A MATHEMATICAL MODEL OF MEASLES DISEASE DYNAMICS

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Abstract

In this paper a Mathematical model was proposed for measles disease dynamics. The model is a system of first order ordinary differential equations with three compartments: Susceptible S(t); Infected I(t) and Recovered R(t). The equilibrium state for both Disease Free and Endemic equilibrium are obtained. Conditions for stability of the Disease Free and Endemic equilibrium are obtained from characteristics equation and Bellman and Cooke theorem respectively. The hypothetical values were used to analyze the Endemic Equilibrium and the result was presented in tabular form. The results from the Disease Free and Endemic Equilibrium state showed that once the epidemic breaks out, the population cannot sustain it.

Introduction

Measles, also known as rubeola or morbilli, is an infection of the respiratory system caused by a virus, specifically a paramyxovirus of the genus Morbillivirus. Morbilliviruses, like other paramyxoviruses, are enveloped, single-stranded, negative-sense RNA viruses. Humans are the natural hosts of the virus; no animal reservoirs are known to exist. This highly contagious virus is spread by coughing and sneezing via close personal contact or direct contact with secretions. The outbreak and spread of disease have been closely investigated for many years. The ability to make predictions about diseases could enable scientists to evaluate inoculation or isolation plans and may have a significant effect on the mortality rate of a particular epidemic. The modeling of infectious diseases is a tool which has been used to study the mechanisms by which diseases spread, to predict the future course of an outbreak and to evaluate strategies for the control an epidemic (Daley & Gani, 2005). In 1927, W. O. Kermack and A. G. McKendrick created a model in which they considered a fixed population with only three compartments, susceptible: S(t), infected, I(t), and recovered, R(t). The compartments used for this model consist of three classes: S(t) is used to represent the number of individuals not yet infected with the disease at time t, or those susceptible to the disease; I(t) denotes the number of individuals who have been infected with the disease and are capable of spreading the disease to those in the susceptible category; R(t) is the compartment used for those individuals who have been infected and then recovered from the disease. Those in this category are not able to be infected again or to transmit the infection to others. As implied by the variable function of *t*, the model is dynamic in that the numbers in each compartment may fluctuate over time. The importance of this dynamic aspect is most obvious in an endemic disease with a short infectious period, such as measles. Such diseases tend to occur in cycles of outbreaks due to the variation in number of susceptibles (S(*t*)) over time. During an epidemic, the numbers of susceptible individual falls rapidly as more of them are infected and thus enter the infectious and recovered compartments. The disease cannot break out again until the number of susceptible has built back up as a result of babies being born into the susceptible compartment. Each member of the population typically progresses from susceptible to infectious to recover. In this paper the birth rate and death rate are consider differently.

Model Equations

The model equations are given as follows:

$$\frac{dS}{dt} = \beta - \alpha SI - \mu S \qquad (1.1)$$

$$\frac{dI}{dt} = \alpha SI - (\gamma + \delta + \mu)I \qquad (1.2)$$

$$\frac{dR}{dt} = \gamma I - \mu R \qquad (1.3)$$

The parameters are defined as follows:
β= Birth rate
α= contact rate
μ = Natural death rate
S = Susceptible
γ = Recovery rate
I = Infected
δ = Death rate due to disease
R = Removed with immunity/ Recovery

Equilibrium State of the Model

At equilibrium $\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$

Let $S = x, I = y$ and $R = z$

$$\beta - \alpha xy - \mu x = 0 \qquad (2.1)$$

$$\alpha xy - (\gamma + \delta + \mu)y = 0 \qquad (2.2)$$

$$\gamma y - \mu z = 0 \qquad (2.3)$$

From (2.3)

$$y = \frac{\mu z}{\gamma} \qquad (2.4)$$

From (2.2)

$$[\alpha x - (\gamma + \delta + \mu)]y = 0 \qquad (2.5)$$

Either $y = 0$ or $\alpha x - (\gamma + \delta + \mu) = 0$

But $y \neq 0$

$\therefore \alpha x - (\gamma + \delta + \mu) = 0$

$$x = \frac{\gamma + \delta + \mu}{\alpha} \qquad (2.6)$$

Substituting (2.4) and (2.6) into (2.1) we obtained,

$$\beta - \alpha \left[\frac{\gamma + \delta + \mu}{\alpha}\right]\left[\frac{\mu z}{\gamma}\right] - \mu \left[\frac{\gamma + \delta + \mu}{\alpha}\right] = 0$$

$$\beta - \mu z \left[\frac{\gamma + \delta + \mu}{\gamma}\right] - \mu \left[\frac{\gamma + \delta + \mu}{\alpha}\right] = 0$$

$$\mu z \left[\frac{\gamma + \delta + \mu}{\gamma}\right] = \beta - \mu \left[\frac{\gamma + \delta + \mu}{\alpha}\right]$$

$$\alpha \mu z (\gamma + \delta + \mu) = \alpha \beta \gamma - \mu \gamma (\gamma + \delta + \mu)$$

$$z = \frac{\alpha \beta \gamma - \mu \gamma (\gamma + \delta + \mu)}{\alpha \mu (\gamma + \delta + \mu)} \qquad (2.7)$$

Substituting (2.7) into (2.4)

$$y = \frac{\mu}{\gamma}\left[\frac{\alpha \beta \gamma - \mu \gamma (\gamma + \delta + \mu)}{\alpha \mu (\gamma + \delta + \mu)}\right]$$

$$y = \frac{\mu \gamma}{\gamma \mu}\left[\frac{\alpha \beta - \mu (\gamma + \delta + \mu)}{\alpha (\gamma + \delta + \mu)}\right]$$

$$y = \frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)} \qquad (2.8)$$

$$x = \frac{\gamma + \delta + \mu}{\alpha}, \quad y = \frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)} \text{ and } z = \frac{\alpha\beta\gamma - \mu\gamma(\gamma + \delta + \mu)}{\alpha\mu(\gamma + \delta + \mu)} \qquad (2.9)$$

The Disease Free Equilibrium (DFE)

The equilibrium state in the absence of infection is known as Disease Free Equilibrium or zero equilibrium and is such that, y = 0,

Hence we substitute y = 0 into equations (2.1), (2.2) and (2.3) we obtain

$\beta = \mu x$

$$x = \frac{\beta}{\mu} \qquad (2.10)$$

y = 0 and z = 0

Therefore the Disease Free equilibrium is:

$$(x, y, z) = \left(\frac{\beta}{\mu}, 0, 0\right) \qquad (2.11)$$

The Endemic Equilibrium (EE) State

The equilibrium state with the presence of infection (i. e. y ≠ 0) is known as endemic equilibrium or non- zero equilibrium.

Therefore, equation (2.9) gives the endemic equilibrium state. That is,

$$(x, y, z) = \left(\frac{\gamma + \delta + \mu}{\alpha}, \frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)}, \frac{\alpha\beta\gamma - \mu\gamma(\gamma + \delta + \mu)}{\alpha\mu(\gamma + \delta + \mu)}\right) \qquad (2.12)$$

Stability of the Equilibrium State

Stability Analysis of the Disease Free Equilibrium (DFE)

$\beta - \alpha xy - \mu x = 0$

$\alpha xy - (\gamma + \delta + \mu)y = 0$

$\gamma y - \mu z = 0$

The Jacobian determinant of this system of equations is given by:

$$J = \begin{bmatrix} -(\alpha y + \mu) & \alpha x & 0 \\ \alpha y & \alpha x - (\gamma + \delta + \mu) & 0 \\ 0 & \gamma & -\mu \end{bmatrix}$$

The characteristic equation

$$\det|J - \lambda I| = \det\begin{bmatrix} -(\alpha y + \mu + \lambda) & \alpha x & 0 \\ \alpha y & \alpha x - (\gamma + \delta + \mu) - \lambda & 0 \\ 0 & \gamma & -(\mu + \lambda) \end{bmatrix} = 0$$

$$-(\alpha y + \mu + \lambda)[-[\alpha x - (\gamma + \delta + \mu) - \lambda](\mu + \lambda)] - \alpha x[-\alpha y(\mu + \lambda)] = 0$$

$$(\alpha y + \mu + \lambda)[[\alpha x - (\gamma + \delta + \mu) - \lambda](\mu + \lambda)] + \alpha^2 xy(\mu + \lambda) = 0 \qquad (3.1)$$

But recall from equation (2.11) the DFE is given as:

$$(x, y, z) = \left(\frac{\beta}{\mu}, 0, 0\right)$$

Then,

$$(\mu+\lambda)^2\left[\alpha\frac{\beta}{\mu}-(\gamma+\delta+\mu)-\lambda\right]=0 \quad (3.2)$$

Either $(\mu+\lambda)^2=0$ or $\alpha\frac{\beta}{\mu}-(\gamma+\delta+\mu)-\lambda=0$

Therefore, $\lambda_1=-\mu,\ \lambda_2=-\mu,$ and $\lambda_3=\alpha\frac{\beta}{\mu}-(\gamma+\delta+\mu)$ (3.3)

From (3.3)

$\lambda_1<0$ and $\lambda_2<0$

$\lambda_3<0$ if $\alpha\frac{\beta}{\mu}<(\gamma+\delta+\mu)$

$\lambda_3>0$ if $\alpha\frac{\beta}{\mu}>(\gamma+\delta+\mu)$

Hence, the DFE is stable if $\alpha\frac{\beta}{\mu}<(\gamma+\delta+\mu)$ and unstable if $\alpha\frac{\beta}{\mu}>(\gamma+\delta+\mu)$

Stability Analysis of the Endemic Equilibrium (EE)

At non- zero equilibrium we have

$$(x,y,z)=\left(\frac{\gamma+\delta+\mu}{\alpha},\frac{\alpha\beta-\mu(\gamma+\delta+\mu)}{\alpha(\gamma+\delta+\mu)},\frac{\alpha\beta\gamma-\mu\gamma(\gamma+\delta+\mu)}{\alpha\mu(\gamma+\delta+\mu)}\right)$$

Expanding (3.1) we have

$$(\alpha y+\mu+\lambda)[[\alpha x-(\gamma+\delta+\mu)-\lambda](\mu+\lambda)]+\alpha^2xy(\mu+\lambda)=0$$

$$\left(\alpha^2\mu xy+\alpha\mu^2x+\alpha\mu x\lambda+\alpha^2xy\lambda+\alpha\mu x\lambda+\alpha x\lambda^2\right)-\alpha\gamma\mu y-\alpha\delta\mu y-\alpha\mu^2y-\mu^2\gamma-\mu^3$$
$$-2\mu\gamma\lambda-2\mu\delta\lambda-2\mu^2\lambda-\alpha\gamma y\lambda-\alpha\delta y\lambda-\alpha\mu y\lambda-\gamma\lambda^2-\delta\lambda^2-\mu\lambda^2-\alpha\mu y\lambda-\mu^2\lambda-\mu\lambda^2$$
$$-\alpha y\lambda^2-\mu\lambda^2-\lambda^3+\alpha^2\mu xy+\alpha^2xy\lambda=0 \quad (3.4)$$

Collect the like terms of λ

$$-\lambda^3+[\alpha(x-y)-(\gamma+\delta+3\mu)]\lambda^2+\left[\alpha\mu(2x-y)+2\alpha^2xy-(\gamma+\delta+\mu)(\alpha y+2\mu)-\mu^2\right]\lambda+2\alpha^2\mu xy$$
$$-(\gamma+\delta+\mu)\left(\alpha\mu y+\mu^2\right)+\alpha\mu^2x=0 \quad (3.5)$$

We apply Bellman and Cooke theorem of stability.

Let (3.5) take the form:

$$H(\lambda)=-\lambda^3+[\alpha(x-y)-(\gamma+\delta+3\mu)]\lambda^2+\left[\alpha\mu(2x-y)+2\alpha^2xy-(\gamma+\delta+\mu)(\alpha y+2\mu)-\mu^2\right]\lambda+2\alpha^2\mu xy$$
$$-(\gamma+\delta+\mu)\left(\alpha\mu y+\mu^2\right)+\alpha\mu^2x \quad (3.6)$$

Setting λ = iw we have

$$H(iw)=F(w)+iG(w) \quad (3.7)$$

Substituting λ = iw into (3.6) we have

$$H(iw)=(-iw)^3+[\alpha(x-y)-(\gamma+\delta+3\mu)](iw)^2+\left[\alpha\mu(2x-y)+2\alpha^2xy-(\gamma+\delta+\mu)(\alpha y+2\mu)-\mu^2\right]iw+2\alpha^2\mu xy$$
$$-(\gamma+\delta+\mu)\left(\alpha\mu y+\mu^2\right)+\alpha\mu^2x$$

$$H(iw)=iw^3-w^2[\alpha(x-y)-(\gamma+\delta+3\mu)]+\left[\alpha\mu(2x-y)+2\alpha^2xy-(\gamma+\delta+\mu)(\alpha y+2\mu)-\mu^2\right]iw+2\alpha^2\mu xy$$
$$-(\gamma+\delta+\mu)\left(\alpha\mu y+\mu^2\right)+\alpha\mu^2x \quad (3.8)$$

Separating the real and imaginary parts of (3.8) we have

$$F(w)=2\alpha^2\mu xy-(\gamma+\delta+\mu)\left(\alpha\mu y+\mu^2\right)+\alpha\mu^2x-w^2[\alpha(x-y)-(\gamma+\delta+3\mu)] \quad (3.9)$$

$$G(w)=w^3+\left[\alpha\mu(2x-y)+2\alpha^2xy-(\gamma+\delta+\mu)(\alpha y+2\mu)-\mu^2\right]w \quad (3.10)$$

Differentiate (3.9) and (3.10) with respect to *w* we have

$$F'(w)=-2w[\alpha(x-y)-(\gamma+\delta+3\mu)] \quad (3.11)$$

$$G'(w) = 3w^2 + \left[\alpha\mu(2x - y) + 2\alpha^2 xy - (\gamma + \delta + \mu)(\alpha y + 2\mu) - \mu^2\right] \quad (3.12)$$

Setting $w = 0$

$$F'(0) = 0 \quad (3.13)$$

$$G'(0) = \alpha\mu(2x - y) + 2\alpha^2 xy - (\gamma + \delta + \mu)(\alpha y + 2\mu) - \mu^2 \quad (3.14)$$

$$F(0) = 2\alpha^2 \mu xy - (\gamma + \delta + \mu)\left(\alpha\mu y + \mu^2\right) + \alpha\mu^2 x \quad (3.15)$$

$$G(0) = 0 \quad (3.16)$$

Recall

$$(x, y, z) = \left(\frac{\gamma + \delta + \mu}{\alpha}, \frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)}, \frac{\alpha\beta\gamma - \mu\gamma(\gamma + \delta + \mu)}{\alpha\mu(\gamma + \delta + \mu)}\right) \quad (3.17)$$

Substituting *x* and *y* into (3.15)

$$F(0) = 2\alpha^2\mu\left(\frac{\gamma+\delta+\mu}{\alpha}\right)\left[\frac{\alpha\beta-\mu(\gamma+\delta+\mu)}{\alpha(\gamma+\delta+\mu)}\right] + \alpha\mu^2\frac{(\gamma+\delta+\mu)}{\alpha} - (\gamma+\delta+\mu)\left\{\alpha\mu\left[\frac{\alpha\beta-\mu(\gamma+\delta+\mu)}{\alpha(\gamma+\delta+\mu)}\right] + \mu^2\right\}$$

$$F(0) = \mu\left[\alpha\beta - \mu(\gamma + \delta + \mu)\right] \quad (3.18)$$

Substituting *x* and *y* into (3.14)

$$G'(0) = \alpha\mu\left\{\frac{2(\gamma + \delta + \mu)}{\alpha} - \left[\frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)}\right]\right\} + 2\alpha^2\left(\frac{\gamma + \delta + \mu}{\alpha}\right)\left[\frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)}\right]$$
$$- (\gamma + \delta + \mu)\left\{\alpha\left[\frac{\alpha\beta - \mu(\gamma + \delta + \mu)}{\alpha(\gamma + \delta + \mu)}\right] + 2\mu\right\} - \mu^2$$

$$G'(0) = \frac{3\mu(\gamma + \delta + \mu)^2 - \alpha\beta(\gamma + \delta)}{(\gamma + \delta + \mu)} \quad (3.19)$$

Since, $F(0)G'(0) - F'(0)G(0) > 0$

We multiply (3.18) by (3.19) we obtain

$$\mu\left[\alpha\beta - \mu(\gamma + \delta + \mu)\right]\left[\frac{3\mu(\gamma + \delta + \mu)^2 - \alpha\beta(\gamma + \delta)}{(\gamma + \delta + \mu)}\right] > 0$$

$$\frac{\left[\alpha\beta\mu - \mu^2(\gamma + \delta + \mu)\right]\left[3\mu(\gamma + \delta + \mu)^2 - \alpha\beta(\gamma + \delta)\right]}{(\gamma + \delta + \mu)} > 0 \quad (3.20)$$

Let $J_1 = \frac{\left[\alpha\beta\mu - \mu^2(\gamma + \delta + \mu)\right]\left[3\mu(\gamma + \delta + \mu)^2 - \alpha\beta(\gamma + \delta)\right]}{(\gamma + \delta + \mu)}$ (3.21)

$J_1 > 0$ implies stability otherwise instability.

Table 3.1: Stability Analysis of Endemic Equilibrium (EE) State

α	β	δ	γ	μ	J_1	REMARK
0.001	0.2	0.01	0.15	0.015	-0.00031	UNSTABLE
0.002	0.2	0.01	0.15	0.015	-0.0003	UNSTABLE
0.003	0.2	0.01	0.15	0.015	-0.0003	UNSTABLE
0.004	0.2	0.01	0.15	0.015	-0.0003	UNSTABLE
0.005	0.2	0.01	0.15	0.015	-0.00029	UNSTABLE
0.006	0.2	0.01	0.15	0.015	-0.00029	UNSTABLE
0.007	0.2	0.01	0.15	0.015	-0.00029	UNSTABLE
0.008	0.2	0.01	0.15	0.015	-0.00029	UNSTABLE
0.009	0.2	0.01	0.15	0.015	-0.00028	UNSTABLE
0.01	0.2	0.01	0.15	0.015	-0.00028	UNSTABLE

Conclusion

The Disease Free equilibrium state will be stable if $\frac{\alpha\beta}{\mu} < (\gamma + \delta + \mu)$ that is the population is sustainable.

We apply Bellman and Cooke theorem to analyze the stability of Endemic Equilibrium (EE) state. The hypothetical values were used on equation (3.21) to test for the stability and it shows unstable. Therefore, both zero and non-zero equilibrium is unstable. The implication of instability is that, the population cannot withstand the epidemics. The limitation of this paper is that, it does not include the aged-structured; infants are not separated in any way from the adults.

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