Jour. of Mathematical Sciences
Vol.23, No.1 (2011) 59-66

A TWO POINT BLOCK HYBRID METHOD FOR SOLVING STIFF INITIAL VALUE PROBLEMS

Muhammad. R

Department of Mathematics/Statistics
Federal University of Technology
Minna, Nigeria
(e-mail: raihanfut@yahoo.com)

ABSTRACT

In this paper, self starting hybrid block method of order $(3,3,3)^T$ is proposed for the solution of stiff initial value problem of the form $y' = f(x, y)$. The continuous formation of the integrator enables us to differentiate and evaluate at grid and off grid points. The schemes compared favourably with exact results and results obtained from Okunuga (2008).

Introduction

We consider the Initial Value Problem (IVP)

$$y' = f(x, y)\ y(x) = y_0 \qquad (1)$$

Lambert (1973, 1991) adopted the continous finite difference (C FD) approximation by the idea of interpolation and collocation. Later Lie and Norsett (1989), Onumanyi (1994, 1999) referred to it as Multistep Collocation (MC).

Conventional Multistep Method including hybrid ones have been made continous through the idea of Multistep Collocation (MC). The continuous form has more ability to solve the ODE'S than the discrete ones. The initial-value problem (IVP) is solved, for instance, without looking for any other methods to start the integration process. Moreover, the desired the whole integration can be achieved with one matrix difference equation over sub-intervals which do not overlap (Jennings (1987), p. 59 and Lambert (1991), p. 53). The continous multistep method produces piece-wise polynomial solutions over k-step $[X_n, X_{n+k}]$ for the first order system/scalar Ordinary Differential Equations. Of note is that the continous scheme is not to be directly use as the numerical integrator but the resulting discrete multistep schemes derived from it which will be self starting and can be applied for solution of both initial value problem and boundary value problem.

In this paper, we attempt to reformulate the hybrid two- step Backward Differentiation Formula (BDF) for efficient and accurate implementation on initial value problem (IVP).

The Multistep Collocation (MC)

Lambert (1973, 1991) adopted the continous finite difference (CFD) approximation method by the idea of interpolation and collocation. Later, lie and norsett (1989), Onumanyi (1994, 1999) referred to it as Multistep Collocation (MC). The method is presented below

$$\underline{a} = (a_0, a_1, \ldots, a_{(t+m-1)})^T,\ \varphi(x) = ((\varphi_0(x), \varphi_1(x), \ldots, \varphi_{(t+m-1)})^T \quad (2)$$

where a_r, $r = 0, \ldots, t + m - 1$ are undetermined constants, $\varphi_r(x)$ are specified basis functions, T denotes transpose of, t denotes the number of interpolation points and m denotes the number of distinct collocation points. We consider a continuous approximation (interpolant) $Y(x)$ to $y(x)$ in the form

$$Y(x) = \sum_{r=0}^{t+m-1} a_r \varphi_r(x) = \underline{a}^T \varphi(x) \quad (3)$$

which is valid in the sub-intervals $X_n \le X \le X_{n+k}$, where $n = 0, k \ldots\ldots N - k$. The quantities $x_0 = a$, $x_N = b$, k, m, n, t and $\varphi_r(x)$, $r = 0, 1, \ldots, t + m - 1$ are specified values. The constant co-efficient a_r of (3) can be determined using the conditions

$$Y(x_{n+j}) = Y_{n+j}, \qquad j = 0, 1, \ldots, t - 1 \quad (4)$$

$$Y'(\bar{x}_j) = f_{n+j} \qquad j = 0, 1, \ldots, m - 1 \quad (5)$$

Where

$$f_{n+j} = f(X_{n+j}, Y_{n+j}) \quad (6)$$

The distinct collocation points $X_0 \ldots\ldots X_{m-1}$, can be chosen freely from the set $[X_n, X_{n+k}]$ Equations (4), (5) and (6) are denoted by a single set of algebraic equations of the form

$$Da = F \quad (7)$$

$$\underline{F} = (Y_n, Y_{n+1}, \ldots, Y_{n+t}, f_n, f_{n+1}, f_{n+m-1})^T \quad (8)$$

$$\underline{a} = D^{-1}\underline{F} \quad (9)$$

where D □ is the non-singular matrix of dimension $(t + m)$ below

$$D\square = \begin{pmatrix} \varphi_0(x_n) & \cdots & \varphi_{t+m-1}(x_n) \\ \vdots & \vdots & \vdots \\ \varphi_0(x_{n+t-1}) & \cdots & \varphi_{t+m-1}(x_{n+t-1}) \\ \varphi_0'(\bar{x}_0) & \cdots & \varphi_{t+m-1}'(\bar{x}_0) \\ \vdots & \vdots & \vdots \\ \varphi_0'(\bar{x}_{m-1}) & \cdots & \varphi_{t+m-1}'(\bar{x}_{m-1}) \end{pmatrix} \quad (10)$$

By substituting (9) into (3), we obtain the MC formula

$$Y(x) = F^T C^T \varphi(x),\ X_n \le X_{n+k}\ n = 0, k, \ldots, N - k \quad (11)$$

where

$$C\square \equiv D\square^{-1} = (c_{ij}),\ i, j = 1, \ldots, t + m - 1$$

$$C\square = \begin{pmatrix} c_{11} & \cdots & c_{1t} & c_{1,t+1} & \cdots & c_{1,t+m} \\ c_{21} & \cdots & c_{2t} & c_{2,t+1} & \cdots & c_{2,t+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{t+m1} & \cdots & c_{t+m,t} & c_{t+m,t+1} & \cdots & c_{t+m,t+m} \end{pmatrix}$$

with the numerical elements denoted by c_{ij}, $i, j = 1, \ldots, k + m$. By expanding $C\square^{T}\varphi(x)$ in (11) yields the following

$$y(x) = (F)^{T}\begin{pmatrix} \sum_{r=0}^{t+m-1} C_{r+1,1}\varphi_r(x) \\ \vdots \\ \sum_{r=0}^{t+m-1} C_{r+1,k+m}\varphi_r(x) \end{pmatrix} \tag{12}$$

$$y(x) = \sum_{j=0}^{t+m-1}\left(\sum_{r=0}^{t+m-1} C_{r+1,j+1}\varphi_r(x)\right) + \sum_{j=0}^{m-1} h\left(\sum_{r=0}^{k+m-1}\frac{c_{r+1,j+1}}{h}\varphi_r(x)\right) f_{n+j} \tag{13}$$

$$y(x) = \sum_{j=0}^{t-1}\alpha_j(x)y_{n+j} + h\sum_{j=0}^{m-1}\beta_j(x)f_{n+j} \tag{14}$$

where we construct $\alpha_j(x)$ and $\beta_j(x)$ explicitly by

$$\alpha_j(x) = \sum_{r=0}^{t+m-1} C_{r+1,j+1}\varphi_r(x) \qquad j = 0, 1, \ldots, t-1 \tag{15}$$

$$\beta_j(x) = \sum_{r=0}^{k+m-1}\left(\frac{c_{r+1,j+1}}{h}\varphi_r(x)\right). \qquad j = 0, 1, \ldots, m-1 \tag{16}$$

a_r can be determined as follows:

$$y(x) = \left\{\sum_{r=0}^{t-1}\alpha_{j,r+1}y_{n+j} + h\sum_{j=0}^{m-1}\beta_{j,r+1}f_{n+j}\right\}\varphi_r(x) \tag{17}$$

Derivation Of The Two- Point Block Hybrid Backward Differentiation Formulae (BHBDF)

The general form of the method upon addition of one off grid point is expressed as:

$$\overline{y}(x) = \alpha_1(x)y_n + \alpha_2(x)y_{n+1} + \alpha_3 y_{n+\frac{1}{2}} + h\beta_0(x)f_{n+2} \tag{18}$$

The matrix D of the proposed method is expressed as:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 \\ 1 & x_n + \frac{1}{2}h & (x_n + \frac{1}{2}h)^2 & (x_n + \frac{1}{2}h)^3 \\ 0 & 1 & 2x_n + 4h & 3(x_n + 2h)^2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+1/2} \\ f_{n+2} \end{bmatrix} \tag{19}$$

i.e. $Da = F$

The matrix D in equation (19) which when solved by matrix inversion technique or Gaussian Elimination method will yield the continuous coefficients substituted in (18) to obtain continous form of the two step block hybrid BDF with one off step interpolation point.

$$\bar{y}(x)\left\{\left(\frac{1}{13}\frac{13h^3+44x_nh^2+41x_n^2h+10x_n^3}{h^3}-\frac{2}{13}\frac{22h^2+41x_nh+15x_n^2}{h^3}x+\frac{1}{13}\frac{41h+30x_n}{h^3}+\frac{-10}{13h^3}x^3\right)y_n\right.$$

$$+\left(\frac{1}{13}\frac{x_n\left(20^2+47x_nh+14x_n^2\right)}{h^3}-\frac{2}{13}\frac{10h^2+47x_nh+21x_n^2}{h^3}x+\frac{1}{13}\frac{47h+42x_n}{h^3}x^2-\frac{14}{13h^3}x^3\right)y_{n+1}$$

$$+\left(\frac{1}{3}\frac{x_n(2x_n+h)}{h}+\frac{8}{13}\frac{9x_n^2+22x_nh+8h^2}{h^3}x+\frac{-8}{13}\frac{9x_n+11h}{h^3}x^2+\frac{24}{13h^3}x^3\right)y_{n+\frac{1}{2}} \tag{20}$$

$$\left.+\left(\frac{-1}{13}\frac{x_n\left(2x_n^2+3x_nh+h^2\right)}{h^2}+\frac{1}{13}\frac{6x_n^2+6x_nh+h^2}{h^2}x-\frac{3}{13}\frac{2x_n+h}{h^2}+\frac{2}{13h^2}x^3\right)f_{n+2}\right\}$$

Evaluating (20) at points at $x = x_{n+2}$ and its derivative at $x = x_{n+1}$, $x = x_{n+1/2}$ yields the following three discrete hybrid schemes which are used as a block integrator:

$$\frac{-36}{13}y_{n+1}+y_{n+2}+\frac{32}{13}y_{n+\frac{1}{2}}=\frac{9}{13}y_n+\frac{6}{13}hf_{n+2}$$

$$\frac{33}{12}y_{n+1}+y_{n+\frac{1}{2}}=\frac{21}{12}y_n+\frac{26}{12}hf_{n+\frac{1}{2}}+\frac{1}{12}hf_{n+2} \tag{21}$$

$$y_{n+1}-\frac{40}{32}y_{n+\frac{1}{2}}=-\frac{8}{32}y_n+\frac{13}{32}hf_{n+1}-\frac{1}{32}hf_{n+2}$$

Equation (21) constitute the members of a zero-stable block integrators of order $(3,3,3)^T$ with $C_4=\left[-\frac{3}{52},\frac{17}{832},-\frac{19}{624}\right]^T$ as the error constants respectively. To start the integration process with $n = 0$, we use (21) and this produces $y_1, y_{1/2}$, and y_2 simultaneously without the need of any starting method (predictor).

Stability Analysis

Following Fatunla (1992; 1994),that defined the block method to be zero-stable provided the roots $R_{ij} = 1(1)k$ of the first characteristic polynomial $\rho(R)$ specified as

$$\rho(R)=\det\left|\sum_{i=0}^{k}A^{(i)}R^{k-i}\right|=0 \tag{22}$$

satisfies $|Rj| \leq 1$,the multiplicity must not exceed 2.

The block methods proposed in equations (21) for $k = 2$ are put in the matrix equation form and for easy analysis the result was normalized to obtain

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

And

$$B^0 = \begin{bmatrix} \frac{8}{9} & \frac{-11}{24} & \frac{5}{72} \\ \frac{10}{9} & \frac{-1}{6} & \frac{1}{18} \\ \frac{-8}{9} & \frac{2}{3} & \frac{52}{117} \end{bmatrix}, \qquad B^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first characteristic polynomial of the block method is given by

$\rho(R) = \det(RA^0 - A^1)$ .Substituting the A^0 and A^1 into the function above gives

$$= \det\left[R\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right] = \det\left[\begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \det\begin{bmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R-1 \end{bmatrix} = R(R(R-1)) - 0 = 0$$

$$\Rightarrow R_1 = R_2 = 0 \text{ or } R_3 = 1$$

From the definition (22) the hybrid method is zero stable and consistent since the order of the method $p = 3 > 1$. And by Henrici (1962); the hybrid method is convergent.

Numerical Example

To illustrate the performance of our proposed methods we will compare their performance with exact results and results obtained from Okunuga et al(2008). Consider the initial value problem

$$y' = \lambda(y - x) + 1, \; y(0) = 1$$

The problem is stiff in nature for negative λ values and it has exact solution $y(x) = e^{\lambda X} + x$.

This problem is considered for $\lambda = -5$, and $\lambda = -20$ with steplength $h = 0.01$. The problem is solved using the Block Hybrid Backward Differentiation Formula (BHBDF) for $k = 2$.

Table 1 : Proposed (BHBDF) For $k = 2$, $\lambda = -5$

N	X	Exact Value	Approximate Value	Error
0	0.00	1.000000000	1.0000000000	0
1	0.01	0.961229424	0.96122958	1.56E – 07
2	0.02	0.924837418	0.924837418	3.07961E – 07
3	0.03	0.890707976	0.890708411	4.34357E – 07
4	0.04	0.858730753	0.858731311	5.5792E – 07
5	0.05	0.828800783	0.828801442	6.58926E – 07
6	0.06	0.80081822	0.800818979	7.58317E – 07
7	0.07	0.774688089	0.774688926	8.3628E – 07
8	0.08	0.750320046	0.75032096	9.14697E – 07
9	.0.09	0.727628151	0.727629121	9.69377E – 07
10	0.1	0.706530659	0.706531694	1.034286E – 06

Table 2 : Proposed (BHBDF) For $k = 2$, $\lambda = -20$

N	X	Exact Value	Approximate Value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.828730753	0.828759936	2.918292E – 05
2	0.02	0.690320046	0690376258	5.6212612E – 05
3	0.03	0.578811636	0.578877222	6.558681E – 05
4	0.04	0.489328964	0.489404328	7.536411E – 05
5	0.05	0.417879441	0.417954259	7.4818082E – 05
6	0.06	0.361194211	0.361269992	7.518096E – 05
7	0.07	0.316596963	0.316667799	7.083649E – 05
8	0.08	0.281896518	0.281964012	6.7494E – 05
9	0.09	0.255298888	0.255469469	1.70581E – 04
10	0.1	0.235335283	0.235701539	3.66256E – 04

Table 3 : Comparison With Okunuga(2008) With $\lambda = -5$

X	Exact Value	BHBDF2	BBDF1
0.02	0.924837418	0.924837418	0.92481216
0.03	0.890707976	0.890708411	0.89065136
0.04	0.858730753	0.858731311	0.85864299
0.05	0.828800783	0.828801442	0.82868407
0.06	0.80081822	0.800818979	0.80067525
0.07	0.774688089	0.774688926	0.77452159
0.08	0,750320046	0.75032096	0.75013262
0.09	0.727628151	0.727629121	0.72742222
0.1	0.706530659	0.706531694	0.70630847

Table 4 : Error In Comparison With Okunuga(2008) With $\lambda = -5$

Error BHBDF2	Error BBDFI
3.07961E – 07	2.53E – 05
4.34357E – 07	5.66E – 05
5.5792E – 07	8.78E – 05
6.58926E – 07	1,17E – 04
7.58317E–07	1.43E–04
8.3628E – 07	1.66E– 04
9.14697E – 07	1.87E – 04
9.69377E – 07	2.06E – 04
1.034286E – 06	2.22E – 04

BBDF1: CF: Okunuga (2008)

$$y_{n+1} = y_{n-1} + 2hf_n$$

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}hf_{n+1}$$

Okunuga (2008) proposed 1-block 1-point method. while the present method is a 1-block 2-point method that produces simultaneously y_1, $y_{1\ 2}$ and y_2.

Conclusion

This paper demonstrated a successful application of the proposed method to solve stiff initial value problem of the form $y' = f(x, y)$. The proposed block method is self-starting and eliminates the overlap of solution model. All the discrete schemes of the method were derived from single continuous formula and its derivatives making use of both grid and off-grid points in the formulation. In addition, the schemes compared favourably with theoretical solution and results are more accurate and converge faster than okunuga (2008).

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