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An Orthogonal-based Self-starting Numerical Integrator for Second Order Initial and Boundary Value Problems ODEs

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Abstract. The direct integration of second order initial and boundary value problems is considered in this paper. We employ a new class of orthogonal polynomials constructed as basis function to develop a two-step hybrid block method (2SHBM) adopting collocation technique. The recursive formula of the class of polynomials have been constructed, and then we give analysis of the basic properties of 2SHBM as findings show that the method is accurate and convergent. The boundary locus of the proposed 2SHBM shows that the new scheme is A-stable.

1. Introduction

Second order differential equation of the form

$$y'' = f(x, y, y'), \quad x \in [a, b] \tag{1}$$

with initial conditions

$$y'(a) = \alpha, \quad y'(b) = \beta \quad \text{or} \tag{2a}$$

$$y(a) = \alpha, \quad y(b) = \beta \tag{2b}$$

arise frequently in areas of science, engineering and technology [1]. Some of these equations (1)-(2) have no analytical solution, thereby numerical schemes were and are being developed to solve these problems. Earlier methods involves reducing the second order differential equation to a system of first order before solving them with existing method [2–5] were able to solve problems of this type by reducing the problem of second order to first order, however the process is time consuming and rigours to implement. But direct method which are self-starting and take less computation time are developed in terms of linear multi-step methods (LMMs) [6] which are called block method. In the paper [7] authors used the self-starting scheme to derive a class of one-step hybrid methods for the numerical solution of second order differential equation with power series. In the study [8] a family of second derivative block methods for stiff initial value problems (IVPs) for ordinary differential equations (ODEs) is proposed.

In this work, we develop a two-step hybrid block method (2SHBM) with orthogonal polynomials as a basis function using collocation technique.

Our derived scheme yield very good results compared to the existing methods in the literature [9–14] and is also able to solve IVPs and BVPs.

2. Development of the Method

2.1. Construction of Orthogonal Polynomial Basis Functions

According to [15] two functions are said to be orthogonal to one another if their inner product is zero, hence a family of functions forms an orthogonal system on an interval (a, b) with a weight function $w(x)$ if for any two distinct members of the family

$$\langle \varphi_1, \varphi_2 \rangle = \int_a^b \varphi_1(x)\varphi_2(x)w(x)dx = 0. \tag{3}$$

An orthogonal system can be written as a sequence of functions $\{\varphi_n\}_{n=0}^{\infty}$ and the corresponding orthogonal property can be expressed as $\langle \varphi_i, \varphi_j \rangle = 0$ for $i \neq j$.

We defined the orthogonal polynomials $\varphi_r(x)$ over the interval $(-1, 1)$ with respect to the weight function $w(x) = x^2$ as

$$\varphi_r(x) = \sum_{r=0}^{n} C_r^n x^r. \tag{4}$$

In order to calculate the real coefficients C_r^n we use the additional property

$$\varphi_i(1) = 1, \quad i = 0, 1, \ldots, n. \tag{5}$$

Denote by A_n the set of indexes for two distinct members of the orthogonal system:

$$A_n = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i + 1 \leq j \leq n\}. \tag{6}$$

The set A_n is the finite set and depends on the value of n. Fixing $n = 5$ in Equation (4) the following orthogonal system is obtained:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x, \quad \varphi_2(x) = \frac{1}{2}\left(5x^2 - 3\right), \quad \varphi_3(x) = \frac{1}{2}\left(7x^3 - 5x\right), \tag{7}$$

$$\varphi_4(x) = \frac{1}{8}\left(63x^4 - 70x^2 + 15\right), \qquad \varphi_5(x) = \frac{1}{8}\left(99x^5 - 126x^3 + 35x\right).$$

These polynomials are employed as the basis functions for the derived scheme. In the same vein, the orthogonal polynomials $\varphi_r(x)$ for $n > 5$ can be obtained.

2.2. Numerical Scheme

We seek to derive numerical scheme for the numerical solution of the problem (1)-(2) with using the LMMs form [6]:

$$\sum_{i=k-2}^{k} \alpha_i y_{n+1} = h^2 \sum_{i=0}^{k} \beta_i f_{n+1} + h^2 \beta_v f_{k-\frac{1}{2}} \tag{8}$$

where k is the number of blocks, h is step of the method, α_i, β_v are the real unknown parameters to be determined, $i = 0, 1, \ldots, r + s - 1$, r is the number of collocation points, s is the number of interpolation points, and $v = \left\{0, 1, \frac{3}{2}, 2\right\}$.

We express the approximation of the analytical solution of the problem (1)-(2) with the the derived orthogonal polynomials (7) of the form

$$y_n(x) = \sum_{i=0}^{n} \alpha_i \varphi_i(x) \tag{9}$$

where $n = r + s - 1$. We interpolate at the interval $[0, 1]$ and collocate at points $v = \left\{0, 1, \frac{3}{2}, 2\right\}$. From the $s = 2$ interpolation and $r = 4$ collocation points we obtained a system of six equations each of order $n = r + s - 1 = 5$. From equations (8) and (9) we obtain the continuous scheme

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 f_n + \alpha_3 f_{n+1} + \alpha_4 f_{n+\frac{3}{2}} + \alpha_5 f_{n+2}. \tag{10}$$

From the scheme (10), taking $t = x - x_n$ the value of the α_i, $i = 0, 1, \ldots, n$ were obtained using the matrix inversion algorithm.

$$\alpha_0 = 1 - \frac{t}{h}, \tag{11}$$

$$\alpha_1 = \frac{t}{h},$$

$$\begin{aligned}\alpha_2 = {} & \frac{1}{2520}\frac{756h^3 + 135h}{h^3} - \frac{1}{7560}\frac{(1869h^4 + 1950h^2 + 70)t}{h^3} + \frac{1}{270}\frac{\left(54h^3 + 15h\right)\left(\frac{5}{2}t^2 - \frac{3}{2}\right)}{h^3} \\ & - \frac{1}{6930}\frac{\left(715h^2 + 42\right)\left(\frac{7}{2}t^3 - \frac{5}{2}x + \frac{5}{2}x_n\right)}{h^3} - \frac{1}{63}\frac{\frac{63}{8}t^4 - \frac{35}{4}t^2 + \frac{15}{8}}{h^2} \\ & - \frac{2}{185}\frac{\frac{99}{8}t^5 - \frac{63}{4}t^3 + \frac{35}{8}x - \frac{35}{8}x_n}{h^3},\end{aligned}$$

$$\begin{aligned}\alpha_3 = {} & -\frac{1}{4h} - \frac{1}{1260}\frac{\left(651h^4 - 900h^2 - 70\right)t}{h^3} - \frac{7}{27}\frac{\frac{5}{2}t^2 - \frac{3}{2}}{h^2} + \frac{2}{1155}\frac{\left(165h^2 + 21\right)\left(\frac{7}{2}t^3 - \frac{5}{2}t\right)}{h^3} \\ & - \frac{2}{27}\frac{\frac{63}{8}t^4 - \frac{35}{4}t^2 + \frac{15}{8}}{h^2} + \frac{4}{495}\frac{\frac{99}{8}t^5 - \frac{63}{4}t^3 + \frac{35}{8}t}{h^3},\end{aligned}$$

$$\begin{aligned}\alpha_4 = {} & \frac{2}{7h^2} + \frac{2}{945}\frac{\left(168h^4 - 300h^2 - 35\right)}{h^3}t + \frac{8}{27}\frac{\frac{5}{2}t^2 - \frac{3}{2}}{h^2} - \frac{8}{34665}\frac{\left(110h^2 + 21\right)\left(\frac{7}{2}t^3 - \frac{5}{2}t\right)}{h^3} \\ & + \frac{16}{189}\frac{\frac{63}{8}t^4 - \frac{35}{4}t^2 + \frac{15}{8}}{h^2} - \frac{16}{1485}\frac{\frac{99}{8}t^5 - \frac{63}{4}t^3 + \frac{35}{8}t}{h^3},\end{aligned}$$

$$\begin{aligned}\alpha_5 = {} & -\frac{5}{56h^2} - \frac{1}{2520}\frac{\left(231h^4 - 450h^2 - 70\right)t}{h^3} - \frac{5}{54}\frac{\frac{5}{2}t^2 - \frac{3}{2}}{h^2} + \frac{1}{2310}\frac{\left(165h^2 + 42\right)\left(\frac{7}{22}t^3 - \frac{5}{2}t\right)}{h^3} \\ & - \frac{5}{189}\frac{\frac{63}{8}t^4 - \frac{35}{4}t^2 + \frac{15}{8}}{h^3} + \frac{2}{495}\frac{\frac{99}{8}t^5 - \frac{63}{4}t^2 + \frac{35}{8}t}{h^3}.\end{aligned}$$

Substituting the value of α_i, $i = 0, 1, \ldots, n$ into the scheme (10) and evaluating at $x = \frac{3}{2}$ and $x = 2$ yields the following implicit scheme:

$$y_{n+\frac{3}{2}} = -\frac{1}{2}y_n + \frac{3}{2}y_{n+1} + \frac{1}{24}h^2 f_n + \frac{13}{32}h^2 f_{n+1} - \frac{5}{48}h^2 f_{n+\frac{3}{2}} + \frac{1}{32}h^2 f_{n+2}, \tag{12a}$$

$$y_{n+2} = -y_n + 2y_{n+1} + \frac{1}{12}h^2 f_n + \frac{5}{6}h^2 f_{n+1} + \frac{1}{12}h^2 f_{n+2}. \tag{12b}$$

Differentiating the continuous scheme (12) with respect to x and evaluating at x_n, x_{n+1}, $x_{n+\frac{3}{2}}$, and x_{n+2} yields the following discrete scheme:

$$z_n = -\frac{y_n}{h} + \frac{y_{n+1}}{h} - \frac{89}{360}hf_n - \frac{189}{360}hf_{n+1} + \frac{128}{360}hf_{n+\frac{3}{2}} - \frac{33}{360}hf_{n+2}, \tag{13a}$$

$$z_{n+1} = -\frac{y_n}{h} + \frac{y_{n+1}}{h} + \frac{31}{360}hf_n + \frac{234}{360}hf_{n+1} - \frac{112}{360}hf_{n+\frac{3}{2}} + \frac{27}{360}hf_{n+2}, \tag{13b}$$

$$z_{n+\frac{3}{2}} = -\frac{y_n}{h} + \frac{y_{n+1}}{h} + \frac{233}{2880}hf_n + \frac{2562}{2880}hf_{n+1} - \frac{56}{2880}hf_{n+\frac{3}{2}} + \frac{141}{2880}hf_{n+2}, \tag{13c}$$

$$z_{n+2} = -\frac{y_n}{h} + \frac{y_{n+1}}{h} + \frac{31}{360}hf_n + \frac{294}{360}hf_{n+1} + \frac{128}{360}hf_{n+\frac{3}{2}} + \frac{87}{360}hf_{n+2}. \tag{13d}$$

Equations (12), (13) yield our desired block method which is self-starting method. Now we present equations (12), (13a) in matrix notation form:

$$A^{(0)}Y_m = hBF(Y_m) + A^{(1)}Y_{n-1} + hDF(Y_{n-1}), \tag{14}$$

where h is affixed mesh size within a block, vectors $Y_m = (y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2})^\top$, $F(Y_m) = (f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2})^\top$, $Y_{n-1} = (y_{n-2}, y_{n-1}, y_n)^\top$, $F(Y_{n-1}) = (f_{n-2}, f_{n-1}, f_n)^\top$, and matrices

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} \frac{31}{60} & -\frac{16}{45} & \frac{11}{120} \\ \frac{189}{160} & -\frac{51}{80} & \frac{27}{160} \\ \frac{28}{15} & -\frac{32}{45} & \frac{4}{15} \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & \frac{89}{360} \\ 0 & 0 & \frac{193}{480} \\ 0 & 0 & \frac{26}{45} \end{pmatrix}.$$

2.3. Order and Error Constant of Proposed Method

We define local truncation error associated with a second order differential equation (1) by the difference operator

$$L[y(x); h] = \sum_{i=0}^{k} [\alpha_i y(x_n + ih) - h^2 \beta_i f(x_n + ih)] \tag{15}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$. Expanding the expression (15) in the Taylors series about the point x, we obtain:

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+2} h^{p+2} y^{(p+2)}(x) \tag{16}$$

where vectors

$$C_0 = \sum_{i=0}^{k} \alpha_i, \quad C_1 = \sum_{i=0}^{k} i\alpha_i, \quad C_2 = \frac{1}{2!} \sum_{i=0}^{k} i\alpha_i - \beta_i, \ldots,$$

$$C_q = \frac{1}{q!} \sum_{i=0}^{k} i^q \alpha_i - q(q-1)(q-2) i^{q-2} \beta_i, \quad \text{where} \quad q = 3, 4, 5 \ldots.$$

According to Lambert [3] the method's order is p if

$$C_0 = C_1 = C_2 = \ldots = C_p = C_{p+1} = 0 \quad \text{and} \quad C_{p+2} \neq 0.$$

Therefore, C_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ is the principal local truncation error at the point x_n. The equations (12a) and (12b) are of order $p = 4$ with the error constant $C_{p+2} = C_6 = \left[-\frac{1}{240}, -\frac{21}{10240}\right]^\top$ and $C_{p+2} = C_6 = \left[-\frac{9}{4}, -\frac{61}{8}, -\frac{171}{16}, -\frac{7}{4}\right]^\top$ respectively.

2.4. Zero Stability of the Method

According to Lambert [3], a linear multi-step method is said to be zero-stable if no root of its characteristic polynomial $\rho(R)$ has no modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the methodwe use the matrix notation (14) of the proposed block method and the characteristic polynomial

$$\rho(R) = det(RA^{(0)} - A^{(1)}). \tag{17}$$

Substituting $A^{(0)}$ and $A^{(1)}$ in equation (17), we obtain the characteristic polynomial

$$\rho(R) = R(R-1)^2$$

which implies that its roots are $R_1 = 0$, $R_2 = R_3 = 1$. According to Fatunla [2] the proposed two-step block hybrid method is zero-stable since from $\rho(R) = 0$ satisfies $|R_j| \leq 1$, $j = 1, 2, 3$ and the multiplicity of roots does not exceed two.

2.5. Region of Absolute Stability of the Method
Stability regions are a standard tool in the analysis of numerical formulas for ODE problems. To evaluate and plot the region of absolute stability of 2SHBM, the methods were reformulated as general linear method [4] expressed as:

$$\begin{bmatrix} Y \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hF(Y) \\ y_{i+1} \end{bmatrix}, \tag{18}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s1} & b_{s2} & \cdots & b_{ss} \end{bmatrix},$$

$$Y = \begin{bmatrix} y \\ y_{n+1} \\ y_{n+2} \end{bmatrix}, \quad y_{i+1} = \begin{bmatrix} y_{n+k} \\ y_{n+k-1} \end{bmatrix}, \quad y_{i-1} = \begin{bmatrix} y_{n+k-1} \\ y_{n+k-2} \end{bmatrix}$$

Also the elements of the matrices A, B, U and V were obtained from interpolation and collocation points and then substituted into the stability matrix as

$$M(z) = V + zB(I - zA)^{-1}U, \quad z \in \mathbb{C}. \tag{19}$$

The stability matrix $M(z)$ (19) was substituted into the stability function

$$\rho(\eta, z) = \det(\eta I - M(z)), \tag{20}$$

where I is identity matrix, and then computed with Maple software to yield the stability polynomial. The coefficients of the block method in (A, B, U, V) formulation is shown below:

$$\begin{bmatrix} A & U \\ B & V \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{89}{360} & \frac{31}{60} & -\frac{16}{45} & \frac{11}{120} & 0 & 1 \\ \frac{1}{24} & \frac{13}{32} & -\frac{5}{48} & \frac{1}{32} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{12} & \frac{5}{6} & 0 & \frac{1}{12} & 2 & -1 \\ \frac{1}{12} & \frac{5}{6} & 0 & \frac{1}{12} & 2 & -1 \\ \frac{89}{360} & \frac{31}{60} & -\frac{16}{45} & \frac{11}{120} & 0 & 1 \end{bmatrix} \tag{21}$$

By substituting the entries of the matrix (21) into Equations (19) and (20), the stability polynomial of the block method is

$$f(z) = \frac{1}{2} \frac{1}{9z^3 - 70z^2 + 714z - 1440} \left(\begin{array}{c} 18\eta^2 z^3 - 140\eta^2 z^2 - 3\eta z^3 + 1428\eta^2 z - 827\eta z^2 \\ -2880\eta^2 - 1356\eta z - 191z^2 + 8640\eta - 1512z - 5760 \end{array} \right).$$

The region of absolute stability for the block method are shown in Figure 1. In this case the stability region is the exteriors of the curve drawn and the proposed method is *A*-stable.
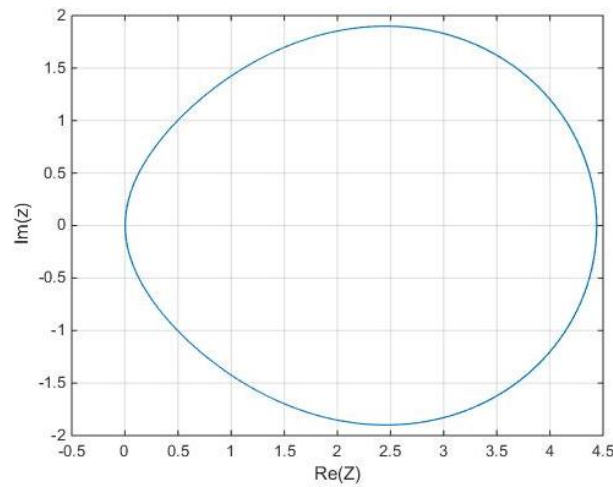
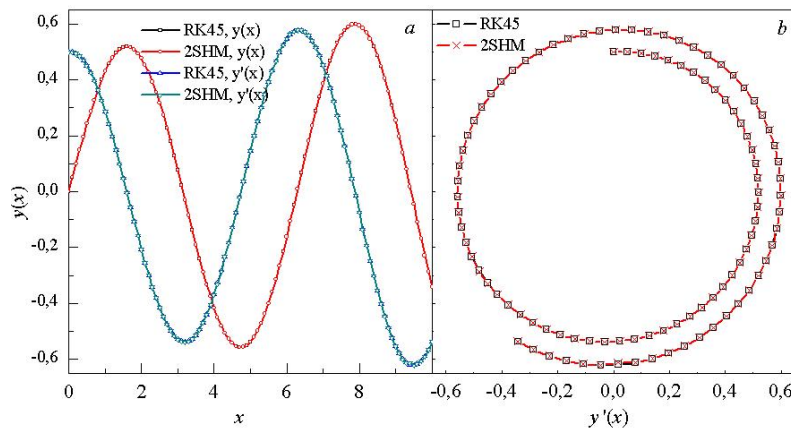**Figure 1.** Stability region of the proposed 2SHBM is the exteriors of the blue curve

Figure 2. Comparison numerical solutions of proposed 2SHBM with RK45 method for the Problem 1 [9]: a) relaxation oscillation, b) phase portrait.

3. Numerical Examples

We consider four numerical examples: the Van Der Pol Oscillator Problem [9], the IVP of Bratu-type [10], the Troesch's Problem [11–13] and the nonlinear system of BVP [14] to test the efficiency of the derived orthogonal-based two-step hybrid block method.

Problem 1. Van Der Pol oscillator [9]

$$y'' - 2\xi \left(1 - y^2\right) y' + y = 0, \quad y(0) = 0, \quad y'(0) = 0.5, \quad x \in [0, 10], \quad \xi = 0.025.$$

Problem 2. Consider the second order initial value problem of Bratu type [10]

$$y'' - 2\exp(y) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad 0 \le x \le 1.$$

The exact solution is $y(x) = -2\ln(\cos x)$. The comparison absolute error of proposed 2SHBM with the method [18] is given in Fig. 3.

Problem 3. Consider the Troesch's Problem [13]

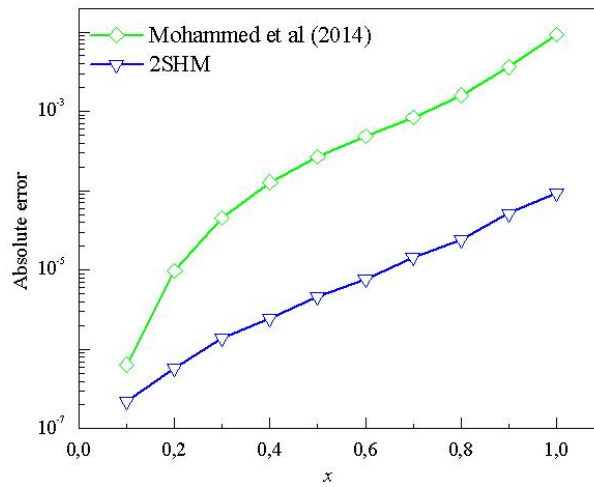$$y'' = n\sinh(ny), \quad y(0) = 0, \quad y(1) = 1, \quad 0 \le x \le 1.$$

Figure 3. Comparison absolute error of proposed 2SHBM with the method [18] for Problem 2 [10].
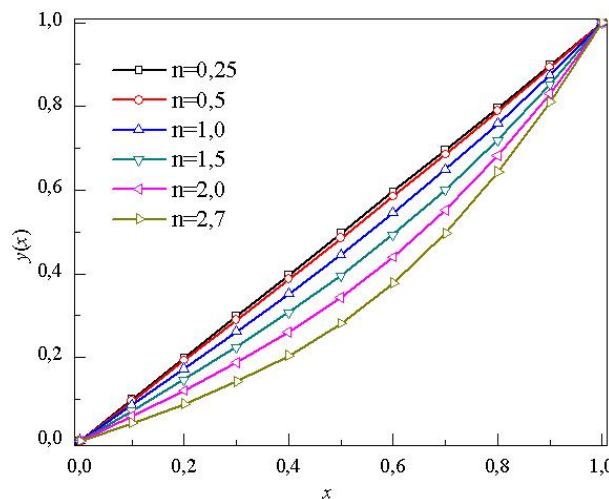

Figure 4. Numerical solution of Problem 3 [13] for different values of n.

Fig. 4 shown the comparison between approximate solutions obtained with the proposed method, using the Troesch's parameter $n = \{0.25, 0.5, 1, 1, 5, 2, 2.7\}$.

Problem 4. Nonlinear system of boundary value problem [14]. The equations governing the free convective boundary-layer flow above a heated impermeable horizontal surface are

$$f'' + mh + \left(\frac{m-2}{3}\right)\eta h' = 0,$$

$$h'' + \left(\frac{m+1}{3}\right)fh' - mf'h = 0,$$

with mixed boundary conditions:

$$f(0) = 0, \quad f' \to 0 \quad as \quad \eta \to \infty,$$
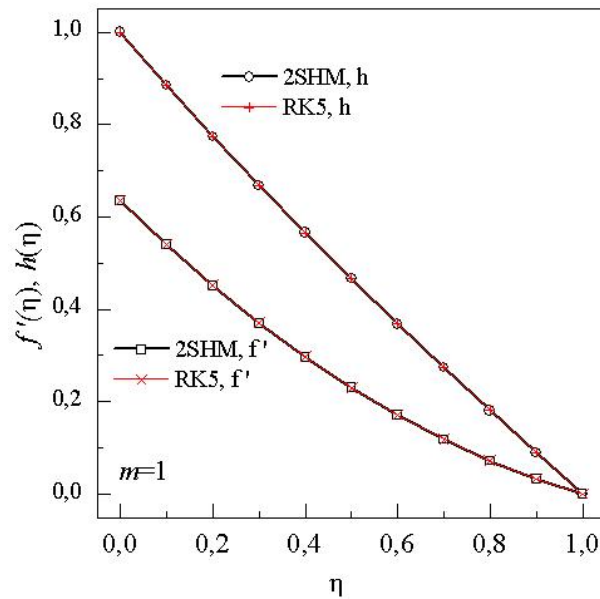$$h(0) = 1, \quad h \to 0 \quad as \quad \eta \to \infty.$$

Figure 5. The comparison of numerical solutions of Problem 4 [14], $m = 1$.

The comparison of 2SHBM and Runge-Kutta method of the value h, f' against η for values of $m = 1$ is presented in Fig. 5.

Conclusion
In this work, we obtained an approximate solution for different second order initial and boundary value problems: the Van der Pol Oscillator problem, the Bratu's type problem, the Troeschs problem, and nonlinear system of BVP. Besides, we presented a comparison between the exact solution, the proposed solution, and other approximations reported in the literature.

Acknowledgements

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