MODELING THE TOPOLOGICAL RELATIONSHIP OF SPATIAL OBJECTS USING EGENHOFER MATRICES

BY

IJIEBOR, Otaigboria Godwin
MTech/SPS/2016/6158

DEPARMENT OF MATHEMATICS
FEDERAL UNIVERSITY OF TECHNOLOGY
MINNA

JULY, 2019

ABSTRACT

The analytical needs in geographic information systems (GIS) have led to the interpretation of formal methods of modeling the topological relationship of spatial objects using Egenhofer-matrices. As a result, we investigate the algebraic approach for the structural analysis of a spatial topology- Simplices, The Order of Simplices, Faces of Simplices, Simplicial Complex, Skeletons of Simplicial Complex. Oriented Simplex, Connectedness of Spatial Objects. The matrix interpretation of the eight spatial topological relations matrices of two Egenhofer 4-intersection models and that of the two Egenhofer 9-intersection models using matrix addition and multiplication modulo 2. We also developed the 8-intersection and The 2x4 Matrix Representation of Topological Relations of Three Objects A, B and C. 16- intersection models and The 2x8 Matrix Representation Of Topological Relations Of Four Objects A, B, C And D. using the Egenhofer's 4 and 9- Intersections Models.

TABLE OF CONTENTS

CHAPTER TWO

2.0 LITERATURE REVIEW

CHAPTER THREE

3.0 RESEARCH METHODOLOGY

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

This work was motivated by the analytical need for a formal understanding of modeling the topological relationship of spatial objects using Egenhofer-matrices within the realm of geographic information systems. To display, process or analyze spatial information, users select data from a Geographic Information System (GIS) by asking queries. Almost any GIS query is based on spatial concepts. Many queries explicitly incorporate spatial relations to describe constraints about spatial objects to be analyzed or displayed. The lack of comprehensive theory of spatial relations has been a major impediment to any GIS implementation. The development of a theory of spatial relations is expected to provide answers to the following questions (Abler, 1987):

(i) What are the fundamental geometric properties of geographic objects needed to describe their relations?

(ii) How can these relations be defined formally in terms of fundamental geometric properties?

(iii) What is a minimal set of spatial relations?

In addition to the purely mathematical aspects, cognitive, linguistic and psychological considerations must also be included if a theory about spatial relations applicable to the real world problems is to be developed (Talmy, 1983 and Herskovits,1986). Within the scope of this thesis, only the formal, mathematical concepts which have been partially provided from point-set topology will be considered.

The variety of spatial relations can be grouped into three different categories as follows:

(i) Topological relations which are invariant under topological transformations of the reference objects (Egenhofer, 1989; Egenhofe and Herring, 1990)

(ii) Metric relations in terms of distances and directions, (Peuquet and Ci-Xiang,1986) and

(iii) Relations concerning the partial and total order of spatial objects (Kainz ,1990) as described by prepositions such as in front of, behind, above and below(Freeman 1975; Chang *et al*., 1989; Hernamdez, 1991)Within the scope of this study, only topological spatial relations are discussed.

1.1.2 Object and object identity

Formally, an object can be defined as an identifiable entity that has a precise role for an application domain (Roy and Clement, 1994; Blaha and Premerlani, 1998). To constitute an entity, something must be identifiable (have identity), relevant (be of interest to the application domain) and describable (have characteristics) (Chen, 1976); cited in (Mattos *et al*, 1993). By means of the modeling process, each entity relevant to an application domain is represented by a corresponding object in the data model. The object in the model should have properties that describe the characteristics of the corresponding entity in the universe of discussion.

In an object-oriented system, each object is unique. This uniqueness of an object is achieved by means of the object identity. Object identity is that property of an object that uniquely distinguishes it from all other objects (Khoshafian and Abnous, 1995). By introducing a unique identity for each object, different objects can be distinguished from each other without the need to compare their attributes and behavior (Ellmer, 1993). The object identity is usually system generated, unique to that object and invariant for the object lifetime (Cooper, 1997).

1.1.3 Spatial relationships

Spatial relationships describe the relationships between spatial objects and geometric elements (Raza, 2001). In spatial databases, spatial relationships are needed for two main purposes as follows:

(a) For performing spatial queries: Queries in spatial databases or GIS are often based on the relationships among spatial objects. For example, “Retrieve all parcels that are adjacent to parcel A”. Such queries involve spatial conditions which standard query definition languages like Standard Query Language (SQL) do not adequately support. Spatial relationships are needed at both the query formulation and processing levels (Clementini *et al,* 1993).

(b) For enforcing consistency of the database. Spatial relationships are also used to formulate consistency constraints in spatial databases. For example, a violation of the constraint that two parcels should not overlap in a cadastral database can be detected by checking the spatial relationship that exists between the two parcels (Kufoniyi *et al*.,1994). The spatial relationships provide the means for defining and monitoring these constraints in the database. Hence the formalization of the basic spatial relationships is an essential component in GIS development.

1.1.4 Topological relationships

Topology is that branch of mathematics that studies the characteristics of geometry that remain invariant under certain transformations (topological mapping or homeomorphism) (Kainz, 1995). A topological property is that which is preserved under topological transformations such as scaling, translation and rotation. Examples of topological properties are connectivity, adjacency and so on. There are two general branches of topology, both of which are applied in spatial data handling (Kainz and Worboys1995).

These are:

(a) Point-set (or analytic) topology: This focuses on set of points and is based on real analysis, using concepts such as open sets, neighborhood and convergence.

(b) Algebraic (or combinatorial) topology: This uses algebraic means to describe the spatial relationships and is based on such concepts as simplified and cell complex and graph theory.

The point-set approach is the most general model for topological relationships (Raza, 2001). Using the point-set approach, topological relationships are defined in terms of three fundamental primitives of object parts, which are interior denoted as (°), boundary denoted as (∂) and exterior or closure denoted as (¯), which themselves are defined based on neighborhood concepts (Egenhofer and Herring,1991). Topological models include:

(i) The 4-Intersection model.

(ii) The 9-Intersection model.

(iii) The dimension extended model.

1.2 Significance of the Study

The significance of this study, modeling the topological relationship between spatial objects using Egenhofer-matrices, cannot be over emphasized due to it applications in the following important areas of the real world:

(a) Natural resource-based like:

(i) Management of areas: agricultural lands, forest, recreation resources, wildlife habitat analysis, migration routes planning.

(ii) Environmental impact analysis.

(iii) Toxic facility sitting.

(iv) Groundwater modeling.

(b) Land parcel-based like:

(i) Zoning, subdivision plan review.

(ii) Environmental impact statements.

(iii) Water quality management.

(iv) Facility management electricity, gaze, clean water, used water and so on.

1.3 Scope and limitation of the Study

The essence of this research is to analytically investigate the topological relationship between spatial objects using Egenhofer-matrices. The study is however, limited to mathematical modeling, sets and matrices.

1.4 Aim and Objectives of the Study

The aim of this study is to analytically model the topological relationship between spatial objects using Egenhofer-matrices.

The objectives of this study are to understand and apply;

(i) The algebraic approach for structural analysis of a spatial topology.

(ii) The matrix interpretation of the spatial topological relations of two Egenhofer 4-Intersection models using matrix addition and multiplication modulo 2.

(iii) The matrix interpretation of the spatial topological relations of two Egenhofer. 9-Intersection models using matrix addition and multiplication modulo 2 .

(iv) Derivation of the 8-intersection and the 16-intersection models and their corresponding eight spatial topological relations using Egenhofer matrices.

1.5.1 Geometry

This is a branch of mathematics concerned with questions of shape, size, relative position of Figs, and the properties of space. A mathematician who works in the field of geometry is called a Geometer (Encyclopedia of Science Clarified, 2013).

1.5.2 Spatial

This is related to space and the position, size, shape etc of things in it, (Encyclopedia of Science Clarified, 2013).

1.5.3 Topological Constraints: These are constraints that satisfy topological conditions. Constraint is a thing that limits or restricts something, (Encyclopedia of science clarified, 2013).

1.5.4 Sets

Cantor (1895) defined a set as any collection M of certain distinct objects of our thought or intuition (called elements of M) into a whole. It is a collection of objects, called the elements or member of the set. The objects could be anything (planets, squirrel, characters in Shakespeare's play, or others) but for us they will be mathematical objects such as numbers, or sets of numbers. We write $x \in X$ if x is an element of the set X and $x \notin X$ if x is not an element of X. Sets are determined entirely by their elements. Thus the sets X, Y are equal written $X = Y$, if $x \in X$ if and only if $x \in Y$. An empty set ($\emptyset$) is a set without an element. If $X \neq \emptyset$, meaning that X has atleast one element, then we say that X is non-empty.

1.5.5 Set operations

The intersection $A \cap B$ of two sets, A, B is the set of all elements that belong to both A and B; that is $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. Two sets A, B are said to be disjoint if $A \cap B = \emptyset$; that is, if A and B have no elements in common.

The union $A \cup B$ is the set of all elements that belong to A or B; that is x ∈ A ∪B if and only if x∈A or x∈B.

Note that we always use "or" in an inclusive sense, so that x ∈ A ∪B if x is an element of A or B, or both A and B.

The difference of two sets A and B is the set of elements of B that do not belong to A, that is $B \setminus A = \{x \in B : x \notin A\}$.

1.5.6 Relations

According to (Science Encyclopedia Clarified, 2013), a binary relation R on set X and Y is a definite relation between elements of X and elements of Y. We write xRy if **x**∈X and y∈Y are related. One can also define relations on more than two sets but we shall consider only

binary relations and refer to them simply as relations. If X = Y then we call R a relation on X. The set of all x-values is called the domain and the set of all y-values is called the range. Relations could be the following:

(a) Equivalence relations: The equivalence relation is a binary relation that is reflexive, symmetric and transitive. For example, for any objects a, b and c
 a = a (reflexive property), if a = b then b = a (symmetric property) and
 if a = b and b = c then a = c (transitive property).

(b) Transitive relations: A relation in a set A is called transitive if and only if (a,b) ∈ R and (b,c) ∈ R then (a,c) ∈ R for all a, b, c∈ A.

(c) Void relations: A relation R in a set A is called void relation or empty relation, if no element of set A is related to any element of A. Hence R = ∅ which is a subset of

 A x A.

(d) Symmetric relations: A relation R in set A is said to be symmetric if and only if aRb implies bRa for all a,b ∈ A.

(e) Identity relations: For a given set A, I = {(a,a), a ∈ A} is called the identity relation in A. In identity relation every element of A is related to itself only.

(f) Reflexive relations: A relation is said to be reflexive if and only if aRa, for all a ∈ A. It means every element of A is related to itself.

1.5.7 Matrices

(Kreyszig, 2004) defines a matrix (plural: matrices) as a rectangular array of numbers, symbols or expressions, arranged in rows and columns enclosed in brackets: There are m rows which are horizontal and the n columns are vertical. Each element of a matrix is often denoted by a variable with two subscripts. For example, $a_{2,1}$ represents the element at the second row and first column of a matrix A. In mathematics a matrix is a rectangular array of numbers, symbols or expressions, arranged in rows and columns. The size of a matrix is defined by the number of rows and columns. A matrix with m rows and n columns is called m x n matrix or m by n matrix, while m and n are called its dimensions.

1.5.8 Mathematical modelling

(Bellomo *et al*, 1995), defines mathematical modeling as the process of using various mathematical structures such as graphs, equations and diagrams to represent real world situations. The process of developing a mathematical model is termed mathematical modeling. A mathematical model may help to study the effects of different components and to make a prediction about a behavior.

1.5.9 Simplex

According to (Giblin 1977), a simplex is a minimal object that exists for each dimension in the spatial dimensions in which spatial objects are classified.

1.5.10 Simplicial Complex

A simplicial complex is a finite collection of simplices and their faces. Simplices is the plural of simplex.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Related Literatures

This section reviews some related literatures on modeling the topological relationship of spatial objects using Egenhofer matrices. Some of which are considered and compared in this work.

2.2 Spatial Relations

According to (Egenhofer, 1989), spatial relation is a means of modeling a particular property of the spatial relationship which exists between two or more objects. Spatial relations may be characterized as topological, metric and other relations. Topological relations model properties which are invariant under consistent topological transformations such as rotation, translation and scaling. Metric relations model properties concerning distance and direction. Order relations model properties concerning the partial and total order of objects as described by prepositions such as in front of, behind, above and below. Many spatial relations cannot be classified as exclusively topological, metric or order. Such relations include the align-along-road relation existing between a set of buildings and a road. A number of authors have considered the effect splitting and merging objects has on topological relations.

Research on the topic of spatial relations is motivated by a broad spectrum of possible application areas. Spatial relations can be used to describe constraints which specify a subset of spatial objects. For example one may specify the subset of objects which fall within a given radius of a point using a metric relation. Spatial relations can also be used as a platform for spatial inference and qualitative spatial reasoning. For example if it is specified using spatial relations that an object A is contained within an object B which in turn is contained within an object C, it is straight forward to infer that A is contained within C. Some spatial relations (transitive spatial relations) have a corresponding easily interpretable natural language expression which offers the potential for the linguistic interaction with spatial data. Other applications of spatial relations include robotics and high-level computer version. Many set of spatial relations have been proposed but the most predominant are the intersection models of (Egenhofer, 1991) and the Region Connected Calculus (RCC) of (Randell *et al*, 1992). Due to their ubiquitous nature we do not describe these in detail suffice to say that each consists entirely of binary topological relations and both sets are in fact equivalent.

According to (Cohn and Hazarika 2001) not all sets of spatial relations are equally useful and the actual set must be relevant to the task been performed. One of the main goals in the research field of spatial relations is to determine if a single universal set of relations can be defined which is pragmatic with respect to many applications. A promising approach towards achieving this goal proposes to model the aspects of spatial relationships which the human cognition models. This has led to the use of the term cognitively adequate model to describe a set of spatial relations which are believed to be an accurate model of these aspects. Initial studies focused on the role topological relations play in defining cognitively adequate models. The study of (Mark and Egenhofer, 1995), suggested that topological relations alone and in particular the intersection models of (Egenhofer, 1991) are sufficient to achieve cognitive adequacy.

This lead to the famous expression “topological matters, metric refines” by (Egenhofer and Mark, 1995). This claim was supported by the works of (Clementini *et al.*, 1993) and (Renz *et al.*, 2000) but these authors claimed that a finer level of granularity than the intersection models was necessary to achieve cognitive adequacy. However a study by (Sheriff *et al*, 2016) suggests that topological relations alone may not be sufficient for cognitive adequacy; the authors propose that a combination of topological and metric relations are necessary. The recent work of (Klippel, 2012) suggests that semantics must also be considered if one wishes to define a set of spatial relations which are cognitively adequate.

Spatial relations may also be categorized as qualitative or non-qualitative relations. Qualitative relations model properties which are of a vague or fuzzy nature and possibly context dependent. Determining the existence of such relations generally requires one to model some aspect of human cognition. Examples of qualitative relations include a relation which indicates if an object is nearly completely contained inside another or a relation which indicates if an object is between two others. The spatial relation indicating if a road entering a housing estate, is also a qualitative relation. On the other hand, non-qualitative spatial relation model properties which are not of a vague or fuzzy nature and not context dependent. Such relations have a precise geometrical definition. Examples include binary relations which indicate if two lines intersect or if an object is completely contained inside another.

2.3 Topological Spatial Relations (Point-Set Topology)

According to (Egenhofer and. Franzosa 1990), the model of topological spatial relations is based on the point-set topological notions of interior and boundary. Let X be a set. A topology on X is a collection $\mathcal{A}$ of subsets of X that satisfies three conditions as follows:

(i) The empty set and X are in $\mathcal{A}$

(ii) $\mathcal{A}$ is closed under arbitrary unions and

(iii) $\mathcal{A}$ is closed under finite intersections.

A topological space is a set X with a topology A on X. The sets in a topology on X are called open sets, and their complements in X are called closed sets. The collection of closed sets:

(i) Contains the empty set and X;

(ii) Is closed under arbitrary intersections; and

(iii) Is closed under finite unions.

Via the open sets in a topology on a set X, a set-theoretic notion of closeness is established. If U is an open set and x $\in U$, then U is said to be a neighborhood of x. This set-theoretic notion of closeness generalizes the metric notion of closeness. A metric (d) on a set X induces a topology on X, called the metric topology defined by d. This topology is such that U$\subset X$ is an open set if for each x $\in U$ $there\ existan\ \varepsilon > 0\ such$ that the d-ball of radius ε around x is contained in U. A d-ball is the set of points whose distance from x in the metric d is less than ε, that is {y$\in X' d(x,y) < \varepsilon$}.

Suppose X is a set with a topology $\mathcal{A}$. If S is a subset of X, then S inherits a topology from $\mathcal{A}$. This topology is called the subspace topology and is defined such that U⊂S is open in the subspace topology, if and only if U-S∩ $Vfor$some set V $\in \mathcal{A}. U$ $u$nder such circumstance, S is called a subspace of X. Some vital topological properties are:

(a) Interior: Given Y⊂X, the interior of Y denoted by Y°, is defined to be the union of all open sets that are contained in Y, that is the interior of Y is the largest open set contained in Y. y is in the interior of Y if and only if there is a neighborhood of y contained in Y, that is
y∈Y°$, if and\ only\ if\ , there\ is\ an\ open\ set\ Usuch\ that\ y \in U \subset Y.$ The interior of a set could be empty, for example the interior of the empty set is empty. The interior of X is X itself. If U is open then U° $= U. If\ Z \subset Y\ then\ Z° \subset Y°$.

(b) Closure/Exterior: The closure of Y, denoted by $\overline{Y}$, is defined to be the intersection of all closed sets that contain Y, that is the closure of Y is the smallest closed set containing Y. It follows that y is in the closure of Y if and only if every neighborhood of y intersects Y, that is y $\in \overline{Y}$ if and only if U $\cap Y \neq \emptyset$ for every open set U containing y. The empty set is the only set with empty closure. The closure of X is X itself. If C is closed then $\overline{C}$ = C. If Z⊂Y then $\overline{Z} \subset \overline{Y}$.

(c) Boundary: The boundary of Y denoted by ∂Y, is the intersection of the closure of Y and the closure of the complement of Y, that is $\partial Y = \overline{Y} \cap \overline{X - Y}$. The boundary is a closed set. It follows that y is in the boundary of Y if and only if every neighborhood of y intersects both Y and its complement, that is y $\in \partial Y$ *if and only if* $U \cap Y \neq \emptyset$ *and* $U \cap (X - Y) \neq \emptyset$ for every open set U containing y. The boundary can be empty, for example, the boundaries of both X and the empty sets are empty.

2.3.1 Topological spatial relations between two sets

According to (Egenhofer and Franzosa, 1990), this model of describing the topological spatial relations between two subsets A and B, of a topological space X is based on a consideration of the four intersections of the boundaries and interiors of the two sets A and B, that is $\partial A \cap \partial B, A^\circ \cap B^\circ, \partial A \cap B^\circ, and A^\circ \cap \partial B$.

Definition 2.3 Let A, B be a pair of subsets of a topological space X. A topological spatial relation between A and B is described by a four-tuple of values of topological invariants associated to each of the four sets $\partial A \cap \partial B, A^\circ \cap B^\circ, \partial A \cap B^\circ and A^\circ \cap \partial B$ respectively.

A topological spatial relation between two sets is preserved under homeomorphism of the underlying space X. Specifically, if f:X $\rightarrow Y$ is a homeomorphism and A,B⊂X, then $\partial A \cap \partial B, A^{\circ} \cap B^{\circ}, \partial A \cap B^{\circ} and\ A^{\circ} \cap \partial B$ are mapped homeomorphically onto $\partial f(A) \cap \partial f(B), f(A^{\circ}) \cap f(B^{\circ}), \partial f(A) \cap f(B) and\ f(A^{\circ}) \cap \partial f(B)$respectively.

Since the topological spatial relation is defined in terms of topological invariants of these intersections, it follows that the topological spatial relation between A and B in X is identical to the topological spatial relation between f(A) and f(B) in Y.

A topological spatial relation is denoted here by the four-tuples above. The entries correspond in order to the values of topological invariants associated to the four set-intersections. The first intersection is called the boundary-boundary intersection, the second is the interior-interior intersection, the third is the boundary – interior intersection and the fourth is the interior-boundary intersection.

2.4 Spatial Topology

According to (Jiang and Claramunt 2004), spatial topology, is a model used to identify spatial relationships of geographic objects via common objects is introduced. The model is introduced by starting with an example from social networks as follows. Two persons A and B, are not acquainted in a general social sense as colleagues or friends, but they are considered as “adjacent”, as both, for instance belong to the same geographic information community. The community or organization people belong to is a context that keeps people “adjacent”. Extending the case into a geographic context, we can say that for instance, two houses are adjacent if they are both situated along the same street, or within the same district. It is further stated that a pair of objects sharing more common objects is more “adjacent”. The model is aimed at representing spatial relationships of various objects as a simplicial complex. To this end, some formal definitions are also presented in the context of GIS.

According to (Jiang and Itzhak, 2005), a definition of map layer according to set theory is used for a start. Map layer is defined as a set of spatial objects at a certain scale in a database or on a map. For example, $M = \{o_1, o_2, \ldots, o_n\}$, or $M = \{0_i, |i=1,2,\ldots n\}$ denotes a map layer (using a capital letter) that consist of multiple objects (using small letters). The objects are put into four categories: point, line, area and volume objects in terms of basic graph primitives. The definition of objects must be appropriate with respect to the modeling purpose. For instance, a street layer can be considered as a set of interconnected street segments, or an interconnected named street depending on the modeling purpose (Jiang and Claramunt, 2004), a city layer can be represented as a point or area object depending on the representation scale.

A spatial topology (T) is defined as a subset of the Cartesian product of two map layers, L and M, denoted by T = L x M. To set up a spatial topology, we examine the relationship of every pair of objects from one map layer to another (As distinct to topological relationships based on possible intersections of internal, external and boundary of spatial extended objects, (Egenhofer, 1991), we simply take a binary relationship). That is, if an object ℓ is within, or intersects, another object m, we say there is a relationship $\lambda = (\ell, m)$, otherwise no relationship, $\lambda = \emptyset$ [Note: the pair (ℓ, m) is ordered and (m, ℓ) represent an inverse relation denoted by λ^{-1}]. The relationship can be simply expressed as "an object has a relationship to a contextual object". If a set of primary objects shares a common contextual object, we say the set objects are adjacent or proximate. Thus two types of map layers can be distinguished: primary layer for the primary objects, with which a spatial topology is to be explored, and contextual layer, whose objects constitute a context for the primary objects. It is important to note that the contextual layer can be given in a rather abstract way with a set of features (instead of map objects). This way, the relationship from the primary to contextual objects can be expressed as "an object has certain features". For the sake of convenience and with notation T = L x M, we refer to the first letter as the primary layer and the second the contextual layer.

The notion of spatial topology represent a network view as how the primary objects become interconnected via the contextual objects. A spatial topology can be represented as a simplicial complex. Before examining the representation, we turn to the definition of simplicial complex (Atkin, 1977). A simplicial complex is the collection of relevant simplicies. We assume the elements of a set A from simplicies (or polyhedral, denoted by σ^d where d is the dimension of the simplex); and the elements of a set B form vertices according to the binary relation λ, indicating that a pair of elements (a_i, b_j) from two different sets A and B, $a_i \in A$ and $bj \in B$, are related. The simplicial complex can be denoted as KA(B;λ). In general, each individual simplex is expressed as q-dimensional geometric Fig with q+1 vertices. The collection of all the simplices forms the simplicial complex. For every relation λ it is feasible to consider the conjugate relation, λ-1, by reversing the relations between two sets A and B by transposing the original incidence matrix. The conjugate structure is denoted as $K_B(A;\lambda^{-1})$.

A spatial topology can be represented as a simplicial complex where the simplices are primary objects, while vertices are contextual objects. Formally, the simplicial complex for the spatial topology T = L x M is denoted by K_L (M;λ), where L represents the primary layer and M the contextual layer, the relation between a primary object and a contextual object λ = (ℓ, m). A spatial topology can be represented as an incidence matrix A, where A simplicial complex with simplices as primary objects and vertices as contextual objects the columns represent objects with primary layer and the rows represent the objects with contextual layer. Formally, it is represented as follows,

$$\mathbf{A_{ij}} = \begin{cases} \mathbf{1}: \boldsymbol{if\lambda} = & (\boldsymbol{i}, \boldsymbol{j}) \\ \mathbf{0}: \boldsymbol{if\lambda} = & \boldsymbol{\phi} \end{cases}$$

The entry 1 of the matrix indicates that a pair of objects (i, j) respectively from the two different layers L and M (that is $i \in L$ and $j \in M$) is related, while the entry 0 represents no relationship between the pair of objects. The incidence matrixA_{ij} is not symmetric as $\lambda \neq \lambda^{-1}$ in general.

Spatial topology is not intended to replace the existing topology or spatial relationship representation, but to extend and enhance the existing ones for more advanced spatial analysis and modeling. In this respect, the simplicial complex representation provides a powerful tool for exploring structural properties of spatial topology. We take a look at a simple example of simplicial complex as follows:

They examine with environmental GIS, three pollution sources whose impact areas are identified through a buffer operation, as a polygon layer. It is likely that the three polluted zones overlap each other. This constitute a map layer, denoted by $Y = \{y_j \mid j = 1, 2, 3\}$.

To assess the pollution impact on a set of locations with another map layer $X = \{x_i \mid I = 1, 2, \ldots 6\}$, it is not sufficient to just examine which location is within which pollution zones. They took a step further to put all locations within an interconnected context (a network view) using the concept of spatial topology. For instance, location x_1 is out of pollution zone of y_2, but it may get polluted through x_2 and x_5, assuming the kind of pollution is transmittable. Only under the network view are we able to investigate the pollution impact thoroughly and matrix equation by (Jiang and Claramunt, 2004).

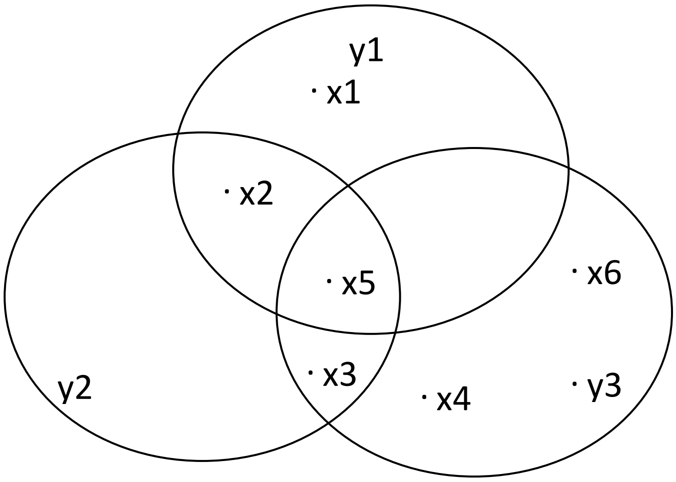


Fig 2.1: A Simple Example of Spatial Topology

In the above example, the locations under pollution impact are of primary interest. The spatial topology can be represented as an incidence matrix as follows:

$$\Lambda_{36} = \begin{bmatrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & 1 & 1 & 0 & 0 & 1 & 0 \\ y_2 & 1 & 1 & 0 & 0 & 1 & 0 \\ y_3 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For the primary layer, the six locations in Fig 2.1 and with respect to the columns of the matrix can be represented as six simplices as follows:

σ^0 (x_1) = $< y_1 >$

σ^1 (x_2) = $< y_1, y_2 >$

σ^1 (x_3) = $< y_2, y_3 >$

σ^0 (x_4) = $< y_3 >$

σ^2 (x_5) = $< y_1, y_2, y_3 >$

σ^0 (x_6) = $< y_3 >$

where the right hand side of the equation represents vertices that consist of a given simplex. The dimension of a simplex is represented by a superscript. For instance, σ^2 (y_5) denotes a two dimensional simplex or a two dimensional face that consists of three vertices $y_{1,} y_{2,}$ and y_3.

Note that the primary and contextual objects are relative and transversal (interchangeable) depending on different application interests. If for instance, we take Y as the primary layer (whether it makes sense is another issue which we will not consider here), that is to transpose the incidence matrix defined in equation (1), then we would have three simplices as follows:

$$\sigma^2\ (y_1) = < x_1, x_{2,} x_5 >$$

$$\sigma^2\ (y_2) = < x_2, x_{3,} x_5 >$$

$$\sigma^3\ (y_3) = < x_3, x_{4,} x_5, x_6 >$$

We have noted that a spatial topology can be represented as a geometric form (the simplicial complex). It is important to note that the geometric representation makes little sense when the dimension of the simplicial complex exceeds three because of a human perceptual constraint, but no such constraint for the algebraic approach.

2.5 Formalism for Spatial Relationships

Three classes of spatial relationships are discriminated which are based upon different spatial concepts, (Pullar and Egenhofer, 1988). It appears natural for each class to develop an independent formalism describing the relationships.

(i) Topological relationships are invariant under topological transformations such as translation, scaling and rotation. Examples are terms like neighbor and disjoint.

(ii) Spatial order and strict order relationships rely upon the definition of order and strict order respectively. In general, each order relation has a converse relationship. For example behind is a spatial order relation based upon the order of preference with the converse relationship in front of.

(iii) Metric relationships exploit the existence of measurements, such as distances and directions. For instance "within 5 miles from the interstate highway 195" describes a corridor based upon a specific distance.

This classification is not complete since it does not consider fuzzy relationships such as close and next to, (Robinson and Wong, 1987), or relationships which are expressions about the motion of one or several objects such as through and into, (Talmy, 1983). These types of relationships can be considered as combinations of several independent concepts. Motion for example may be seen as a combination of spatial and temporal aspects. So far three different formal approaches for the definition of spatial relationships exist in this literature. The first one is based upon distance and direction in combination with the logical connectors AND, OR and NOT, (Peuquet, 1986). The relationship disjoint (A,B) for example, is defined by the constraint that the distance from any point of object A to any point of object B is greater than 0. This approach has two severe deficiencies as follows:

(i) It is not possible to model inclusion or containment unless negative distances are introduced (Peuquet, 1986) defines the relationship touching, for example, by the distance which equals to zero at a single location and is never less than zero; however, by definition, distances are symmetric and a violation of this principle would lead to strange geometries.

(ii) The lack of appropriate computer numbering systems for geometric applications, (Franklin, 1984) impedes the immediate application of coordinate geometry and distance-based formalisms for spatial relationships. The assumption that every space has a metric is unnecessarily complex and promotes the confusion about two different concepts like metric and topology. The formal definition of spatial relationships in the context of a geo-relational algebra is based upon the representation of spatial data in the form of point sets, (Guting, 1988).Binary relationships are described by comparing the points of two objects with conventional set operators such as equal and less than or equal. For example, the relationship inside (x,y) is expressed by points (x) ⊆ points (y). This point set approach is in favour of raster representations in which each object can be represented as a set of pixels, but it is not easily applicable to vector representation. A serious deficiency inherent to the point set approach is that only a subset of topological relationships is covered with this formalism. While equality, inclusion and intersection can be described, the point set model does not provide the necessary power to define neighborhood relationships. A crucial characteristic of neighborhood is that the boundaries of two objects have common parts, while the interior do not. These distinct object parts cannot be distinguished with the point set model; therefore, pure point set theory is not applicable for the description of those relationships which rely upon interior or bounding parts only.

(iii) A third approach was developed for the representation of relationships among 1-dimensional intervals in a 1-dimensional space (Egenhofer, 1987), (Pullar and Egenhofer, 1988). It is based upon the intersection of the boundary and interior of the two objects to be compared and distinguishes only between “empty” and “non-empty” intersection.

Table 2.5: The Minimal Set of Topological Relationships among Intervals in a One- Dimensional Space described by the Intersection Of Boundaries ($\partial \cap \partial$), Interiors ($^\circ \cap {}^\circ$), Boundary with Interior ($\partial \cap {}^\circ$) and Interior with Boundary ($^\circ \cap \partial$).

(i1,i2)	$\partial \cap \partial$	$^\circ \cap {}^\circ$	$\partial \cap {}^\circ$	$^\circ \cap \partial$
Disjoint	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Meet	$\neq \emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Overlap	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
Inside	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
Contains	$\emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
Covers	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
Covered by	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
Equal	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\emptyset$

Source: Pullar and Egenhofer (1988).

This method is superior to the other two formalism because it describes topological relations by purely topological properties.

CHAPTER THREE

3.0 RESEARCH METHODOLOGY

3.1 The Algebraic Approach for Structural Analysis of Spatial Topology

Here we analytically and structurally investigate the following concepts:

3.1.1 Simplices

An n-simplex (σ^n) is a spatial (geometric) object with (n+1) vertices which leave in an n-dimensional space and cannot fit in any space of smaller dimension (less than zero), (Casey and Alessandra, 2016). A vertex is a point in space where two lines or edges meet to form an angle. It is also called the node of a graph and the point at which the sides of an angle intersect.

3.1.2 Formation of Some Simplices

Generally, simplices are formed by using the (n+1) vertices which generate an object of dimension n where n is the number of simplex.

(i) Zero-Simplex: This is a n=0 simplex with 0+1= 1 vertex that generate a point P_o. Thus a 0-simplex is a point P_o. For example, the origin or another point in the coordinate axis is a 0-simplex.

It is denoted by $\sigma^0 = < P_0 >$

Figs 3.1a – 3.1d formation of some simplices by (Author).

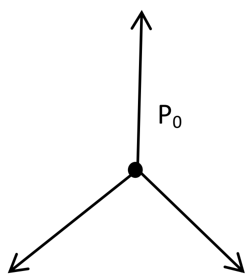


Fig 3.1a: 0-Simplex

a) 1-Simplex: This is a n=1 simplex with 1+1=2 vertices which generate a segment by connecting the two vertices. The 1-simplex is a line segment with two end-points P_o and P_1 which in construction produces two 0-subsimplices. It is denoted by

σ¹= $< P_{0,}P_1 >$

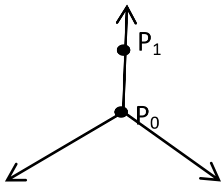


Fig 3.1b: 1-Simplex

b) 2-Simplex: This is a n=2 simplex with 2+1=3 vertices which generate a triangle. The 2-simplex produces two 1-subsimplices by connecting all possible pairs of two points $< P_{0,}P_1 >$ and $< P_{0,}P_2 >$ and three 0-subsimplices $< P_{0,}P_{1,}P_2 >$. It is denoted by σ² = $< P_{0,}P_{1\,,}P_2 >$

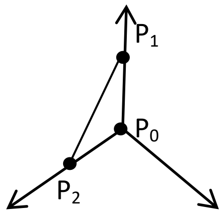


Fig 3.1c: 2-Simplex

3-Simplex: This is a n=3 simplex with 3+1=4 vertices which produce a solid tetra-hedron (4-sided shape) including its border. The 3-simplex produces four 0-subsimplices $< P_{0,}P_{1,}P_2,P_3 >$ three 1-subsimplices $< P_{0,}P_1 >,< P_{0,}P_2 >$ and$< P_{0,}P_3 >$ and two 2-subsimplices$< P_0,P_{1,}P_2 >$and $< P_{0,}P_{1,}P_3 >$.. It is denoted by σ³= $< P_{0,}P_{1,}P_{2,}P_3 >$.

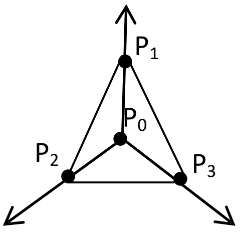


Fig 3.1d: 3-Simplex

Simplices can be rotated, translated, dilated and even stretched but cannot be crushed. Neither can an n-simplex be turned into an (n - 1) simplex by deforming it.

3.1.3 The Order of simplices

The order in which a simplex is generated does not matter. For example by (Author)

(i) For a 1-simplex, $< P_{0,}P_1 > = < P_{1,}P_0 >$

(ii) For a 2-simplex, $< P_{0,}P_{1,}P_2 > = < P_{1,}P_{2,}P_0 > = < P_{2,}P_{1,}P_0 > = < P_{2,}P_{0,}P_1 >$

(iii)For a 3-simplex,$< P_{0,}P_{1,}P_{2,}P_3 > = < P_{1,}P_{2,}P_{3,}P_0 > = < P_{2,}P_{1,}P_{3,}P_0 > =$

$< P_{3,}P_{2,}P_{1,}P_0 >$.

3.1.4 Faces of simplices

The face of an n- dimensional simplex (σ^n) is a sub simplex of (σ^n) which is the simplex generated by a subset of the vertices of (σ^n)

if$\sigma^n = < P_0, P_1, - - -, P_n >$

To get a face of dimension $m \leq n$, we choose m+1 points among P_0, $P_{1,}$-,-,-,P_n and take the corresponding simplex.

A table that counts the number of m-simplices needed to construct an n-simplex using the Fig below by (Author):

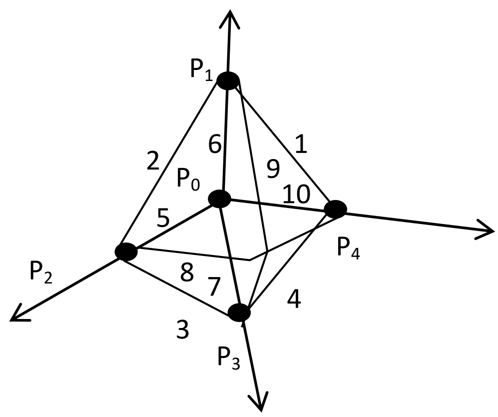


Fig 3.1e: 4-Simplex

Table 3.2: The Number Of m-Simplices Needed To Construct An n- Simplex.

Number of m-simplices contained in an n-simplex	**m =0 Number of points (1 point)**	**m =1 Number of lines (2 points)**	**m=2 Number of triangles (3 points)**	**m =3 Number of tetrahedrons (4 points)**	**m =4 5 points**
n =0	1	0	0	0	0
n =1	2	1	0	0	0
n =2	3	3	1	0	0
n =3	4	6	4	1	0
n =4	5	10	10	2	1

3.2 Simplicial Complex

A simplicial complex K is a collection of simplices such that:

(i) If K contains a simplex σ, then K also contains every face of σ, for example, for the 3-simplex $\sigma^3 = < P_{0,}P_{1,}P_{2,}P_3 >$, the simplicial complex contains every faces of σ. That is K is a set of every face of σ as below.

$$K = \{< P_{0,}P_{1,}P_{2,}P_3 >, < P_{1,}P_{2,}P_3 >, < P_{0,}P_{2,}P_3 >, < P_{0,}P_{1,}P_3 >, < P_{0,}P_{1,}P_2 >, < P_{0,}P_1 >, < P_{0,}P_2 >, < P_{0,}P_3 >, < P_{1,}P_2 >, < P_{1,}P_3 >, < P_{2,}P_3 >, < P_0 >, < P_1 >, < P_2 >, < P_3 >\}$$

The cardinality of this set (number of elements in the set) is 15.

(ii) If any two simplices in the simplicial complex K intersect, then their intersection is a face of each of them. For example, for a 5-simplex, the simplicial complex is given by

K = {$< P_{0,}P_{1,}P_2 >$, $< P_{0,}P_1 >$, $< P_{0,}P_2 >$, $< P_{1,}P_2 >$, , $< P_{2,}P_3 >$, $< P_{2,}P_4 >$, $<$ $P_{3,}P_4 >$, $< P_{4,}P_5 >$, $< P_0 >$, $< P_1 >$, $< P_2 >$, $< P_3 >$, $< P_4 >$, $< P_5 >$}

The cardinality of the set (complex) is 14.

Note:

(i) Since $< P_{2,}P_4 >$ is in the complex K then $< P_2 >$ and $< P_4 >$ are also in K.

(ii) Also since σ_3 = $< P_{0,}P_{1,}P_{2,}P_3 >$ is in K then all its subsets minus the empty set is in K.

3.2.1 Skeletons of simplicial complex

The g-skeleton of a simplicial complex K is denoted by $K^{(g)}$ and is the set of all of the simplices in K of dimension g or less.

Examples

From the 3-simplex below, we obtain the following skeletons

$\sigma^3 = < P_{0,}P_{1,}P_{2,}P_3 >$

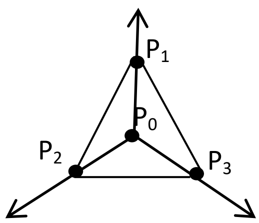


Fig 3.2d: 3-Simplex

(i) The 0-skeleton is given for the 3-simplex as

$K^{(o)}$ = {$< P_0 >$, $< P_1 >$, $< P_2 >$, $< P_3 >$}

The 0- skeleton is the set of all points in a simplex.

(ii) The 1-skeleton is the set of all the lines and points in a simplex. It is given for the 3-simplex $\sigma^3 = < P_{0,}P_{1,}P_{2,}P_3 >$ as $K^{(1)}$ = {$< P_{0,}P_1 >$, $< P_{0,}P_2 >$, $<$

$P_{0,}P_3 >$, $< P_{1,}P_2 >$, $< P_{1,}P_3 >$, $< P_{2,}P_3 >$, $< P_0 >$, $< P_1 >$, $< P_2 >$, $< P_3 >\}$.

(iii) The 2-skeleton is the set of all the points, lines and the triangles in a simplex. It is given for the 3-simplex $\sigma^3 = < P_{0,}P_{1,}P_{2,}P_3 >$ as $K^{(2)} = \{< P_{1,}P_{2,}P_3 >, < P_{0,}P_{2,}P_3 >, < P_{0,}P_{1,}P_3 >, < P_{0,}P_{1,}P_2 >, < P_{0,}P_1 >, < P_{0,}P_2 >, < P_{0,}P_3 >, < P_{1,}P_2 >, < P_{1,}P_3 >, < P_{2,}P_3 >, < P_0 >,< P_1 >, < P_2 >, < P_3 >\}$

(iv) The 3-skeleton is the set of all the points, lines, triangles and the tetrahedron in a simplex. It is given for the 3-simplex $\sigma^3 = < P_{0,}P_{1,}P_{2,}P_3 >$ as $K^{(3)} = \{< P_{0,}P_{1,}P_{2,}P_3 >,< P_{1,}P_{2,}P_3 >, < P_{0,}P_{2,}P_3 >, < P_{0,}P_{1,}P_3 >,< P_{0,}P_{1,}P_2 >, < P_{0,}P_1 >, < P_{0,}P_2 >, < P_{0,}P_3 >, < P_{1,}P_2 >, < P_{1,}P_3 >, < P_{2,}P_3 >, < P_0 >, < P_1 >, < P_2 >, < P_3 >\}$.

Note that for all n where n is the number of simplex, we have:

(i) The n-skeleton is contained in the n+1 skeleton. $K^{(n)} \subset K^{(n+1)}$

(ii) If the number of simplex (n) is equal to the dimension of the simplicial complex (K) then the n-skeleton is equal to the simplicial complex. If n = dim(K), then $K^{(n)} = K$.

(iii) If the number of simplex (n) is greater than the dimension of the simplicial complex then the n-skeleton is equal to the empty set. If n> dim(K), then $K^{(n)} = \emptyset$.

3.3 Oriented Simplex

The oriented simplex is derived from the word orientation which means direction of motion. The oriented simplex is an oriented g-simplex if it is a g-simplex and has a fixed orientation (that is the order of the points is fixed). To denote the oriented simplex, we use the brackets [.] instead of $<.>$ symbols around the generating points. The oriented simplices have the property that switching any two points introduces a minus sign. For example,

$$[P_0, P_{1,-,-,-}, P_{i,-,-,-}, P_{j,-,-,-}, P_{n-1,} P_n] = -[P_0, P_{1,-,-,-}, P_{j,-,-,-}, P_{i,-,-,-} P_{n-1,} P_n]$$

Here P_i and P_j are switched.

The oriented simplices are drawn only by considering n-simplices

for n = 1, 2, 3. (n [1,2,3]).

Examples by (Author)

(i) Diagrammatic representation (model) of 1-oriented simplices

$\sigma^1 = [P_0, P_1]$are

[a, b] [b, a]

Fig 3.3a: Models of 1-oriented simplices

(ii) Diagrammatic representation (model) of 2-oriented simplices

$\sigma^2 = [P_0, P_1, P_2]$ are

[a, b, c] [a, c, b]

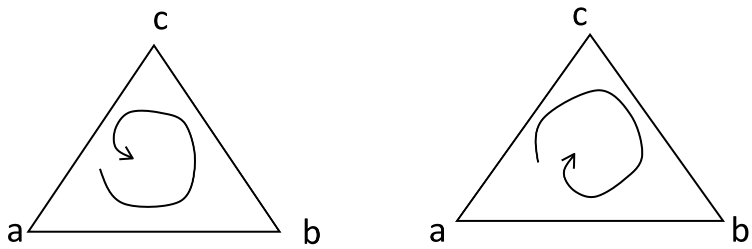


Fig 3.3b: Models of 2-oriented simplices

(i) Model of 3-oriented simplices $\sigma_3 = [P_0, P_1, P_2, P_3]$ are

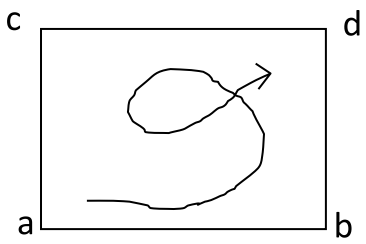


Fig 3.3c: Model of 3-Oriented Simplices

3.4 Connectedness of Spatial Objects

3.4.1 Two objects connection

Two objects placed beside each other can only be connected once. For example, the 1-simplex with two points below.

$$< P_0, P_1 >$$

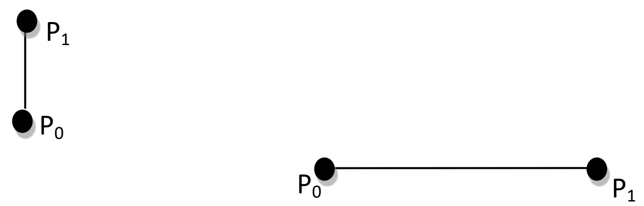
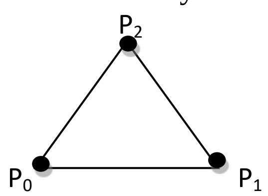


Fig 3.4a: Two Objects Connection

It has zero- intersection.

3.4.2 Three objects connection

Three objects can be connected 3 times only. For example, the 2-simplex with 3 points below.



$$< P_0, P_1, P_2 >$$

Fig 3.4b: Three Objects Connection

It has zero- intersection.

3.4.3 Four objects connection

Four objects can be connected 6 times only. For example, the 3-simplex with 4 points below:

$< P_0, P_1, P_2, P_3 >$

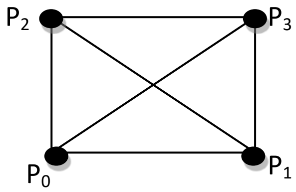


Fig 3.4c: Four Objects Connection

It has 2-Intersections $P_1 \cap P_2$ and $P_0 \cap P_3$.

3.4.4 Five objects connection

Five objects can be connected in 10 ways only. For example, the 4-simplex with 5 vertices. The table below summarizes the connectedness of spatial objects.

Table 3.4: Summary of the Connectedness of Spatial Objects by (Author)

Number of simplex (n)	**Number of points = number of vertices, n(v)**	**Number of connections n(C)**	**Number of intersection (T)**
1	2	0	0
2	3	1	0
3	4	3	0
4	5	6	1
5	6	10	5
6	7	15	13
Formula			

Note that the above concepts are part of spatial topology which according to (Jiang and Claramunt, 2004) represents a social network view. The algebraic approach for structural analysis of a spatial topology is representation of values in table 3.4 using formula.

The relationship between number of vertices n(V) and simplex (σ^n) geometrically is that for each n, n(V) = n+1. This is easily observed from the table. Also, number of connections, n(C) in n-gon shape is determined combinatorially as:

n(C)= ( ), $n \geq 2$. Note that when $n < 2$, then n(C) = 0.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSIONS

4.1 The Matrix Interpretations of the Eight Spatial Topological Relations Matrices of two Egenhofer4-Intersection Models using Matrix Addition and Multiplication

The binary topological relations between two objects A and B, are defined in terms of the four intersections of A's boundary (∂A) and interior (A°) with B's boundary (∂B) and interior (B°) ,(Egenhofer and Franzosa, 1991). It is the topological relationship between the boundary and interior parts of objects A and B given by p^n, where p is the object parts and n is the number of objects (2^2 = 4 intersection model).This model is concisely represented by a 2x2 matrix, called the 4-intersection matrix and is given in matrix form by;

$$\Im_4(A,B) = \begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^\circ \\ A^\circ \cap B^\circ & A^\circ \cap \partial B \end{pmatrix} \qquad (4.1)$$

Note:

a) The boundary-boundary intersection is denoted by $\partial A \cap \partial B$

b) The boundary-interior intersection is denoted by $\partial A \cap B^\circ$

c) The interior-interior intersection is denoted by A° $\cap$ B°

d) The interior-boundary intersection is denoted by A° $\cap$ ∂B

Topological invariants of these four intersections (that is properties that are preserved under topological transformations), are used to categorize topological relations. Examples of topological invariants applicable to the four intersection, are the content (that is emptiness or non-emptiness) of a set, the dimension and the number of separations, (Franzosa and Egenhofer, 1992). By considering the values empty (0) and non-empty (1) for the four intersections, one can distinguish 2^4 = 16 binary topological relations. Eight of these sixteen relations can be realized for homogenously 2-dimensional objects with connected boundaries, called regions, if the objects are embedded in R^2 (Egenhofer and Herring, 1990). The Fig 4.1 below shows the matrix representation of topological relations of two objects A and B by (Egenhofer and Herring, 1990).

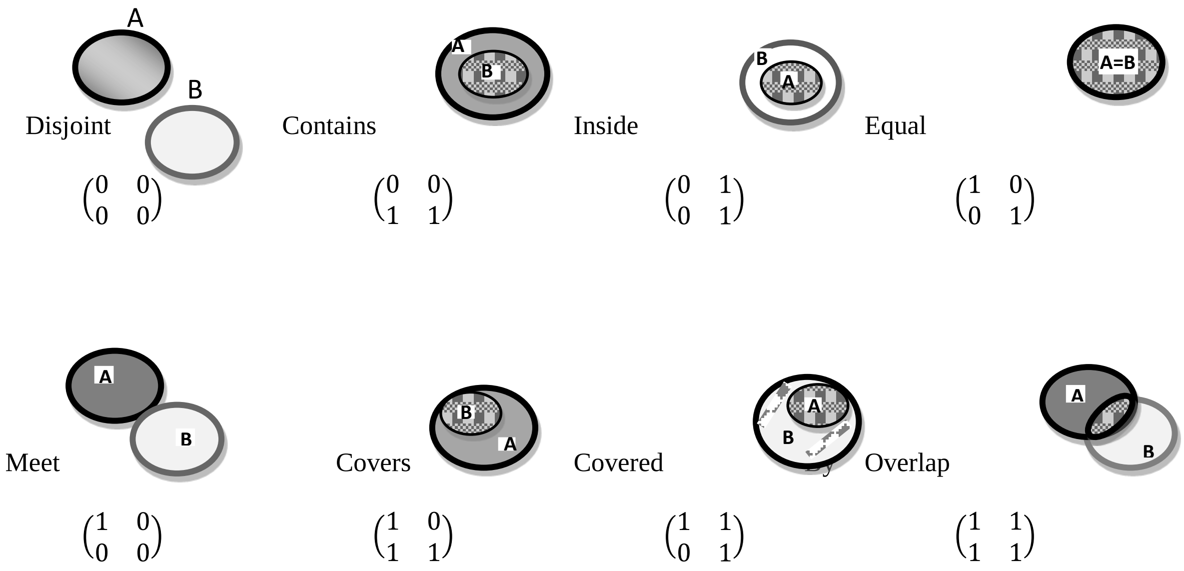


Fig 4.1: The Egenhofer2x2 Matrix Representation of Topological Relations of Two Objects A and B

The topological relations matrices are obtained from the p by p^{n-1} matrix, where p is the object parts and n is the number of objects

Table 4.1.1: Matrix addition of the eight spatial topological relations matrices of two Egenhofer 4-intersection models using modulo 2 by (Author)

Matrix Addition (+)	Disjoint $\begin{pmatrix}0&0\\0&0\end{pmatrix}$	Contains $\begin{pmatrix}0&0\\1&1\end{pmatrix}$	Inside $\begin{pmatrix}0&1\\0&1\end{pmatrix}$	Equal $\begin{pmatrix}1&0\\0&1\end{pmatrix}$	Meet $\begin{pmatrix}1&0\\0&0\end{pmatrix}$	Covers $\begin{pmatrix}1&0\\1&1\end{pmatrix}$	Coveredby $\begin{pmatrix}1&1\\0&1\end{pmatrix}$	Overlap $\begin{pmatrix}1&1\\1&1\end{pmatrix}$
Disjoint $\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$
Contains $\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\end{pmatrix}$
Inside $\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&1\\0&2\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$
Equal $\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
Meet $\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$
Covers $\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$
Coveredy $\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$
Overlap $\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$

It is observed by (Author) from table 4.1.1 that the matrix addition of the eight spatial topological relations matrices of two Egenhofer 4-intersection models gave a total of 64 2x2 matrices with 32 possible Egenhofer 2x2matrices and 32Egenhofer 2x2 complement matrices. These 2x2 complement matrices are formed from the set X = {0, 1} which also give the Egenhofer 2x2 matrices. The matrix addition of two similar Egenhofer 2x2 complement matrices using modulo 2 gives an Egenhofer 2x2 matrices while the addition of two different Egenhofer 2x2 complement matrices using modulo 2 gives an Egenhofer 2x2 compliment matrix.

4.1.2 The Matrix Multiplication of the Eight Spatial Topological Relations Matrices of two Egenhofer 4-Intersection Models

Table 4.1.2: Matrix Multiplication of the Eight Spatial Topological Relations Matrices of two Egenhofer 4-Intersection Models Using Modulo 2 by (Author)

Matrix Multiplication (x)	**Disjoint** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	**Contains** $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	**Inside** $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	**Equal** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	**Meet** $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	**Covers** $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	**Coveredby** $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	**Overlap** $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Disjoint $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Contains $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Inside $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Equal $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Meet $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
Covers $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
Coveredby $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$
Overlap $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

It is observed by (Author) from table 4.1.2, that the matrix multiplication of the eight spatial topological relations matrices of two Egenhofer 4-intersection models using modulo 2, gave a total of 64 2x2 matrices with 49 Egenhofer 2x2 matrices and 15 2x2 complement matrices. The matrix multiplication of two similar or different Egenhofer 2x2 complement matrices using modulo 2 gives an Egenhofer 2x2 matrices.

4.2 The Matrix Interpretation of the Eight Spatial Topological Relations Matrices of two Egenhofer 9-Intersection Models Using Matrix Addition and Multiplication

The nine intersections between the six object parts describe a topological relation and can be concisely represented by $\mathfrak{I}_9$ called the 9-intersection model.

$$\mathfrak{I}_9(A,B) = \begin{pmatrix} A^o \cap B^o & A^o \cap \partial B & A^o \cap B^- \\ \partial A \cap B^o & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^o & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \tag{4.2}$$

It is obtained from the topological relationship between the interior , boundary and exterior of two spatial objects A and B.

In analog to the 4-intersection, each intersection will be characterized by a value empty (0) or non-empty (1), which allows one to distinguish $2^9 = 512$ different configurations. Only a small subset of them can be realized between two objects in R^2.

The Fig 4.2 below shows the Egenhofer 3x3 matrix representation of topological relations of two objects A and B.

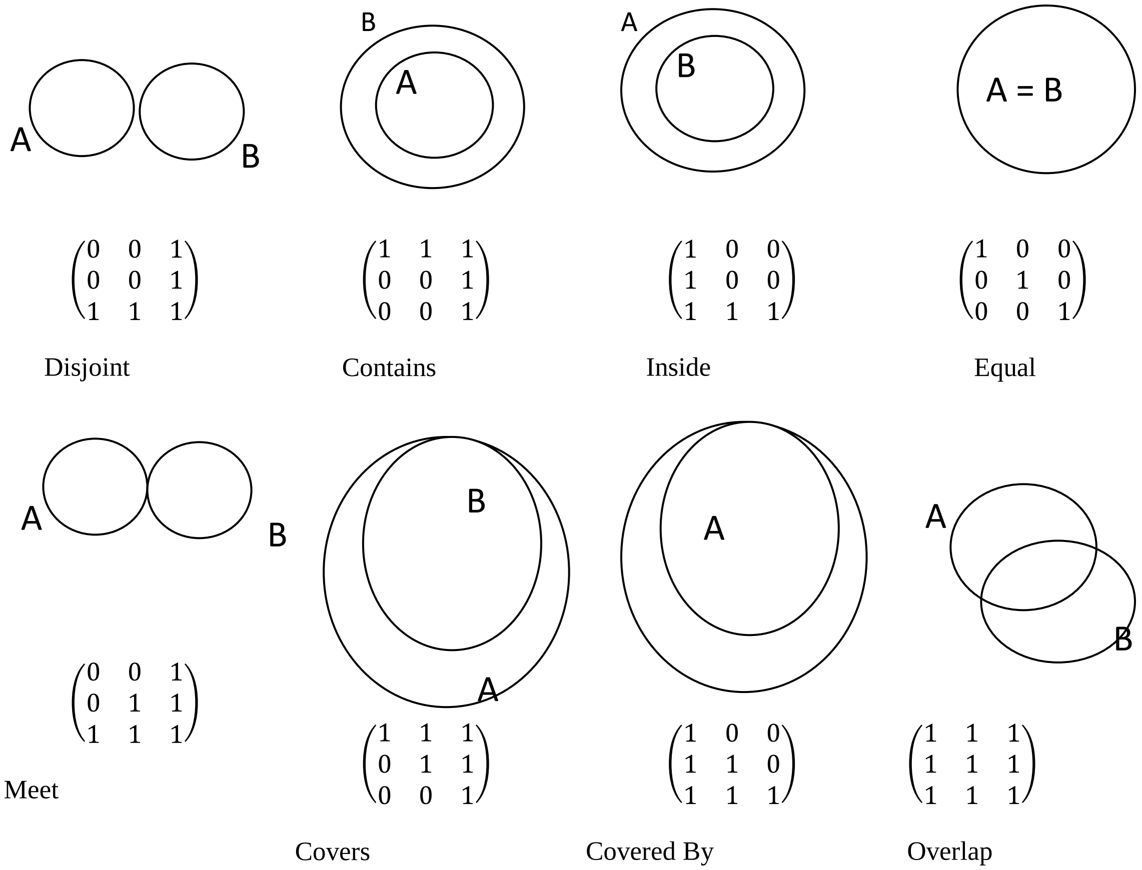


Fig 4.2: The Egenhofer3x3 Matrix Representation Of Topological Relations Of Two Objects A And B by (Egenhofer and Herring,1990).

The topological relations matrices are obtained from the p x p^{n-1} matrix, where p is the object parts and n is the number of objects.

4.2.1 The Matrix Addition of the Eight Spatial Topological Relations Matrices of two Egenhofer 9- Intersection Models by (Author)

Table 4.2.1: Matrix Addition of the Eight Spatial Topological Relations Matrices of two Egenhofer9-Intersection Models Using Modulo 2

Matrix Addition (+)	**Disjoint** $\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{pmatrix}$	**Contains** $\begin{pmatrix}\mathbf{1}&\mathbf{1}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}$	**Inside** $\begin{pmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{pmatrix}$	**Equals** $\begin{pmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}$	**Meet** $\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{pmatrix}$	**Covers** $\begin{pmatrix}\mathbf{1}&\mathbf{1}&\mathbf{1}\\\mathbf{0}&\mathbf{1}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}$	**Coveredby** $\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{pmatrix}$	**Overlap** $\begin{pmatrix}\mathbf{1}&\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{pmatrix}$
Disjoint $\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$
Contains $\begin{pmatrix}1&1&1\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&0\\1&1&0\end{pmatrix}$
Inside $\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\0&0&0\end{pmatrix}$
Equals $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&0\\1&1&0\end{pmatrix}$
Meet $\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$
Covers $\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\end{pmatrix}$
Coveredby $\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$
Overlap $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

It is observed from table 4.2.1, that the matrix addition of the spatial topological relations matrices of two Egenhofer 9-intersection models using modulo 2 gave a total of 64 9-Intersection matrices with no possible Egenhofer 3x3 matrix but all 64 Egenhofer3x3 complement matrices. These complement 3x3 matrices are formed from the set X = {0, 1}, which also gave the Egenhofer 3x3 matrices. The matrix addition of two similar or different Egenhofer 3x3 complement matrices using modulo 2 will always give an Egenhofer 3x3 complement matrix and not an Egenhofer 3x3 matrix.

4.2.2 The Matrix Multiplication of the Eight Spatial Topological Relations Matrices of two Egenhofer 9-Intersection Models.

Table 4.2.2: Matrix Multiplication of the Eight Spatial Topological Relations Matrices of two Egenhofer 9-Intersection Models Using Modulo 2

Matrix Multiplication (x)	Disjoint $\mathbf{\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}}$	Contains $\mathbf{\begin{pmatrix}1&1&1\\0&0&1\\0&0&0\end{pmatrix}}$	Inside $\mathbf{\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}}$	Equals $\mathbf{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}$	Meet $\mathbf{\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}}$	Covers $\mathbf{\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}}$	Coveredby $\mathbf{\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}}$	Overlap $\mathbf{\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}}$
Disjoint $\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
Contains $\begin{pmatrix}1&1&1\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
Inside $\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\0&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
Equals $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
Meet $\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\1&1&1\end{pmatrix}$
Covers $\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\1&1&1\end{pmatrix}$
Coveredby $\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\1&1&1\end{pmatrix}$
Overlap $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\0&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&0&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

It is observed from the table that the matrix multiplication of the eight spatial topological relations matrices of two Egenhofer 9-intersection models using modulo 2 gave a total of 64 9-Intersection matrices with 16 Egenhofer 3x3 matrices and 48 Egenhofer 3x3 complement matrices. The matrix multiplication of two similar or different Egenhofer 3x3 complement matrices using modulo 2 will always give an Egenhofer 3x3 complement matrices and not an Egenhofer 3x3 matrix.

4.3 Derivations of the 8-Intersection and the 16-Intersection Models and their Corresponding Eight Spatial Topological Relations Matrices and Figs by (Author) using the Egenhofer 4 and 9- Intersections Models

4.3.1 The 8-Intersection model

For the topological relationship between the interiors and boundaries of three objects, we obtain the 8- Intersection Model. It is generally derived from the topological relationships between any two objects parts of three objects. using p^n where p is the object parts and n is the number of objects, we have $2^3 = 8$intersection model. By considering the values empty (0) and non-empty (1) for the eight intersection model, we can distinguish $2^8 = 64$ topological relations. 32 of these 64 relations can be realized for homogenously 2-dimensional objects with connected boundaries called regions, if the objects are embedded in two-dimensional space (R^2) (Egenhofer and Herring, 1990).

$$\mathfrak{Z}_8(A,B,C) = \begin{pmatrix} \partial A \cap \partial B \cap \partial C & \partial A \cap B^o \cap C^o \partial A \cap \partial B \cap C^o & \partial A \cap B^o \cap \partial C \\ A^o \cap B^o \cap C^o & A^o \cap \partial B \cap C^o A^o \cap B^o \cap \partial C & A^o \cap \partial B \cap \partial C \end{pmatrix} \quad (4.3)$$

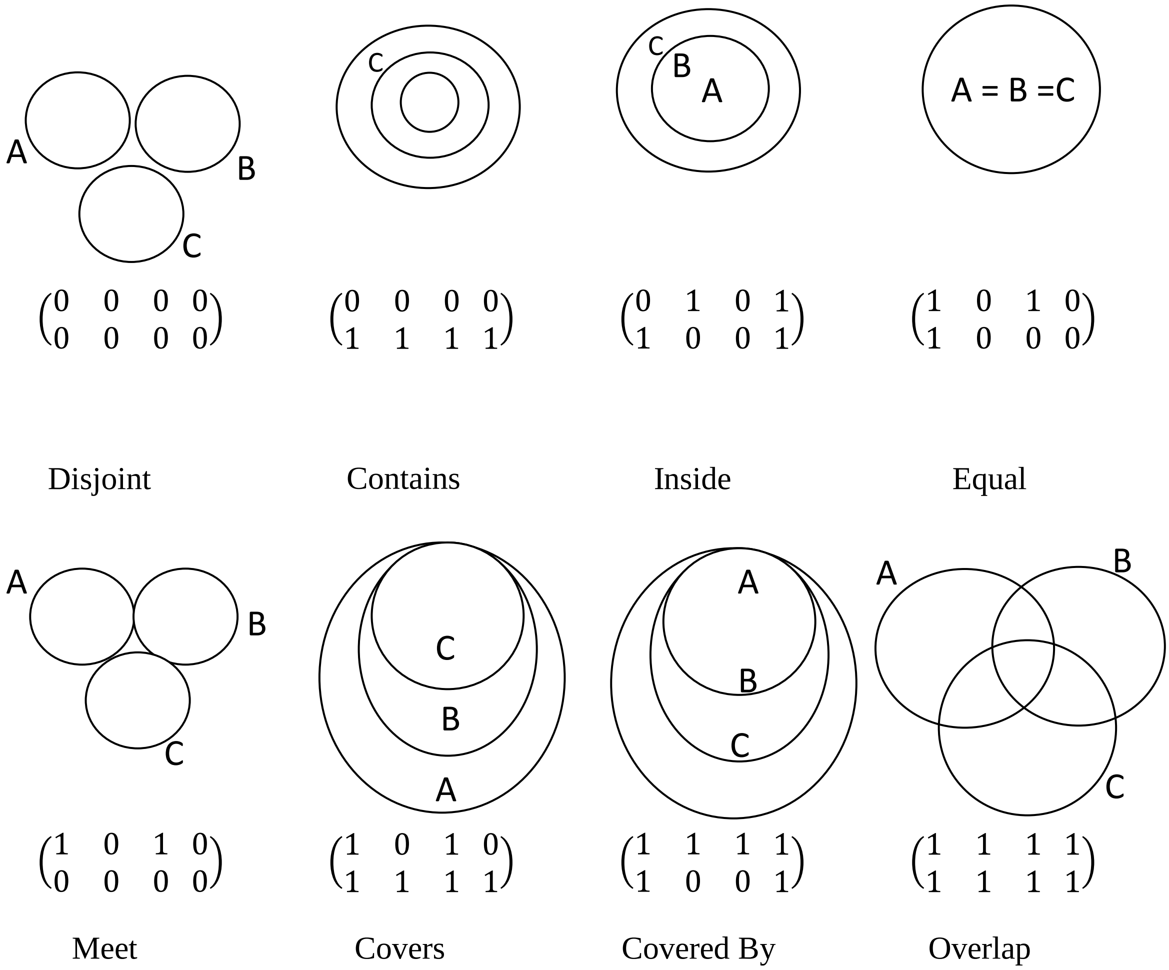


Fig 4.3.1: The 2x4 Matrix Representation of Topological Relations of Three Objects A, B and C

It is observed that the 2x4 eight spatial topological relations matrices are obtained from p by p^{n-1} (pxp^{n-1}) matrix, where p is the object parts and n is the number of objects.

4.3.2 The 16-Intersection model

For the topological relationship between any two object parts of four objects we obtain the 16-Intersection model. It can be obtain from the topological relationship between the interior and exterior of four objects .using p^n where p is the object parts and n is the number of objects, we obtain $2^4 = 16$ intersection model. By considering the values empty (0) and non-empty (1) for the 16-intersection, we can distinguish $2^{16} = 256$

topological relations. 128 of these 256 relations can be realized for homogenously 2-dimensional objects with connected boundaries called regions, if the objects are embedded in R^2 (Egenhofer and Herring, 1990). It is denoted by

Ȝ16(A,B,C,D)=

$$\begin{pmatrix} A^0 \cap B^0 \cap C^0 \cap D^O & A^0 \cap B^- \cap C^0 \cap D^O & A^0 \cap B^0 \cap C^- \cap D^O & A^0 \cap B^0 \cap C^0 \cap D^O \\ A^- \cap B^0 \cap C^0 \cap D^O & A^- \cap B^- \cap C^0 \cap D^O & A^- \cap B^0 \cap C^- \cap D^O & A^- \cap B^0 \cap C^0 \cap D^- \end{pmatrix}$$

$$\begin{pmatrix} A^0 \cap B^- \cap C^- \cap D^O & A^0 \cap B^0 \cap C^0 \cap D^- & A^0 \cap B^0 \cap C^- \cap D^- & A^0 \cap B^- \cap C^- \cap D^- \\ A^- \cap B^- \cap C^- \cap D^O & A^- \cap B^- \cap C^- \cap D^- & A^- \cap B^0 \cap C^- \cap D^- & A^- \cap B^- \cap C^- \cap D^- \end{pmatrix}$$

(4.4)

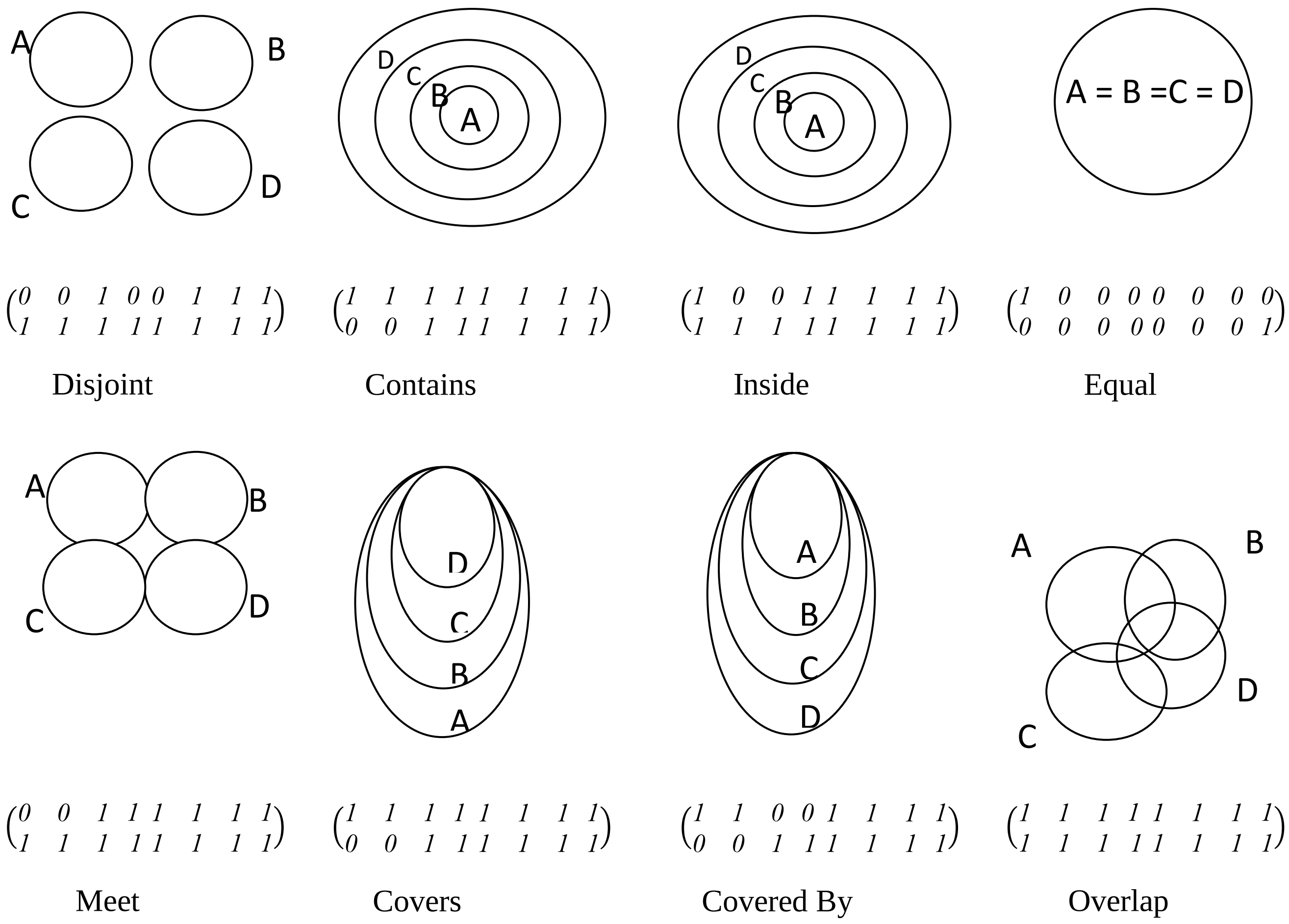


Fig 4.3.2: The 2x8 Matrix Representation of Topological Relations Of Four Objects A, B, C And D

From the above Fig, it is observed for the spatial topological relation meet, that objects A and D does not meet likewise objects C and D. The 2x4 topological relations matrices are obtained from p by p^{n-1} (pxp^{n-1}) matrix, where p is the object parts and n is the number of objects.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

We have figure out and understood analytically a mathematical model of the topological relationship of spatial objects using Egenhofer matrix. Besides we brought about the following:

1. The topological relationship between spatial objects is the power set (p^n) of the objects where p is the object parts and n is the number of objects related. For example, the topological relationship between

a) Two object parts by 2 objects gives $2^2 = 4$ which is the 4-intersection model.

b) Two object parts by 3 objects gives $2^3 = 8$ which is the 8-intersection model.

c) Two object parts by 4 objects gives $2^4 = 16$ which is the 16-intersection model.

d) Three object parts by 2 objects gives $3^2 = 9$ which is the 9-intersection model.

e) Three object parts by 3 objects gives $3^3 = 27$ which is the 27-intersection model and so on.

2. The topological relationship between spatial objects is given by the p by p^{n-1}($p \times p^{n-1}$) matrix where p is the object parts and n is the number of objects.

3. The topological relationship between spatial objects can be denoted by $T_{Sobj} = p^n$ which is the power set of the object.

4. The topological relationship between spatial objects can be expressed in logarithmic equation as $\log_p T_{SObj} = n$ where p is the object parts and n is the number of objects related. That is the logarithm to base p of T_{SObj} is the number of objects related.

5. The algebraic approach for structural analysis of a spatial topology is representation of values using the formulae $n(V) = n+1$ and $n(C) = \frac{n(n+1)}{2}$.

6. The relationship between number of vertices n(V) and simplex (σ^n) geometrically is that for each n, n(V) = n+1. Also, the number of connections, n(C) in n-gon shape is determined combinatorially.
7. The matrix addition of the eight spatial topological relations matrices of two Egenhofer 4-intersection models gave a total of 64 2x2 matrices with 32 possible Egenhofer 2x2matrices and 32Egenhofer 2x2 complement matrices. These 2x2 complement matrices are formed from the set X = {0, 1} which also gave the Egenhofer 2x2 matrices. The matrix addition of two similar Egenhofer 2x2 complement matrices using modulo 2 gives an Egenhofer 2x2 matrices while the addition of two different Egenhofer 2x2 complement matrices using modulo 2 gives an Egenhofer 2x2 compliment matrix.
8. The matrix multiplication of the eight spatial topological relations matrices of two Egenhofer 4-intersection models using modulo 2, gave a total of 64 2x2 matrices with 49 Egenhofer 2x2 matrices and 15Egenhofer 2x2 complement matrices. The matrix multiplication of two similar or different Egenhofer 2x2 complement matrices using modulo 2 gives an Egenhofer 2x2 matrices.
9. The matrix addition of the spatial topological relations matrices of two Egenhofer 9-intersection models using modulo 2 gave a total of 64 9-Intersection matrices with no possible Egenhofer 3x3 matrix but all 64 Egenhofer 3x3 complement matrices. These complement 3x3 matrices are formed from the set X = {0, 1}, which also gave the Egenhofer 3x3 matrices. The matrix addition of two similar or different Egenhofer 3x3 complement matrices using modulo 2 will always give an Egenhofer 3x3 complement matrix and not an Egenhofer 3x3 matrix.

10. The matrix multiplication of the eight spatial topological relations matrices of two Egenhofer 9-intersection models using modulo 2 gave a total of 64 9-Intersection matrices with 16 Egenhofer 3x3 matrices and 48Egenhofer 3x3 complement matrices. The matrix multiplication of two similar or different Egenhofer 3x3 complement matrices using modulo 2 will always give an Egenhofer 3x3 complement matrices and not an Egenhofer 3x3 matrix.
11. The 8-Intersection model is obtained from the topological relationship between the interiors and boundaries of three objects.

    It is generally derived from the topological relationships between any two objects parts of three objects. By considering the values empty (0) and non-empty (1) for the eight intersection model, we can distinguish $2^8 = 64$ topological relations.
12. The 16-Intersection model is obtained from the topological relationship between any two object parts of four objects. It can be obtain from the topological relationship between the interior and exterior of four objects. By considering the values empty (0) and non-empty (1) for the 16-intersection, we can distinguish $2^{16} = 256$ topological relations.

5.2 Contribution to knowledge

From our findings, we have brought the following to knowledge:

1. The algebraic approach for structural analysis of a spatial topology.
2. Matrix interpretation of the spatial topological relations of two Egenhofer 4-intersection matrices using matrix addition and multiplication modulo 2.
3. Matrix interpretation of the spatial topological relations of two Egenhofer 9-intersection matrices using matrix addition and multiplication modulo 2.

4. The 8-intersection model and its corresponding 8 spatial topological relations matrices developed from the Egenhofer matrices.
5. The 16-intersection model and its corresponding 16 spatial topological relations matrices developed from the Egenhofer matrices.

5.3 Recommendations

Here we recommend that spatial objects and their topological relations should be studied analytically for decision making in Qualitative Spatio-Temporal Representation and Reasoning. As a result, this work is prescribed for a further research on the division, subtraction and possibly the inverse of the Egenhofer 4 and 9-Intersections matrices using modulo 2.

REFERENCES

Abler, R (1987). What shall we say? to whom shall we speak? Annals of The Association of American Geographers, Volume TI, Issue 4.Among Spatial Objects. In: D. Marble, editor, Third International Symposium on Spatial objects.

Atkin, D. (1977). Combinatorial Connectives in Social System: An Application of Simplicial Complex Structures to the Study of Large Organizations Based Birkhauser. (Ph. D) Thesis-Cornell University, bibliographical references, pages 179-181.

Bellomo N. and Preziosi L. (1995). Modeling mathematical methods and scientific computation, CRC Press, Boca Raton

Blaha M. and Premerlani W. (1998) Object Oriented Modeling and Design for Database Applications. Text book. Richardson publishing office.

Cantor, G. (1895)"A history of set theory "Mac Tutor History of Mathematics archive, University of Saint Andrew, Italy.

Casey K. and Alessandra P. (2016). *Introduction to Simplicial Complexes*. 45-57

Chang, D'Zurilla and Maydeu-Olivares,(1994). Optimism and Pessimism as Partially Independent constructs: Relationship to positive and negative affectivity and psychological well-being. Department of statistics and Economics, university of Carlos III de Madrid, Madrid 126 28903, Spain.

Chen, P. (1976). Value-oriented Information Systems Design: The Concept of Potential Modeling and its Application to Service –oriented Architectures. PDF available in Business and Information Systems Engineering 1(3):223-233.

Clementini, E., Di felice, P. and Van Orsterom, (1993). A Small Set of Formed Topological Relationships for End User Interaction, Advances In Spatial BataBased Third International Symposium, Malta.

Cohn and Hazarika (2001). Qualitative Spatial Representation and Reasoning: An overview. in Fundamental Informaticae 46(1-2):1-29.

Cooper, R. (1997). *An Object DataBases: An ODMG approach*. International Thompson Computer Press. 72-80

Corbett, J. (1975). Topological Principles in Cartography Proceeding of 2nd International Symposium on Computer-Assisted Cartography 61,-65.

Egenhofer M, Frank A. and Jackson J. (1989). A Topological Data Model for Spatial Database systems. In Computational Geometry and its applications. New York. Smart Publishers.

Egenhofer M. (1987). Relations Between Intervals In A One-Dimensional Space. Internal Databases in Symposium on the Design and Implementation of Large Databases. Department of Surveying Documentation, University of Maine, Orono,

Egenhofer M. and Franzosa R. (1991).Point-set topological Spatial Relations. 14th International Conference on Very Large Databases Handling Communication Research Centre University of Glasgow, 56 Hillhead Street, Computer-Assisted Cartography.

Egenhofer M. J and Herring J.R. (1990). A Mathematical Framework for the Definition of concept of simplicial complex. D. DeWitt and F. Bancillion, publishers. 34-51

Egenhofer, M. (1989).Appropraite conceptual Data-base Scheme Design for Two – Dimensional Spatial Structures in: ASPRS Annual convention, 167-179, Baltimore, MD.

Egenhoter M. J. and Franoza, R.d., (1990). Point Set Topological Spatial Relations *International Journal of Geographical Information Systems* 5(2), 161-174.

Egenhoter, M. J, and Mark, D, (1995). Naïve Geography, Spatial information Theory: A Theoretical Basis for G/S – 6th *International Conference, University of Ontario, Ontario*

Ellmer, E (1993) Basic Principles and Concepts of Object Orientation.University of Viencna, Institute of Applied Computer Science and Information System.*Conference Paper*. 9 Reads 6 Citation.Engineering, Orono, ME.

Freeman, J. (1975) ‘’The Modeling of Spatial Relations’’ Computer Graphics and Images Processing, Elsevier. DOI 10.10.1016/SO146-664X (75) 80007 – 4. Geometry. In: D. Marble, editor, Second International Symposium on Spatial Data.

Giblin.P. (1977).*Graphs, Surfaces and Homology*. Halsted Press, John Wiley and Sons, Glasgow G12 9YR, UK.

Guting R. (1988). Geo-Relational Algebra: A Model and Query Language for Geometric. *International Journal of Geographic Information Systems,5(2):161-174.*

Herring J. (1987). *TIGRIS: Topological Integrated Geographic Information Systems*. intersections of diagonals in the interior of regular n-gon. Razor Publisher. 54-65.

Herskovits (1986) Investigating the Functional Geometry of Spatial Prepositions. *Human International Symposium on Spatial Data Handling*. California, United State of America

Jiang B. and Itzhak O. (2005). Spatial topology and its structural analysis based on the Topological Relationships. Management and Organising 31:151-162.

Jiang, B. and Claramunt, C. (2004).Topological Analysis of Urban Street Networks. Environment and Planning B; Planning and Design Vol. 31:151-162.

Kainz and Worboys (1995) Logical Consistency, In: Guptil, C,S and Morrision, J. L (eds), Elements of Spatial Data Quality Published on behalf of the *International Cartographic Conference* pergamon press, 108 – 137.

Kainz W. (1990).Spatial Relationships-Topology versus Order. In: *Proceedings of the 4th International Symposium* in New Delhi, India

Khoshafians, S and Abnous, R (1995).*Object Orientation*. Second Edition, John Wiley and Sons Inc, New York.

Klippel (2012). *Qualitative Spatio-Temporal Presentation and Reasoning*: Trends Future Directions, Publisher: IGI al, Editors: S M Hazarika, page 168-202.

Kreyzig E. (2004). *Advanced Engineering Mathematics*,8Th Edition Textbook.

Kufoniyi O. Molenar, M. and Bovloucost, T (1994).Topology as a Tool for Consistency Operation in Vector Maps.*International Archieve of Photo Grammetry and Remote Sensing*. 30 (3-1), 455 – 462.

Mattos, N.M, Meyer-Wegener, K and Mitsshache, B (1993). Grand Tour of Concepts of Object Orientation from a Database Point View Data Knowledge Engineering Volume 9, pp 321-352. N.R. Chrisman, editor, AUTO-CARTO 8.

Peuguent and Ci –xiang, (1987).The Use of Spatial Relationship to Aid Spatial Database Retrieval. In:D. Marble, editor, *Second International Symposium on Spatial Data Handling.*

Puller D. and Egenhofer M. (1988). Towards Formal Definitions of Topological Relations. *Eight International Symposium on Topological Relation* in New York, NY.

Randell D, Cui Z and Cohn A (1992) A Spatial Logic based on Regions and Connection, *Proceedings of the Third International Conference on Knowledge Representation and Reasoning*. San Mateo Kaufmann, 165-176.

Raza, A (2001). Object Oriented temporal G/I for Urban Applications *Phd Thesis* Information and Communication Technology (ICT) Publication.

Renz ,Rauh and Knauff (2000). Towards Cognitive Adequacy of Topological Spatial Relations. *Conference Paper*, January 2000. DOI: 10.1007/3-540-45460-8_14. Publisher: 0302-974.

Renz J (2001). Qualitative Spatial Reasoning with Topological Information, Volume 2293 of Lecture Notes in Computer Science. Springer, 2002.ISBN 3-540-43346-5.

Robinson and Wong (1987). Strengths of Green and fired Spherical Aluminosilicate Aggregates. *Journal of the American Ceramic Society*, 70(10), 320-329.

Roy and Clement (1994). Object-Oriented Data-Base Technology for Geographic Applications: A Case Study of Otawa.

Sheriff, M. A., DreBler, K., Smeros, P., and Ngonga Ngomo, A. C. (2016). *Annex: Radon-Rapid Discovery of Topological Relations*. ArXiv e-prints.37-58..

Tamy, L. (1983) How Language Structure Space.In H. Pick and L. Acredolo (Eds). *Spatial Orientation Theory Research and Application*. New York: plenum press.