\left(\tfrac{3}{4}z\right) - 6331z\sin\left(\tfrac{1}{4}z\right) + 2501z\sin\left(\tfrac{1}{2}z\right)
\end{aligned}$$

$$\begin{aligned}
M_6 = z\Big[& -\sin\left(\tfrac{1}{4}z\right) - \sin\left(\tfrac{3}{4}z\right) + 2\sin\left(\tfrac{1}{2}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& - 4\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{2}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& + 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cos(z)\sin\left(\tfrac{1}{4}z\right) + \cos(z)\sin\left(\tfrac{3}{4}z\right) - 2\cos(z)\sin\left(\tfrac{1}{2}z\right)\Big]
\end{aligned}$$

$$\begin{aligned}
M_7 = {} & 2610 + 5040\sin\left(\tfrac{1}{2}z\right) z\cos\left(\tfrac{3}{4}z\right) - 5040\sin\left(\tfrac{1}{2}z\right) z\cos\left(\tfrac{1}{4}z\right) - 328z\cos(z)\sin\left(\tfrac{1}{2}z\right) \\
& - 5220\cos(z) + 1469z\cos(z)\sin\left(\tfrac{3}{4}z\right) + 1469z\cos(z)\sin\left(\tfrac{1}{4}z\right) \\
& + 1740z\cos\left(\tfrac{3}{4}z\right)\sin(z) - 6780z\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) - 6780z\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) \\
& - 1740z\cos\left(\tfrac{1}{4}z\right)\sin(z) + 6780z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) + 6780z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) \\
& - 6576\sin\left(\tfrac{1}{2}z\right)\sin(z) - 2610\sin(z)^2 + 2610\cos(z)^2 + 5898\sin(z)\sin\left(\tfrac{3}{4}z\right) \\
& + 5898\sin(z)\sin\left(\tfrac{1}{4}z\right) + 5220\cos\left(\tfrac{1}{4}z\right)\cos(z) - 5220\cos\left(\tfrac{3}{4}z\right)\cos(z) \\
& + 5220\cos\left(\tfrac{3}{4}z\right) - 5220\cos\left(\tfrac{1}{4}z\right) + 2610z\sin(z) - 7367z\sin\left(\tfrac{3}{4}z\right) \\
& + 6904z\sin\left(\tfrac{1}{2}z\right) - 7367z\sin\left(\tfrac{1}{4}z\right)
\end{aligned}$$

$$\begin{aligned}
M_8 = z\Big[& -\sin\left(\tfrac{1}{4}z\right) - \sin\left(\tfrac{3}{4}z\right) + 2\sin\left(\tfrac{1}{2}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& - 4\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{2}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& + 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cos(z)\sin\left(\tfrac{1}{4}z\right) + \cos(z)\sin\left(\tfrac{3}{4}z\right) - 2\cos(z)\sin\left(\tfrac{1}{2}z\right)\Big]
\end{aligned}$$

$$\begin{aligned}
M_9 = {} & -6 - 3z\cos(z)\sin\left(\tfrac{3}{4}z\right) - 3z\cos(z)\sin\left(\tfrac{1}{4}z\right) - 4z\cos\left(\tfrac{3}{4}z\right)\sin(z) + 10z\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& + 10z\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) + 4z\cos\left(\tfrac{1}{4}z\right)\sin(z) - 10z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& - 10z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 6\cos(z)^2 + 6\sin(z)^2 - 6\sin(z)\sin\left(\tfrac{3}{4}z\right) - 6\sin(z)\sin\left(\tfrac{1}{4}z\right) \\
& - 12\cos\left(\tfrac{1}{4}z\right)\cos(z) + 12\cos\left(\tfrac{3}{4}z\right)\cos(z) - 12\cos\left(\tfrac{3}{4}z\right) + 12\cos(z) - 6z\sin(z) \\
& + 9z\sin\left(\tfrac{3}{4}z\right) + 9z\sin\left(\tfrac{1}{2}z\right) + 12z\cos\left(\tfrac{1}{4}z\right)
\end{aligned}$$

$$\begin{aligned}
M_{10} = z\Big[& -\sin\left(\tfrac{1}{4}z\right) - \sin\left(\tfrac{3}{4}z\right) + 2\sin\left(\tfrac{1}{2}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& - 4\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{2}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
& + 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cos(z)\sin\left(\tfrac{1}{4}z\right) + \cos(z)\sin\left(\tfrac{3}{4}z\right) - 2\cos(z)\sin\left(\tfrac{1}{2}z\right)\Big]
\end{aligned}$$

$$\begin{aligned}
M_{11} = {} & 3\sin(z)^2 - 6\sin(z)\sin\left(\tfrac{1}{4}z\right) - 6\sin(z)\sin\left(\tfrac{3}{4}z\right) + 2z\cos\left(\tfrac{1}{4}z\right)\sin(z) - 3z\sin(z) \\
& - 2z\cos\left(\tfrac{3}{4}z\right)\sin(z) + 6\sin\left(\tfrac{1}{2}z\right)\sin(z) - 3 + 10\sin\left(\tfrac{1}{2}z\right) z\cos(z)\cos\left(\tfrac{3}{4}z\right) \\
& - 10\sin\left(\tfrac{1}{2}z\right) z\cos\left(\tfrac{1}{4}z\right) - 7z\cos(z)\sin\left(\tfrac{1}{2}z\right) + 2z\cos(z)\sin\left(\tfrac{3}{4}z\right) \\
& + 2z\cos(z)\sin\left(\tfrac{1}{4}z\right) - 3\cos(z)^2 - 6\cos\left(\tfrac{1}{4}z\right)\cos(z) + 6\cos\left(\tfrac{3}{4}z\right)\cos(z) \\
& - 6\cos\left(\tfrac{3}{4}z\right) + 6\cos(z) + 4z\sin\left(\tfrac{3}{4}z\right) - 4z\sin\left(\tfrac{1}{4}z\right) + z\sin\left(\tfrac{1}{2}z\right) + 6\cos\left(\tfrac{1}{4}z\right)
\end{aligned}$$

$$
\begin{aligned}
M_{12} = z[&-\sin\left(\tfrac{1}{4}z\right) - \sin\left(\tfrac{3}{4}z\right) + 2\sin\left(\tfrac{1}{2}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
&- 4\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{2}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{4}z\right) - 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\
&+ 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cos(z)\sin\left(\tfrac{1}{4}z\right) + \cos(z)\sin\left(\tfrac{3}{4}z\right) - 2\cos(z)\sin\left(\tfrac{1}{2}z\right)]
\end{aligned}
$$

$$
\left.
\begin{aligned}
\gamma_2 &= \cos\left(\tfrac{1}{4}z\right) \\
\gamma_3 &= \cos\left(\tfrac{1}{4}z\right) + \tfrac{259}{1000} z\sin\left(\tfrac{1}{4}z\right) \\
\gamma_4 &= \cos\left(\tfrac{1}{2}z\right) + \tfrac{113}{1000} z\sin\left(\tfrac{1}{4}z\right) \\
\gamma_5 &= \cos\left(\tfrac{3}{4}z\right) - \tfrac{1}{125} z\sin\left(\tfrac{1}{4}z\right) + \tfrac{69}{125} z\sin\left(\tfrac{1}{2}z\right)
\end{aligned}
\right\} \tag{38}
$$

The Taylor series expansion of (37) and (38) are described, respectively, by

$$
\left.
\begin{aligned}
b_{-1} &= \tfrac{1}{45} + \tfrac{13}{3780} z^2 + \tfrac{17}{172800} z^4 + O(z^6) \\
b_1 &= \tfrac{46}{45} + \tfrac{13}{3780} z^2 + \tfrac{17}{172800} z^4 + O(z^6) \\
b_2 &= \tfrac{997}{23310} - \tfrac{437617}{15664320} z^2 + \tfrac{465161}{5012582400} z^4 + O(z^6) \\
b_3 &= -\tfrac{2551}{23310} + \tfrac{124403}{7832160} z^2 + \tfrac{225641}{2506291200} z^4 + O(z^6) \\
b_4 &= -\tfrac{1}{10} + \tfrac{139}{10080} z^2 + \tfrac{227}{3225600} z^4 + O(z^6) \\
b_5 &= \tfrac{29}{45} - \tfrac{313}{60480} z^2 + \tfrac{271}{19353600} z^4 + O(z^6)
\end{aligned}
\right\} \tag{39}
$$

$$
\left.
\begin{aligned}
\gamma_1 &= 1 \\
\gamma_2 &= 1 - \frac{1}{32} z^2 + \frac{1}{6144} z^4 + O(z^6) \\
\gamma_3 &= 1 + \frac{67}{2000} z^2 - \frac{131}{256000} z^4 + O(z^6) \\
\gamma_4 &= 1 - \frac{387}{4000} z^2 + \frac{887}{384000} z^4 + O(z^6) \\
\gamma_5 &= 1 - \frac{29}{4000} z^2 + \frac{1309}{768000} z^4 + O(z^6)
\end{aligned}
\right\} \tag{40}
$$

From (39) and (40) it is evident that the original method, IRK55 is recovered as z approaches zero which validates the exponentially fitted method.

Next, we confirm that EFIRK5 method is of order five as claimed by substituting the coefficients into the order conditions (26) and the Taylor expansion of each of the order conditions is obtained thus:

$$
\left.
\begin{aligned}
\textit{order } 1:&\quad b_1 - b_{-1} = 1 + O(z^6) \\
\textit{order } 2:&\quad b_{-1} + \textstyle\sum_{i=2}^{s} b_i = \tfrac{1}{2} + O(z^6) \\
\textit{order } 3:&\quad \textstyle\sum_{i=2}^{s} b_i c_i = \tfrac{5}{12} + O(z^6) \\
\textit{order } 4:&\quad \textstyle\sum_{i=2}^{s} b_i c_i^2 = \tfrac{1}{3} - \tfrac{1}{4608} z^2 + \tfrac{29}{2064384} z^4 + O(z^6) \\
&\quad \textstyle\sum_{i=2,j=1}^{s} b_i a_{ij} c_j = \tfrac{1}{6} + O(z^6) \\
\textit{order } 5:&\quad \textstyle\sum_{i=2}^{s} b_i c_i^3 = \tfrac{31}{120} - \tfrac{209}{322560} z^2 + \tfrac{1223}{103219200} z^4 + O(z^6) \\
&\quad \textstyle\sum_{i=2,j=1}^{s} b_i c_i a_{ij} c_j = \tfrac{186119}{1440000} - \tfrac{295927}{48384000} z^2 + \tfrac{373969}{154828800000} z^4 + O(z^6) \\
&\quad \textstyle\sum_{i=2,j=1}^{s} b_i a_{ij} c_j^2 = \tfrac{323}{3750} - \tfrac{7199}{20160000} z^2 + \tfrac{6233}{6451200000} z^4 + O(z^6) \\
&\quad \textstyle\sum_{i=2,j=1}^{s} b_i a_{ij} a_{jk} c_k = \tfrac{3879067}{90000000} + \tfrac{11818103}{60480000000} z^2 + \tfrac{76672159}{19353600000000} z^4 + O(z^6)
\end{aligned}
\right\} \tag{41}
$$

Again, (41) reduces to the order conditions (26) of the IRK5 -5, which further shows that coefficients of the EFIRK5 method satisfies the order five conditions and hence the method is of order five.

Choice of Frequency

Following Berghe et al . (2000), the frequency (ω) for the EFIRK5 is calculated in each integration interval $x_n, x_{n+1}]$ by means of the algorithm

$$\omega = \sqrt{-\frac{y^{(2)}(x_n)}{y(x_n)}}, \quad n = 0, \dots \ if\ y(x_n) \neq 0\ and\ \omega = 0\ otherwise \quad (3.51)$$

Numerical Experiments

We present a standard set of initial value problems to show the efficiency and accuracy of the proposed method. The approximate solutions are sought on the partition $[x_0, X]$, and the errors are calculated at the endpoints a $\mathrm{Error} = |y_n - y(x_n)|$. The following problems are solved.

Problem 1: $y'(x) = x\,e^{-3x} + 2x, \quad y(0) = -\frac{1}{9}$

Exact solution $y(x) = x^2 - \frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}$

Problem 2: $y'(x) = y\cos(x), \ y(0) = 1$

Exact solution $y(x) = e^{\sin(x)}$

The results of the numerical examples are presented in Table 1 to 4.

Table 1 Results of EFIRK5 with IRK5 for problem 1 in [0,1], $h = 0.05$, $\omega = 5$

x	Exact	EFIRK5	Error	IRK5	Error
0.05	-0.1074793525	-0.1074793525	2.47189770E-10	-0.1074799552	6.026722408E-07
0.10	-0.0970070763	-0.0970070763	1.35991889E-09	-0.0970094064	2.330095927E-06
0.15	-0.0802289799	-0.0802289825	2.47722396E-09	-0.0802334515	4.471536354E-06
0.20	-0.0575665131	-0.0575665165	3.39311803E-09	-0.0575729771	6.464005662E-06
0.25	-0.0293490519	-0.0293490561	4.14203276E-09	-0.0293573917	8.339793139E-06
0.30	0.00416862738	0.0041686226	4.75271549E-09	0.0041585023	1.012508317E-05
0.35	0.04279195715	0.0427919519	5.24914482E-09	0.0427801161	1.184102949E-05
0.40	0.08637474819	0.0863747425	5.65130128E-09	0.0863612436	1.350464858E-05
0.45	0.13480948749	0.1348094815	5.97581595E-09	0.1347943579	1.512956184E-05
0.50	0.18801939996	0.1880193937	6.23651583E-09	0.1880026733	1.672661121E-05
0.55	0.24595197135	0.2459519649	6.44488239E-09	0.2459336669	1.830436927E-05
0.60	0.30857367922	0.3085736726	6.61043669E-09	0.3085538097	1.986956097E-05
0.65	0.37586572098	0.3758657142	6.74106276E-09	0.3758442936	2.142741215E-05
0.70	0.44782056360	0.4478205568	6.84327891E-09	0.4477975817	2.298193659E-05
0.75	0.52443916891	0.5244391620	6.92246526E-09	0.5244146327	2.453617235E-05
0.80	0.60572877320	0.6057287662	6.98305431E-09	0.6057026808	2.609237565E-05
0.85	0.69170112063	0.6917011136	7.02869059E-09	0.6916734685	2.765217978E-05
0.90	0.78237106698	0.7823710599	4.65599044E-09	0.7823418503	2.921672486E-05
0.95	0.87775548496	0.8777554779	7.08652209E-09	0.8777246982	3.078676349E-05
1.00	0.97787241406	0.9778724070	7.10316205E-09	0.9778400513	3.236274667E-05

The results of Table 1 established that the proposed EFIRK5 method produces more accurate results than the classical IRK5 method with the same number of function evaluations.

Table 2 Results of EFIRK5with IRK5 for problem2 in [0,1], $h = 0.05$, $\omega = 5$

x	Exact	EFIRK5-5	Error	IRK5-5	Error
0.05	1.0512491979	1.0512491979	1.07576978E-13	1.0512489897	2.082045233E-07
0.10	1.1049868303	1.1049868303	3.13239222E-12	1.1049859991	8.312331255E-07
0.15	1.1611816292	1.1611816292	3.78701070E-13	1.1611799752	1.653935747E-06
0.20	1.2197785560	1.2197785560	9.12379897E-12	1.2197760956	2.460392373E-06
0.25	1.2806963574	1.2806963574	2.61481801E-11	1.2806931154	3.242064657E-06
0.30	1.3438252437	1.3438252437	5.13294764E-11	1.3438212541	3.989626919E-06
0.35	1.4090247627	1.4090247627	8.51138548E-11	1.4090200697	4.693024921E-06
0.40	1.4761219464	1.4761219463	1.27707721E-10	1.4761166049	5.341559978E-06
0.45	1.5449098113	1.5449098112	1.79029628E-10	1.5449038873	5.923999986E-06
0.50	1.6151462964	1.6151462962	2.38668232E-10	1.6151398677	6.428718241E-06
0.55	1.6865537216	1.6865537213	3.05849794E-10	1.6865468778	6.843860083E-06
0.60	1.7588188458	1.7588188454	3.79418689E-10	1.7588116882	7.157536430E-06
0.65	1.8315935966	1.8315935962	4.57834221E-10	1.8315862386	7.358042139E-06
0.70	1.9044965344	1.9044965338	5.39186447E-10	1.9044891003	7.434095994E-06
0.75	1.9771150961	1.9771150954	6.21232974E-10	1.9771077210	7.375097829E-06
0.80	2.0490086502	2.0490086495	7.01457546E-10	2.0490014788	7.171397175E-06
0.85	2.1197123704	2.1197123696	7.77149973E-10	2.1197055559	6.814566622E-06
0.90	2.1887419126	2.1887419118	8.45505443E-10	2.1887356149	6.297672153E-06
0.95	2.2555988538	2.2555988529	9.03739727E-10	2.2555932383	5.615531945E-06
1.00	2.3197768247	2.3197768238	9.49215311E-10	2.3197720598	4.764954617E-06

Similarly, in Table 2 the proposed EFIRK5 method gives more accurate approximations than the existing IRK5 method with the same computational efficiency.

Table 3 Maximum Errors for Problem1 in [0, 100], with $\omega = 5$

h	EFIRK5-5	IRK5-5	NFEs
0.05	7.12276483049E-09	3.33146902525E03	10000
0.025	2.48840446639E-10	1.66619898825E03	20000
0.0125	8.20344237904E-12	8.33216270746E04	40000
0.00625	2.63098712622E-13	4.16637385124E04	80000
0.003125	1.31058212168E-12	2.08326011023E04	160000
0.0015625	2.73435862373E-10	1.04164835850E04	320000

Table 3 compares the maximum errors obtained from solving problem 1 using the proposed EFIRK5 method with the existing IRK5 method in the interval $[0, 100]$ with varied step sizes. It can be seen that the accuracy of the EFIRK5 method decreases as the step size grows smaller which shows that the EFIRK5 method approaches the original IRK5 method a $h \to 0$ to further validate our earlier assertion that the original method has to be recovered from our new method.

Table 4 Maximum Errors for Problem 2 in [0, 100], with $\omega = 1$

h	EFIRK5-5	IRK5-5	NFEs
0.05	9.93219493678E10	4.11833349814E05	10000
0.025	3.08764811836E11	2.05398168695E05	20000
0.0125	1.92752108713E12	1.02568893991E-05	40000
0.00625	1.42279405657E09	5.12518956850E06	80000
0.003125	4.50477462771E08	2.56178100088E06	160000
0.0015625	2.23331141225E06	1.28068706241E06	320000

In Table 4 we present the maximum errors gotten from solving problem 2 using the EFIRK5 the existing IRK5 methods in the interval $[0, 100]$ for different step sizes. It is evident that the EFIRK5 method exhibited greater degree of accuracy by comparing the errors generated from both methods. More so, it is observed that the accuracy of the new method equally diminishes with a decreasing step size thus indicating that the method derived approaches the original method as $h \to 0$.

Conclusion

We have constructed a fifth order five stage Exponentially-fitted Improved Runge-Kutta method. The method was used to solve oscillatory and exponential problems and the results were compared with those of existing Improved Runge-Kutta method of the same order. The results of the numerical examples considered indicate that the Exponentially-fitted Improved Runge-Kutta method is far more accurate for numerically integrating oscillatory or exponential initial value problems than the Improved Runge-Kutta method of the same order with the same number of function evaluations.

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