A Third Order Trigonometrically-Fitted Improved Runge-Kutta Method for Oscillatory Problems

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Abstract
This research is concerned about improving the numerical solution of initial value problems with oscillatory solutions by trigonometrically – fitting a third order Improved Runge – Kutta (IRK) method with coefficients that are dependent on the frequency and step – size. Numerical experiments with sample problems compared the results of the proposed trigonometrically – fitted method with those of the non – fitted method. It was established that the trigonometrically-fitted third order three stage IRK method, IRK32, is more efficient in terms of fewer number of function evaluations and accurate in terms of small errors than non-fitted IRK method of same order.

Keywords: Oscillatory solution, Initial value problem, Improved Runge – kutta method, Trigonometrically – fitted

Introduction

Consider the initial value problem

$$y'(x) = f\big(x, y(x)\big), \qquad y(x_0) = y_0 \tag{1}$$

One numerical method of solution is the Runge – Kutta method

$$\left.\begin{array}{c} y_{n+1} = y_n + h\displaystyle\sum_{i=1}^{s} b_i k_i \\ \text{where} \\ k_i = f\left(x_n + c_i h, y_n + h\displaystyle\sum_{j=1}^{s} a_{i,j} k_j\right), \quad i = 1, \dots, s \end{array}\right\} \tag{2}$$

The Improved Runge -Kutta methods (IRK) originate from the classical RK methods. They can be considered as a special class of two step methods (Rabiei *et al*., 2013). That is, the approximate solution y_{n+1} is calculated using the values of y_n and y_{n-1}. The Improved Runge -Kutta method (IRK) with s – stages for solving (1) has the general form:

$$y_{n+1} = (1-\alpha)y_n + \alpha y_{n-1} + h\left(b_1 k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i\,(k_i - k_{-i})\right) \tag{3}$$

for $0 \le \alpha \le 1, 1 \le n \le N-1$

where,

$$\left.\begin{array}{c} k_1 = f(x_n, y_n), \quad k_{-1} = f(x_{n-1}, y_{n-1}) \\ k_i = f\left(x_n + c_i h, y_n + h\displaystyle\sum_{j=1}^{i-1} a_{i,j} k_j\right), \qquad 2 \le i \le s \\ k_{-i} = f\left(x_{n-1} + c_i h, y_{n-1} + h\displaystyle\sum_{j=1}^{i-1} a_{i,j} k_{-j}\right), \qquad 2 \le i \le s \end{array}\right\} \tag{4}$$

By assigning $\alpha = 0$, the IRK method (3) is written as

$$y_{n+1} = y_n + h\left(b_1k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i(k_i - k_{-i})\right) \quad (5)$$

for $1 \leq n \leq N-1$
An alternative form of (5) is

$$y_{n+1} = y_n + hb_1f(x_n, y_n) - hb_{-1}f(x_{n-1}, y_{n-1}) + h\sum_{i=2}^{s} b_i\big(f(x_n + c_ih, Y_i) - f(x_{n-1} + c_ih, Y_{-i})\big) \quad (6)$$

where,

$$Y_i = y_n + h\sum_{j=1}^{i-1} a_{i,j}f(x_n + c_jh, Y_j) \quad (7)$$

$$Y_{-i} = y_{n-1} + h\sum_{j=1}^{i-1} a_{i,j}f(x_{n-1} + c_jh, Y_{-j}) \quad (8)$$

where y_{n+1} and y_n are approximations to $y(x_{n+1})$ and $y(x_n)$ respectively.

Trigonometrically-fitted methods are generally derived to exactly approximate the solution of IVPs whose solutions are linear combinations of the functions $\{^je^{\alpha x}, {}^je^{-i\alpha x}\}$, where α can be complex or a real number (Jikantoro *et al* ., 2015). Other re searchers who have contributed to development of trigonometrically – fitted methods include, Gautschi (1961), Simos (2004), Anastassi and Simos (2008), Fang and Wu (2007), Ngwane and Jator (2014), Ndukum, *et al.* (2015).

Materials and Methods

Suppose each stage equation Y_i exactly integrate $y(x)$, i.e., $Y_i = y(x_n + c_ih)$, and the final step equation exactly integrates $y(x)$, i.e., $y_n = y(x_n)$, then the TFIRK scheme itself also integrates $y(x)$ exactly. If a function $y(x)$ is integrated exactly by TFIRK method for all problems whose solution is $y(x)$ then,

$$y_n = y(x_n) = e^{i\omega x_n} \quad (9)$$
$$y_{n+1} = y(x_{n+1}) = y(x_n + h) = e^{i\omega(x_n+h)} \quad (10)$$
$$y_{n-1} = y(x_{n-1}) = y(x_n - h) = e^{i\omega(x_n-h)} \quad (11)$$

Thus

$$y'_n = i\omega e^{i\omega x_n} = f(x_n, y_n) \quad (12)$$
$$y'_{n-1} = i\omega e^{i\omega(x_n-h)} = f(x_{n-1}, y_{n-1}) \quad (13)$$

Consequently,

$$Y_i = e^{i\omega(x_n+c_ih)} \quad (14)$$
$$Y_{-i} = e^{i\omega(x_{n-1}+c_ih)} \quad (15)$$
$$f(x_n + c_ih, Y_i) = y'(x_n + c_ih, Y_i) = i\omega e^{i\omega(x_n+c_ih)} \quad (16)$$
$$f(x_{n-1} + c_ih, Y_{-i}) = y'(x_{n-1} + c_ih, Y_{-i}) = i\omega e^{i\omega(x_{n-1}+c_ih)} \quad (17)$$
$$f(x_n + c_jh, Y_j) = y'(x_n + c_jh, Y_j) = i\omega e^{i\omega(x_n+c_jh)} \quad (18)$$
$$f(x_{n-1} + c_jh, Y_{-j}) = y'(x_{n-1} + c_jh, Y_{-j}) = i\omega e^{i\omega(x_{n-1}+c_jh)} \quad (19)$$

Thus, (14) becomes

$$e^{i\omega(c_ih)} = 1 + (i\omega)h\sum_{j=1}^{i-1} a_{ij}\, e^{i\omega(c_jh)} \quad (20)$$

By letting $\omega h = z$, where ω is the frequency, h the step size, we have,

$$cos(c_iz) + i\, sin(c_iz) = 1 + iz\sum_{j=1}^{i-1} a_{ij}[\cos(c_jz) + i\sin(c_jz)] \quad (21)$$

Equating the real and imaginary parts of Equation (14) we obtain the recursive relations

$$cos(c_iz) = 1 - z\sum_{j=1}^{i-1} a_{ij}\sin(c_jz), \qquad i = 2,3,\dots,s \quad (22)$$

and

$$sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\, cos(c_j z), i = 2,3,\dots,s \qquad (23)$$

And Equation (6) becomes

$$e^{i\omega(x_n+h)} = e^{i\omega(x_n)} + hb_1(i\omega)e^{i\omega(x_n)} - hb_{-1}(i\omega)e^{i\omega(x_n-h)} + h\sum_{i=2}^{s} b_i\,(i\omega)\left[e^{i\omega(x_n+c_ih)} - e^{i\omega(x_n-h+c_ih)}\right] \qquad (24)$$

$$e^{i\omega(h)} = 1 + hb_1(i\omega) - hb_{-1}(i\omega)e^{-i\omega(h)} + h(i\omega)\sum_{i=2}^{s} b_i\left[e^{i\omega(c_ih)}\left(1 - e^{-i\omega(h)}\right)\right] \qquad (25)$$

By letting $\boldsymbol{\omega h} = \boldsymbol{z}$ in (25),

$$e^{iz} = 1 + izb_1 - izb_{-1}e^{-iz} + iz\sum_{i=2}^{s} b_i\left[e^{ic_iz} - e^{iz(c_i-1)}\right] \qquad (26)$$

Thus,

$$cos(z) + i\,sin(z) = 1 + izb_1 - izb_{-1}[cos(z) - i\,sin(z)] + iz\sum_{i=2}^{s} b_i[\,cos(c_iz) + i\,sin(c_iz) - (\cos(z(c_i-1)) + i\sin(z(c_i-1)))] \qquad (27)$$

Equating the real and imaginary parts of Equation (27) we obtain the recursive relations

$$cos(z) = 1 - zb_{-1}\,sin(z) - z\sum_{i=2}^{s} b_i\,sin(c_iz) + z\sum_{i=2}^{s} b_i\,sin(z(c_i-1)) \qquad (28)$$

and

$$sin(z) = zb_1 - zb_{-1}\,cos(z) + z\sum_{i=2}^{s} b_i\,cos(c_iz) - z\sum_{i=2}^{s} b_i\,cos(z(c_i-1)) \qquad (29)$$

The relations (22), (23), (28) and (29) are the relations of order conditions of the trigonometrically -fitted method. These relations replace the equations of order conditions of two -step Imp roved Runge -Kutta (IRK) method, which can be solved to give the coefficients of a particular method based on existing coefficients.

In order to derive the third order two stage trigonometrically – fitted IRK (TFIRK) method, that is, a method with $s = 2, p = 3$, we consider the order conditions up to order three for IRK methods, according to Rabiei *et al.* (2013) is

$$\left.\begin{array}{ll} \text{First order:} & b_1 - b_{-1} = 1 \\ \text{Sercond order:} & b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} \\ \text{Third order 3 :} & \sum_{i=2}^{s} b_i\,c_i = \frac{5}{12} \\ & \sum_{i=2}^{s} b_i\,c_i^2 = \frac{1}{3} \end{array}\right\} \qquad (30)$$

The IRK32 method is thus described by the its Butcher tableau represented by Table 1 thus

TABLE 1 Butcher tableau of IRK32

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & \frac{1}{3} & \frac{2}{3} & \frac{5}{6} \end{array}$$

From the recursive relations (22), (23), (28) and (29) substitute $s = 2, c_1 = 0$ to obtain

$$\cos(c_2 z) - 1 = 0 \quad (31)$$

$$sin(c_2 z) - z\, a_{2,1} = 0 \quad (32)$$

$$cos(z) - 1 + zb_{-1}\, sin(z) + zb_2\, sin(c_2 z) - zb_2\, sin(z(c_2 - 1)) = 0 \quad (33)$$

$$sin(z) - zb_1 + zb_{-1}\, cos\, z - z\, b_2 cos(c_2 z) + zb_2\, cos(z(c_2 - 1)) = 0 \quad (34)$$

Equations (31) – (34) are now the equations of order conditions for third order two stage trigonometrically-fitted IRK method that replaces order conditions (30) of the original method. To obtain the coefficients of the new TFIRK32 method, the system of two equations (33)and (34) is solved together with an additional equation from the order conditions (30) namely,

$$b_1 - b_{-1} = 1 \quad (35)$$

These sum up to three equations with four unknown parameters $(b_{-1}, b_1, b_2 \text{and } c_2)$. The equations are solved in terms of one free parameter c_2 whose value $(c_2 = \frac{1}{2})$ is obtained from Table 1. Equation (35) is chosen to augment the updated (33)and(34) so that $b_{-1}, b_1 \text{and} b_2$ are not taken as free parameters. Solving equations (33), (34)and(35) the following expressions are obtained for $b_{-1}, b_1 \text{and} b_2$.

$$\left.\begin{aligned} b_{-1} &= -\frac{sin(z) - z}{z(cos(z) - 1)} \\ b_1 &= -\frac{sin(z) - z\,cos(z)}{z(cos(z) - 1)} \\ b_2 &= \frac{1}{2}\frac{sin(z)^2 - cos(z)^2 + 2\,cos(z) - 1 - z\,sin(z)}{z\,sin\left(\frac{z}{2}\right)(cos(z) - 1)} \end{aligned}\right\} \quad (36)$$

The Taylor series expansions of (36) are obtained thus

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{3} - \frac{1}{90}z^2 - \frac{19}{2520}z^4 - \frac{187}{75600}z^6 + o(z^7) \\ b_1 &= \frac{2}{3} - \frac{1}{90}z^2 - \frac{1}{2520}z^4 - \frac{1}{75600}z^6 + o(z^7) \\ b_2 &= \frac{5}{6} - \frac{37}{720}z^2 + \frac{11}{80640}z^4 + o(z^6) \end{aligned}\right\} \quad (37)$$

It is clear from (37) that as z approaches zero the original method, IRK32, is recovered.

In order to verify the order of the method to be three as claimed, the coefficients of the method are substituted into order conditions(30) and take the Taylor series expansions obtained thus

$$\left.\begin{aligned} &\text{First order 1:} && b_1 - b_{-1} = 1 + o(z^7) \\ &\text{Second order 2 :} && b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} - \frac{1}{16}z^2 - \frac{1}{3840}z^4 + o(z^6) \\ &\text{Third order :} && \sum_{i=2}^{s} b_i c_i = \frac{5}{12} - \frac{37}{1440}z^2 + \frac{11}{161280}z^4 + o(z^6) \end{aligned}\right\} \quad (38)$$

From Equation (38) it is evident that as z tends to zero the order conditions of the Improved Runge Kutta (IRK) method up to order three are recovered, which implies that the coefficients of the trigonometrically-fitted third order two stage method satisfy the IRK order three conditions.

Numerical Experiments

Problems 1 and 2 are sample initial value problems that are solved using the proposed TFIRK32 method. The solutions are obtained using Maple software package and the results are shown in Tables 2 – 5.

Problem 1:
$y'(x) = 10\cos(10x)$, $y(0) = 0$; Exact solution: $y(x) = \sin(10x)$, $\omega = 10$
Problem 2:
$y'(x) = -\pi\sin(\pi x)$, $y(0) = 0$; Exact solution: $y(x) = \cos(\pi x)$, $\omega = \pi$

TABLE 2: Results of Problem 1 on [0, 1], $h = 0.05$, $\omega = 10$

x	Exact	TFIRK32	Error	IRK32	Error
0.05	0.4794255380	0.4387912810	4.0634257000E-02	0.4387912809	4.0634257100E-02
0.10	0.8414709848	0.8419337803	4.6279548461E-04	0.8397765789	1.6944059222E-03
0.15	0.9974949866	0.9982430856	7.4809902152E-04	0.9926550635	4.8399231467E-03
0.20	0.9092974268	0.9100834852	7.8605835459E-04	0.9006310075	8.6664193530E-03
0.25	0.5984721441	0.5990395238	5.6737971520E-04	0.5862351093	1.2237034816E-02
0.30	0.1411200081	0.1412656113	1.4560326096E-04	0.1264424497	1.4677558341E-02
0.35	-0.3507832277	-0.3511592331	3.7600542217E-04	-0.3361736923	1.5390464653E-02
0.40	-0.7568024953	-0.7576722336	8.6973833681E-04	-0.7710037047	1.4201209423E-02
0.45	-0.9775301177	-0.9787448301	1.2147124459E-03	-0.9889310815	1.1400963808E-02
0.50	-0.9589242747	-0.9602507407	1.3264660562E-03	-0.9665996003	7.6753255971E-03
0.55	-0.7055403256	-0.7076179636	1.1776379863E-03	-0.7094767865	3.9364609597E-03
0.60	-0.2794154999	-0.2802201647	8.0466653834E-04	-0.2805152726	1.0997743571E-03
0.65	0.2151199881	0.2148211100	2.9886813055E-04	0.2152602025	1.4021439778E-04
0.70	0.6569865987	0.6572025189	2.1592014648E-04	0.6564665115	5.2008718855E-04
0.75	0.9379999768	0.9386136369	6.1366016866E-04	0.9350809625	2.9190142589E-03
0.80	0.9893582466	0.9901552179	7.9697130690E-04	0.9828890208	6.4692258009E-03
0.85	0.7984871126	0.7992080852	7.2097260135E-04	0.7881856064	1.0301506211E-02
0.90	0.4121184852	0.4125227564	4.0427118568E-04	0.3986409057	1.3477579590E-02
0.95	-0.0751511205	-0.0752267139	7.5593388200E-05	-0.0903709529	1.5219832404E-02
1.00	-0.5440211109	-0.5446222444	6.0113353670E-04	-0.5591228113	1.5101700402E-02

TABLE 3: Results of Problem 2 on [0, 1], $h = 0.05$, $\omega = \pi$

x	Exact	TFIRK32	Error	IRK32	Error
0.05	0.9876883400	-0.0245726683	1.0122610080+000	-0.0245726683	1.0122610080+000
0.10	0.9510565163	0.9510562574	2.5891479115E-07	0.9510189067	3.7609627904E-05
0.15	0.8910065242	0.8910057538	7.7043584939E-07	0.8909326158	7.3908421536E-05
0.20	0.8090169944	0.8090154724	1.5219678286E-06	0.8089089918	1.0800258413E-04
0.25	0.7071067812	0.7071042862	2.4950055174E-06	0.7069677286	1.3905260424E-04
0.30	0.5877852523	0.5877815867	3.6655894984E-06	0.5876189584	1.6629392734E-04
0.35	0.4539904997	0.4539854948	5.0048961090E-06	0.4538014440	1.8905578163E-04
0.40	0.3090169944	0.3090105144	6.4799471758E-06	0.3088102167	2.0677769473E-04
0.45	0.1564344650	0.1564264106	8.0544220459E-06	0.1562154417	2.1902329431E-04
0.50	0.0000000000	-0.0000096896	9.6895519226E-06	-0.0002254911	2.2549105300E-04
0.55	-0.1564344650	-0.1564458101	1.1345074482E-05	-0.1566604868	2.2602171334E-04
0.60	-0.3090169944	-0.3090299746	1.2980225264E-05	-0.3092375966	2.2060220848E-04
0.65	-0.4539904997	-0.4540050500	1.4554741429E-05	-0.4541998657	2.0936598470E-04
0.70	-0.5877852523	-0.5878012821	1.6029853165E-05	-0.5879778420	1.9258971511E-04
0.75	-0.7071067812	-0.7071241504	1.7369238324E-05	-0.7072774677	1.7068648716E-04
0.80	-0.8090169944	-0.8090355343	1.8539916799E-05	-0.8091611900	1.4419563100E-04
0.85	-0.8910065242	-0.8910260373	1.9513062601E-05	-0.8911202936	1.1376943944E-04
0.90	-0.9510565163	-0.9510767810	2.0264713649E-05	-0.9511366734	8.0157106282E-05
0.95	-0.9876883406	-0.9877091170	2.0776361802E-05	-0.9877325269	4.4186278729E-05
1.00	-1.0000000000	-0.0000210354	2.1035408583E-05	-1.0000067427	6.7426779328E-06

TABLE 4: Maximum errors for Problem 1 on [0, 100], $\omega = 10$

h	TFIRK32	IRK32	NFEs
$\frac{1}{20}$	1.3271676650E-03	1.5406317410E-02	4000
$\frac{1}{40}$	7.5907710628E-05	1.9462564478E-03	8000
$\frac{1}{80}$	4.4979855147E-06	2.4392570975E-04	16000
$\frac{1}{160}$	2.7310678531E-07	3.0510860031E-05	32000
$\frac{1}{320}$	1.6814312713E-08	3.8144873117E-06	64000
$\frac{1}{640}$	1.0428682507E-09	4.7683059731E-07	128000

TABLE 5: Maximum errors for Problem 2 on [0, 100], $\omega = \pi$

h	TFIRK3-2	IRK3-2	NFEs
$\frac{1}{20}$	2.1035475100E-05	2.5727342372E-04	4000
$\frac{1}{40}$	1.3195730949E-06	3.1238569154E-05	8000
$\frac{1}{80}$	8.2549351157E-08	3.8454714391E-06	16000
$\frac{1}{160}$	5.1605230828E-09	4.7691926388E-07	32000
$\frac{1}{320}$	3.2255126755E-10	5.9377908847E-08	64000
$\frac{1}{640}$	2.0159744464E-11	7.4073731306E-09	128000

Discussion of Results

Table 2 shows the results of solving Problem 1 using TFIRK32 and IRK32. A close look at the errors generated by the respective methods indicate that the TFIRK32 performs better that the existing non-fitted IRK32 with the same number of function evaluations. In Table 3, the two methods are applied to Problem 2 with the observation that the TFIRK32 started off on a better footing than the IRK32, however, subsequently, accuracy fluctuates between the two methods within the interval of definition of the problem. Furthermore, Table 4 exhibits the result of solving Problem 1 using the two methods on the interval [0, 100]. It was observed that TFIRK32 performs much better than the IRK32 as the step size grows smaller. And lastly, in Table 5, it is evident that TFIRK32 produces lesser errors than the non-fitted IRK32.

Conclusion

Conclusively, a third order two stage Improved Runge – Kutta method has been trigonometrically – fitted to solve initial value problems with oscillatory solutions. The TFIRK32 method requires only two function evaluations at each integration step and in general requires $(2\cdot(\frac{T}{h}))$ number of function evaluations (NFEs) on the entire interval of integration. Numerical examples established that the trigonometrically – fitted third order IRK method is more accurate than the non- fitted IRK method of the same order.

References

Fang, Y. and Wu, X. (2007). A trigonometrically explicit hybrid method for the numerical integration of orbital problems. *Applied Mathematics and Computation* 189(1): 178-185.

Gautschi, W. (1961). Numerical integration of ordinary differential equations based on trigonometric polynomials. *Numerische Mathematik,* 3: 381 - 397.

Ndukum, P. L., Biala, T. A., Jator, S. N. and Adeniyi, R. B. (2015). A fourth order trigonometri-cally fitted method with the block unification implementation approach for oscillatory initi-al value problems. *International Journal of Pure and Applied Mathematics,* 103(2): 201-213.

Ngwane, F. and Jator, S.N. (2014). Trigonometrically-fitted second derivative method for oscillatory problems. *SpringerPlus,* 3: 304.

Rabiei, F., Ismail, F. and Suleiman, M. (2013). Improved Runge-Kutta methods for solving ordinary differential equations. *Sains Malaysiana,* 42(11): 1679 - 1687.

Simos, T. E. (2004). A trigonometrically - fitted method for long-time integration of orbital problems. *Mathematical and Computer Modelling,* 40:1263-1272.

Jikantoro, Y. D., Ismail, F. and Senu, N. (2015). Zero-dissipative trigonometrically-fitted hybr-id method for numerical solution of oscillatory problems. *Sains Malaysiana,* 44(3): 473-482.