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Formulation Of A Standard Runge-Kutta Type Method For The Solution First And Second Order Initial Value Problems

Muhammad R.

Department of Mathematics/ Statistics , Federal University of Technology, Minna

Y. A Yahaya

Department of Mathematics/ Statistics , Federal University of Technology, Minna

A.S Abdulkareem

Department of Chemical Engineering, Federal University of Technology, Minna

ABSTRACT

In this paper, we present a standard Runge-Kutta Type Method (RKTM) for $k = 3$. The process produces Backward Differentiation Formula (BDF) scheme and its hybrid form which combined together to form a block method. The method is reformulated into a Runge-Kutta Type of the same step number ($k = 3$) for the solution of first and second order (special or general) initial value problem of Ordinary Differential Equation (ODE).

Keywords: Backward Differentiation Formula (BDF), Runge-Kutta Type Method (RKTM), Block, Hybrid

1. INTRODUCTION

Differential equations play an important role in modeling virtually every physical, technical or biological process, from celestial motion to bridge design to interaction between neurons. As an example, the propagation of light and sound in the atmosphere may be described by second order differential equation. (Adegboye 2013).

The Backward Differentiation formulae (BDF) were discovered in 1952 by Curtiss and Hirschfelder for the solution to stiff problems.

In an attempt to develop more efficient and accurate methods for the solution of stiff problems, the hybrid forms of some known BDF schemes collated from literature were derived. The idea of Multistep Collocation (MC) was used to reformulate the derived hybrid schemes into continous form which were employed as block methods for direct solution of $y' = f(x, y)$. The existing methods can only solve first order Ordinary Differential Equation (ODE).

The idea propose in this paper is to reformulate the block Backward Differentiation formulae (Hybrid and Non-Hybrid) for $K = 3$ into Runge Kutta Type method of the same step number ($k = 3$) for initial value problems in Ordinary Differential Equations (ODE). The proposed method intend to solve both first and second order (special or general form) ODE. Our present method differs from conventional Backward Differentiation Formulae (BDF) as derivation is done only once. It can also be extended to solve higher order ODE. Such method will compete reasonably with some known standard Runge Kutta Type Methods.

Butcher defined an s-stage Runge Kutta method for the first order differential equation in the form

$$y_{n+1} = y_n + h\sum_{i,j=1}^{s} a_{ij}k_i \qquad (1)$$

where for $i = 1,2 \ldots \ldots \ldots . s$

$$k_i = f(x_i + \alpha_j h, y_n + h\sum_{i,j=1}^{s-1} a_{ij}k_j) \qquad (2)$$

The real parameters α_j, k_i, a_{ij} define the method. The method in Butcher array form can be written as

α	β
	b^T

Where $a_{ij} = \beta$

The Runge Kutta Nystrom (RKN) method is an extension of Runge Kutta method for second order ODE of the form

$$y'' = f(x, y, y') \qquad y(x_0) = y_0 \quad y'(x_0) = y_0' \qquad (3)$$

An S-stage implicit Runge Kutta Nystrom for direct integration of second order initial value problem is defined in the form

$$y_{n+1} = y_n + \alpha_i h y_n' + h^2 \sum_{i,j=1}^{s} a_{ij} k_j \qquad (4a)$$

$$y_{n+1}' = y_n' + h \sum_{i,j=1}^{s} \bar{a}_{ij}\, k_j \qquad (4b)$$

where for $i = 1,2 \ldots \ldots \ldots s$

$$k_i = f(x_i + \alpha_j h, y_n + \alpha_i h y_n' + h^2 \sum_{i,j=1}^{s} a_{ij} k_j\,, y_n' + h \sum_{i,j=1}^{s} \bar{a}_{ij}\, k_j) \qquad (4c)$$

The real parameters $\alpha_j, k_j, a_{ij}, \bar{a}_{ij}$ define the method, the method in butcher array form is expressed as

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2 \qquad \bar{A} = \bar{a}_{ij} = \beta \qquad \beta = \beta e$

2. CONSTRUCTION OF THE BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULA WHEN $K = 3$ (BHBDF3)

Consider the approximate solution to

$$y' = f(x, y) \quad y(x_0) = y) \qquad (5)$$

in the form of power series

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j \qquad (6)$$

$\alpha \in R, j = 0(1)t + m - 1, y \in C^m(a, b) \subset P(x)$

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$$y'(x) = \sum_{j=0}^{t+m-1} j\alpha_j x^{j-1} \quad (7)$$

Where α_j's are the parameters to be determined, t and m are the points of interpolation and collocation respectively.

When $K = 3$, we interpolate $(t = 4)$ at $j = 0, \frac{1}{2}, 1, 2$ and collocate $(m = 1)$ at $j = 3$. The equation can be expressed as

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j = y_{n+j} \qquad j = 0, \frac{1}{2}, 1, 2 \quad (8)$$

$$y'(x) = \sum_{j=0}^{t+m-1} j\alpha_j x^{j-1} = f_{n+j} \qquad j = 3 \quad (9)$$

The general form of the proposed method upon addition of one off grid point is expressed as;

$$\bar{y}(x) = \alpha_1(x) y_n + \alpha_2(x) y_{n+1} + \alpha_3(x) y_{n+2} + \alpha_4(x) y_{n+\frac{1}{2}} + h\beta_0(x) f_{n+3} \quad (10)$$

The matrix D of dimension $(t + m) * (t + m)$ of the proposed method is expressed as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 \\ 1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 \\ 1 & x_n + \frac{1}{2}h & \left(x_n + \frac{1}{2}h\right)^2 & \left(x_n + \frac{1}{2}h\right)^3 & \left(x_n + \frac{1}{2}h\right)^4 \\ 0 & 1 & 2x_n + 6h & 3(x_n + 3h)^2 & 4(x_n + 3h)^3 \end{bmatrix}$$

Using the maple software package, we invert the matrix D, to obtain columns which form the matrix C. The elements of C are used to generate the continous coefficients of the method as:

$$\alpha_1(x) = C_{11} + C_{21}x + C_{31}x^2 + C_{41}x^3 + C_{51}x^4$$

$$\alpha_2(x) = C_{12} + C_{22}x + C_{32}x^2 + C_{42}x^3 + C_{52}x^4$$

$$\alpha_3(x) = C_{13} + C_{23}x + C_{33}x^2 + C_{43}x^3 + C_{53}x^4 \quad (11)$$

$$\alpha_4(x) = C_{14} + C_{24}x + C_{34}x^2 + C_{44}x^3 + C_{54}x^4$$

$$\beta_0(x) = C_{15} + C_{25}x + C_{35}x^2 + C_{45}x^3 + C_{55}x^4$$

The values of the continuous coefficients (11) are substituted into (10) to give as the continuous form of the three step block hybrid BDF with one off step interpolation point.

$$(x) = \{(\frac{1}{134}\frac{134h^4+507x_nh^3+602x_n^2h^2+267x_n^3h+38x_n^4}{h^4} - \frac{1}{134}\frac{507h^3+1204x_nh^2+801x_n^2h+152x_n^3}{h^4}x +$$

$$\frac{1}{134}\frac{602h^2+801x_nh+228x_n^2}{h^4}x^2-\frac{1}{134}\frac{267h+152x_n}{h^4}x^3+\frac{19}{67h^4}x^4)y_n+\left(\frac{1}{67}\frac{x_n\,(186h^3+517x_nh^2+316x_n^2h+52x_n^3}{h^4}-\frac{2}{67}\frac{93h^3+517x_nh^2+474x_n^2h+104x_n^3}{h^4}x+\frac{1}{67}\frac{517h^2+948x_nh+312x_n^2}{h^4}x^2-\frac{4}{67}\frac{79h+52x_n}{h^4}x^3+\frac{52}{67h^4}x^4\right)y_{n+1}+$$
$$\left(-\frac{1}{402}\frac{x_n(460x_nh^2+393x_n^2h+74x_n^3+141h^3)}{h^4}+\frac{1}{402}\frac{920x_nh^2+1179x_n^2h+296x_n^3+141h^3}{h^4}x-\frac{1}{402}\frac{460h^2+1179x_nh+444x_n^2}{h^4}x^2+\frac{1}{402}\frac{393h+296x_n}{h^4}\mathrm{x}^3-\frac{37}{201h^4}x^4\right)y_{n+2}+$$
$$\left(-\frac{16}{201}\frac{x_n\,(11x_n^3+72x_n^2h+139x_nh^2+78h^3)}{h^4}+\frac{32}{201}\frac{22x_n^3+108x_n^2h+139x_nh^2+39h^3}{h^4}x-\frac{16}{201}\frac{66x_n^2+216x_nh+139h^2}{h^4}x^2+-\frac{1}{67}\frac{8x_n+7h}{h^3}\frac{64}{201}\frac{11x_n+18h}{h^4}x^3-\frac{176}{201h^4}x^4\right)y_{n+\frac{1}{2}}+$$
$$\left(\frac{1}{67}\frac{x_n(2x_n^3+7x_n^2h+7x_nh^2+2h^3)}{h^3}-\frac{1}{67}\frac{8x_n^3+21x_n^2h+14x_nh^2+2h^3}{h^3}x+\frac{1}{67}\frac{12x_n^2+21x_nh+7h^2}{h^3}x^2-\frac{1}{67}\frac{8x_n+7h}{h^3}x^3+\frac{2}{67h^3}x^4\right) \qquad (12)$$

Evaluating (12) at point $x=x_{n+3}$ and its derivative at $x=x_{n+1}$, $x=x_{n+2}$

$x=x_{n+1/2}$ yields the following four discrete hybrid schemes which are used as

block integrator;

$$\frac{225}{67}y_{n+1}-\frac{150}{67}y_{n+2}+y_{n+3}-\frac{192}{67}y_{n+1/2}=\frac{-50}{67}y_n+\frac{30}{67}hf_{n+3}$$

$$y_{n+1}+\frac{52}{324}y_{n+2}-\frac{448}{324}y_{n+1/2}=\frac{-72}{324}y_n+\frac{201}{324}hf_{n+1}+\frac{3}{324}hf_{n+3} \qquad (13)$$

$$\frac{1476}{649}y_{n+1}-y_{n+2}-\frac{1088}{649}y_{n+1/2}=-\frac{261}{649}y_n-\frac{402}{649}hf_{n+2}+\frac{36}{649}hf_{n+3}$$

$$-\frac{2880}{1600}y_{n+1}+\frac{245}{1600}y_{n+2}+y_{n+1/2}=-\frac{1035}{1600}y_n-\frac{1608}{1600}hf_{n+1/2}+\frac{18}{1600}hf_{n+3}$$

It is of order $[4,4,4,4]^T$ with error constants $[-\frac{15}{268},\frac{41}{4020},-\frac{97}{2680},-\frac{79}{8576}]^T$

Re-arranging the block implicit hybrid scheme simultaneously we obtained the following block scheme

$$y_{n+\frac{1}{2}}=y_n+\frac{h}{960}\left\{952f_{n+\frac{1}{2}}-625f_{n+1}+190f_{n+2}-37f_{n+3}\right\}$$

$$y_{n+1}=y_n+\frac{h}{30}\left\{36f_{n+\frac{1}{2}}-10f_{n+1}+5f_{n+2}-f_{n+3}\right\}$$

$$(14)$$

$$y_{n+2}=\ y_n+\frac{h}{2010}\left\{2144f_{n+\frac{1}{2}}+670f_{n+1}+1340f_{n+2}+221f_{n+3}\right\}$$

$$y_{n+3}=\ y_n+\frac{h}{10}\left\{12f_{n+\frac{1}{2}}+0f_{n+1}+15f_{n+2}+3f_{n+3}\right\}$$

3. REFORMULATION OF THE BHBDF3INTO RUNGE KUTTA TYPE METHOD (RKTM)

Reformulating the block hybrid method with the coefficients as characterized by the butcher array

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $a_{ij} = \beta$

Gives

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{119}{120} & -\frac{125}{192} & \frac{19}{96} & -\frac{37}{960} \\ 3 & 0 & \frac{6}{5} & 0 & \frac{3}{2} & \frac{3}{10} \\ 2 & 0 & \frac{16}{15} & \frac{1}{3} & \frac{2}{3} & \frac{-1}{15} \\ 1 & 0 & \frac{6}{5} & \frac{-1}{3} & \frac{1}{6} & \frac{-1}{30} \\ \hline & 0 & \frac{6}{5} & \frac{-1}{3} & \frac{1}{6} & \frac{-1}{30} \end{array}$$

The butcher table is being rearranged with the off grid point appearing first, followed by the $c_{i's}$ in descending order. This is done in other to satisfy the consistency condition.

Using equation (1) we obtained an implicit 5-stage block Runge kutta type method of uniform order 4 everywhere on the interval of solution

$$y_{n+\frac{1}{2}} = y_n + h(0k_1 + \frac{119}{120}k_2 - \frac{125}{192}k_3 + \frac{19}{96}k_4 - \frac{37}{960}k_5)$$

$$y_{n+3} = y_n + h\left(0k_1 + \frac{6}{5}k_2 + 0k_3 + \frac{3}{2}k_4 + \frac{3}{10}k_5\right)$$

$$y_{n+2} = y_n + h\left(0k_1 + \frac{16}{15}k_2 + \frac{1}{3}k_3 + \frac{2}{3}k_4 - \frac{1}{15}k_5\right)$$

$$y_{n+1} = y_n + h\left(0k_1 + \frac{6}{5}k_2 - \frac{1}{3}k_3 + \frac{1}{6}k_4 - \frac{1}{30}k_5\right) \quad (15)$$

Where

$k_1 = f(x_n, y_n)$

$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + h\left(0k_1 + \frac{119}{120}k_2 - \frac{125}{192}k_3 + \frac{19}{96}k_4 - \frac{37}{960}k_5\right)\right)$$

$$k_3 = f\left(x_n + h, y_n + h\left(0k_1 + \frac{6}{5}k_2 - \frac{1}{3}k_3 + \frac{1}{6}k_4 - \frac{1}{30}k_5\right)\right)$$

$$k_4 = f\left(x_n + 2h, y_n + h\left(0k_1 + \frac{16}{15}k_2 + \frac{1}{3}k_3 + \frac{2}{3}k_4 + \frac{221}{2010}k_5\right)\right)$$

$$k_5 = f\left(x_n + 3h, y_n + h\left(0k_1 + \frac{6}{5}k_2 + 0k_3 + \frac{3}{2}k_4 + \frac{3}{10}k_5\right)\right)$$

Extending the method (15) in the butcher array

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2$ $\qquad$ $\bar{A} = \bar{a}_{ij} = \beta$ $\qquad$ $\beta = \beta e$

gives

0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{119}{120}$	$-\frac{125}{192}$	$\frac{19}{96}$	$-\frac{37}{960}$	0	$\frac{1057}{2880}$	$-\frac{557}{1536}$	$\frac{373}{2304}$	$-\frac{317}{7680}$
3	0	$\frac{6}{5}$	0	$\frac{3}{2}$	$\frac{3}{10}$	0	$\frac{63}{20}$	$-\frac{9}{32}$	$\frac{27}{16}$	$-\frac{9}{160}$
2	0	$\frac{16}{15}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{-1}{15}$	0	$\frac{94}{45}$	$-\frac{7}{12}$	$\frac{11}{18}$	$\frac{-7}{60}$
1	0	$\frac{6}{5}$	$\frac{-1}{3}$	$\frac{1}{6}$	$\frac{-1}{30}$	0	$\frac{167}{180}$	$-\frac{59}{96}$	$\frac{35}{144}$	$-\frac{9}{160}$
	0	$\frac{6}{5}$	$\frac{-1}{3}$	$\frac{1}{6}$	$\frac{-1}{30}$	0	$\frac{167}{180}$	$-\frac{59}{96}$	$\frac{35}{144}$	$-\frac{9}{160}$

NOTE:

The butcher table is being rearranged with the off grid points appearing first, followed by the $c_{i's}$ in descending order. This is done in other to satisfy the consistency condition.

Using Equation (4), we obtained an implicit 5 stage block Runge kutta Type method of uniform order 4.

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy_n' + h^2\left(0k_1 + \frac{1057}{2880}k_2 - \frac{557}{1536}k_3 + \frac{373}{2304}k_4 - \frac{317}{7680}k_5\right),$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(0k_1 + \frac{119}{120}k_2 - \frac{125}{192}k_3 + \frac{19}{96}k_4 - \frac{37}{960}k_5\right)$$

$$y_{n+3} = y_n + 3hy'_n + h^2\left(0k_1 + \frac{63}{20}k_2 - \frac{9}{32}k_3 + \frac{27}{16}k_4 - \frac{9}{160}k_5\right),$$

$$y'_{n+3} = y'_n + h\left(0k_1 + \frac{6}{5}k_2 + 0k_3 + \frac{3}{2}k_4 + \frac{3}{10}k_5\right)$$

(16)

$$y_{n+2} = y_n + 2hy'_n + h^2\left(0k_1 + \frac{94}{45}k_2 - \frac{7}{12}k_3 + \frac{11}{18}k_4 - \frac{7}{60}k_5\right),$$

$$y'_{n+2} = y'_n + h\left(0k_1 + \frac{16}{15}k_2 + \frac{1}{3}k_3 + \frac{2}{3}k_4 - \frac{1}{15}k_5\right)$$

$$y_{n+1} = y_n + hy'_n + h^2\left(0k_1 + \frac{167}{180}k_2 - \frac{59}{96}k_3 + \frac{35}{144}k_4 - \frac{9}{160}k_5\right),$$

$$y'_{n+1} = y'_n + h\left(0k_1 + \frac{6}{5}k_2 - \frac{1}{3}k_3 + \frac{1}{6}k_4 - \frac{1}{30}k_5\right)$$

where

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_2 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hy'_n + h^2\left(0k_1 + \frac{1057}{2880}k_2 - \frac{557}{1536}k_3 + \frac{373}{2304}k_4 - \frac{9757}{1543680}k_5\right),$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(0k_1 + \frac{119}{120}k_2 - \frac{125}{192}k_3 + \frac{19}{96}k_4 - \frac{37}{960}k_5\right))$$

$$k_3 = f(x_n + h, y_n + hy'_n + h^2\left(0k_1 + \frac{167}{180}k_2 - \frac{59}{96}k_3 + \frac{35}{144}k_4 - \frac{2587}{96480}k_5\right),$$

$$y'_{n+1} = y'_n + h\left(0k_1 + \frac{6}{5}k_2 - \frac{1}{3}k_3 + \frac{1}{6}k_4 - \frac{1}{30}k_5\right))$$

$$k_4 = f(x_n + 2h, y_n + 2hy'_n + h^2\left(0k_1 + \frac{6937}{3015}k_2 - \frac{7}{12}k_3 + \frac{2113}{2412}k_4 + \frac{163}{3015}k_5\right),$$

$$y'_n + h\left(0k_1 + \frac{16}{15}k_2 + \frac{1}{3}k_3 + \frac{2}{3}k_4 + \frac{221}{2010}k_5\right))$$

$$k_5 = f(x_n + 3h, y_n + 3hy'_n + h^2\left(0k_1 + \frac{63}{20}k_2 - \frac{9}{32}k_3 + \frac{27}{16}k_4 + \frac{2237}{10720}k_5\right),$$

$$y'_n + h\left(0k_1 + \frac{6}{5}k_2 + 0k_3 + \frac{3}{2}k_4 + \frac{3}{10}k_5\right))$$

4. NUMERICAL EXAMPLES

To study the efficiency of the method we present some numerical examples widely used by renown authors such as Yahaya and Adegboye (2013), Sunday , Odekunle & Adesanya (2013).

Problem 1: (Mixture Model)

In an oil refinery, a storage tank contains 2000 gal of gasoline that has $100lb$ of an additive dissolve in it. In the preparation for winter weather, gasoline containing $2lb$ of additive per gallon solution is pumped into the tank at a rate of 40 gal/min. The well mixed solution is pumped out at a a rate of 45 gal/min. Using a numerical integrator, how much of the additive is in the tank 0.1 min, 0.5 min and 1 min after the pumping process begins?.

Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. Thus, the IVP modeling the mixture process is,

$$y' = 80 - \frac{45y}{(2000-5t)}, \qquad y(0) = 100$$

With the theoretical solution

$$y(t) = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9$$

Applying the Runge Kutta Type Method (RKTM) to problem1 yields the following result as indicated in table 1 at different values of time t.

Problem 2

$$y'' = -y \qquad y(0) = 1, \quad y'(0) = 1 \qquad h = 0.1 \;\; 0 \le x \le 0.4$$

Exact Solution

$$y(x) = \cos x + \sin x$$

Applying the Runge Kutta Type Method (RKTM) to problem 2 yields the following result as indicated in table 2.

t	Exact Solution	Computed Solution	Error
0.1	107.766230117	107.7681482	-0.001918083
0.2	115.514940919	115.5130074	0.001933519
0.3	123.2461630501	123.2461594	3.6501E-06
0.4	130.959927109	130.9620430	-0.002115891
0.5	138.6562636455	138.6547810	0.001482645
0.6	146.3352031660	146.3358772	-0.000674034
0.7	153.99677613051	153.9998481	-0.003071969
0.8	161.6410129533	161.6408155	0.000197453

0.9	169.26794400299	169.2702062	-0.002262197
1.0	176.87759960259	176.8826137	-0.005014097

Table 1: Performance of RKTM on problem 1

x	Numerov Onumanyi (2002)	RKTM
0.1	2.E-07	1.47E-07
0.2	4.E-07	1.99E-07
0.3	6.E-07	-4E-09
0.4	7.E-06	4.6E-08

Table 2: Absolute Error on Performance of RKTM on problem 2

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