A 4- STAGE RUNGE-KUTTA TYPE METHOD FOR SOLUTION OF STIFF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, a 2 step implicit block hybrid linear multistep method was reformulated into a 4-stage block hybrid Runge-Kutta Type Method via the butcher analysis. The method can be use to solve first order stiff ordinary differential equation. A numerical example solved with the proposed method showed a better result in comparison with an existing method.

Keywords:hybrid,implicit, runge-kutta type

1. Introduction

Ordinary Differential Equations (ODEs) arise frequently in the study of physical problems. Unfortunately many of these equations cannot be solved exactly. This is why solving these equations numerically is important. Traditionally, mathematicians have used one of two classes of methods for solving numerically ordinary differential equations. These are one step methods and Linear Multistep Methods (LMMs) (Rattenbury, 2005).

Runge-Kutta methods are very popular because of their simple coefficients, efficiency and numerical stability (Agam, 2013). The methods are fairly simple to program, easy to implement and their truncation error can be controlled in a more straight forward manner than multistep methods (Kendall,1989). The application of Runge-Kutta methods have provided many satisfactory solutions to many problems that have been regarded as insolvable. The popularity and growth of these methods, coupled with the amount of research effort being undertaken, are further evidence that the applications are still the leading source of inspiration for mathematical creativity (Adegboye, 2013). With the advancement in computer technology, numerical methods are now an increasingly attractive and efficient way to obtain approximate or nearly accurate solutions to differential equations which have hitherto proved difficult or even impossible to solve analytically.

2. Literature Review

Yahaya and Adegboye (2011) reformulated the block hybrid Quade's method into Runge-Kutta type method of order 6. The method was extended to the case in which the approximate to a second order (special or general) as well as first order initial value problem can be calculated. The method was A-stable, possessed the Runge-Kutta stability property, however it was limited to the step number $k = 4$. Chollom et *al.* (2012) constructed a class of A-stable Block Adams Bashforth Explicit Method (BABE) including their hybrid forms. The method tested on non-linear initial value problem performed well and compete favourably with the block hybrid Adams Moulton of higher order, but the step number was restricted to 2. Yahaya and Ajibade (2012) reformulated the two step hybrid linear multistep method into a 3-Stage Runge-Kutta Type method through the idea of general linear method. The method was used to solve only first order initial value problem.

Sofoluwe *et al.* (2012) derived some Backward Differentiation Formulae (BDF) capable of generating solution to stiff initial value problem using lagrangian interpolation technique. The BDF derived were implemented on some standard initial value problem. The region of absolute stability was constructed and the nature so obtained established some facts about the choice of BDF for numerical treatment of stiff problems. The non-stiff problems were not considered. Okunuga *et al.* (2012) presented a direct integration of second order ordinary differential equation using only Explicit Runge-Kutta Nystrom (RKN) method with higher derivative. They derived and tested various numerical schemes on standard problems. Due to the limitations of Explicit Runge-Kutta (ERK) in handling stiff problems, the extension to higher order Explicit Runge-Kutta Nystrom (RKN) was considered and results obtained showed an improvement over conventional Explicit Runge-Kutta schemes. The Implicit Runge-Kutta scheme was however not considered.

3. Methodology

Butcher defined an S-stage Runge-Kutta methods for the first order differential equation in the form

$$y_{n+1} = y_n + h\sum_{i,j=1}^{s} a_{ij}k_i \tag{1}$$

where for $i = 1,2 \ldots\ldots s$

$$k_i = f(x_i + \alpha_j h, y_n + h\sum_{i,j=1}^{s} a_{ij}k_j) \tag{2}$$

The real parameters α_j, k_i, a_{ij} define the method. The method in Butcher array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $a_{ij} = \beta$

Consider the approximate solution to (3.1) in the form of power series given as

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j = y_{n+j} \tag{3}$$

$$\alpha \in R, j = 0(1)t+m-1, y \in C^m(a,b) \subset P(x) \tag{4}$$

$$y'(x) = \sum_{j=1}^{t+m-1} j\alpha_j x^{j-1} = f(x,y) \tag{5}$$

Where α_j's are the parameters to be determined, t and m are the points of interpolation and collocation respectively

For $K = 2$, we choose $t = 3$ and $m = 1$ at (3). Also interpolate (3) at $x = x_{n+i}, i = 0, \frac{1}{2}, 1$ and collocate (4) at $x = x_{n+i}, i = 2$ to have the following system of linear equations of the form

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j = y_{n+i} \qquad i = 0, \frac{1}{2}, 1 \tag{6}$$

$$y'(x) = \sum_{j=0}^{t+m-1} j\alpha_j x^{j-1} = f_{n+i} \qquad i = 2 \tag{7}$$

The general form of the method upon addition of one off grid point is expressed as;

$$\bar{y}(x) = \alpha_1(x)y_n + \alpha_2(x)y_{n+1} + \alpha_3 y_{n+\frac{1}{2}} + h\beta_0(x)f_{n+2} \tag{8}$$

The matrix D of dimension $(t+m)*(t+m)$ of the proposed method is expressed as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 \\ 1 & x_n + \frac{1}{2}h & \left(x_n + \frac{1}{2}h\right)^2 & \left(x_n + \frac{1}{2}h\right)^3 \\ 0 & 1 & 2x_n + 4h & 3(x_n + 2h)^2 \end{bmatrix}$$

We invert the matrix D, to obtain columns which form the matrix C i:e $C = D^{-1}$. The elements of C are used to generate the continous coefficients of the method equations:

$$\left.\begin{aligned} \alpha_1(x) &= C_{11} + C_{21}x + C_{31}x^2 + C_{41}x^3 \\ \alpha_2(x) &= C_{12} + C_{22}x + C_{32}x^2 + C_{42}x^3 \\ \alpha_3(x) &= C_{13} + C_{23}x + C_{33}x^2 + C_{43}x^3 \\ \beta_0(x) &= C_{14} + C_{24}x + C_{34}x^2 + C_{44}x^3 \end{aligned}\right\} \tag{9}$$

The values of the continuous coefficients (9) are substituted into (8) to give the continuous form of the two step block hybrid BDF with one off step interpolation point.

$$
\left.
\begin{aligned}
y(x) = &\left[1 - \frac{44}{13h}(x - x_n) + \frac{41}{13h^2}(x - x_n)^2 - \frac{10}{13h^3}(x - x_n)^3\right] y_n + \\
&\left[-\frac{20}{13h}(\dot{x} - x_n) + \frac{47}{13h^2}(x - x_n)^2 - \frac{14}{13h^3}(x - x_n)^3\right] y_{n+1} + \\
&\left[\frac{64}{13h}(x - x_n) - \frac{88}{13h^2}(x - x_n)^2 + \frac{10}{13h^3}(x - x_n)^3\right] y_{n+\frac{1}{2}} + \\
&\left[\frac{1}{13h}(x - x_n) - \frac{3}{13h}(x - x_n)^2 + \frac{2}{13h^2}(x - x_n)^3\right] f_{n+2}
\end{aligned}
\right\} \quad (10)
$$

Evaluating (10) at point $x = x_{n+2}$ and its derivative at $x = x_{n+1/2}, x = x_{n+1}$ yields the following three discrete hybrid schemes which are used as block integrator;

$$
\left.
\begin{aligned}
&\frac{-36}{13} y_{n+1} + y_{n+2} + \frac{32}{13} y_{n+\frac{1}{2}} = \frac{9}{13} y_n + \frac{6}{13} h f_{n+2} \\
&\frac{33}{12} y_{n+1} + y_{n+\frac{1}{2}} = \frac{21}{12} y_n + \frac{26}{12} h f_{n+\frac{1}{2}} + \frac{1}{12} h f_{n+2} \\
&y_{n+1} - \frac{40}{32} y_{n+\frac{1}{2}} = -\frac{8}{32} y_n + \frac{13}{32} h f_{n+1} - \frac{1}{32} h f_{n+2}
\end{aligned}
\right\} \quad (11)
$$

The equation (11) is of order $[3,3,3]^T$ with error constant $\left[-\frac{3}{52}, \frac{17}{832}, -\frac{19}{624}\right]^T$ respectively.

Equation (11) is transformed as

$$
\left.
\begin{aligned}
&y_{n+\frac{1}{2}} = y_n + \frac{h}{72}(0f_n + 64f_{n+\frac{1}{2}} - 33f_{n+1} + 5f_{n+2}) \\
&y_{n+1} = y_n + \frac{h}{72}\left(0f_n + 80f_{n+\frac{1}{2}} - 12f_{n+1} + 4f_{n+2}\right) \\
&y_{n+2} = y_n + \frac{h}{9}\left(0f_n + 8f_{n+\frac{1}{2}} + 6f_{n+1} + 4f_{n+2}\right)
\end{aligned}
\right\} \quad (12)
$$

Reformulating the block hybrid method with the coefficient as characterized by the Butcher array form as

$$
\begin{array}{c|c}
\alpha & \beta \\
\hline
 & b^T
\end{array}
$$

Where $a_{ij} = \beta$ Gives Table 3.1

Table 3.1: The Butcher Table for method (12)

0	0	0	0	0
$\frac{1}{2}$	0	$\frac{8}{9}$	$\frac{-11}{24}$	$\frac{5}{72}$
2	0	$\frac{8}{9}$	$\frac{2}{3}$	$\frac{4}{9}$
1	0	$\frac{10}{9}$	$\frac{-1}{6}$	$\frac{1}{18}$
	0	$\frac{10}{9}$	$\frac{-1}{6}$	$\frac{1}{18}$

NOTE:

The Butcher table is being rearranged with the off grid points appearing first, followed by the $c_{i's}$ in descending order. This is done in other to satisfy the consistency condition.

Using equation (1), we obtained an implicit 4-stage block Runge-Kutta Type method of uniform order 3.

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + h(0k_1 + \tfrac{8}{9}k_2 - \tfrac{11}{24}k_3 + \tfrac{5}{72}k_4) \\ y_{n+2} &= y_n + h\left(0k_1 + \tfrac{8}{9}k_2 + \tfrac{2}{3}k_3 + \tfrac{4}{9}k_4\right) \\ y_{n+1} &= y_n + h\left(0k_1 + \tfrac{10}{9}k_2 - \tfrac{1}{6}k_3 + \tfrac{1}{18}k_4\right) \end{aligned}\right\} \quad (13)$$

Where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + h\{\tfrac{8}{9}k_2 - \tfrac{11}{24}k_3 + \tfrac{5}{72}k_4\}) \\ k_3 &= f(x_n + h, y_n + h\{\tfrac{10}{9}k_2 - \tfrac{1}{6}k_3 + \tfrac{1}{8}k_4\}) \\ k_4 &= f(x_n + 2h, y_n + h\{\tfrac{8}{9}k_2 + \tfrac{2}{3}k_3 + \tfrac{4}{9}k_4\}) \end{aligned}\right\} \quad (14)$$

4. Results and Discussion

The newly derived block integrators were used to solve this problem within the interval $0 \leq x \leq 0.1$. Okunuga *et al.* (2013) solved this stiff problem by adopting a new 3-point block method of order five. Consider the highly stiff Ordinary Differential Equation (ODE)

$$y' = -10(y-1)^2, \; y(0) = 2, \; h = 0.01$$

Exact Solution: $y(x) = 1 + \frac{1}{1+10x}$

Applying the proposed Runge-kutta Type Method (RKTM) to this problem yields the following results in Table 4.1

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Table 4.1: Absolute Error and Comparison of Result with Okunuga *et al.* (2013) for Problem 1 Using method (13)

t	Exact Solution	Computed Solution	Error RKTM	Error Okunuga [2013]
0.01	1.909090909	1.909125964	3.51E-05	1.07E-04
0.02	1.833333333	1.833397888	6.46E-05	2.38E-04

0.03	1.769230769	1.769301368	7.06E-05	4.51E-04
0.04	1.714285714	1.714362172	7.65E-05	6.20E-04
0.05	1.666666667	1.666741036	7.44E-05	8.84E-04
0.06	1.625000000	1.625073119	7.31E-05	1.03E-03
0.07	1.588235294	1.588304277	6.90E-05	1.27E-03
0.08	1.555555556	1.555621291	6.57E-05	1.53E-03
0.09	1.526315789	1.526377230	6.14E-05	1.75E-03
0.1	1.500000000	1.500057888	5.79E-05	1.81E-03

Results of the proposed method of order 3 with fewer function evaluations displayed in the table able showed a better than accuracy than results obtained by okunuga *et al.(*2013) of order five .Also there is an increase in accuracy as the computed solution moves closer to the exact solution.

5. Conclusion

This research work shows the link between a k-step linear multistep methods and Runge-Kutta methods which leads to a more accurate Block implicit Runge-Kutta Type Method (RKTM) for solving first order stiff ordinary differential equation (ODE).The method can also be extended to second and higher order as derivation is done only once.

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