Nigerian Research Journal of Engineering and Environmental Sciences 2(1) 2017 pp. 203-214

Nigerian Research Journal of Engineering and Environmental Sciences
Journal homepage: www.rjees.com

RJEES

Original Research Article

A GENERALIZED FORMULATION FOR INTEGRATED VARIANT OF TAU METHOD FOR OVERDETERMINED M-TH ORDER ORDINARY DIFFERENTIAL EQUATIONS

[1]Badeggi, A.Y., [2]Ma'ali, A.I., [1]Abubakar, A. [1]Abubakar, A.W. and [2]Mohammed, U.

[1]Department of Mathematics and Computer Science, Ibrahim Badamasi Babangida University, Lapai Niger State, Nigeria.
[2]Department of Mathematics, Federal University of Technology Minna, Niger State, Nigeria.
*aybadeggi@ibbu.edu.ng

ARTICLE INFORMATION

Article history:
Received 25 February 2017
Revised 02 May 2017
Accepted 03 May 2017
Available online 01 June 2017

Keywords:
Integrated formulation
Tau method
Variant
Recurrence
Generalization

ABSTRACT

The tau method approximates the solution of a differential equation with a polynomial. This is the exact solution of a differential equation obtained by adding to the right hand side of the given equation a perturbation term, consisting of suitably chosen linear combination of polynomials. In this paper, a generalized formula or recurrence relation for the integrated variant of the tau method was derived which captures the general class of problems involving m-th order ordinary differential equations (ODEs) To validate this result, the generalized form was casted as a theorem for which mathematical induction principle was used to prove the result.

1. INTRODUCTION

The essence of the tau method is to perturb the given differential problem in such a way that its exact solution becomes a polynomial (Lanczos, 1938; Lanczos, 1956; Ortiz, 1969). This is achieved by using a polynomial perturbation term, added to the right hand side of the differential equation. The desired Tau approximation is written in terms of a special polynomial basis called the canonical polynomial basis. This is uniquely associated with the given differential operator *D* which defines the given problem such basis does not depend on the degree of approximation (Ortiz, 1974). The order of the approximation can be increased by just adding one or more canonical polynomials to those already generated and updating the coefficients affecting them (Adeniyi and Ma'ali, 2008; Yisa and Adeniyi, 2012).

A.Y. Badeggi et al. / Nigerian Research Journal of Engineering and Environmental Sciences
2(1) 2017 pp. 203-214

An attempt to improve upon accuracy of the tau approximant of $y(x)$ resulting from the differential form led to the integrated formulation. By this method, the differential equation is integrated through as many times as its order, which consequently leads to higher perturbation than the original differential formation. To give more flexibility in computation of Tau solution, Lanczos (1956) introduced a systematic use of the so-called canonical polynomials in the Tau method. A recursive generation of Lanczos canonical polynomials was proposed by Ortiz (1969) and extended in Namasivayam and Ortiz (1981) and more recently, Adeniyi and Aliyu (2012a), Adeniyi and Aliyu (2012b) and Ma'ali (2012).

Let us consider the following m-th order linear differential equation:

$$Ly(x) := \sum_{r=0}^{m-1} p_r(x) y^r(x) = \sum_{r=0}^{n} f_r x^r, \quad a \le x \ge b \tag{1}$$

$$Ly(x) := \sum_{r=0}^{m-1} a_{rk}\, y^{(r)}(x_{rk}) = \alpha_k, \quad k = 1(1)m. \tag{2}$$

The integrated form of Equation (2) is given by:

$$I_L y(x) := \iiint \overset{m}{\ldots\ldots} \int f(x)dx + C_m(x) \tag{3}$$

Where $C_m(x)$ denotes an arbitrary polynomial of degree (m-1),

arising from constants of integration, and equation (4) is the m time indefinite integration of $L(.)$

$$I_L y(x) := \iiint \overset{m}{\ldots\ldots} \int L(.)dx + \tag{4}$$

The tau approximant $y_n(x)$ of Equation (1), satisfied the perturbed problem:

$$I_L y(x) := \iiint \overset{m}{\ldots\ldots} \int f(x)dx + C_m(x) + H_{m+n}(x) \tag{5}$$

$$Ly_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \tag{6}$$

Where:

$$H_{n+m}(x) = \sum_{r=0}^{m+s-1} C_{m+s-r} T_{-m+r+1}(x) \tag{7}$$

and

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \cong y(x), n + \infty \tag{8}$$

The tau problem (Equation 5) gives a more accurate approximate of $y(x)$ than Equation (1) due to the higher perturbation term.

2. THE GENERALIZED RECURRENCE RELATION FOR THE INTEGRATED VARIANT FOR M-TH ORDER ORDINARY DIFFERENTIAL EQUATIONS

The generalized recurrence relation for the initial value problem (Equation 1) will be obtained for the cases $m = 1, 2, 3, \ldots$ and $s = 0, 1, 2, \ldots$. The following individual cases are considered and from which the general case will be obtained.

The case $m = 1, s = 0$

From Equation (1):

$$(P_{10} + P_{11}x)y'(x) + P_{00}y(x) = \sum_{r=0}^{f} f_r x^r \qquad a \le \alpha \le b \tag{9}$$

$$y(a) = \alpha_0 \tag{10}$$

From the method, we have:

A.Y. Badeggi et al. / Nigerian Research Journal of Engineering and Environmental Sciences
2(1) 2017 pp. 203-214

$$\int_0^x [(P_{10} + P_{11}u)(e_n'(u))_{n+1} + P_{00}\,(e_n(u))_{n+1}]du = -\tau_1 \int_0^x H_n(u)du + \tilde{\tau}_1 H_{n+2}(x) \quad (11)$$

Where:

$$(e_n(x))_{n+1} = \frac{V_m(x)\emptyset_n \tau_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\widehat{\emptyset_n}(x)\tau_n(x)}{C_n^{(n)}} == \frac{\widehat{\emptyset_n}(x)\tau_n(x)}{2^{2n-1}} \quad (12)$$

$$(e_n(x))_{n+1} = \frac{\widehat{\emptyset}}{K_1}[K_1x^{n+1} + K_2x^n + K_3x^{n-1} + + + \cdots] \quad (13)$$

Where:

$K_1 = C_n^{(n)}$, $K_2 = C_{n-1}^{(n)}$, $K_3 = C_{n-2}^{(n)}$ etc.

and

$$H_n(x) = \tau_1 \sum_{r=0}^{n} C_r^{(n)} \quad and \quad H_{n+1}(x) = \tau_1 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r \quad (14)$$

Thus, from Equation (13) we have:

$$\int_0^x (e_n(u))_{n+1}\,du = \frac{\emptyset}{K_1}\left[\frac{K_1x^{n+2}}{n+2} + \frac{K_2x^{n+1}}{n+1} + \frac{K_3x^n}{n} + + + \cdots\right] \quad (15)$$

Substitute Equation (13), (14) and (15) into (11), expanding it and collecting the like terms results in:

$$\frac{\widehat{\emptyset}}{K_1}[\lambda_1 x^{n+2} + \lambda_2 x^{n+1} + \cdots] = \hat{\tau}_1 C_{n+2}^{(n+2)} x^{n+2}\left[\hat{\tau}_1 C_{n+1}^{n+2} - \frac{\tau_1 C_n^{(n)}}{n+1}\right] x^{n+1} + \ldots \quad (16)$$

Where:

$$\lambda_1 = \left[\frac{P_{00} + (n+1)P_{11}}{n+2}\right] K_1,\ \lambda_2 = p_{10} + \left[\frac{P_{00} + nP_{11}}{n+1}\right] K_2 \quad (17)$$

Equating the corresponding coefficients of x^{n+2} and x^{n+1} in Equation (17) and solving the resulting system of equations results in:

$$\hat{\tau}_1 = \frac{\widehat{\emptyset}\lambda_1}{K_1 C_{n+2}^{(n+2)}} \quad (18)$$

and

$$\widehat{\emptyset} = \frac{K_1^2 \tau_1}{(n+1)R_2} \quad (19)$$

where:

$$R_1 = \lambda_2 - \frac{C_{n+1}^{(n+2)} R_1}{C_{n+2}^{(n+2)}} \quad (20)$$

The case $m = 1, s = 1$

$$Ly(x) = (P_{10} + P_{11} + P_{12}x^2)y'(x) + (P_{00} + P_{01}\alpha)y(x) = \sum_{r=0}^{f} f_r\, x^r \quad (21)$$

$$a \le x \le b$$

With the initial condition as given in Equation (10). The perturbed integral form of Equation (21) becomes:

$$\int_0^x \left[(P_{10} + P_{11}u + P_{12}u^2)(e_n^{|}(u))_{n+1} + (P_{00} + P_{01}u)(e_n(u))_{n+1}\right] du$$

$$= -\tau_1 \int_0^x H_{n+1}(u)du + \tilde{\tau}_1 \widetilde{H}_{n+2}(x) \quad (22)$$

where:

$$H_{n+1}(x) = \tau_1 \tau_{n+1}(x) + \tau_2 \tau_n(x) \quad (23)$$

and

$$\widetilde{H}_{n+2}(x) = \tilde{\tau}_1 \tau_{n+3}(x) + \tilde{\tau}_2 \tau_{n+2}(x) \quad (24)$$

Inserting Equation (24), (13) and (15) into (22) gives:

A.Y. Badeggi et al. / Nigerian Research Journal of Engineering and Environmental Sciences
2(1) 2017 pp. 203-214

$$\frac{\widehat{\Phi}}{K_1}[\lambda_1 x^{n+3} + \lambda_2 x^{n+2} + \lambda_3 x^{n+1} + + + \cdots]$$
$$= \tilde{\tau}_1 C_{n+3}^{(n+3)} x^{n+3} + \left[\tilde{\tau}_1 C_{n+2}^{(n+3)} + \tilde{\tau}_1 C_{n+2}^{(n+2)} - \frac{\tau_1 C_{n+1}^{(n+1)}}{n+2}\right] x^{n+2}$$
$$+ \left[\tilde{\tau}_1 C_{n+1}^{(n+3)} + \tilde{\tau}_1 C_{n+1}^{(n+2)} - \frac{\tau_1 C_n^{(n+1)}}{n+1} - \frac{\tau_2 C_n^{(n)}}{n+1}\right] x^{n+1} \tag{25}$$

where:

$$\lambda_1 = \left[\frac{P_{01} + (n+1)P_{12}}{n+3}\right] K_1$$
$$\lambda_2 = \left[\frac{P_{00} + (n+1)P_{11}}{n+2}\right] K_1 + \left[\frac{P_{01} + nP_{12}}{n+2}\right] K_2$$
$$\lambda_3 = P_{10} K_1 + \left[\frac{P_{00} + nP_{11}}{n+1}\right] K_2 + \left[\frac{P_{01} + (n-1)P_{12}}{n+1}\right] K_3$$
$$K_1 = C_n^{(n)}, \quad K_2 = C_{n-1}^{(n)}, \; K_3 = C_{n-2}^{(n)} \qquad etc. \tag{26}$$

Equating the coefficients of powers of $x^{n+3}, x^{n+2}, and\ x^{n+1}$ from both sides of Equation (25) we obtain the following values for $\tilde{\tau}_1, \tilde{\tau}_2, and\ \widetilde{\Phi}$:

$$\tilde{\tau}_1 = \frac{\widetilde{\Phi}\lambda_1}{K_1 C_{n+3}^{(n+3)}}$$
$$\tilde{\tau}_2 = \frac{\tau_1 C_{n+1}^{(n+1)}}{(n+2) C_{n+2}^{(n+2)}} + \frac{\widetilde{\Phi}}{K_1}\left[\lambda_2 - \frac{C_{n-2}^{(n+3)}\lambda_1}{C_{n-3}^{(n+3)}}\right]$$
$$\widehat{\emptyset}_n = \frac{-K_1 \tau_2}{(n+1) R_3} \tag{27}$$

Where:

$$R_3 = \lambda_3 - \frac{C_{n+1}^{(n+3)} R_1}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_2}{C_{n+2}^{(n+2)}} \tag{28}$$

Which gives the following recursive form:

$$R_1 = \lambda_1$$
$$R_2 = \lambda_2 - \frac{C_{n+2}^{(n+3)} R_1}{C_{n+3}^{(n+3)}}$$
$$R_3 = \lambda_3 - \frac{C_{n+1}^{(n+3)} R_1}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_2}{C_{n+2}^{(n+2)}} \tag{29}$$

Continuing with the process using $m = 1, s = 1, 2, 3, 4, ...$ by expanding Equation (1) we have the following recursive forms:

For $m = 1, s = 2$, we have

$$R_1 = \lambda_1$$
$$R_2 = \lambda_2 - \frac{C_{n+3}^{(n+4)} R_1}{C_{n+4}^{(n+4)}}$$
$$R_3 = \lambda_3 - \frac{C_{n+2}^{(n+4)} R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)} R_2}{C_{n+3}^{(n+3)}}$$

A.Y. Badeggi et al. / Nigerian Research Journal of Engineering and Environmental Sciences
2(1) 2017 pp. 203-214

$$R_4 = \lambda_4 - \frac{C_{n+1}^{(n+4)} R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} R_2}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_3}{C_{n+2}^{(n+2)}} \tag{30}$$

Where:

$$\lambda_1 = \left[\frac{P_{02} + (n+1)P_{13}}{n+4}\right] K_1$$

$$\lambda_2 = \left[\frac{P_{01} + (n+1)P_{12}}{n+3}\right] K_1 + \left[\frac{P_{02} + nP_{13}}{n+3}\right] K_2$$

$$\lambda_3 = \left[\frac{P_{00} + (n+1)P_{11}}{n+2}\right] K_1 + \left[\frac{P_{01} + nP_{12}}{n+2}\right] K_2 + \left[\frac{P_{02} + (n-1)P_{13}}{n+2}\right] K_3$$

$$\lambda_4 = P_{10} K_1 + \left[\frac{P_{00} + nP_{11}}{n+1}\right] K_2 + \left[\frac{P_{01} + (n-1)P_{12}}{n+1}\right] K_3 + \left[\frac{P_{02} + (n-2)P_{13}}{n+1}\right] K_4 \tag{31}$$

For $m = 1, s = 3, , we\ have$

$$R_1 = \lambda_1$$

$$R_2 = \lambda_2 - \frac{C_{n+4}^{(n+5)} R_1}{C_{n+5}^{(n+5)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+3}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+3}^{(n+4)} R_2}{C_{n+3}^{(n+3)}}$$

$$R_4 = \lambda_4 - \frac{C_{n+2}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+1}^{(n+3)} R_2}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)} R_3}{C_{n+3}^{(n+3)}}$$

$$R_5 = \lambda_5 - \frac{C_{n+1}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+1}^{(n+4)} R_2}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} R_3}{C_{n+3}^{(n+3)}} \tag{32}$$

Where:

$$\lambda_1 = \left[\frac{P_{03} + (n+1)P_{14}}{n+5}\right] K_1$$

$$\lambda_2 = \left[\frac{P_{02} + (n+1)P_{13}}{n+4}\right] K_1 + \left[\frac{P_{03} + nP_{14}}{n+4}\right] K_2$$

$$\lambda_3 = \left[\frac{P_{01} + (n+1)P_{12}}{n+3}\right] K_1 + \left[\frac{P_{02} + nP_{13}}{n+3}\right] K_2 + \left[\frac{P_{03} + (n-1)P_{14}}{n+3}\right] K_3$$

$$\lambda_4 = \left[\frac{P_{00} + (n+1)P_{11}}{n+2}\right] K_1 + \left[\frac{P_{01} + nP_{12}}{n+2}\right] K_2 + \left[\frac{P_{02} + (n-1)P_{13}}{n+2}\right] K_3 + \left[\frac{P_{03} + (n-2)P_{14}}{n+2}\right] K_4$$

$$\lambda_5 = P_{10} K_1 + \left[\frac{P_{00} + nP_{11}}{n+1}\right] K_2 + \left[\frac{P_{01} + (n-1)P_{12}}{n+1}\right] K_3 + \left[\frac{P_{02} + (n-2)P_{13}}{n+1}\right] K_4 + \left[\frac{P_{02} + (n-1)P_{13}}{n+2}\right] K_5 \tag{33}$$

Similarly, for $m = 1, s = 4$:

$$R_1 = \lambda_1$$

$$R_2 = \lambda_2 - \frac{C_{n+5}^{(n+6)} R_1}{C_{n+6}^{(n+6)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+4}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+4}^{(n+5)} R_2}{C_{n+5}^{(n+5)}}$$

A.Y. Badeggi et al. / Nigerian Research Journal of Engineering and Environmental Sciences
2(1) 2017 pp. 203-214

$$R_4 = \lambda_4 - \frac{C_{n+3}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+3}^{(n+5)} R_2}{C_{n+5}^{(n+5)}} - \frac{C_{n+3}^{(n+4)} R_3}{C_{n+4}^{(n+4)}}$$

$$R_5 = \lambda_5 - \frac{C_{n+2}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+2}^{(n+5)} R_2}{C_{n+5}^{(n+5)}} - \frac{C_{n+2}^{(n+4)} R_3}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)} R_4}{C_{n+3}^{(n+3)}}$$

$$R_6 = \lambda_6 - \frac{C_{n+1}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+1}^{(n+5)} R_2}{C_{n+5}^{(n+5)}} - \frac{C_{n+1}^{(n+4)} R_3}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} R_4}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_5}{C_{n+2}^{(n+2)}} \qquad (34)$$

Where:

$$\lambda_1 = \left[\frac{P_{04} + (n+1)P_{15}}{n+6}\right] K_1$$

$$\lambda_2 = \left[\frac{P_{03} + (n+1)P_{14}}{n+5}\right] K_1 + \left[\frac{P_{04} + nP_{15}}{n+5}\right] K_2$$

$$\lambda_3 = \left[\frac{P_{02} + (n+1)P_{13}}{n+4}\right] K_1 + \left[\frac{P_{03} + nP_{14}}{n+4}\right] K_2 + \left[\frac{P_{04} + (n-1)P_{15}}{n+4}\right] K_3$$

$$\lambda_4 = \left[\frac{P_{01} + (n+1)P_{12}}{n+3}\right] K_1 + \left[\frac{P_{02} + nP_{13}}{n+3}\right] K_2 + \left[\frac{P_{03} + (n-1)P_{14}}{n+3}\right] K_3 + \left[\frac{P_{04} + (n-2)P_{15}}{n+3}\right] K_4$$

$$\lambda_5 = \left[\frac{P_{00} + (n{+}1)P_{11}}{n+2}\right] K_1 + \left[\frac{P_{01} + nP_{12}}{n+2}\right] K_2 + \left[\frac{P_{02} + (n-1)P_{13}}{n+2}\right] K_3 + \left[\frac{P_{03} + (n-2)P_{14}}{n+2}\right] K_4 + \left[\frac{P_{04} + (n-3)P_{15}}{n+2}\right] K_5$$

$$\lambda_6 = P_{10}K_1 + \left[\frac{P_{00} + nP_{11}}{n+1}\right] K_2 + \left[\frac{P_{01} + (n-1)P_{12}}{n+2}\right] K_3 + \left[\frac{P_{02} + (n-2)P_{13}}{n+1}\right] K_4 + \left[\frac{P_{03}+(n-3)P_{14}}{n+1}\right] K_5 + \left[\frac{P_{04}+(n-4)P_{15}}{n+1}\right] K_6 \qquad (35)$$

The case $\boldsymbol{m = 2, s = 0}$

From Equation (1) we have for $m = 2, s = 0$:

$$Ly(x) := (P_{20} + P_{21} + P_{22}x^2)y^{ii}(x) + (P_{10} + P_{11}x)y^{i}(x) + P_{00}y(x) = \sum_{r=0}^{F} f_r x^r,\ \ a \leq x \leq b \qquad (36)$$

$$y(a) = \alpha_0, y^{i}(x) = \alpha_1 \qquad (37)$$

which yields:

$$\int_0^x \int_0^u \left[(P_{20} + P_{21}t + P_{22}t^2)(e_n^{ii}(t))_{n+1} + (P_{10} + P_{11}t)(e_n^{i}(t))_{n+1} + P_{00}(e_n(t))_{n+1}\right] dtdu = -\int_0^x \int_0^u H_n(t)dtdu + \widetilde{H}_{n+m+1}(x) \qquad (38)$$

$$(e_n(x))_{n+1} = \frac{\widehat{\Phi}_n (x-\alpha)^m \tau_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\widehat{\Phi}_n x^2 \tau_{n-1}(x)}{C_{n-1}^{(n-1)}} = \frac{\widehat{\Phi}_n x^2 \tau_{n-1}(x)}{2^{2n-3}} \qquad (39)$$

$$(e_n(x))_{n+1} = \frac{\widehat{\Phi}_n}{K_1}\left[K_1 x^{n+1} + K_2 x^n + K_3 x^{n-1} + ++..\right] \qquad (40)$$

A.Y. Badeggi et al. / Nigerian Research Journal of Engineering and Environmental Sciences
2(1) 2017 pp. 203-214

Where:
$K_1 = C_{n-1}^{(n-1)}, \qquad K_2 = C_{n-2}^{(n-1)}, K_3 = C_{n-3}^{(n-1)}$
and
$H_n(x) = \tau_1\tau_n(x) + \tau_2\tau_{n-1}(x)$
$\widetilde{H}_{n+m+1}(x) = \tilde{\tau}_1\tau_{n+3}(x) + \tilde{\tau}_2\tau_{n+2}(x)$ (41)
Inserting Equation (39) and (40) into (36), collecting the like terms and equating the corresponding coefficients of $x^{n+3}, x^{n+2}, and\ x^{n+1}$ yields:

$$\widehat{\Phi}_n = \frac{-K_1^2\tau_2}{n(n+1)R_3} \quad (42)$$

Where R_is are obtained recursively by

$$R_1 = \lambda_1$$
$$R_2 = \lambda_2 - \frac{C_{n+2}^{(n+3)}R_1}{C_{n+3}^{(n+3)}}$$
$$R_3 = \lambda_3 - \frac{C_{n+1}^{(n+3)}R_1}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)}R_2}{C_{n+2}^{(n+2)}} \quad (43)$$

and

$$\lambda_1 = \left[\frac{P_{00} + (n+1)P_{11} + n(n+1)P_{22}}{(n+2)(n+3)}\right]K_1$$
$$\lambda_2 = \left[\frac{P_{10} + nP_{21}}{n+2}\right]K_1 + \left[\frac{P_{00} + nP_{11} + n(n-1)P_{22}}{(n+1)(n+2)}\right]K_2$$
$$\lambda_3 = P_{20}K_1 + \left[\frac{P_{10}+(n-1)P_{21}}{n+1}\right]K_2 + + \left[\frac{P_{00}+(n-1)P_{11}+(n-1)(n-2)P_{22}}{n(n+1)(n+2)}\right]K_3 \quad (44)$$

Continuing with the process, using m = 2 and varying the value of s i.e. (s = 1, 2, 3, …) and expanding Equation (36) the following recursive form is obtained:

$$R_1 = \lambda_1$$
$$R_2 = \lambda_2 - \frac{C_{n+3}^{(n+4)}R_1}{C_{n+4}^{(n+4)}}$$
$$R_3 = \lambda_3 - \frac{C_{n+2}^{(n+4)}R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)}R_2}{C_{n+3}^{(n+3)}}$$
$$R_4 = \lambda_4 - \frac{C_{n+1}^{(n+4)}R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)}R_2}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)}R_3}{C_{n+2}^{(n+2)}} \quad (45)$$

Where:

$$\lambda_1 = \left[\frac{P_{01} + (n+1)P_{12} + n(n+1)P_{23}}{(n+2)(n+3)}\right]$$
$$\lambda_2 = \left[\frac{P_{00} + (n+1)P_{11} + n(n+1)P_2}{(n+2)(n+3)}\right]K_1 + \left[\frac{P_{01} + nP_{12} + n(n-1)P_{23}}{(n+2)(n+3)}\right]K_2$$
$$\lambda_3 = \left[\frac{P_{10} + nP_{21}}{n+2}\right]K_1 + \left[\frac{P_{00} + nP_{11} + n(n-1)P_{22}}{(n+1)(n+2)}\right]K_2 + \left[\frac{P_{01} + (n-1)P_{12} + (n-1)(n-2)P_{23}}{n(n+1)(n+2)}\right]K_3$$

$$\lambda_4 = P_{20}K_1 + \left[\frac{P_{10} + (n-1)P_{21}}{n+1}\right]K_2 + \left[\frac{P_{00} + (n-1)P_{11} + (n-1)(n-2)P_{22}}{n(n+1)}\right]K_3 + \left[\frac{P_{01} + (n-2)P_{12} + (n-2)(n-3)P_{23}}{n(n-1)(n+1)}\right]K_4 \quad (46)$$

and for m = 2, s = 2, we obtain:

$$R_1 = \lambda_1$$

$$R_2 = \lambda_2 - \frac{C_{n+4}^{(n+5)} R_1}{C_{n+5}^{(n+5)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+3}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+3}^{(n+4)} R_2}{C_{n+4}^{(n+4)}}$$

$$R_4 = \lambda_4 - \frac{C_{n+2}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+2}^{(n+4)} R_2}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)} R_3}{C_{n+3}^{(n+3)}}$$

$$R_5 = \lambda_5 - \frac{C_{n+1}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+1}^{(n+4)} R_2}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} R_3}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_4}{C_{n+2}^{(n+2)}} \quad (47)$$

Similarly, the recursive form for m = 2, s = 3 were obtained as follows: $R_1 = \lambda_1$

$$R_2 = \lambda_2 - \frac{C_{n+5}^{(n+6)} R_1}{C_{n+6}^{(n+6)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+4}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+4}^{(n+5)} R_2}{C_{n+5}^{(n+5)}}$$

$$R_4 = \lambda_4 - \frac{C_{n+3}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+3}^{(n+5)} R_2}{C_{n+5}^{(n+5)}} - \frac{C_{n+3}^{(n+5)} R_3}{C_{n+4}^{(n+4)}}$$

$$R_5 = \lambda_5 - \frac{C_{n+2}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+2}^{(n+5)} R_2}{C_{n+5}^{(n+5)}} - \frac{C_{n+2}^{(n+4)} R_3}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)} R_4}{C_{n+3}^{(n+3)}}$$

$$R_6 = \lambda_6 - \frac{C_{n+1}^{(n+6)} R_1}{C_{n+6}^{(n+6)}} - \frac{C_{n+1}^{(n+5)} R_2}{C_{n+5}^{(n+5)}} - \frac{C_{n+1}^{(n+4)} R_3}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} R_4}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_5}{C_{n+2}^{(n+2)}} \quad (48)$$

Continuing with the procedure for m = 3, 4, . . . and varying s i.e. (s = 0,1,2,3,…) and expanding Equation (1) fully, the following general expression for λ_v *and* R_v is obtained as shown:

$$R_v = \lambda_v - \sum_{i=1}^{v-1} \frac{C_{n+m+s+2-v}^{(n+m+s+2-i)}}{C_{n+m+s+2-i}^{(n+m+s+2-i)}} R_i \qquad v = 1,2,3, \ldots m+s. \quad (49)$$

Where λ_v:

$$\lambda_v = \frac{\sum_{i=1}^{v}\left\{\sum_{j=0}^{m} (P_{j,s-v+i+j})(j!)\binom{n+2-i}{j}\right\}K_i}{\pi_{r=1}^{m+i-v}(n+s+m+3-v-r)} \qquad v = 1,2,3, \ldots m+s+1 \quad (50)$$

Provided $i \geq v - m + 1$

In order to establish the validity of Equations (49) and (50), the following theorems were stated.

2.1. Theorem

The parameters R_{m+s+1} of the error estimate is given by the recurrence relation:

$$R_v = \lambda_v - \sum_{v-1}^{i=1} \frac{C_{n+m+s+2-v}^{(n+m+s+2-i)}}{C_{n+m+s+2-i}^{(n+m+s+2-i)}} R_i \qquad ,v = 1,2,3, \dots m+s \tag{51}$$

Where: λ_v:

$$\lambda_v = \frac{\sum_{i=1}^{v}\left\{\sum_{j=0}^{m}(P_{j,s-v+i+j})(j!)\binom{n+2-i}{j}\right\}K_i}{\pi_{r=1}^{m+s+i-v}(n+s+m+3-v-r)} \qquad v = 1,2,3, \dots m+s+1 \tag{52}$$

2.2. Proof

The principle of mathematical induction was employed over the summation variables m and v to establish the validity of Equation (50). This was done by varying one of these variables at a time while the other was fixed.

$$\lambda_v = \sum_{i=1}^{v} \frac{\left\{\sum_{j=0}^{m}(P_{j,s-v+i+j})(j!)\binom{n+2-i}{j}\right\}K_i}{\pi_{r=1}^{m+s+i-v}(n+s+m+3-v-r)}$$

Firstly, let v = 1 in Equation (50), that is, it is assumed that Equation (50) is true for v = 1, so that:

$$\lambda_1 = \frac{\left\{\sum_{j=0}^{m}(P_{j,s+j})(j!)\binom{n+1}{j}\right\}K_1}{\pi_{r=1}^{1}(n+s+m+2-r)} \qquad i.e\ i = v = 1 \tag{53}$$

Now, using induction on m for fixed v = 1, it is shown that Equation (50) holds for m=1, that is, for m=1:

$$\lambda_1 = \frac{\left\{\sum_{j=0}^{1}(P_{j,s+j})(j!)\binom{n+1}{j}\right\}K_1}{\pi_{r=1}^{1}(n+s+3-r)} \qquad i.e\ m = 1$$

$$= \left\{\frac{P_{0,s} + P_{1,,s+1}(n+1)}{(n+s+2)}\right\} K_1$$

$$= \left\{\frac{P_{0,s} + (n+1)P_{1,,s+1}}{(n+s+2)}\right\} K_1 \tag{54}$$

Which is the same as $\boldsymbol{\lambda}_1$ in equation (54) when $s = 0, s = 1, s = 2\ \ s = 3, and\ \ s = 4$ respectively.

Hence, Equation (53) is true for m=1. Thus we now assume that Equation (53) is true for *m*= *l* which gives:

$$\lambda_1 = \frac{\left\{\sum_{j=0}^{l}(P_{j,s+j})(j!)\binom{n+1}{j}\right\}K_1}{\pi_{r=1}^{l}(n+s+c+2-r)} \qquad m = \quad l \tag{55}$$

It is then shows that Equation (53) holds for $m = l + 1$.

From the construction of $\boldsymbol{\lambda}_1$, for m=1 up to m= l:

$$\lambda_1 = \frac{\left\{\sum_{j=0}^{l}(P_{j,s+j})(j!)\binom{n+1}{j}\right\}K_1}{\pi_{r=1}^{1}(n+s+l+3-r)} + \frac{\left\{\left(P_{l+1,s+l+1}\right)[l+1)!]\binom{n+1}{l+1}\right\}K_1}{\pi_{r=1}^{l+1}(n+s+l+2-r)}$$

$$\left\{\frac{\sum_{j=0}^{l}(P_{j,s+j})(j!)\binom{n+1}{j}}{\pi_{r=1}^{1}(n+s+l+3-r)} + \frac{\left\{\left(P_{l+1,s+l+1}\right)[l+1)!]\binom{n+1}{l+1}\right\}}{\pi_{r=1}^{l+1}(n+s+l+2-r)}\right\}K_1$$

$$= \frac{\left\{\sum_{j=0}^{l+1}(P_{j,s+j})(j!)\binom{n+1}{j}\right\}K_1}{\pi_{r=1}^{l+1}(n+s+l+2-r)} \tag{56}$$

Thus, since equation (49) holds for $m = l + 1$, hence it holds for all positive values of m

Next, it is assumed that Equation (49) holds for $v = l$, that is:

$$\lambda_l = \frac{\sum_{j=1}^{l}\left\{\sum_{j=0}^{m}(P_{j,s-l+i+j})(j!)\binom{n+2-i}{j}\right\}K_i}{\pi_{r=1}^{m}(n+s+m+3-l-r)} \tag{57}$$

and then it is shown that it holds for $v = l + 1$. That is:

$$\lambda_{l+1} = \frac{\sum_{i=1}^{l+1}\left\{\sum_{j=0}^{m}(P_{j,s-l+i+j})(j!)\binom{n+2-i}{j}\right\}K_i}{\pi_{r=1}^{m+i-l-1}(n+s+m+3-l-r)} \tag{58}$$

But by the construction of $\boldsymbol{\lambda}_{l+1}$:

$$\lambda_{l+1} = \frac{\sum_{i=1}^{l}\left\{\sum_{j=0}^{m}\left(P_{j,s-l+i+j})(j!)\binom{n+2-i}{j}\right)\right\}K_i}{\pi_{r=1}^{m+i-l}(n+s+m+3-l-r)} + \; + \frac{\left\{\sum_{j=0}^{m}(P_{j,s-l+1})(j!)\binom{n+2-i}{j}\right\}K_{l+1}}{\pi_{r=1}^{m}(n+s+m+3-l-r)} \tag{59}$$

$$\lambda_{l+1} = \frac{\sum_{i=1}^{l+1}\left\{\sum_{j=0}^{m}(P_{j,s-l+i+j+1})(j!)\binom{n+2-i}{j}\right\}K_i}{\pi_{r=1}^{m+i-l}(n+s+m+3-l-r)} \tag{60}$$

Thus, from the foregoing it can be concluded that Equation (49) and (50) hold for all values of m and v and this confirms the validity of the theorem.

4. CONCLUSION

The derivation of a general formula for the integrated formulation of the tau method for m-th order linear ODEs has been presented. The formulae are recursive and hence makes for easy determination of particular cases for which m will be specified. The present error estimation technique shows a remarkable improvement over these works done on the subject of error analysis of the Tau method as it leads to error estimation formula with wider scope of application. Also, the estimate proposed here does not involve any iteration for linear problems nor matrix inversion. This is desirable in handling non-linear problems, where s, the number of over determination of the equation being considered, depends on n, the degree of the Tau approximant being sought and for large value of n, (m + s) then becomes very large. All these features are desirable and render the error estimation technique and the formula attractive for use with software design for the Tau method.

5. CONFLICT OF INTEREST

There is no conflict of interest associated with this work.

REFERENCES

Adeniyi R. B. (1991). On the Tau method for numerical solution of ordinary differential equations, Ph.D thesis, University of Ilorin, Nigeria.

Adeniyi, R. B. and Ma'ali, A. I. (2008). On the Tau Method for a Class of Non-overdetermined second Order Differential Equation. *Journal of the Nigerian association of mathematical physics,* 12, pp. 387-398.

Adeniyi, R. B. and Aliyu, A. I. M. (2012a). An Error Estimation of the Tau method for some classes of ordinary differential equations. *IOSR journal of mathematics,* 2(1), pp. 32-40.

Adeniyi, R.B. and Aliyu, A. I. M. (2012b). On the integrated formulation of the Tau method involving at most two Tau parameters, for IVPS in ODEs. *IOSR journal of mathematics,* 2(1), pp. 23-31.

Adeniyi, R. B. and Onumanyi, P. (1991), Error estimation in the numerical solution of the ordinary differential equation with Tau method. *Computer mathematics Application,* 21(9), pp. 19-27

Fox, I. and Parker, I.B. (1981). *Chebyshev polynomials in numerical analysis,* Oxford, University Press

Lanczos, C. (1938). Trigonometric interpolation of empirical and analytic Function. *Journal of Mathematical Physics*, 17, pp.123-199

Lanczos, C. (1956). *Applied Analysis*. Prentice Hall, New Jersey.

Ma'ali, .A.I. (2008). *On the Tau method for solution of a class of second order differential equations*. M.Sc. Thesis, University of Ilorin, Ilorin.

Ma'ali, A.I. (2012): A Computational Error Estimation of the Integral Formulation of the Tau Method for some class of Ordinary Differential Equations. (Unpublished), Doctoral Thesis, University of Illorin, Illorin.

Namasivayam , S and Ortiz , E.L(1981) Approximate Coefficients and Approximation Error in the Numerical Solution of Differential Equations with an application to the Tau Method. Research Report, NAS 04-08-82

Ortiz, E.L. (1969) The Tau method. *SIAM Journal of Numerical Analysis*, 6, pp. 480-492

Ortiz, E.L. (1974). *Canonical polynomials in the Lanzcos Tau method.* In: Scaife, B.K.P (Ed), Studies in numerical analysis. Academic Press, New York

Yisa, B.M. and Adeniyi, R.B. (2012). A generalized formula for canonical polynomials for m-th order non-overdetermined ordinary differential equations (ODEs). *International Journal of Engineering Research & Technology (IJERT,* 1(6), pp. 1-22.