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DERIVATION OF THREE VARIANTS OF τ-METHOD FOR THE CLASS OF NON-OVER DETERMINED ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper compared the three variants of the tau methods, namely the differential, integrated and the recursive form for the initial value problems in non-over determined ordinary differential equations. The fifth degree approximant for the three variants of the tau methods was derived and their corresponding error estimations were obtained. The numerical examples confirms that the order of the approximants was accurately captured and the integrated form performed better than the other two variants.

1. INTRODUCTION

The Tau method was initially formulated as a tool for the approximation of special functions of mathematical physics which could be expressed in terms of simple differential equations (Ortiz, 1969; Adeniyi, 1985; Adeniyi and Aliyu, 2011). It later developed into a powerful and accurate tool for the numerical solution of complex differential and functional equations. The main idea in it is to approximate the solution of a given problem by solving exactly an approximate problem. The method is related to the principle of economization of a differentiable function implicitly defined by a linear differential equation with polynomial coefficients.

1.1. The Differential Formulation of Lanczos Tau Method

Consider the m-th order linear differential system as shown in Equation (1).

$$\mathrm{Ly}(x) := \sum_{r=0}^{m} p_r(x) y^{(k)}(x) = f(x), \quad a \leq x \leq \mathrm{b} \tag{1}$$

$$\mathrm{Ly}^{*}(x_{rk}) := \sum_{r=0}^{m-1} a_{rk}\, \mathrm{y}^{(r)}(x_{rk}) = \alpha_{k,\ k=1(1)m} \tag{2}$$

The idea of Lanczos, C. (1938) as in Adeniyi (1991) and Aliyu, (2007) is to approximate the solution of the differential Equation (1) by an n-th degree polynomial function.

$$y_n(x) := \sum_{r=0}^{n} a_r\, \mathrm{x}^r, \qquad \mathrm{n} < \infty \tag{3}$$

To obtain the exact solution of a perturbed equation, the polynomial perturbation term is added to the right hand side of Equation (1). The polynomial $y_n(x)$ satisfies the condition given in Equation (4).

$$Ly_n(x) := \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = f(x) + H_n\ (x) \tag{4}$$

Where:

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1}\ T_{n-m+r+1\,(x)} \tag{5}$$

$H_n(x)$ is taken as a linear combination of powers of x multiplied by Chebyshev Polynomial (Fox, 1962; Adeniyi and Aliyu, 2007). The corresponding coefficients of x in Equation (5) are equated and using the initial conditions, the system of equations are solved by Gaussian elimination method.

1.2. The Integrated Formulation of Lanczos Tau Method

Consider the m-th order linear differential system of Equation (1) and let $\int\int\int_{\ldots}^{i}\int g(x)dx$ denote the indefinite integration i times applied to the function $g(x)$, and let $\mathrm{I_L}$ be
defined as follows:

$$\mathrm{I_L} = \int\int\int_{\ldots}^{m}\int l(\cdot)dx \tag{6}$$

The integral form of Equation (1), is then:

$$\mathrm{I_L}\,(y(x)) = \int\int\int_{\ldots}^{m}\int f(x)dx + c_{m-1}(x) \tag{7}$$

Where $c_{m-1}(x)$ denotes an arbitrary polynomial of degree $(m-1)$ arising from the constants of integration. The Tau approximate $y_n(x)$ of the solution $y(x)$ of Equation (1) thus satisfies the perturbed problem:

$$\mathrm{I_L}\,(y_n(x)) = \int\int\int_{\ldots}^{m}\int f(x)dx + c_m(x)\, H_{n+m}(x) \tag{8}$$

$$\mathrm{L}^*\, y_n\ \ (x_{rk})\ \ \alpha_{k,}\ k = 1(1)m \tag{9}$$

where $H_{n+m}(x)$) is as in Equation (5).

1.3. The Ortiz Recursive Formulation of the Tau Method

It was noted in Lanczos.C (1938) as in Aliyu A. I. (2007), that if a sequence of polynomials $Q_n(x)$, n = 0, 1 such that $DQ_n(x) = x^n\ for\ all\ n \in N$ can be found for any linear differential operator with polynomial coefficients D, then since the chebyshev polynomial

$$T_n(x) = \sum_{k=0}^{n} c_k^{(r)} x^r \quad (10)$$

The solution of the Tau problem would be immediately given by:

$$y_n(x) = \tau_1 \sum_{k=0}^{n} c_k^{(r)} Q_n(x) \quad (11)$$

where the parameter τ is fixed using the initial condition. A recursive generation of Lanczos canonical polynomials was proposed by Ortiz (1969). Let $y(x)$ be a known function which satisfies

$$\mathrm{L}y(x) = f(x) \quad (12)$$

where L is m-th order linear differential operator with polynomial coefficients and

$$f(x) = \sum_{r=0}^{f} a_r\, \mathrm{x}^r \quad (13)$$

is a given polynomial of degree n with real coefficient f_i, $i = 0(1)n$. In addition, it is assumed that $y(x)$ $a \le x \le$ b, satisfies Equation (1).

A sequence $Q_r(x), r \ge 0$ of canonical polynomial $Q_r(x)$ is uniquely associated with the operator L in Equation (1) such that:

$$\mathrm{L}Q_r(x) = x^r \quad (14)$$

A few members of the sequence may not be defined for certain operators L. Let S denote the set of indices for which members of the sequence are not defined and let s denote the number of elements of S. For the generation of the sequence, the generating polynomials $\mathrm{L}x^r$ is utilised. The Tau method involves seeking a polynomial solution of the perturbed equation:

$$Ly_n(x) = \sum_{r=0}^{f} f_r\, \mathrm{x}^r + \sum_{r=0}^{m+s-1} \tau_{m+s-r}\; T_{n-m+r+1}(x), \;\; \mathrm{n} \ge \mathrm{F} \quad (15)$$

$$Ly_n(x) = \sum_{r=0}^{f} f_r\, \mathrm{x}^r + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} \sum_{k=0}^{(n-m+r+1)} x^k \quad (16)$$

Using Equations (14) and (16), we have:

$$Ly_n(x) = \sum_{r=0}^{f} f_r\, Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{(n-m+r+1)} \mathrm{L}Q_k(x) \quad (17)$$

Since L is linear, this becomes:

$$Ly_n(x) = L\{\sum_{r=0}^{f} f_r\; Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} \sum_{k=0}^{(n-m+r+1)} Q_k(x)\} \quad (18)$$

Assuming that L^{-1} exits, this further gives Equation (19) as the approximation:

$$Ly_n(x) = \sum_{r=0}^{f} f_r\; Q_r(\mathrm{x}) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} \sum_{k=0}^{(n-m+r+1)} Q_k(x) \quad (19)$$

At this stage, all the quantities on the right hand sides of Equation (18), except $\tau_1, \ldots, \tau_{m-s}$ are known. To determine these parameters, the Gaussian elimination is applied to m+s algebraic equations.

(a). The coefficient of any $Q_r(x)$, $r \epsilon S$ in (18) is set to zero to give n conditions and

(b). $y_n(x)$ given by (18) satisfies (a) and also m conditions $\mathrm{L}^* y_n(x_{rk}) = \alpha_r$, k = 1,2,…,m.

2. NUMERICAL EXAMPLE

Consider the solution of an ordinary differential equation with the Tau method for the three variants described above.

2.1. Example 2.1

Consider the first-order IVP

$Ly(x) := (x - 1)\, y'(x) + y(x) = 0\ , y(0) =$ (20)
with exact solution $y(x) = (1 + x)^{-1}$, $0 \le x \le 1$. From (2.1), m =1, s = 0 and $f(x) = 0$

2.1.1. First variant (differential form)

From Equations (4) and (5), the Tau approximant (Equation 20) satisfies the condition of Equation (20).

$y_n(x) = \sum_{r=0}^{f} a_r\, \mathrm{x}^r, n < \infty$ (21)

$(x + 1) y_n'(x) + y_n(x) = \tau_1 T_n^*(x)$ (22)

y_n' (0) =1 (23)

Equation (22) is then solved for n=5, so the Tau problem we are concerned with is:

$(x + 1)\, y_n'\,(x) + \ y_5\,(x) = \tau_1\, T_5^*(x)$ (24)

$y_5(0) = 1$ (25)

where:

$T_5^*(x) = -1 + 50x - 400x^2 + 1120x^3 - 1280x^4 + 512x^5$ (26)

Substituting Equation (21), ($n = 5$) into (24) gives:

$$\sum_{r=0}^{5} (r + 1)a_r\, \mathrm{x}^r + \sum_{r=0}^{5} r a_r\, \mathrm{x}^{r-1}$$
$$= \tau_1(-1 + 50x - 400x^2 + 1120x^3 - 1280x^4 + 512x^5) \quad (27)$$

Equating corresponding coefficients of powers of x in (27) and $a_r = 1$ from the condition (25) we obtain the Tau system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 & 0 & -50 \\ 0 & 3 & 3 & 0 & 0 & 400 \\ 0 & 0 & 4 & 4 & 0 & -1120 \\ 0 & 0 & 0 & 5 & 5 & 1280 \\ 0 & 0 & 0 & 0 & 6 & -512 \end{bmatrix} \begin{bmatrix} a_2 \\ a_2 \\ a_2 \\ a_2 \\ a_2 \\ \tau_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (28)$$

The solution of which gives:

τ_1= -3/2342 , a_1= - 2339/2342 , a_2 = 2264/2342 , a_3 = -1864/2342 ,
a_4=1024/2342, a_5 = -256/2342

Hence, the 5-th degree Tau approximant of Equation (20) is

$y_5(x) = -(256x^5 - 1024x^4 + 1864x^3 - 2264x^2 + 2339x - 2342)/2342$ (29)

2.1.2. Second variant (integrated formulation)

From Equation (4), we have for the given problem:

$I_L\big(y(x)\big) = \int[(1 + x)y'(x) + y(x)]dx + c_1 = 0$ (30)

$\Rightarrow I_L\big(y(x)\big) = (1 + x)y(x) - \int y(x)dx + \int y(x)dx + c_1 = 0$ (31)

consider Equation (31) in the range $[0, x]$ to obtain:

$(1 + x)y(x) - 1 = 0$ (32)

Thus, the perturbed integrated problem becomes

$(1 + x)y_n(x)$=1 + $\tau_1\, T_{n+1}^*(x)$ (33)

$y_n(0)$ = 1

Substituting Equation (21) and (n = 5) into Equation (33) gives

$$\sum_{r=0}^{5} a_r \, \mathrm{x}^r + \sum_{r=0}^{5} a_r \, \mathrm{x}^{r+1} = 1 + \tau_1 T_6(x) \qquad (34)$$

Where:

$$T_6(x) = 1 - 72x + 840x^2 - 3584x^3 + 6912x^4 - 6144x^5 + 2048x^6 \qquad (35)$$

Equating corresponding coefficients of powers of x in (32) and the proceeding as in the first variant to obtain, for the case $n = 5$:

a_0 = 19600/19601, a_1 = 19528/19601, a_2 = 18688/19601, $a_3 = 15104/19601$ a_4 = 8192/19601, a_5 = 2048/19601, τ_1 = -1/19601

Hence the 5-th degree Tau approximant of (20) becomes

$$y_5(x) = (19600 - 19528x + 18688x^2 - 15104x^3 + 8192x^4 - 2048x^5)/19601 \qquad (36)$$

It is observed that the value of τ_1 in (36) is smaller than that obtained for the first variant due to the higher order perturbation term used in Equation (33). If we define the error of the Tau method at the point $x = \alpha$, $0 \le \alpha \le 1$, as |y(α)- y_n (α)|, then the error at the point $x = 0.5$, for example, from the use of Equation (29) is approximately 7.12×10^{-5} while the one obtained from the use of Equation (36) is approximately 3.40×10^{-5}, justifying the claim that the integrated formulation of the Tau method gives more accurate approximants than the differential form.

2.1.3 Third variant (recursive formulation)

From Equation (19), we have, for the IVP (Equation 20)

$$y^n(x) = \tau_1 \sum_{r=0}^{n} C_r^{(x)} Q_r(x) \qquad (37)$$

where the members of the sequence $Q_r(x)$,r =0(1)n, of canonical polynomials are generated recursively as follows:

From equation (20):

$$Lx^r = (1 + x)rx^{r-1} + x^r \qquad (38)$$

i.e. $Lx^r = (r + 1)x^r + x^{r-1}$ (39)

Using Equation (14) in Equation (39) we have:

$$Lx^r = (\mathrm{r} + 1)\, \mathrm{L}Q_r(x) + rLQ_{r-1}(x) \qquad (40)$$

and since L is linear, Equation (40) becomes:

$$x^r = rQ_{r-1}(x) + (r + 1)Q_r(x) \qquad (41)$$

which gives the recursive formula:

$$Q_r(x) = \frac{x^r - rQ_{r-1}(x)}{r+1} \;, \; r\epsilon N \qquad (42)$$

From Equation (42), we obtain, for $r = 0,1,2,3,4,5$

$$Q_0(x) = 1, \quad Q_1(x) = \frac{x-1}{2}, \quad Q_2(x) = \frac{x^2-x-1}{3}, \; Q_3(x) = \frac{x^3-x^2+x-1}{4}$$

$$Q_4(x) = \frac{x^4-x^3-x^2-x+1}{5}, \; Q_5(x) = \frac{x^5-x^4+x^3-x^2+x+1}{6} \qquad (43)$$

Thus, the 5-th degree Tau solution of Equation (22) is given by:

$$y_5(x) = \tau_1(-Q_0(x) + 50Q_1(x) - 400Q_2(x) + 1120Q_3(x) - 1280Q_4(x) + 512Q_5(x)) \qquad (44)$$

where $Q_r(x)$, $r = 0(1)5$ are defined by Equation (43).Using condition of Equation (25) in Equation (44), we obtain:

τ_1 = -3/1830. Thus, Equation (44) now becomes:

$$y_5(x) = -(256x^5 - 1024x^4 + 1864x^3 - 2264x^2 + 1827x - 1830)/1830 \quad (45)$$

After substituting the values of τ_1 and the expression for $Q_r(x)$ and then simplifying the resulting expression, thus, we observe that Equation (45) is almost the same as Equation (29) obtained for the differential formulation.

3. ERROR ESTIMATION OF THE LANCZOS TAU METHOD

The subject of error estimation is of central concern in numerical analysis since most numerical methods provide approximations to the true desired solutions of mathematical problems; Namasivayam, S. and Ortiz, E. L. (1981) and Adeniyi R. B. (1991). It is important to be able to bound or estimate the resulting error and a numerical method therefore that fails to provide a suitable procedure for doing this is incomplete. The error estimation for the three variants are thus reviewed hereunder.

3.1. Error Estimation for the Differential Form

The error function $e_n(x)$ defined

$$e_n(x) = y(x) - y_n(x) \quad (46)$$

which satisfies the error equation:

$$Le_n(x) - H_n(x) \quad (47)$$

and

$$E_{n+1}(x) = \frac{\emptyset_n \mu_m(x) T^*_{n-m+1}(x)}{2^{2(n-m)+1}} \quad (48)$$

To determine $\emptyset_n$ in Equation (48), we consider the perturbed equation:

$$LE_{n+1}(x) = -H_n(x) + \widetilde{H}_{n+1} \quad (49)$$

Where:

$$\widetilde{H}_{n+1}(x) = \tau_1 T^*_{n+s+1}(x) + \tau_2 T^*_{n+s}(x) + \ldots + \bar{\tau}_{m+s} T^*_{n-m+2}(x) \quad (50)$$

That is:

$$LE_{n+1}(x) = + \bar{\tau}_1 T^*_{n+s+1}(x) + (\bar{\tau}_2 - \tau_1) T^*_{n+s}(x) + (\bar{\tau}_3 - \tau_2) T_{n+s-1}(x) + \ldots + (\bar{\tau}_{m+s} - \tau_{n+s-1}) T^*_{n-m+2}(x) - \tau_{m+s} - \tau_{n-m+1}(x) \quad (51)$$

The necessary (m+s+1) equations to determine $\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_{m+s}$ and $\emptyset_n$ are given by equating the coefficients of $x^{n+s+1}, x^{n+s}, \ldots, x^{n-m-1}$ in Equation (51). Then the value of $\emptyset_n$ is obtained by a forward substitution which depends on n and τ_{m+s}. Substituting the expression so obtained into m+s+1 (48) gives $E_{n+1}(x)$ in terms of known quantities and for large n, Equation (52) is obtained as an estimate of the maximum error ε, in the interval $0 \le x \le 1$.

$$\varepsilon_1^* = \max_{0\le x\le 1} |E_{n+1}(x)| = \frac{|\emptyset_n|}{2^{2n-2m+1}} \quad (52)$$

3.2. Error Estimation for Integrated Formulation

The integrated formulation of the Tau method often leads to better accuracy of the Tau solution (Fox, 1962; Ortiz, 1969; Aliyu, 2007). The integrated error equation is therefore considered here with the aim of improving the accuracy of the estimate Equation (52). To the $\int\int\int^{i}_{\ldots}\int g(x)dx$ denote the indefinite integration i times applied to the function g(x) and let

$I_L = \int\int_{.\,..}^{i}\int L(\cdot)dx$ (53)

be the integrated form of Equation (47). Therefore:

$I_L(E_n(x)) = -\int\int_{.\,..}^{m}\int H_n(x)dx$ (54)

The perturbed form of Equation (54) that is the perturbed error equation:

$I_L(E_{n+1}(x)) = -\int\int_{.\,..}^{m}\int H_n(x)dx + \widetilde{H}_{n+m+1}(x)$ (55)

which is satisfied by $E_{n+1}(x)$ given by Equation (48) with $\emptyset_n$ replaced by $\hat{\emptyset}_n$.

$\widetilde{H}_{n+m+1}(x) = \overline{T}_{n+m+s+1}(x) + \overline{\tau}_2 T^*_{(n+m+s)}(x) + \overline{T}_{m+s}T_{n+2}(x)$ (56)

Equating corresponding coefficient of like powers of x in Equation (55) and solving the resulting algebraic equations lead to the values of $\hat{\emptyset}_n$ which yields Equation (57) as an estimate of ε.

$\varepsilon_1^* = \frac{|\widetilde{\emptyset}_n|}{2^{2n-2m+1}}$ (57)

3.3. Error Estimation for the Ortiz Recursive Formulation

The expression of the approximate solution of $y_n(x)$ in terms of canonical polynomials offers several advantages as canonical polynomials neither depend on the interval in which the solution is sought (Crisci and Russo, 1982). Also, when an approximation of a higher degree $n+k,\ k \geq 1$ is required, it is only necessary to compute $Q_{n+1}(x)$, $Q_{n+2}(x)$,… $Q_{n+k}(x)$, canonical polynomials, and then weight these and the ones already computed with a different set of coefficients $C_r^{(n+k)}$, r = 0(1)$(n+k)$, to get the desired approximation.

Once the canonical polynomials are generated, they can be used for an error estimation of the Tau method (Namasivayam and Ortiz, 1981; Crisci and Russo, 1982; Adebiyi and Aliyu, 2007). Here, we consider a slight perturbation of the given boundary conditions, by ε_1^* given by Equation (52) to obtain an estimate of the Tau parameter T_{m+s} in terms of canonical polynomials which is then substituted back into the expression for ε_1^* in Equation (52) for a new estimate ε_3^*.

4.0. NUMERICAL EXAMPLES

In this section, two examples are considered. The error was estimated for the three variants of the Tau method. The effect of perturbing some of the homogenous conditions of the error function $e_n(x)$ for the error polynomial $E_{n+1}(x)$ for the case of boundary value problems was also examined.

4.1. Example 4.1

We consider the IVP:

$Ly(x) := (1+x)\frac{dy}{dx} + y(x) = 0,\ y(0) = 1$ (58)

with theoretical solution $y(x) = (1+x)^{-1},\ 0 \leq x \leq 1$

For the problem, m = 1, s = 0, $f(x) = 0$:

$H_n(x) = T_n^*(x) = \bar{\tau}_1 \sum_{r=0}^{n} C_r^{(n)} x^r$, $\widetilde{H}_{n+1}(x) = \tilde{\tau}_1 T_{n+1}^*(x) = \tilde{\tau}_1 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r$

$E_{n+1}(x) = \frac{\emptyset_n x T_n^*(x)}{C_{n-m+1}^{(n-m-1)}} = \frac{\emptyset_n x T_n^*(x)}{2^{2n-1}} = \frac{\emptyset_n}{2^{2n-1}} \sum_{r=0}^{n+1} C_r^n x^{r+1}$ (59)

4.1.1. Differential formulation

From Equation (49), we have:

$$LE_{n+1}(x) = \bar{\tau}_1 - T_n^*(x) + \bar{\tau}_1 T_{n+1}^*(x) \quad (60)$$

Substituting Equation (59) into Equation (60) results in:

$$(1+x)\,(E'_{n+1}(x)) + E_{n+1}\ (x) = -\sum_{r=0}^{n+1} C_r^n\ x^r + \bar{\tau}_1 \sum_{r=0}^{n} C_r^{n+1}\ x^r \quad (61)$$

Where:

$$E'_n(x) = \frac{\emptyset_n}{2^{2n-1}} \sum_{r=0}^{n} (r+1)\, C_r^{(n)} x^r = \frac{\emptyset_n}{2^{2n-1}} [(n+1) C_{n-1}^n x^{n-1} + (n-1) C_{n-2}^n x^{n-2} + \ldots] \quad (62)$$

Expanding both sides of Equation (61) results in:

$$\frac{\emptyset_n}{2^{2n-1}}[2+n)\ C_n^{(n)}\ x^{n+1} + [(n+1)\ C_n^{(n)} + (n+1)\ C_{n-1}^{(n)}]\ x^n + \ldots..] =$$
$$\bar{\tau}_1 C_{n+1}^{(n+1)} x^{n+1} + (\bar{\tau}_1\ C_{n+1}^{(n+1)} - \tau_1 C_{n+1}^{(n+1)}) x^n + \ldots.. \quad (63)$$

Equating corresponding coefficients of x^{n+1} and x^n in Equation (63) gives the system of equations:

$$\bar{\tau}_1 C_{n+1}^{(n+1)} = \frac{\emptyset_n}{2^{2n-1}} (2+n)\ C_n^{(n)} \quad (64)$$

$$\bar{\tau}_1 C_{n+1}^{(n+1)} - \bar{\tau}_1 C_n^{(n)} = \frac{\emptyset_n}{2^{2n-1}} [(1+n)\ C_n^{(n)} + (n+1)\ C_{n-1}^{(n)}] \quad (65)$$

Equation (65) was solved for $\bar{\tau}_1$ and $\emptyset_n$ by a forward substitution. From Equation (64):

$$\bar{\tau}_1 = \frac{(2+n)\ C_n^{(n)}}{2^{2n-1} - C_{n+1}^{(n+1)}} \emptyset_n \quad (66)$$

$$C_n^{(n)} = 2^{2n-1}, \quad C_{n+1}^{(n+1)} = 2^{2n-1}, \quad \frac{C_n^{(n-1)}}{C_n^{(n)}} = \frac{-1}{2} n, \quad \frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} = \frac{-1}{2}(n+1) \quad (67)$$

Substituting Equation (67) in (66) results in:

$$\bar{\tau}_1 = \frac{(2+n)\emptyset_n}{2^{2n-1}} \quad (68)$$

Substituting Equation (68) into (65) and simplifying results in:

$$\emptyset_n = \frac{2\char`^(2n-2)}{(n+1)} \quad (69)$$

From Equation (49), we have for $m = 1$:

$$\varepsilon_1^* = \frac{|\tau_1|}{2(n+1)} \quad (70)$$

4.1.2. Integrated formulation

From Equation (53) we have:

$$\int_0^x [(1+t)E'_{n+1}(t) + E_n(t)]dt = -\tau_1 \int_0^x H_n(t)dt + \bar{\tau}_1 H_{n+2}(x) \quad (71)$$

Replacing $\emptyset_n$ by $\widetilde{\emptyset}_n$ and $\widetilde{H}_{n+2}(x) = \tilde{\tau}_1 T_{n+2}(x) = \tilde{\tau}_1 \sum_{r=0}^{n+2} C_r^{n+2} x^r$

Thus:

$$(1+t)E_{n+1}(t)]_0^x - \int_0^x E_{n+1}\ (t)dt + \int_0^x E_{n+1}(t)dt = -\bar{\tau}_1 \int_0^x T_n(t)dt + T_{n+2}(x) \quad (72)$$

Expanding Equation (72) and collecting like terms results in:

$$\frac{\emptyset_n}{2^{2n-1}}[C_n^{(n)} x^{n+2} + (C_n^{(n)} + C_{n-1}^{(n)}) x^{n+1} + \cdots] = \bar{\tau}_1 C_{n+2}^{(n+2)}\ x^{n+2} +$$

$$[\bar{\tau}_1 C_{n+1}^{(n+2)} - \frac{\bar{\tau}_1 C_n^{(n)}}{n+1]}]x^{n+1} + \cdots \tag{73}$$

Equating the corresponding of x^{n+2} and x^{n+1} in Equation (73) gives the following system of equations.

$$\bar{\tau}_1 C_{n+2}^{(n+2)} = \frac{C_n^{(n)} \widetilde{\emptyset}_n}{2^{2n-1}} \tag{74}$$

$$\bar{\tau}_1 C_{n+1}^{(n+2)} - \frac{\tau_1 C_n^{(n)}}{n+1} = \frac{\widetilde{\emptyset}_n}{2^{2n-1}} [C_n^{(n)} + C_{n\ -1}^{(n)}] \tag{75}$$

Solving for $\bar{\tau}_1$and $\widehat{\emptyset}_n$ using forward substitution results in:

$$\bar{\tau}_1 = \frac{C_n^{(n)} \widehat{\emptyset}}{2^{2n-1} C_{n+1}^{(n+1+}} = \frac{\widehat{\emptyset}}{2^{2n+1}} \text{ and } \widehat{\emptyset}_n = \frac{-2^{2n} \tau_1}{(n+1)(3n+10)} \tag{76}$$

From Equation (57):

$$\varepsilon_2^* = \frac{2|\bar{\tau}_1|}{(n+1)(3n+10)} \tag{77}$$

4.1.3. Recursive formulation

The Tau approximant $L_y(x) = (1+x)\frac{dy}{dx} + y(x) = 0, y(0) = 0$ is defined as:

$$y_n(x) = \bar{\tau}_1 \sum_{r=0}^{n} C_r^n \ Q_r(x) \tag{78}$$

where the members of the set of canonical polynomials $Q_r(x)$, r= 0 (1) n, are generated recursively from:

$$Q_r(x) = \frac{x^r - rQ_{r-1}(x)}{r+1} \tag{79}$$

$$Q_0(x) = 1,\ Q_1(x) = \frac{x-1}{2}, Q_2(x) = \frac{x^2-x+1}{3},\ Q_3(x) = \frac{x^3-x^2+x-1}{4}$$

and we let

$$R_r^k(x) = \sum_{r=0}^{v} C\ {}^{(v)}_{r} Q\ {}^{(k)}_{r}(x), \quad K = 0\ (1)\ m \tag{80}$$

denote the derivative of $R_v\ (x)$ where $Q_r(x)$, $r \geq 0$ is set of canonical polynomials associated with conditions. Using the initial condition $y_n(0) = 1$, we have from Equation (79) and (80):

$\bar{\tau}_1 R_n(0) = 1$

Hence, $|\bar{\tau}_1| \, |R_n(0)| \leq 1 + \varepsilon_1^*$ *Since* $\varepsilon_1^* \geq 0$

and this becomes:

$$|\tilde{\tau}_1| \leq \frac{2(n+1)}{2\,(n+1)|\,R_n\,(0)|-1} \tag{81}$$

This estimate of $|\tilde{\tau}_1|$ is substituted into Equation (70) to give:

$$\varepsilon_1^* \leq \frac{1}{2\,(n+1)|\,R_n\,(0)|-1} = \varepsilon_3^* \tag{82}$$

This is dependent on the canonical polynomials but independent on $\bar{\tau}_1$ .The main attraction of Equation (82) is that it is possible to obtain an estimate of the error prior to computation of $y_n(x)$, assuming $C_r^{(n)}$ and $Q_r(x)$, $r = 0(1)n$ are known.

4.2. Example 4.2

Consider the second Order IVP:

$$L_y(x) = y''(x) - 4y = 0 \ , \ y(0) = 1 \ , y'(0) = 1 \tag{83}$$

with analytic solution $y(x) = \frac{1}{4}(3e^{2x} - e^{-2x}), \quad 0 \leq x \leq 1.$

For this problem $m = 2, s, = 0, \ f(x) = 0$

$$H_n(x) = \bar{\tau}_1 T^*_{n-}(x) = \bar{\tau}_1 \sum_{r=0}^{(n)} C_r^{(n)} x^r + \bar{\tau}_2 \sum_{r=0}^{(n-1)} C_r^{(n-1)} x^r$$

$$\hat{H}_{n+1}(x) = \bar{\tau}_1 T^*_{n+1}(x) + \bar{\tau}_2 T^*_n(x) = \bar{\tau}_1 \sum_{r=0}^{(n+1)} C_r^{(n+1)} x^r + \bar{\tau}_2 \sum_{r=0}^{(n)} C_r^{(n)} x^r$$

$$E_{n+1}(x) = \frac{\emptyset_n x^2 T^*_{n-1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\emptyset_n x^2 T^*_{n-1}(x)}{2^{2n-3}} = \frac{\emptyset_n}{2^{2n-3}} \sum_{r=0}^{(n)} C_r^{(n-1)} x^{r+2} \tag{84}$$

4.2.1. Differential formulation

From Equation (49) we have:

$$LE_{n+1}(x) = -\bar{\tau}_1 T^*_n(x) - \bar{\tau}_2 T^*_{n-1}(x) + \bar{\tau}_1 T^*_{n+1}(x) + \bar{\tau}_2 T^*_n(x) \tag{85}$$

which after expanding Equation (85) results in:

$$\frac{\emptyset_n}{2^{2n-3}} \left[-4C_{n-1}^{(n-1)} x^{n+1} - 4C_{n-2}^{(n-1)} x^n + \left[n(n+1)C_{n-1}^{(n-1)} - 4C_{n-3}^{(n-1)}\right] x^{n-1} + \cdots\right]$$
$$= \bar{\tau}_1 C_{n+1}^{(n+1)} x^{n+1} + \left[\bar{\tau}_1 C_n^{(n+1)} + \bar{\tau}_2 C_n^{(n)} - \tau_1 C_n^{(n)}\right] x^n +$$
$$\left[\bar{\tau}_1 C_{n-1}^{(n+1)} + \bar{\tau}_2 C_{n-1}^{(n)} - \tau_1 C_{n-1}^{(n)} - \tau_2 C_{n-1}^{(n-1)}\right] x^{n-1} + \cdots \tag{86}$$

Equating the corresponding coefficients of x^{n+1}, x^n and x^{n-1} in (86) and solving the resulting systems equations we obtain the value of $\emptyset_n$.

$$\emptyset_n = \frac{2^{2n-4}\tau_1}{k_1} \tag{87}$$

where $k_1 = 16\, C_{n-3}^{(n+1)} - C_{n-1}^{(n+1)} - n(n-1)2^{2n}$

Hence:

$$\varepsilon_1^* = \frac{2^{2n-1}|\tau_2|}{k_1} \tag{88}$$

4.2.2. Integrated formulation

From Equation (55):

$$I_L(E_{n+1}(\mathrm{x})) = -\int_0^x \int_0^t H_n(u)dudt + \widetilde{H}_{n+3}(x) \tag{89}$$

Where:

$$\widetilde{H}_{n+3}(x) = \bar{\tau}_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \bar{\tau}_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r\}$$
$$H_n(x) = \tau_1 \sum_{r=0}^{n} C_r^{(n)} x^r + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} x^r$$
$$E_{n+1}(x) = \frac{\emptyset_n x^2 T_{n-1}(x)}{2^{2n-3}} \tag{90}$$

Substituting Equation (89) into (90) results in:

$$\int_0^x \int_0^t E''_{n+1}(u)dudt - \int_0^x \int_0^t E_{n+1}(u))\, dudt = -\tau_1 \int_0^x \int_0^t H_n(u)\, dudt + \widetilde{H}_{n+3}(x) \tag{91}$$

Equating corresponding coefficients of x^{n+3}, x^{n+2} and x^{n+1} in (91) and solving the resulting systems of equations results in the following:

$$\widetilde{\emptyset}_n = \frac{(n+2)(n+3)\times 2^{6n-3}\tau_2}{k_2} \tag{92}$$

$$k_2 = (n+2)(n+3)\left[n \times 2^{4n+1} - n(n+1)2^{4n} + 2^{2n+5}C_{n-3}^{(n-1)}\right] - n(n+1)C_{n+3}^{(n+1)} \times 2^{2n-3} \tag{93}$$

Thus:

$$\varepsilon_1^* = \frac{(n+2)(n+3)\times 2^{4n}\,|\tau_1|}{k_2} \tag{94}$$

4.2.3. Recursive formulation

The Tau approximation of Equation (58) is defined as:

$$y_n\,(x) = \tau_1 \sum_{r=0}^{n} C_r^{(n)}\, Q_r\,(x) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)}\, Q_r\,(x) \tag{95}$$

where the members of the set of canonical polynomials $Q_r(x),\ r = 0\,(1)n$ are generated recursively from:

$$Q_r\,(x) = \frac{r(r-1)Q_{r-2}\,(x) - x^r}{4},\ \text{r = 0 (1) n}$$

That is:

$$Q_0\,(x) = \frac{-1}{4}, Q_1\,(x) = \frac{-x}{4}, Q_2\,(x) = \frac{-1}{8} - \frac{x^2}{4}, Q_3\,(x) = \frac{-3x}{8} - \frac{x^3}{4},$$

$$Q_4\,(x) = \frac{-3}{8} - \frac{3x^2}{4} - \frac{x^4}{4},\ Q_5(x) = \frac{-15x}{8} - \frac{5x^3}{4} - \frac{x^5}{4}$$

Substituting the initial conditions Equation (83) in (95) results in:

$$\tau_1 \sum_{r=0}^{n} C_r^{(n)}\ Q_r\,(0) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)}\ Q_r\,(0) = 1 \tag{96}$$

and

$$\tau_1 \sum_{r=0}^{n} C_r^{(n)}\ Q_r'(0) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)}\ Q_r'(0) = 1 \tag{97}$$

In the notation of Equation (80), these yields:

$$\tau_1 R_n\,(0) + \tau_2 R_{n-1}\,(0) = 1 \tag{98}$$

$$\tau_1\, R_r'(0) + \tau_2 R_r'(0) = 1 \tag{99}$$

From Equation (99):

$$\tau_1 = [\ 1 - \tau_2 R_{n-1}'(0)]/R_n'(0) \tag{100}$$

Inserting Equation (100) into (98) gives:

$$[R_{n-1}(0)]R_n'(0) - R_{n-1}'(0)]\,\tau_2 = R_n'\,(0) - R_n\,(0)$$

Thus:

$$|[R_{n-1}(0)R_n'(0) - R_{n-1}'(0)\,R_n\,(0)]\big||\tau_2| \le R_n'(0) - R_n\,(0)\big| + \varepsilon_1^*$$

since $\varepsilon_1^* \ge 0$ from Equation (88), thus:

$$|\tau_2| \le \frac{K_1|R_n'(0) - R_n\,(0)|}{|k_1|\,||[R_{n-1}(0)R_n'(0) - R_{n-1}'(0)\,R_n\,(0)| - 2^{2n-1}} \tag{101}$$

But: $|\tau_2| = \frac{k_1\varepsilon_1^*}{2^{2n-1}}$

This estimate of $|\tau_2|$ is substituted back into Equation (101) to give:

$$\varepsilon_1^* \le \frac{|R_n'(0) - R_n\,(0)|}{2^{-(2n-1)}|k_1||R_{n-1}(0)R_n'(0) - R_{n-1}'(0)\,R_n\,(0)| - 1} \tag{102}$$

where $k_1 = 16C_{n-3}^{(n+1)} - C_{n-1}^{(n+1)} - n(n-1)\,2^{2n}$

This is the error estimate which is dependent on the canonical polynomials but independent of τ_2

Table 1: Error and error estimate for Example 1

No.of opproximant	2	3	4	5
ε_1^*	2.17×10^{-0}	3.68×10^{-1}	6.25×10^{-2}	9.94×10^{-4}
ε_2^*	2.45×10^{-3}	2.66×10^{-4}	3.15×10^{-5}	3.96×10^{-6}
ε_3^*	2.22×10^{-2}	3.69×10^{-3}	6.28×10^{-4}	1.07×10^{-4}
Exact Error	2.29×10^{-2}	4.95×10^{-3}	7.51×10^{-4}	1.12×10^{-4}

Comment*:* ε_1^* has smaller error than ε_2^*

Table 2: Error and error estimate for Example 2

No.of approximant	2	3	4	5
ε_1^*	4.16×10^{0}	1.70×10^{-1}	1.19×10^{-2}	9.94×10^{-4}
ε_2^*	1.96×10^{-2}	1.11×10^{-3}	5.70×10^{-5}	3.36×10^{-6}
ε_3^*	4.86×10^{0}	1.71×10^{-1}	1.19×10^{-2}	9.94×10^{-4}
Exact Error	1.55×10^{0}	4.92×10^{-1}	3.61×10^{-2}	4.86×10^{-4}

Comment: ε_2^* shows significant improvement over ε_1^* for example 2

4. CONCLUSION

It is observed in this work that perturbing the integrated error equation appears to improve the accuracy of the error estimate significantly. Also the representation of the estimate in terms of canonical polynomials does not lead to appreciable loss of accuracy. In Examples 4.1 and 4.2, it is noted that both the differential and recursive forms yield approximations of the same order when all the parameters involved in the general form took value.

5. CONFLICT OF INTEREST

There is no conflict of interest associated with this work.

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