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LITERATIVE METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Iterative methods are just approximate methods, applied in the solution of partial differential equations (pdes) of elliptic, parabolic and hyperbolic types. In this paper, we analyze the basic theory, convergence, and other properties of iterative methods for elliptic pdes. We examine the three basic iterative methods, Jacobi, Gauss-Seidel and Successive Overrelaxation (SOR) methods, and perform numerical experiments with them, with a view to establishing the most efficient of the methods in terms of rate of convergence, simplicity, and ease of implementation on the computer. It was discovered that the SOR has the fastest convergence rate, followed by the Gauss-Seidel and then the Jacobi method. In terms of simplicity, however, the Jacobi method is far simpler than the more complicated Gauss-Seidel and SOR methods, in view of the fact that it involves lesser computational rigour than them.

Keywords: *Iterative matrix, Spectral radius, convergence rate, ordering*

INTRODUCTION

Equilibrium problems are problems of steady state in which the equilibrium configuration u in a domain D is to be determined by solving the differential equation

$$L[u] = f \qquad (1)$$

within D, subject to certain boundary conditions

$$B_1[u] = g_1 \qquad (2)$$

on the boundary of D, Ames (1977). The mathematical modeling of equilibrium problems usually results in elliptic partial differential equations. Typical of this type of equations is the Poisson equation

$$\Delta u = f \qquad (3)$$

where Δ is the laplace operator, and f and u are real or complex-valued functions on a manifold. For vanishing f, equation (3) becomes the Laplace equation

$$\Delta u = 0 \qquad (4)$$

Kopchenova and Maron (1981), stated the Dirichlet problem for Poisson equation (3) as the problem of finding the function $u = u(x, y)$ satisfying inside a certain domain D the equation (3), and at the boundary Γ, the condition

$$u|_{\Gamma} = g(x, y) \qquad (5)$$

where $g(x, y)$ is the given continuous function. Young (1950), asserts that finite difference methods afford a powerful tool for obtaining approximate numerical solutions for many differential equations whose analytic solutions are not known. The differential equation is replaced by a difference equation which must be satisfied by the values of the unknown function u at a finite set of points in the domain, D, of the independent variable. This set of points usually consists of the nodes, or net points, of a square network D_h contained in D. The mesh size is denoted by $h (> 0)$. The finite difference discretisation of elliptic partial differential equations normally results in a system of linear algebraic equations with respect to the values of the function $u(x, y)$ at points (x_i, y_j)

$$Au = b \qquad (6)$$

where $A \in R^{n \times n}$ $b \in R^n$ and u is the unknown column vector. Traditionally, exact solutions to the linear system (6) are obtained through the application of direct methods such as Gaussian elimination. However, direct methods have proved to be too laborious, especially when the dimension of the matrix is too large. Iteration entails repeating a process over and over until an approximation of the solution is reached. An iterative method for the solution of the linear system (6) is of the form

$$u^{(k)} = Gu^{(k-1)} + c \qquad (7)$$

where G is a matrix called the *iterative matrix.*

A minimum property for any acceptable iterative method is that the method converges. Convergence demands that the sequence of vectors $u^{(k)} \to A^{-1}b$ (the solution) as $k \to \infty$. Saad (2000), stated that the iteration (7) converges for any initial vector $u^{(0)}$ iff

$$\rho(G) < 1 \qquad (8)$$

where $\rho(G)$, referred to as the *spectral radius* of the iterative matrix G, is the maximum modulus of the eigenvalues of G, i.e.,

$$\rho(G) = \max_{\lambda \in \sigma(G)} |\lambda| \qquad (9)$$

and $\sigma(G)$, referred to as the *spectrum* of G, is the set of eigenvalues of G.

Besides convergence, an iterative method must be seen to converge fast enough for it to be of practical significance. The *Convergence rate* (r) of an iterative method is a measure of the speed at which it converges to the solution. It is defined as

$$r = -\log \rho(G) \qquad (10)$$

RESEARCH METHODOLOGY

All the iterative methods we would consider revolve around a decomposition of the matrix A of the linear system (6) into the form

$$A = L + D + U \qquad (11)$$

where L, D, and U are the strictly lower triangular, diagonal and strictly upper triangular matrices, respectively. Although many iterative methods are in existence, we present analyse the derivation of the three basic iterative methods, the Jacobi, Gauss-Seidel and Successive overrelaxation (SOR). The Jacobi method is derived by substituting the value of A in equation (11) into equation (6) thus:

$$(L + D + U)u = b$$
$$(L + U)u + Du = b$$
$$Du = -(L + U)u + b$$
$$u = -D^{-1}(L + U)u + D^{-1}b$$

If we iterate the above equation we have the scheme

$$u^{(k)} = -D^{-1}(L + U)u^{(k-1)} + D^{-1}b \qquad (12)$$

Equation (12) is the Jacobi iterative method. Comparing it with the general linear iterative scheme (7) we see that

$$G = -D^{-1}(L + U), \quad r = D^{-1}b.$$

The matrix $G = -D^{-1}(L + U)$ is called the Jacobi iterative matrix. We would denote it by G_J. The Jacobi iterative method (12) can be expressed in algebraic form as

$$u_i^{(k)} = \sum_{j=1}^{n} g_{ij} u_j^{(k-1)} + c_i \qquad (13)$$

where the u_i denote the elements of u and the c_i the elements of $c = D^{-1}b$.

We derive the Gauss-Seidel method in a similar way by employing the following steps

$$(L + D + U)u = b$$
$$(L + D)u + Uu = b$$
$$(L + D)u = -Uu + b$$
$$u = -(L + D)^{-1}Uu + (L + D)^{-1}b$$

Therefore, the Gauss-Seidel iteration formula would be,

$$u^{(k)} = -(L + D)^{-1}Uu^{(k-1)} + (L + D)^{-1}b \qquad (14)$$

where $G = -(L + D)^{-1}U$ and $r = (L + D)^{-1}b$. We denote the Gauss-Seidel iterative matrix $G = -(L + D)^{-1}U$ by G_{GS}.

Equation (14) has the following algebraic form

$$u_i^{(k)} = \sum_{j=1}^{i-1} g_{ij} u_j^{(k)} - \sum_{j=i-1}^{n} g_{ij} u_j^{(k-1)} + c_i \qquad (15)$$

The Gauss-Seidel iterative method is based upon immediate use of the improved values. To systematize such a computation the order in which one solves for the components of the kth approximation $u^{(k)}$ must be established beforehand. Such a sequential arrangement is called an *ordering* of the mesh points. The Successive Overrelaxation (SOR) method is a modified version of the Gauss-Seidel method. Axelsson (1994), stated the SOR theorem thus:

Theorem 1 *Assuming that*

(a) A *has property* (A^{τ})

(b) *The Jacobi iterative matrix* G_J *has only real eigenvalues*

Then the SOR method converges for any initial vector iff $\rho(G_J) < 1$ *and* $0 < \omega < 2$. *Furthermore,*

$$\omega_{OPT} = \frac{2}{1 - \sqrt{1 - [\rho(G_J)]^2}} \qquad (16)$$

for which the spectral radius is given by

$$\rho(G_{SOR}, \omega_{OPT}) = \omega_{OPT} - 1 \qquad (17)$$

We begin derivation of the SOR by substituting the value of A in equation (11) back into equation (6)

$$(L - D - U)u = b \qquad (18)$$

We multiply both sides of equation (18) by the *relaxation parameter*, ω

$$\omega(L - D - U)u = \omega b$$

$$Du - \omega(L - D - U)u = Du - \omega b$$

$$(D - \omega L)u = Du - \omega Du - \omega Uu - \omega b$$

$$(D - \omega L)u = [(1 - \omega)D - \omega U]u + \omega b$$

$$u = (D - \omega L)^{-1}[(1 - \omega)D - \omega U]u + (D - \omega L)^{-1}\omega b$$

i.e.,

$$u^{(k)} = (D - \omega L)^{-1}[(1 - \omega)D - \omega U]u^{(k-1)} - (D - \omega L)^{-1}\omega b \qquad (19)$$

Equation (19) is the SOR iterative method, where $G_{SOR} = (D - \omega L)^{-1}[(1 - \omega)D - \omega U]$ is the SOR iterative matrix and $r = (D - \omega L)^{-1}\omega b$. If $\omega = 1$ the method reduces to that of Gauss-Seidel. The SOR method (19) has the following algebraic form

$$u_i^{(k)} = (1 - \omega)u_i^{(k-1)} + \omega\left\{\sum_{j=1}^{i-1} g_{ij}u_j^{(k)} + \sum_{j=i+1}^{n} g_{ij}u_j^{(k-1)} + c_i\right\} \qquad (20)$$

Problem 1: We wish to determine the steady state heat distribution in a thin plate bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. This gives rise to the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (21)$$

with boundary conditions

$$\left.\begin{aligned} u(0, y) &= 0 \\ u(a, y) &= 0 \\ u(x, 0) &= 0 \\ u(a, b) &= 200 \end{aligned}\right\} \qquad (22)$$

To solve this equation, we construct a net, with 2 rows and 3 columns, over the plane region as follows.

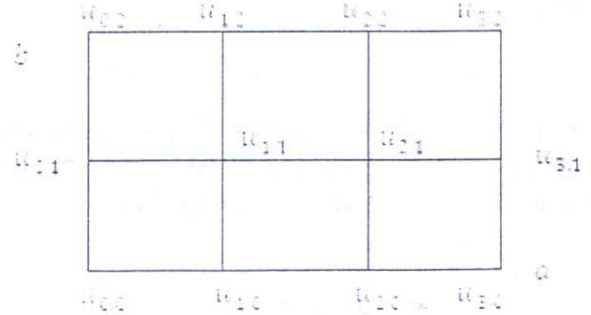

Figure 1 Discretization of plane region with 2 × 3 net density

From figure 1, we see that we have 2 internal points, u_{11} and u_{21}; and 10 boundary points, u_{00}, u_{10}, u_{20}, u_{30}, u_{01}, u_{31}, u_{02}, u_{12}, u_{22} and u_{32}.

We replace equation (21) by the following finite difference relations

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2} = 0 \qquad (23)$$

where h and k are equal to the horizontal and vertical distances between the nodes, respectively. If we let $h = k$ the above equation becomes

$$4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0 \qquad (24)$$

Using equation (24) above, we write the finite difference equations for the 2 interior points u_{11} and u_{21} to obtain the following linear system.

$$\left.\begin{aligned} 4u_{11} - u_{21} - u_{01} - u_{12} - u_{10} &= 0 \\ 4u_{21} - u_{31} - u_{11} - u_{22} - u_{20} &= 0 \end{aligned}\right\} \qquad (25)$$

From the boundary conditions (22), we have that $u_{01} = u_{10} = u_{31} = u_{20} = 0$ and $u_{12} = u_{22} = 200$. Hence, equations (25) become

$$\left.\begin{aligned} 4u_{11} - u_{21} &= 200 \\ 4u_{21} - u_{11} &= 200 \end{aligned}\right\} \qquad (26)$$

and in matrix from we have

$$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = \begin{pmatrix} 200 \\ 200 \end{pmatrix} \qquad (27)$$

which is the form of equation (6), $Au = b$.

where, $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$, $u = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}$ and $b = \begin{pmatrix} 200 \\ 200 \end{pmatrix}$.

And, from the matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$, we have

$$L = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad (28)$$

We apply the three basic iterative methods to solve system (27) thus:

JACOBI METHOD

Substituting the values of equations (28) in the Jacobi iterative scheme (12) we have

$$\begin{pmatrix} u_{11}^{(k)} \\ u_{21}^{(k)} \end{pmatrix} = -\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}^{-1}\left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} u_{11}^{(k-1)} \\ u_{21}^{(k-1)} \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 200 \\ 200 \end{pmatrix}$$

$$\begin{pmatrix} u_{11}^{(k)} \\ u_{21}^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}\begin{pmatrix} u_{11}^{(k-1)} \\ u_{21}^{(k-1)} \end{pmatrix} - \begin{pmatrix} 50 \\ 50 \end{pmatrix} \qquad (29)$$

where $G_J = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$ is the Jacobi iterative matrix, and $r = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$

From equation (29) we obtain the following relations to be solved iteratively.

$$\left.\begin{aligned} u_{11}^{(k)} &= \frac{1}{4}u_{21}^{(k-1)} - 50 \\ u_{21}^{(k)} &= \frac{1}{4}u_{11}^{(k-1)} - 50 \end{aligned}\right\} \qquad (30)$$

We fix the initial approximations to be $u_{11}^{(0)} = u_{21}^{(0)} = 0$. Ten iterations of the Jacobi method are performed for problem 1, the results of which are presented in Table 1, for comparison with results of other methods. The spectral radius of the Jacobi iterative matrix is also calculated as,

$$\rho(G_J) = \rho\begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} = \frac{1}{4} \qquad (31)$$

The convergence rate of the Jacobi method if therefore,

$$r = -\log \rho(G_J) = -\log\left(\frac{1}{4}\right) = 0.6020599914 \qquad (32)$$

GAUSS-SEIDEL METHOD

Substituting the values of equations (28) in the Gauss-Seidel method (14) gives

$$\begin{pmatrix} u_{11}^{(k)} \\ u_{21}^{(k)} \end{pmatrix} = -\left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right]^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{11}^{(k-1)} \\ u_{21}^{(k-1)} \end{pmatrix} - \left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right]^{-1} \begin{pmatrix} 200 \\ 200 \end{pmatrix}$$

The above equation simplifies to

$$\begin{pmatrix} u_{11}^{(k)} \\ u_{21}^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{16} \end{pmatrix} \begin{pmatrix} u_{11}^{(k-1)} \\ u_{21}^{(k-1)} \end{pmatrix} - \begin{pmatrix} 50 \\ \frac{125}{2} \end{pmatrix} \qquad (33)$$

where $G_{GS} = \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{16} \end{pmatrix}$ is the Gauss-Seidel iterative matrix, and $r = \begin{pmatrix} 50 \\ \frac{125}{2} \end{pmatrix}$

It implies

$$\left.\begin{aligned} u_{11}^{(k)} &= \frac{1}{4} u_{21}^{(k-1)} - 50 \\ u_{21}^{(k)} &= \frac{1}{4} u_{11}^{(k)} - 50 \end{aligned}\right\} \qquad (34)$$

We let $u_{11}^{(0)} = u_{21}^{(0)} = 0$. Results of the first ten iterations of the Gauss-Seidel method are presented in Table 1 alongside other methods'.

$$\rho(G_{GS}) = \rho\begin{pmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{16} \end{pmatrix} = \frac{1}{16} \qquad (35)$$

$$r = -\log \rho(G_{GS}) = -\log\left(\frac{1}{16}\right) = 1.204119983 \qquad (36)$$

SUCCESSIVE OVERRELAXATION (SOR) METHOD

The matrix form of the SOR method is obtained from equation (19) thus:

$$\begin{pmatrix} u_{11}^{(k)} \\ u_{21}^{(k)} \end{pmatrix} = \left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - w\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right)^{-1} \left[(1-w)\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - w\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} u_{11}^{(k-1)} \\ u_{21}^{(k-1)} \end{pmatrix} - \left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - w\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right)^{-1} w\begin{pmatrix} 200 \\ 200 \end{pmatrix}$$

It implies

$$\begin{pmatrix} u_{11}^{(k)} \\ u_{21}^{(k)} \end{pmatrix} = \begin{pmatrix} 1-w & \frac{1}{4}w \\ \frac{1}{4}w(1-w) & \frac{1}{16}w^2 - 1 - w \end{pmatrix} \begin{pmatrix} u_{11}^{(k-1)} \\ u_{21}^{(k-1)} \end{pmatrix} - \begin{pmatrix} 50w \\ \frac{25}{2}w^2 - 50w \end{pmatrix} \qquad (37)$$

where $G_{SOR} = \begin{pmatrix} 1-w & \frac{1}{4}w \\ \frac{1}{4}w(1-w) & \frac{1}{16}w^2 - 1 - w \end{pmatrix}$ is the SOR iterative matrix,

and $\omega = \left(\frac{25}{2}\omega^2 - 50\omega\right)$

From equation (16) the value of ω is calculated as $\omega = 1.016133231$. We equally obtain the spectral radius ρ_{SOR} from equation (17) thus:

$\rho(G_{SOR}) = \omega_{OPT} - 1 = 0.016133231$ (38)

$r = -\log_{10}\rho(G_{SOR}) = -\log(0.016133231) = 1.792278643$ (39)

FINDINGS

Table 1 illustrates results of the 1st ten iterations of the Three methods applied to solve problem 1.

Table 1 Ten iterations of the three basic iterative methods

Iteration	$u_{2,1}$ Exact solution: $\frac{3333333333}{50000000}$			$u_{2,1}$ Exact solution: $\frac{3333333333}{50000000}$		
	Jacobi	Gauss-seidel	SOR	Jacobi	Gauss-Seidel	SOR
1st	50	50	50.80666155	50	62.5	63.71324584
2nd	62.5	65.62500000	66.17227253	62.5	66.40625000	66.58872231
3rd	65.625	66.60156250	66.65484238	65.625	66.65039062	66.66492040
4th	66.40625	66.66259766	66.66641382	66.40625	66.66564942	66.66663061
5th	66.601562	66.66641236	66.66666158	66.601562	66.66660309	66.66666596
6th	66.65039	66.66665077	66.66666657	66.65039	66.66666269	66.66666665
7th	66.662598	66.66666567	66.66666666	66.662598	66.66666642	66.66666666
8th	66.66565	66.66666660	66.66666666	66.66565	66.66666665	66.66666666
9th	66.666412	66.66666666	66.66666666	66.666412	66.66666666	66.66666666
10th	66.666603	66.66666666	66.66666666	66.666603	66.66666666	66.66666666

DISCUSSION OF RESULTS

The spectral radii of all the three iterative methods satisfy the necessary condition for convergence, i.e., $\rho(G) < 1$ (equations (31), (35) and (38)). This is further corroborated by the results of Table 1, i.e., the results converge to the exact solutions. From the results of equations (32), (36) and (39), it is observed that the Gauss-Seidel method has a convergence rate approximately twice as fast as the Jacobi method, while convergence rate of the SOR method is approximately one and a half times faster than the Gauss-Seidel method.

CONCLUSION

The SOR method converges faster than the Jacobi and Gauss-Seidel methods, even though the Gauss-Seidel method exhibits a faster convergence rate than the Jacobi method. In terms of simplicity, however, we say that the Jacobi method is far simpler than the more complicated Gauss-Seidel and SOR methods, in the sense that it involves lesser computational rigour than them. Although, we may sacrifice this computational rigour by virtue of the fact that almost all the methods need to be implemented on the computer, especially when one has to deal with very large linear systems, say for example, a 500×500 matrix. In such situation, manual computation is almost an impossible task. Generally, we conclude that the SOR is the most efficient of the three basic iterative methods.

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