# A Class of Six Step Block Method for Solution of General Second Order Ordinary Differential Equations.

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## ABSTRACT

In this paper, self starting block method of order (7, 7, 7, 7, 7, 7) is proposed for the solution of general second order initial value problem of the form $y'' = f(x, y, y')$ directly without reducing it to first systems of odes. The continuous formation of the integrator enable us to differentiate and evaluate at some grids points to take care of y' in the method. The schemes compare favorably with optimal order four (Fatunla Based) proposed in Yahaya (2004). There is anticipated speed up of computation as a result of admissible parallelism across the method.

(Keywords: general second order, initial value problems, parallel/block method, self starting)

## INTRODUCTION

Linear multistep methods constitute a powerful class of numerical procedures for showing a second order equation of the form:

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \beta \quad \textbf{(1)}$$

It has been well known that an analytical solution to this equation is of little value because many of such problems cannot be solved by analytical approach. In practice, the problems are reduced to systems of first order equations and any methods for first order equations are used to solve them. Awoyemi (1999); Fatunla (1988); and Lambert (1973) extensively discussed that due to dimension of the problem after it has been reduced to a system of first order equations, the approach waste a lot of computer time and human efforts. Some attempts has been made to solve problem (1) directly without reduction to a first order systems of equations; Brown (1977) and Lambert (1991) independently proposed a method known as multi derivative to solve second order initial value problems type (1) directly. In a recent paper of Onumanyi et al. (2008), they proposed direct block Adam Molton Method (BAM) and hybrid block Adam Molton method (IBAM) for accurate approximation to y' appearing in Equation (1) to be able to solve problem (1) directly. The aim of this paper is to demonstrate using the proposed block method of order $(7, 7, 7, 7, 7, 7)^T$ derived to solve Equation (1) directly and compare its performance with the optimal order four schemes (Fatunla) based proposed in Yahaya (2004).

## The Multistep Collocation Method

In the spirit of Onumayi et al. (1994) and Yahaya (2004), we consider the construction of multistep collocation method of constant step size h, though h can be variable and give continuous expression for the coefficient. The values of K and M are arbitrary except for collocation at the mesh points where $0 \prec m \leq k+1$.

Let $y_{n+j}$ be an approximation to $y_{n+j}$ where:

$$y_{n+j} = Y(x_{n+j}) = 0..........K-1$$

Then a K-step multistep collocation method is constructed as follows. We find a polynomial $y(x)$ of degree $p = t + m - 1, t \succ 0, m \succ 0$ and such that it satisfies the conditions:

$$y(x_{n+j}) = y_{n+j}, \ j \varepsilon \{0.......k\} \quad \textbf{(2)}$$

$$y''(x_{n+j}) = f(x_{n+j}), \ j = 0.......... ..m-1 \quad \textbf{(3)}$$

Where $x_1..............x_{m-1}$ are free collocations points, we then take as an approximation to $y_{n+k}, Y_{n+k} = Y(x_{n+k})$.

Let

$$y(x) = \sum_{j=1}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=1}^{m-1} \beta_j(x) f(x_{j+1}, y(x_{j+1})) \quad \textbf{(4)}$$

Where $\alpha_j$ and $\beta_j$ are assumed polynomial of the form,

$$\alpha_j(x) = \sum_{j=0}^{m+t-1} a_j, i+1x^i; h^2\beta_j(x) = \sum_{j=0}^{m+t-1} h^2\beta_j, i+1x^i \quad \textbf{(5)}$$

and the collocation point $x_{j+1}$ in (3) belong to the extended set,

$$Q = \{x_n................x_{n+k}\} U \{x_{n+k-1}................x_{n+k}\} \quad \textbf{(6)}$$

From the interpolation conditions (2) and the expression for $y(x)$ in (4) the following conditions are imposed on $\alpha_j(x) and \beta_j(x)$,

$$\begin{aligned} &\alpha_j(x_{m+i}) = \delta_{ij},\ j = 0......t-1, i = 0......., t-1 \\ &h^2\beta_j(x_{m+i}) = 0,\ j = 0......m-1, i = 0......., t-1 \end{aligned} \quad \textbf{(7)}$$

and

$$\begin{aligned} &\alpha^{11}{}_j(x_{n+i}) = 0,\ j = 0......t-1, i = 0......., m-1 \\ &h^2\beta_j(x_{n+i}) = \delta_{ij},\ j = 0......m-1, i = 0......., m-1 \end{aligned} \quad \textbf{(8)}$$

Next we write (7) – (8) in a matrix equation of the form:

DC=I **(9)**

Where I is the identity matrix of dimension $(t+m)$ The matrices D and C are both of dimensions $(t+m)\times(t+m)$. It follows from (8) that the columns of $C = D^{-1}$ give the continuous coefficient $\alpha_j(x)$ and $\beta_j(x)$.

**<u>Derivation of the proposed Method</u>**

Consider a power series of a single variable x in the form:

$$p(x) = \sum_{J=0}^{\infty} a_j$$

Is used as the basis or trial function to produce our approximate solution to (1) as:

$$p(x) = \sum_{J=0}^{\infty} a_j x^j \quad \textbf{(10)}$$

$$p'(x) = \sum_{J=0}^{m+t-1} j a_j x^{j-1} \quad \textbf{(11)}$$

$$p''(x) = \sum_{J=0}^{m+t-1} j(j-1) a_j x^{j-2} \quad \textbf{(12)}$$

From Equations (10) and (12):

$$p''(x) = \sum_{J=0}^{m+t-1} j(j-1) a_j x^{j-2} = f(x, y, y') \quad \textbf{(13)}$$

Where $a_j$ is the parameter to be determined, t and m are points of interpolation and collocation points. We collocate (13) and interpolate (10) yields the following systems of non linear equations:

$$
\begin{aligned}
y(x) ={}& \left(5-\frac{x-x_n}{h}\right)y_{n+4}+\left(-4+\frac{x-x_n}{h}\right)y_{n+5} \\
&+\left(-\frac{409}{6048}h^2-\frac{12287}{40320}h(x-x_n)+\frac{1}{2}(x-x_n)^2-\frac{49}{120}\frac{(x-x_n)^3}{h}+\right. \\
&\frac{203}{1080}\frac{(x-x_n)^4}{h^2}-\frac{49}{960}\frac{(x-x_n)^5}{h^3}+\frac{7}{864}\frac{(x-x_n)^6}{h^4}-\frac{1}{1440}\frac{(x-x_n)^7}{h^5} \\
&\left.+\frac{1}{40320}\frac{(x-x_n)^8}{h^6}\right)f_n \\
&+\left(-\frac{1061}{1008}h^2-\frac{29369}{20160}h(x-x_n)+\frac{(x-x_n)^3}{h}-\frac{29}{40}\frac{(x-x_n)^4}{h^2}\right. \\
&\left.+\frac{29}{120}\frac{(x-x_n)^5}{h^3}-\frac{31}{720}\frac{(x-x_n)^6}{h^4}+\frac{1}{252}\frac{(x-x_n)^7}{h^5}-\frac{1}{6720}\frac{(x-x_n)^8}{h^6}\right)f_{n+1} \\
&+\left(-\frac{3893}{2016}h^2-\frac{2671}{5760}h(x-x_n)-\frac{5}{4}\frac{(x-x_n)^3}{h}+\frac{39}{32}\frac{(x-x_n)^4}{h^2}\right. \\
&\left.-\frac{461}{960}\frac{(x-x_n)^5}{h^3}+\frac{137}{1440}\frac{(x-x_n)^6}{h^4}-\frac{19}{2016}\frac{(x-x_n)^7}{h^5}+\frac{1}{2688}\frac{(x-x_n)^8}{h^6}\right)f_{n+2} \\
&+\left(\frac{4633}{1512}h^2-\frac{14719}{10080}h(x-x_n)+\frac{10}{9}\frac{(x-x_n)^3}{h}-\frac{127}{108}\frac{(x-x_n)^4}{h^2}\right. \\
&\left.+\frac{31}{60}\frac{(x-x_n)^5}{h^3}-\frac{121}{1080}\frac{(x-x_n)^6}{h^4}+\frac{1}{84}\frac{(x-x_n)^7}{h^5}-\frac{1}{2016}\frac{(x-x_n)^8}{h^6}\right)f_{n+3} \\
&+\left(\frac{7085}{2016}h^2-\frac{5293}{8064}h(x-x_n)-\frac{5}{8}\frac{(x-x_n)^3}{h}+\frac{11}{16}\frac{(x-x_n)^4}{h^2}\right. \\
&\left.-\frac{307}{960}\frac{(x-x_n)^5}{h^3}+\frac{107}{1440}\frac{(x-x_n)^6}{h^4}-\frac{17}{2016}\frac{(x-x_n)^7}{h^5}+\frac{1}{2688}\frac{(x-x_n)^8}{h^6}\right)f_{n+4} \\
&+\left(\frac{389}{1008}h^2-\frac{3481}{20160}h(x-x_n)+\frac{1}{5}\frac{(x-x_n)^3}{h}-\frac{9}{40}\frac{(x-x_n)^4}{h^2}\right. \\
&\left.+\frac{13}{120}\frac{(x-x_n)^5}{h^3}-\frac{19}{720}\frac{(x-x_n)^6}{h^4}+\frac{1}{315}\frac{(x-x_n)^7}{h^5}-\frac{1}{6720}\frac{(x-x_n)^8}{h^6}\right)f_{n+5} \\
&+\left(-\frac{95}{6048}h^2+\frac{13}{896}h(x-x_n)-\frac{1}{36}\frac{(x-x_n)^3}{h}+\frac{137}{4320}\frac{(x-x_n)^4}{h^2}\right. \\
&\left.-\frac{1}{64}\frac{(x-x_n)^5}{h^3}+\frac{17}{4320}\frac{(x-x_n)^6}{h^4}-\frac{1}{2016}\frac{(x-x_n)^7}{h^5}+\frac{1}{40320}\frac{(x-x_n)^8}{h^6}\right)f_{n+6}
\end{aligned}
\qquad \textbf{(14)}
$$

The continuous scheme (14) was evaluated at some selected points it yielded the following discrete schemes. Evaluating (14) at $x=x_{n+6}$, $x=x_{n+2}$, $x=x_{n+3}$, $x=x_{n+1}$ and $x=x_n$ yield respectively four integrator below:

$$y_{n+6}+y_{n+4}-2y_{n+5}=h^2\left[-\frac{221}{60480}f_n+\frac{263}{10080}f_{n+1}-\frac{1609}{20160}f_{n+2}+\frac{1987}{15120}f_{n+3}-\frac{769}{20160}f_{n+4}+\frac{8999}{10080}f_{n+5}+\frac{863}{12096}f_{n+6}\right]$$

See Jain (1984).

$$y_{n+2}-3y_{n+4}+2y_{n+5}=h^2\left[\frac{31}{20160}f_n-\frac{3}{224}f_{n+1}+\frac{257}{2240}f_{n+2}+\frac{4927}{5040}f_{n+3}+\frac{3897}{2240}f_{n+4}+\frac{209}{1120}f_{n+5}-\frac{137}{20160}f_{n+6}\right]$$

$$y_{n+3}-2y_{n+4}+y_{n+5}=h^2\left[\frac{31}{60480}f_n-\frac{31}{10080}f_{n+1}+\frac{71}{20160}f_{n+2}+\frac{1357}{15120}f_{n+3}+\frac{16451}{20160}f_{n+4}+\frac{977}{10080}f_{n+5}-\frac{221}{60480}f_{n+6}\right]$$

$$y_{n+1}-4y_{n+4}+3y_{n+5}=h^2\left[-\frac{11}{10080}f_n+\frac{41}{560}f_{n+1}+\frac{1167}{1120}f_{n+2}+\frac{4927}{2520}f_{n+3}+\frac{2987}{1120}f_{n+4}+\frac{153}{560}f_{n+5}-\frac{19}{2016}f_{n+6}\right]$$

$$y_n - 5y_{n+4} + 4y_{n+5} = h^2\left[\frac{409}{6048}f_n + \frac{1061}{1008}f_{n+1} + \frac{3893}{2016}f_{n+2} + \frac{4633}{1512}f_{n+3} + \frac{7085}{2016}f_{n+4} + \frac{389}{1008}f_{n+5} - \frac{95}{6048}f_{n+6}\right]$$
**(15)**

The block scheme of 15 is of orders (7, 7, 7, 7, 7)$^T$ with error constants:

$$\left(-\frac{19}{6048}, \frac{31}{30240}, \frac{31}{60480}, \frac{31}{30240}, \frac{314}{30240}\right)$$

The first derivative of Equation 14 at $x=x_{n+6}$ is used along with the schemes in 15 to start the integration process, that is"

$$\begin{aligned} &40320hz_{n+6} + 40320y_{n+4} - 40320y_{n+5} = \\ &\quad h^2\left[-479f_n + 3470f_{n+1} - 10921f_{n+2} + 19460f_{n+3} - 18689f_{n+4} + 55246f_{n+5} + 12393f_{n+6}\right] \end{aligned} \quad \textbf{(16)}$$

**Implementation Strategies:** To start the IVP integration on the sub interval $[x_0, x_6]$.We combine (15) and(16), when $n=0$ i.e the 1-block 6-point method as given in equation (18). Thus produces simultaneously values for $y_1, y_2, y_3, y_4, y_5$ along with $y6$ without recourse to any predictor

**Numerical Experiment:** This section deals with the implementation of the algorithm proposed for second order initial value problems. Consider the initial value problem

$$y'' - y' = 0, \qquad y(0) = 0, y'(0) = -1$$

$$\textit{Analytic solution}: \qquad y(x) = 1 - e^x$$

**Table 2:** Accuracy of 6-Step Block Method of Order 7, H=0.1.

| Theoretical/Approximate Solutions, h=0.1 | | | | |
|---|---|---|---|---|
| **X** | **Exact solution y(x)** | **6 step Block Method y-computed** | **Errors Of Optimal (2004)** | **Errors Proposed Method** |
| 0.1 | -0.1051709180 | -0.105165192 | 5.008136E-03 | 5.7260E-06 |
| 0.2 | -0.2214027580 | -02213961189 | 1.101918E-02 | 6.6391E-06 |
| 0.3 | -0.3498588080 | -0.3498517797 | 1.9041146E-02 | 7.0283E-06 |
| 0.4 | -0.4918246980 | -0.4918172441 | 2.8374166E-02 | 7.4539E-06 |
| 0.5 | -0.6487212710 | -0.6487133775 | 4.0041949E-02 | 7.8935E-06 |
| 0.6 | -0.82211880 | -0.8221106058 | 5.339556E-02 | 8.1942E-06 |
| 0.7 | -1.0137527000 | -1.013744519 | 6.9481732E-02 | 8.1810E-06 |
| 0.8 | -1.2255409280 | -1.225532747 | 8.7709919E-02 | 8.1810E-06 |
| 0.9 | -1.4596031110 | -1.459594938 | 1.09158725E-01 | 8.1730E-06 |
| 1 | -1.7182818280 | -1.718273663 | 1.33295713E-01 | 8.1650E-06 |

## CONCLUSION

This paper demonstrated a successful application of linear multi-step method to solve a general second order ordinary differential equation of the form $y'' = f(x, y, y')$ directly without reducing it to first order ones. Numerical results show that the block method converges better than optimal order 4 block method. Furthermore, the proposed block method is self starting and does not call for special predictor to estimate $y'$ in the integrators, all the discrete schemes used in each of the method were derived from a single continuous formula and its derivatives making use of grid point in the formulation.

## REFERENCES


1. Awoyemi, D.O. 1999. "A Class of Continuous Method for General Second Order Initial Value Problems in Ordinary Differential Equations". *Int. J. Comput. Maths.* 72:29-37.

2. Brown, R.L. 1977. "Some Characteristics of Implicitly Multi Derivative Integration Formulas SIAM. *J.Numer. Anal*. 14:982-983.

3. Fatunla, S.O. 1988. "Numerical Methods for Initial Value Problems for Ordinary Differential Equations". Academy Press: New York, NY. 295.

4. Jain, M.K. 1984. *Numerical Solution of Differential Equations*. John Wiley and Sons: New York, NY. 150.

5. Lambert, J.D. 1973. *Computational Methods in Ordinary Differential Equations*. John Wiley and Sons: New York, NY. 278.

6. Lambert, J.D. 1991. *Numerical Methods for Ordinary Differential Systems*. John Wiley and Sons: New York, NY. 293.

7. Onumanyi, P., Awoyemi, D.O., Jator, S.N., and Siresina, U.W. 1994. "New Linear Multistep Methods with Continuous Coefficients for the First Order Initial Value Problem. *Journal. Nig. Maths Soc.* 13:37-51.

8. Onumanyi, P., Fatokun, J., and Adejo, B.O. 2008. "Accurate Numerical Differentiation by Continuous Integration for Ordinary Differential Equations". *J. Math. Soc.* 27:69-90.

9. Yahaya, Y.A. 2004. "Some Theory and Application of Continuous Linear Multi-Step Methods for Ordinary Differential Equations. Ph.D. Thesis (unpublished). University of Jos: Jos, Nigeria.


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