A08: Modified Single-step Methods of Higher Order ofAccuracy for Stiff System of Ordinary Differential Equations

Umaru Mohammed[1*].and Jamiu, Garba[2]
[1,2]Department of Mathematics, Federal University of Technology, Minna
[2]umaru.mohd@futminna.edu.ng,
[1]garbajamiu90@gmail.com
[*]Corresponding author

Abstract

In this paper, amodified single-step method is proposed to integrate stiff systems of ordinary differential equations. In order to obtain higher order A-stable methods, we have used second derivative of the solutions and imposedsome special sets of off-grid points in the formulation process of the algorithms. The consistency, convergence and order of accuracy of the algorithms were successfully established and in addition, the methods are found to be A-stable. The proposed methods which are self-starting were applied as simultaneous numerical integrators on non-overlapping intervals. In order to demonstrate the effectiveness of the proposed algorithms, some stiff systems of IVPs are considered and results obtained are compared with those from related schemes and from other methods in the literature.

Keywords: collocation, interpolation,intra-point, one-step method, second derivative.

1. Introduction

Numerical methods for Ordinary Differential Equations (ODEs) are very important tools for scientific computation, as they are widely used for solution of real life problems. Analytical methods have been used in some literature to solve mathematical problems. But some of the problems that occur in real life cannot be solved using analytical methods. The purposeof using numerical methods is to find approximate solutions tomathematical problems. It is important to note that numerical methods cannot give an exact solution, therefore the errors involved are of great concern of study.

Many fields of application, notably control system, spring and damping system, chemical reaction, electrical circuits, diffusion and control theory, usually lead to initial value problems involving systems of ordinary differential equations which exhibit a phenomenon known as 'stiffness'(Lambert, 1973). This property is disturbing because these systems are characterized by very high stability, which can turn into very high instability when approximated by standard numerical methods (Butcher, 2008). Attempts to use classical numerical methods to solve such systems of ODEs usually encounter very substantial difficulties.Linear multistep methods have been found to produce relatively higher order of accuracy to differential equations in the stiff systems of ordinary differential equations by many researchers – (Awoyemi, 2003), (Mohammed and Yahaya, 2010) (Mohammed, 2010), (Ndanusa and Adeboye, 2008), (Ndadusa and Adebody, 2009), (Ndanusa, 2007), (Yahaya and Mohammed, 2010), (Brass and Jackiewicz, 2020), (Jackiewicz, 2002), (Yu-Kulikov and Weiner, 2020). Some researchers have also attempted the approximate solution to higher order differential equations directly using linear multistep methods without reduction to system of first order ordinary differential equation see (Mohammed *et al.,* 2010, Mohammed *et al.* 2019).

In this paper, a modified form of Single-step method is presented to solve stiff systems of ordinary differential equation. In the derivation process of the proposed methods, some special off-step points $\left(x \in (0,1)\right)$ (which are referred to as intra-points) and the second derivative of the solution are incorporated in order for higher order and zero-stability be guaranteed(see Enright, 1974, Abhulimen and Ukpebor, 2019, Fazeli and Hojjati, 2020). The methods are implemented as block method whereby, there is no requirement for a different strategy for finding starting values. In the implementation process,we obtain initial conditions at $x_{n+1}, n = 0,1,\ldots,N-1$ using the computed values y_{n+1} over sub-intervals $[x_0, x_1],\ldots,[x_{N-1}, x_N]$. For instance when $n=0,\left(y_\eta, y_1\right)$ are obtained simultaneously over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP, for $n=1$, $\left(y_{\eta+1}, y_2\right)$ are also obtained simultaneously over the sub-interval $[x_1, x_2]$, as y_1 is now known from the previous block, and so on. Therefore, the sub-interval $[x_n, x_{n+1}]$ do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional way.

2. Theoretical Procedure of the Method

The proposed one-step second derivative block intra-step point method for the solution of stiff systems of first order ordinary differential equations is of the form:

$$y_{n+1} = y_n + h\sum_{j=0}^{1} \beta_j f_{n+j} + h^2 \sum_{j=0}^{1} \gamma_j g_{n+j} \qquad (1)$$

and the additional method

$$y_{n+\eta} = y_n + h\sum_{j=0}^{1} \beta_j f_{n+j} + h^2 \sum_{j=0}^{1} \gamma_j g_{n+j} \qquad (2)$$

where $\beta_1 \neq 0, \gamma_1 \neq 0$ $\beta_j, \beta_{\eta j}, \gamma_j, \gamma_{\eta j}$ are unknown coefficients, v is the intra-step points. The general approach in the derivation of (1) and (2) involves the use of continuous collocation approach using a trial function of the form:

$$Y(x) = \sum_{j=0}^{r+2s-1} a_j x^j \qquad (3)$$

where a_j's are unknown coefficients to be determined, r and s are numbers of interpolation and collocation points respectively. We interpolate (3) at x_n and collocate its first derivative at x_n, and x_{n+1}, and a countable number of intra-step points defined as $x_{n+\eta} = x_n + h\eta$. Here, $\eta \in (0,1)$ are points generated from the Bhaskara cosine formula (see Orakwe,lu2019). These lead to a system of equations of the form

$$\left.\begin{array}{l} Y(x_n) = y_n \\ Y'(x_{n+jv}) = f_{n+jv} \\ Y''(x_{n+jv}) = g_{n+jv} \\ j = 0,1,\; v = 1,2,\ldots,m \end{array}\right\} \qquad (4)$$

which is solved using matrix inversion method to obtain a_j's and then substituted into (3) to get the continuous scheme of the form

$$y(x)=y_n+h\left(\beta_0(x)f_n+\beta_v(x)f_{n+jv}+\beta_1(x)f_{n+1}\right)+h^2\left(\gamma_0(x)g_n+\gamma_v(x)g_{n+jv}+\gamma_1(x)g_{n+1}\right) \quad (5)$$

The continuous scheme (5) generated produces the main and additional algorithms which are merged to generate approximations simultaneously. In this paper, we consider two different blocks.

The specification of one-step second derivative block method with 5 intra-points is given as

$k=1, m=5, \eta_j=\left(\frac{5}{74},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{69}{74}\right), x\in[x_n,x_{n+1}]$ which results in system of equations

$$Y_\omega = D\Psi_{\omega-n} \quad (6)$$

where
$$Y_\omega=\left(y_n,f_n,f_{n+\frac{5}{74}},f_{n+\frac{1}{4}},f_{n+\frac{1}{2}},f_{n+\frac{3}{4}},f_{n+\frac{69}{74}},f_{n+1},g_n,g_{n+\frac{5}{74}},g_{n+\frac{1}{4}},g_{n+\frac{1}{2}},g_{n+\frac{3}{4}},g_{n+\frac{69}{74}},g_{n+1,}\right)^T,$$
and
$$\Psi_\omega=\left(\alpha_0,\beta_0,\beta_{\frac{5}{74}},\beta_{\frac{1}{4}},\beta_{\frac{1}{2}},\beta_{\frac{3}{4}},\beta_{\frac{69}{74}},\beta_1,\gamma_0,\gamma_{\frac{5}{74}},\gamma_{\frac{1}{4}},\gamma_{\frac{1}{2}},\gamma_{\frac{3}{4}},\gamma_{\frac{69}{74}},\gamma_1\right)$$

The D-matrix for this method is given as:

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12} & x_n^{13} & x_n^{14}\\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 & 11x_n^{10} & 12x_n^{11} & 13x_n^{12} & 14x_n^{13}\\
0 & 1 & 2(x_{n+\frac{5}{74}}) & 3(x_{n+\frac{5}{74}})^2 & 4(x_{n+\frac{5}{74}})^3 & 5(x_{n+\frac{5}{74}})^4 & 6(x_{n+\frac{5}{74}})^5 & 7(x_{n+\frac{5}{74}})^6 & 8(x_{n+\frac{5}{74}})^7 & 9(x_{n+\frac{5}{74}})^8 & 10(x_{n+\frac{5}{74}})^9 & 11(x_{n+\frac{5}{74}})^{10} & 12(x_{n+\frac{5}{74}})^{11} & 13(x_{n+\frac{5}{74}})^{12} & 14(x_{n+\frac{5}{74}})^{13}\\
0 & 1 & 2(x_{n+\frac{1}{4}}) & 3(x_{n+\frac{1}{4}})^2 & 4(x_{n+\frac{1}{4}})^3 & 5(x_{n+\frac{1}{4}})^4 & 6(x_{n+\frac{1}{4}})^5 & 7(x_{n+\frac{1}{4}})^6 & 8(x_{n+\frac{1}{4}})^7 & 9(x_{n+\frac{1}{4}})^8 & 10(x_{n+\frac{1}{4}})^9 & 11(x_{n+\frac{1}{4}})^{10} & 12(x_{n+\frac{1}{4}})^{11} & 13(x_{n+\frac{1}{4}})^{12} & 14(x_{n+\frac{1}{4}})^{13}\\
0 & 1 & 2(x_{n+\frac{1}{2}}) & 3(x_{n+\frac{1}{2}})^2 & 4(x_{n+\frac{1}{2}})^3 & 5(x_{n+\frac{1}{2}})^4 & 6(x_{n+\frac{1}{2}})^5 & 7(x_{n+\frac{1}{2}})^6 & 8(x_{n+\frac{1}{2}})^7 & 9(x_{n+\frac{1}{2}})^8 & 10(x_{n+\frac{1}{2}})^9 & 11(x_{n+\frac{1}{2}})^{10} & 12(x_{n+\frac{1}{2}})^{11} & 13(x_{n+\frac{1}{2}})^{12} & 14(x_{n+\frac{1}{2}})^{13}\\
0 & 1 & 2(x_{n+\frac{3}{4}}) & 3(x_{n+\frac{3}{4}})^2 & 4(x_{n+\frac{3}{4}})^3 & 5(x_{n+\frac{3}{4}})^4 & 6(x_{n+\frac{3}{4}})^5 & 7(x_{n+\frac{3}{4}})^6 & 8(x_{n+\frac{3}{4}})^7 & 9(x_{n+\frac{3}{4}})^8 & 10(x_{n+\frac{3}{4}})^9 & 11(x_{n+\frac{3}{4}})^{10} & 12(x_{n+\frac{3}{4}})^{11} & 13(x_{n+\frac{3}{4}})^{12} & 14(x_{n+\frac{3}{4}})^{13}\\
0 & 1 & 2(x_{n+\frac{69}{74}}) & 3(x_{n+\frac{69}{74}})^2 & 4(x_{n+\frac{69}{74}})^3 & 5(x_{n+\frac{69}{74}})^4 & 6(x_{n+\frac{69}{74}})^5 & 7(x_{n+\frac{69}{74}})^6 & 8(x_{n+\frac{69}{74}})^7 & 9(x_{n+\frac{69}{74}})^8 & 10(x_{n+\frac{69}{74}})^9 & 11(x_{n+\frac{69}{74}})^{10} & 12(x_{n+\frac{69}{74}})^{11} & 13(x_{n+\frac{69}{74}})^{12} & 14(x_{n+\frac{69}{74}})^{13}\\
0 & 1 & 2(x_{n+1})^2 & 3(x_{n+1})^2 & 4(x_{n+1})^3 & 5(x_{n+1})^4 & 6(x_{n+1})^5 & 7(x_{n+1})^6 & 8(x_{n+1})^7 & 9(x_{n+1})^8 & 10(x_{n+1})^9 & 11(x_{n+1})^{10} & 12(x_{n+1})^{11} & 13(x_{n+1})^{12} & 14(x_{n+1})^{13}\\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 & 132x_n^{10} & 156x_n^{11} & 182x_n^{12}\\
0 & 0 & 2 & 6(x_{n+\frac{5}{74}}) & 12(x_{n+\frac{5}{74}})^2 & 20(x_{n+\frac{5}{74}})^3 & 30(x_{n+\frac{5}{74}})^4 & 42(x_{n+\frac{5}{74}})^5 & 56(x_{n+\frac{5}{74}})^6 & 72(x_{n+\frac{5}{74}})^7 & 90(x_{n+\frac{5}{74}})^8 & 110(x_{n+\frac{5}{74}})^9 & 132(x_{n+\frac{5}{74}})^{10} & 156(x_{n+\frac{5}{74}})^{11} & 182(x_{n+\frac{5}{74}})^{12}\\
0 & 0 & 2 & 6(x_{n+\frac{1}{4}}) & 12(x_{n+\frac{1}{4}})^2 & 20(x_{n+\frac{1}{4}})^3 & 30(x_{n+\frac{1}{4}})^4 & 42(x_{n+\frac{1}{4}})^5 & 56(x_{n+\frac{1}{4}})^6 & 72(x_{n+\frac{1}{4}})^7 & 90(x_{n+\frac{1}{4}})^8 & 110(x_{n+\frac{1}{4}})^9 & 132(x_{n+\frac{1}{4}})^{10} & 156(x_{n+\frac{1}{4}})^{11} & 182(x_{n+\frac{1}{4}})^{12}\\
0 & 0 & 2 & 6(x_{n+\frac{1}{2}}) & 12(x_{n+\frac{1}{2}})^2 & 20(x_{n+\frac{1}{2}})^3 & 30(x_{n+\frac{1}{2}})^4 & 42(x_{n+\frac{1}{2}})^5 & 56(x_{n+\frac{1}{2}})^6 & 72(x_{n+\frac{1}{2}})^7 & 90(x_{n+\frac{1}{2}})^8 & 110(x_{n+\frac{1}{2}})^9 & 132(x_{n+\frac{1}{2}})^{10} & 156(x_{n+\frac{1}{2}})^{11} & 182(x_{n+\frac{1}{2}})^{12}\\
0 & 0 & 2 & 6(x_{n+\frac{3}{4}}) & 12(x_{n+\frac{3}{4}})^2 & 20(x_{n+\frac{3}{4}})^3 & 30(x_{n+\frac{3}{4}})^4 & 42(x_{n+\frac{3}{4}})^5 & 56(x_{n+\frac{3}{4}})^6 & 72(x_{n+\frac{3}{4}})^7 & 90(x_{n+\frac{3}{4}})^8 & 110(x_{n+\frac{3}{4}})^9 & 132(x_{n+\frac{3}{4}})^{10} & 156(x_{n+\frac{3}{4}})^{11} & 182(x_{n+\frac{3}{4}})^{12}\\
0 & 0 & 2 & 6(x_{n+\frac{69}{74}}) & 12(x_{n+\frac{69}{74}})^2 & 20(x_{n+\frac{69}{74}})^3 & 30(x_{n+\frac{69}{74}})^4 & 42(x_{n+\frac{69}{74}})^5 & 56(x_{n+\frac{69}{74}})^6 & 72(x_{n+\frac{69}{74}})^7 & 90(x_{n+\frac{69}{74}})^8 & 110(x_{n+\frac{69}{74}})^9 & 132(x_{n+\frac{69}{74}})^{10} & 156(x_{n+\frac{69}{74}})^{11} & 182(x_{n+\frac{69}{74}})^{12}\\
0 & 0 & 2 & 6(x_{n+1}) & 12(x_{n+1})^2 & 20(x_{n+1})^3 & 30(x_{n+1})^4 & 42(x_{n+1})^5 & 56(x_{n+1})^6 & 72(x_{n+1})^7 & 90(x_{n+1})^8 & 110(x_{n+1})^9 & 132(x_{n+1})^{10} & 156(x_{n+1})^{11} & 182(x_{n+1})^{12}
\end{pmatrix}$$

Equation (5) is solved by matrix inversion technique which yield the continuous coefficients

$\alpha_0(x),\beta_j(x),\gamma_j(x)$; which are then substituted into (5)to obtain its equivalent continuous

scheme:

$$y(x)=y_n+h\begin{pmatrix}\beta_0(x)f_n+\beta_{\frac{5}{74}}(x)f_{n+\frac{5}{74}}+\beta_{\frac{1}{4}}(x)f_{n+\frac{1}{4}}+\beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}}+\beta_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}\\ +\beta_{\frac{69}{74}}(x)f_{n+\frac{69}{74}}+\beta_1(x)f_{n+1}\end{pmatrix}+$$
$$h^2\begin{pmatrix}\gamma_0(x)g_n+\gamma_{\frac{5}{74}}(x)g_{n+\frac{5}{74}}+\gamma_{\frac{1}{4}}(x)g_{n+\frac{1}{4}}+\gamma_{\frac{1}{2}}(x)g_{n+\frac{1}{2}}+\gamma_{\frac{3}{4}}(x)g_{n+\frac{3}{4}}+\gamma_{\frac{69}{74}}(x)g_{n+\frac{69}{74}}\\ +\gamma_1(x)g_{n+1}\end{pmatrix} \tag{7}$$

Evaluating (7) at $x=\frac{5}{74}h,\frac{1}{4}h,\frac{1}{2}h,\frac{3}{4}h,\frac{69}{74}h,$ and h gives the following discrete schemes which form the block for the one-step second derivative block intra-points method with $m=5$ (OSDBM5)

$$\left.\begin{aligned}y_{n+\frac{5}{74}}=y_n&+\frac{2290002406715498368470603385}{8197163229070036195593562944}hf_n+\frac{3082196653420347171129603501 5}{7948032095976952087277815726 08}hf_{n+\frac{5}{74}}+\\
&\frac{479906684273373618194618 0000}{790663609705300413921583677 6327}hf_{n+\frac{1}{4}}+\frac{3450076071236439375}{3410275900715750575308 8}hf_{n+\frac{1}{2}}+\\
&\frac{614057441651297725766 0000}{790663609705300413921583677 6327}hf_{n+\frac{3}{4}}-\frac{10731565677694320752577 9655}{794803209597695208727781572608}hf_{n+\frac{69}{74}}+\\
&\frac{22770646151768088090844 15}{819716322907003619559356294 4}hf_{n+1}+\frac{550589251942802755341 25}{237598934175943078133146752}h^2g_n-\\
&\frac{20909102194147642656275}{29304099459426191709044736}h^2g_{n+\frac{5}{74}}-\frac{2957002796078266059650 00}{289938984123689187356649680 1}h^2g_{n+\frac{1}{4}}-\\
&\frac{26786422976951818125}{28236130534597543224934 4}h^2g_{n+\frac{1}{2}}-\frac{899475767427590786050 00}{289938984123689187356649680 1}h^2g_{n+\frac{3}{4}}-\\
&\frac{102495359829189387925}{418629992277517024414924 8}h^2g_{n+\frac{69}{74}}-\frac{133489853165766590507 5}{237598934175943078133146752}h^2g_{n+1}\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}y_{n+\frac{1}{4}}=y_n&+\frac{1943212527496001}{34093872933120000}hf_n+\frac{67276659574651002016833877949 2501}{687397370462871531872675954688 0000}hf_{n+\frac{5}{74}}+\\
&\frac{47060462768187769}{526168068417351360}hf_{n+\frac{1}{4}}+\frac{1963790013}{671759728640}hf_{n+\frac{1}{2}}+\frac{19206102090155}{1052336136834702 72}hf_{n+\frac{3}{4}}-\\
&\frac{641769922504699857049741591517}{274958948185148612749070381875200}hf_{n+\frac{69}{74}}+\frac{6710626995623}{1363754917324800}hf_{n+1}+\\
&\frac{597715882969}{790582560768000}h^2g_n+\frac{4963027059592093628627263}{136995059619632044834816000}h^2g_{n+\frac{5}{74}}-\\
&\frac{669184831493}{154358069209344}h^2g_{n+\frac{1}{4}}-\frac{299379159}{150323855360}h^2g_{n+\frac{1}{2}}-\\
&\frac{445505665001}{771790346046720}h^2g_{n+\frac{3}{4}}-\frac{1197758139000212953365 89}{273990119123926408966963200}h^2g_{n+\frac{69}{74}}-\\
&\frac{2241217229}{22588073164800}h^2g_{n+1}\end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} y_{n+\frac{1}{2}} = y_n &+ \frac{17996522320261}{213086705832000} hf_n + \frac{2397161746783962471673366105871}{2685145978370591921377640448000} hf_{n+\frac{5}{74}} + \\ &\frac{319896474521248}{1644275213804223} hf_{n+\frac{1}{4}} + \frac{153451983}{1312030720} hf_{n+\frac{1}{2}} + \frac{26394146969312}{8221376069021115} hf_{n+\frac{3}{4}} - \\ &\frac{1248260965583234972388189 07979}{1342572989185295960688820224 0000} hf_{n+\frac{69}{74}} + \frac{22213495592711}{1065433529160000} hf_{n+1} + \\ &\frac{1596937613}{1235285251200} h^2 g_n + \frac{3370173164797268362157}{535136951413918767513600} h^2 g_{n+\frac{5}{74}} + \\ &\frac{1551045736}{430686577035} h^2 g_{n+\frac{1}{4}} - \frac{5197001}{293601280} h^2 g_{n+\frac{1}{2}} - \frac{1671184136}{602961207849} h^2 g_{n+\frac{3}{4}} - \\ &\frac{5047051038806095992223}{2675684757069593837568000} h^2 g_{n+\frac{69}{74}} - \frac{2579379871}{6176426256000} h^2 g_{n+1} \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} y_{n+\frac{3}{4}} = y_n &+ \frac{1564933587251}{15589333760000} hf_n + \frac{86235065051280332051276556917}{1047702134526553165481902080000} hf_{n+\frac{5}{74}} + \\ &\frac{15845155729673}{80196321965760} hf_{n+\frac{1}{4}} + \frac{155171040579}{671759728640} hf_{n+\frac{1}{2}} + \frac{26061104016757}{240588965897280} hf_{n+\frac{3}{4}} - \\ &\frac{56251752175435417469664759647}{3143106403579659496445706240000} hf_{n+\frac{69}{74}} + \frac{753115181853}{15589333760000} hf_{n+1} + \\ &\frac{1747267617}{1084475392000} h^2 g_n + \frac{231080940715692766313}{29828871810038366208000} h^2 g_{n+\frac{5}{74}} + \\ &\frac{681768631}{117633035520} h^2 g_{n+\frac{1}{4}} - \frac{299379159}{150323855360} h^2 g_{n+\frac{1}{2}} - \frac{3778927211}{352899106560} h^2 g_{n+\frac{3}{4}} - \\ &\frac{2857204560657354607459}{626406308010805690368000} h^2 g_{n+\frac{69}{74}} - \frac{1034958591}{1084475392000} h^2 g_{n+1} - \end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned} y_{n+\frac{69}{74}} = y_n &+ \frac{20221603651679559569621}{192535718271610673960000} hf_n + \frac{498507078481817039654801}{6222803449116088074240000} hf_{n+\frac{5}{74}} + \\ &\frac{1191607029513799685019841568}{6025480945780372000621732035} hf_{n+\frac{1}{4}} + \frac{39868522991012976378507}{170513795035787528765440} hf_{n+\frac{1}{2}} + \\ &\frac{3563863323867605677819724704}{18076442837341116001865196105} hf_{n+\frac{3}{4}} - \frac{769051431703194651320653}{18668410347348264222720000} hf_{n+\frac{69}{74}} + \\ &\frac{14896308994129235863371}{192535718271610673960000} hf_{n+1} + \frac{131291407474186803729}{77014287308644269584000} h^2 g_n + \\ &\frac{8611510805044157833}{1055389290508320768000} h^2 g_{n+\frac{5}{74}} + \frac{14012917501073299128088}{2209563969849788045699205} h^2 g_{n+\frac{1}{4}} - \\ &\frac{267864229769518181 25}{2823613053459754322493 44} h^2 g_{n+\frac{1}{2}} - \frac{6131490692161136847752}{946955987078480591013945} h^2 g_{n+\frac{3}{4}} - \\ &\frac{28171179896528379749}{3166167871524962304000} h^2 g_{n+\frac{69}{74}} - \frac{113877534642128422479}{77014287308644269584000} h^2 g_{n+1} - \end{aligned}\right\} \quad (12)$$

$$\left.\begin{array}{l} y_{n+1} = y_n + \frac{3506128349813}{33294797786250} hf_n + \frac{26213837325040472133740518 1}{32777660868781639665254400 00} hf_{n+\frac{5}{74}} + \\ \frac{1625876519575552}{8221376069021115} hf_{n+\frac{1}{4}} + \frac{153451983}{656015360} hf_{n+\frac{1}{2}} + \frac{1625876519575552}{8221376069021115} hf_{n+\frac{3}{4}} + \\ \frac{262138373250404721337405181}{327776608687816396652544 0000} hf_{n+\frac{69}{74}} + \frac{3506128349813}{33294797786250} hf_{n+1} + \\ \frac{330127123}{193013320500} h^2 g_n + \frac{10692342218160370021}{1306486697787887616000} h^2 g_{n+\frac{5}{74}} + \frac{19213240832}{3014806039245} h^2 g_{n+\frac{1}{4}} - \\ - \frac{19213240832}{3014806039245} h^2 g_{n+\frac{3}{4}} - \frac{10692342218160370021}{1306486697787887616000} h^2 g_{n+\frac{69}{74}} - \frac{330127123}{193013320500} h^2 g_{n+1} \end{array}\right\} \quad (13)$$

For the case, $m = 6$ the specification is given as

$k = 1, m = 6, \eta_j = \left(\frac{1}{20}, \frac{10}{53}, \frac{45}{116}, \frac{71}{116}, \frac{43}{53}, \frac{19}{20}\right), x \in [x_n, x_{n+1}]$ which results in system of equations

$$Y_\omega = D\Psi_{\omega-n}$$

where

$$Y_\omega = \left(y_n, f_n, f_{n+\frac{1}{20}}, f_{n+\frac{10}{53}}, f_{n+\frac{45}{116}}, f_{n+\frac{71}{116}}, f_{n+\frac{53}{43}}, f_{n+\frac{19}{20}}, f_{n+1}, g_n, g_{n+\frac{1}{20}}, g_{n+\frac{10}{53}}, g_{n+\frac{45}{116}}, g_{n+\frac{71}{116}}, g_{n+\frac{43}{53}}, g_{n+\frac{19}{20}}, g_{n+1}, \right)^T,$$

$$\Psi_\omega = \left(\alpha_0, \beta_0, \beta_{\frac{1}{20}}, \beta_{\frac{10}{53}}, \beta_{\frac{45}{116}}, \beta_{\frac{71}{116}}, \beta_{\frac{43}{53}}, \beta_{\frac{19}{20}}, \beta_1, \gamma_0, \gamma_{\frac{1}{20}}, \gamma_{\frac{10}{53}}, \gamma_{\frac{45}{116}}, \gamma_{\frac{71}{116}}, \gamma_{\frac{43}{53}}, \gamma_{\frac{19}{20}}, \gamma_1 \right)$$

Similar to the above procedure, the coefficients of the discrete one-step second derivative block method with 6 intra-points (OSDBM6) are given as

$$
\left.
\begin{aligned}
y_{n+\frac{1}{20}} = {} & y_n + \frac{847643686742018502407616289619130484 09}{4101996725147397630153600000000000000000} hf_n + \\
& \frac{22721967413854407756598403239794593 6521}{790856019461245089581164687247749693 4400} hf_{n+\frac{1}{20}} + \\
& \frac{8309864308878757040733200141902497870105453942140 8142149}{18852599248678637017096624097862170900520000000000000000000} hf_{n+\frac{10}{53}} \\
& + \frac{223746159443743385615698757876726222787247761100318 0621}{267843804634695123479709444783464321228718750000000000000000} hf_{n+\frac{45}{116}} \\
& - \frac{347349811855041030571390733174356649531235358453 6559}{214275043707756098783767555826771456982975000000000000000000} hf_{n+\frac{71}{116}} - \\
& \frac{1198504877973930208756755775476817045672589525331 5689}{150820793989429096136772992782897367204160000000000000000000} hf_{n+\frac{43}{53}} - \\
& \frac{614592394296913367433882817337280649}{790856019461245089581164687247749693 4400} hf_{n+\frac{19}{20}} + \frac{12314735918315764397225343062922130 7}{82039934502947952603072000000000000000 0} hf_{n+1} \\
& + \frac{153111187368927437788484727 11}{120881276637770880000000000000000} h^2 g_n - \frac{763315543400393644060752017 3}{1956918164836945752102747340 8000} h^2 g_{n+\frac{1}{20}} \\
& - \frac{110748444791834369897239845156563551467888701}{201358355578576304365028382811200000000000000000000} h^2 g_{n+\frac{10}{53}} - \\
& \frac{3067190167814926026545020639000467150569 59}{124070784806442028110615551992500000000000000000} h^2 g_{n+\frac{45}{116}} - \\
& \frac{7601132255791032618505512882114991196951}{496283139225768112442462207970000000000000000000} h^2 g_{n+\frac{71}{116}} - \\
& \frac{8456574360853384607611027477964765178763 1}{732212202103913834054648664768000000000000000000} h^2 g_{n+\frac{10}{53}} - \\
& \frac{62826121818771053030514319}{6523060549456485840342491136000} h^2 g_{n+\frac{19}{20}} - \frac{5447599228449762232070806 9}{24176255327554176000000000000000000} h^2 g_{n+1}
\end{aligned}
\right\} \qquad (14)
$$

$$
\left.\begin{aligned}
y_{n+\frac{10}{53}} &= y_n + \frac{27441347628779617907200170289513875065735}{62973200514393426369953789536622813693214 3}hf_n + \\
&\frac{34438710105495950731334648336705993321093750000000000000}{47426154973799726310302501913458041294546457304362 52237}hf_{n+\frac{1}{20}} + \\
&\frac{19423288443435809543431369122273091340603}{28386015914203629599605712420076564139987 5}hf_{n+\frac{10}{53}} \\
&+\frac{12700879266066244166053769600833686880279397421287644025221 12}{52632261590717182697226904346845470686860887416288059577118487 5}hf_{n+\frac{45}{116}} \\
&+\frac{438305608791142931169834642666003264857249407380331753533 44}{10526452318143436539445380869369094137372177483257611915423697 5}hf_{n+\frac{71}{116}} - \\
&\frac{64014881304329509386355405282696627 64}{56772031828407259199211424840153128279975}hf_{n+\frac{43}{53}} - \\
&\frac{66972588075670136459558333053663269453125000000000000}{47426154973799726310302501913458041294546457304362 52237}hf_{n+\frac{19}{20}} + \\
&\frac{17364257464418460724626595551129149456 60}{62973200514393426369953789536622813693214 3}hf_{n+1} \\
&+\frac{40114660287384726097492167488135}{92787505478477363798789667938135397}h^2g_n + \frac{247584516201592008800497777429687500000000000}{11735271891819841116835486056332769139833646 59}h^2g_{n+\frac{1}{20}} \\
&-\frac{44333388816344454498375351731 11}{17675488150631901217542647728645 65}h^2g_{n+\frac{10}{53}} - \frac{6735955510254528785402246350716842149686486118 4}{12190175558856955671161411239783624694905226134438 5}h^2g_{n+\frac{45}{116}} - \\
&\frac{7357898517627990324583368352302152038615872512}{2438035111771391134232282247956724938981045226887 7}h^2g_{n+\frac{71}{116}} - \frac{768001489211427570466709112 88}{35350976301263802435085295457291 3}h^2g_{n+\frac{43}{53}} - \\
&\frac{69488781572411823542183419531250000000000}{39117572972732803722784953521109230466112155 3}h^2g_{n+\frac{19}{20}} - \frac{383452994056904752217628668 1200}{92787505478477363798789667938135397}h^2g_{n+1}
\end{aligned}\right\} \quad (15)
$$

$$
\left.\begin{aligned}
y_{n+\frac{45}{116}} &= y_n + \frac{19629673882951274012574136032547890390165}{28994202824905806506677886305649583928115 2}hf_n + \\
&\frac{1928964942259961599107980232876921332168579101562 5}{29928744395769861373052857253577244052404311755980 8}hf_{n+\frac{1}{20}} + \\
&\frac{20109569763951180259008046017090635332766182956941057065428 9}{13325610014308667917627746045844234121487076498401841905664000}hf_{n+\frac{10}{53}} \\
&+\frac{9077861091313133811569360932004222554290 3}{95198890606725558773795500466419325747200 0}hf_{n+\frac{45}{116}} + \frac{112060025622679017970211584936384309726143}{31034838337792532160257333152052700193587200}hf_{n+\frac{71}{116}} - \\
&\frac{357797004394792133928861436094727889800546724632975464157}{26651220028617335835255492091688468242974152996803683811328 00}hf_{n+\frac{43}{53}} - \\
&\frac{21075024916541005675752132843588794326782226562 5}{34886266473596770925644434835458137238998891003904}hf_{n+\frac{19}{20}} + \frac{18427505788686264901356564361552435258 5}{15260106749950424477198887529289254699008}hf_{n+1} \\
&+\frac{30030924169128727774687874314513 5}{38449209478571958167751486631785267 2}h^2g_n + \frac{170396352322404835050020788650512695312 5}{44433956713442481669676959330176194320793 6}h^2g_{n+\frac{1}{20}} \\
&+\frac{10735755448280175163941582972283734597136737128 37}{42697925376268742726598210077349027767990198730752 0}h^2g_{n+\frac{10}{53}} - \frac{63045820950236311588873668713 7}{11509602335388526476861975789568 0}h^2g_{n+\frac{45}{116}} - \\
&\frac{20268752308887546186386476142 7}{12938380556333309073989669335859 2}h^2g_{n+\frac{71}{116}} - \frac{85521998219535799309561024441459157523810402879}{85395850752537485453196420154698055535980397461504}h^2g_{n+\frac{43}{53}} - \\
&\frac{40632023663651565694375353240966796875}{51794182672110868203917928053579613011968}h^2g_{n+\frac{19}{20}} - \frac{365051350693965458247874352167 5}{20236426041353662193553414016729088}h^2g_{n+1}
\end{aligned}\right\} \quad (16)
$$

$$
\left.
\begin{aligned}
y_{n+\frac{71}{116}} = y_n &+ \frac{1446543698451751010537145316203794557835503}{16609485732330016792784520108644931993600000} hf_n + \\
&\frac{690685007521106660138413821911807664872888549072265625}{1243629115877425049634465377457895222109556366396270182 4} hf_{n+\frac{1}{20}} + \\
&\frac{419525997664836471369990392436803179954297411909567803151241167623 41}{2732166478246224068986113806462000627221147153312952648220672000000 00} hf_{n+\frac{10}{53}} \\
&+ \frac{17272586214347788825023526991366292702548152 31}{8889237405312389266441603272645720844800000000} hf_{n+\frac{45}{116}} + \\
&\frac{911706528436974152032094995508703695437396481}{888923740531238926644160327264572084480000000 0} hf_{n+\frac{71}{116}} + \\
&\frac{685017726303832999846803958604526150888849943749006833360177656091}{2732166478246224068986113806462000627221147153312952648220672000000 00} hf_{n+\frac{43}{53}} - \\
&\frac{1859862545089097255819031649376932261871776740722656 25}{1243629115877425049634465377457895222109556366396270182 4} hf_{n+\frac{19}{20}} + \frac{5226168536414760937883164615611681561167 53}{16609485732330016792784520108644931993600000} hf_{n+1} \\
&+ \frac{530111460109006899246994173785188 9}{4964553821144996298971794455920640000} h^2 g_n + \frac{28409999813365306893614975400813834584960937 5}{5539092609941026322460260073140434207835809382 4} h^2 g_{n+\frac{1}{20}} \\
&+ \frac{1295584498130926495045025533566658883676700045737410077 1}{2918136337434366885720946419973822616518580144504832000000} h^2 g_{n+\frac{10}{53}} - \frac{252747680396554380369930119800541}{292355169657774446877432210432000000} h^2 g_{n+\frac{45}{116}} - \\
&\frac{67052792164553341926105092925384439}{847829992007545895944553410252800000 0} h^2 g_{n+\frac{71}{116}} - \frac{776461286366680238988666850466537374685938207916777911}{2652851215849424441564496745430747833198709222277120000000} h^2 g_{n+\frac{43}{53}} - \\
&\frac{115139242574992946982034199980374979150390625}{5539092609941026322460260073140434207835809382 4} h^2 g_{n+\frac{19}{20}} - \frac{162336980357985956908472286415944 73}{347518767480149740928025611914444800 00} h^2 g_{n+1}
\end{aligned}
\right\} \quad (17)
$$

$$
\left.
\begin{aligned}
y_{n+\frac{43}{53}} = y_n &+ \frac{763608553697955974727708325045408877402 27}{7920459898423211336102958171811640949000 00} hf_n + \frac{1270744860630510418461734550864960734124251562 5000000}{2496113419673669805805394837550423226028760910755922 23} hf_{n+\frac{1}{20}} + \\
&\frac{5481381114334268254681426093854700254733 7}{3570253677563438389023068713456244625000 00} hf_{n+\frac{10}{53}} \\
&+ \frac{324845287563140269653938715898992418488057555523458736127314970048}{164475817470991195928834076083892095896440273175900186178495273437 5} hf_{n+\frac{45}{116}} + \\
&\frac{164475817470991195928834076083892095896440273175900186178495273437 5}{164475817470991195928834076083892095896440273175900186178495273437 5} hf_{n+\frac{71}{116}} + \\
&\frac{3034389543266566756936001562063230364733 7}{3570253677563438389023068713456244625000 00} hf_{n+\frac{43}{53}} - \\
&\frac{109642836342729541451572752008083720672642203125000000}{4742615497379972631030250191345804129454645730436252237} hf_{n+\frac{19}{20}} + \frac{231739299992292064637396343552566132843 3}{416866310443326912426471482726928471000 00} hf_{n+1} \\
&+ \frac{1730379827288132537777161484950 33}{1433786687452327339856133322075800 00} h^2 g_n + \frac{354272303334395430669870292244876562500000}{617645889043149532465025581912251007359665 61} h^2 g_{n+\frac{1}{20}} \\
&+ \frac{499386898175078232238402489179443}{955948520856241277314367102685000 00} h^2 g_{n+\frac{10}{53}} + \frac{8111368231685972947190962770287314599620526211403 84}{3809429862142798647237941012432382717157883167012031 25} h^2 g_{n+\frac{45}{116}} - \\
&\frac{672545915504135306558894441416185741268123266473 6}{2254100510143667838602331960019161371099339152078125} h^2 g_{n+\frac{71}{116}} - \frac{759924465846260451549970349079443}{955948520856241277314367102685000 00} h^2 g_{n+\frac{43}{53}} - \\
&\frac{148793164868494285511636935609255156250000 0}{3911757297273280372284953521109230466112155 3} h^2 g_{n+\frac{19}{20}} - \frac{4309663839693479753319468542974 9}{5282372006403311252101543818174000 0} h^2 g_{n+1}
\end{aligned}
\right\} \quad (18)
$$

$$
\left.
\begin{aligned}
y_{n+\frac{19}{20}} = y_n &+ \frac{118433324084917460916241918893965 27}{19609176998028798080000000000000000} hf_n + \\
&\frac{571603690354930766691723404020028 11}{115301941895501543895781409425244 1600} hf_{n+\frac{1}{20}} \\
&\frac{3373609093801931407066822345257220019600782457256 9371}{219887438386687703946308489259217622400000000000000000 00} hf_{n+\frac{10}{53}} + \\
&\frac{7725500277999724640483500498349841191703187850107 88281}{3904997880663290909457784586433362315625000000000000000 0} hf_{n+\frac{45}{116}} + \\
&\frac{7725500277999724640483500498349841191703187850107 88281}{3904997880663290909457784586433362315625000000000000000 0} hf_{n+\frac{71}{116}} + \\
&\frac{4204677678057742297444363629758087059781901635308 148689}{274859297983359629932885611574022028000000000000000000 00} hf_{n+\frac{43}{53}} + \\
&\frac{146892851076151813380686668988087}{7073738766595186742072479105843200} hf_{n+\frac{19}{20}} + \\
&\frac{469483092390929246524786847992545 49}{598045884990143990400000000000000000} hf_{n+1} + \\
&\frac{119200578458235852701540 53}{956717662348800000000000000} h^2 g_n + \frac{320037844261503841664566363}{542082594137658103075553 28000} h^2 g_{n+\frac{1}{20}} \\
&+ \frac{1211428696292479388080098290156269572937219}{223111751333602553313050839680000000000000000 0} h^2 g_{n+\frac{10}{53}} + \\
&\frac{20374629629623758954952734892735669643}{843402170565348660745479000000000000000000} h^2 g_{n+\frac{45}{116}} - \\
&\frac{649251131591200503361905697172044464 4163}{264374141927215060964448225000000000000000000} h^2 g_{n+\frac{71}{116}} - \\
&\frac{306569196906476666343242582808346247963099 41}{557779378334006383282627099200000000000000000000} h^2 g_{n+\frac{43}{53}} - \\
&\frac{698782044751078499513437}{1108553362244699597291520 00} h^2 g_{n+\frac{19}{20}} - \frac{375543468439087934130613649}{334851181822080000000000000000} h^2 g_{n+1}
\end{aligned}
\right\} \qquad (19)
$$

$$\left.\begin{aligned}
y_{n+1} = y_n &+ \frac{1815994739933661698357647}{1831248538012231084890000 0}hf_n + \\
&\frac{758479776669747681645312500000 0}{1532381610129210630543861388671 39}hf_{n+\frac{1}{20}} \\
&\frac{1067111232175577109968204771912992330347739}{6955652025043771036414043719695310987500000}hf_{n+\frac{10}{53}} + \\
&\frac{5601392753882553428533167771974771 06468416}{2830133185066516520284334792724686403515625}hf_{n+\frac{45}{116}} + \\
&\frac{560139275388255342853316777197477106468416}{2830133185066516520284334792724686403515625}hf_{n+\frac{71}{116}} + \\
&\frac{1067111232175577109968204771912992330347739}{6955652025043771036414043719695310987500000}hf_{n+\frac{43}{53}} + \\
&\frac{7584797766697476816453125000000}{153238161012921063054386138867139}hf_{n+\frac{19}{20}} + \\
&\frac{1815994739933661698357647}{18312485380122310848900000}hf_{n+1} + \\
&\frac{4715072213856359}{37775398949303400000}h^2 g_n + \frac{20180339729328125000000}{341259445597960685006757}h^2 g_{n+\frac{1}{20}} \\
&+ \frac{48912498051482023594168827652789}{898921230261501358772448137550000 0}h^2 g_{n+\frac{10}{53}} + \\
&\frac{26930925525708524125106903843 2}{1107774864343232393844781714218 75}h^2 g_{n+\frac{45}{116}} - \\
&\frac{269309255257085241251069038432}{110777486434323239384478171421875}h^2 g_{n+\frac{71}{116}} - \\
&\frac{48912498051482023594168827652789}{8989212302615013587724481375500000}h^2 g_{n+\frac{43}{53}} - \\
&\frac{20180339729328125000000}{341259445597960685006757}h^2 g_{n+\frac{19}{20}} - \frac{4715072213856359}{37775398949303400000}h^2 g_{n+1}
\end{aligned}\right\} \quad (20)$$

4. Analysis of the method

4.1. Local truncation error and order. Let the Linear operator defined on the method be

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right) + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\sum_{j=0}^{k}\gamma_j g_{n+j} \quad (21)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (21) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right] = C_0 y(x) + C_1 h y(x) + \ldots + C_q h^q y(x) + \ldots \quad (22)$$

where the constant $C_q, q = 0,1,\ldots$ are given as follows

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j \\ C_1 &= \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \\ C_2 &= \frac{1}{2!}\sum_{j=1}^{k} (j)^2 \alpha_j - \sum_{j=1}^{k} j\beta_j - \sum_{j=0}^{k} \gamma_j \\ &\vdots \\ C_q &= \frac{1}{q!}\sum_{j=1}^{k} (j)^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j \end{aligned}\right\} \qquad (23)$$

$q = 3, 4, \ldots$

According to Henrici (1962), we say the method (1) is of order p if $C_0 = C_1 = \ldots C_p = 0, C_{p+1} \neq 0$. C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(x_n)$ the principal truncation error at the point x_n.

From our analysis, the block methods have the following order and error constants summarized in tables 1 and 2 respectively. It is noted from table 1 that OSDBM5 is of uniform accurate order 14. And the members of the block OSDBM6 are of uniformly accurate order 16.

Table 1. Order and Error Constants for the Proposed One-step Second Derivative block method with 5 intra-points (OSBDM5).

Equation	Order p	Error constants, C_{p+1}
(8)	14	$\frac{21374355910049302497 5}{98073727507265805514550207132752882434048}$
(9)	14	$\frac{4754047589}{141115163246626160693477376000}$
(10)	14	$\frac{222373}{44098488514570675216711680000}$
(11)	14	$\frac{12402993}{6452453737842988600524800 0}$
(12)	14	$\frac{25082462978234246925 3}{1121098851249037557322247452363430297600}$
(13)	14	$\frac{222373}{9843412614859525718016 00}$

Table 2. Order and Error Constants for the Proposed One-step Second Derivative block method with 6 intra-points (ODBM6).

Equation	Order, p	Error constants, C_{p+1}
(14)	16	$\frac{28727435421072805936551660 31}{9156100122216741050207591989248000000000000000000000}$
(15)	16	$\frac{3980654421806922380022906104324 11}{762863464976540318838587800997329156255755159798074572 80}$
(16)	16	$\frac{289647468279310318545692852 2899}{150146212445951165887041672606246013030792227738091520}$
(17)	16	$\frac{609250024915265883890645700510873851893}{15392332810279462490388756468274688929609809471712788480000000}$
(18)	16	$\frac{2046548331663368127001493773853407891049}{381431732488270159419293900498664578127877579899037286400000000}$
(19)	16	$\frac{5361690613291790506563448339 69}{9156100122216741050207591989248000000000000000000000}$
(20)	16	$\frac{86246688779405813}{146497601955467856803321471827968000000}$

4.3. Consistency: The block methods OSBDM5 and OSBDM6 are said to be consistent if the order of the individual block member is greater or equal to one. That is, $p \geq 1$. Therefore, we can infer from tables 1 and 2 that the methods are consistent.

In what follows, the methods OSBDM5 and OSBDM6 can generally be written as a matrix difference equation as follows:

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + h\left[B^{(0)}F_{w-1} + B^{(1)}F_w\right] + h^2\left[C^{(0)}G_{w-1} + C^{(1)}G_w\right] \quad (24)$$

where

$$\left.\begin{aligned}
Y_w &= \left(y_{n+\frac{5}{74}}, y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+\frac{69}{74}}, y_{n+1} \right)^T \\
Y_{w-1} &= \left(y_{n-\frac{69}{74}}, y_{n-\frac{3}{4}}, y_{n-\frac{1}{2}}, y_{n-\frac{1}{4}}, y_{n-\frac{5}{74}}, y_n \right)^T \\
F_w &= \left(f_{n+\frac{5}{74}}, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+\frac{69}{74}}, f_{n+1} \right)^T \\
F_{w-1} &= \left(f_{n-\frac{69}{74}}, f_{n-\frac{3}{4}}, f_{n-\frac{1}{2}}, f_{n-\frac{1}{4}}, f_{n-\frac{5}{74}}, f_n \right)^T \\
G_w &= \left(g_{n+\frac{5}{74}}, g_{n+\frac{1}{4}}, g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+\frac{69}{74}}, g_{n+1} \right)^T \\
G_{w-1} &= \left(g_{n-\frac{69}{74}}, g_{n-\frac{3}{4}}, g_{n-\frac{1}{2}}, g_{n-\frac{1}{4}}, g_{n-\frac{5}{74}}, g_n \right)^T
\end{aligned}\right\}$$

And the matrices $A^{(1)}, A^{(0)}, \mathrm{B}^{(1)}, \mathrm{B}^{(0)}, \mathrm{C}^{(1)}$ and $C^{(0)}$ are matrices whose entries are given by the coefficients of the method OSBDM5. And the method OSBDM6 has similar representation.

4.2. Zero stability: Zero-stability is concerned with the stability of the difference system in the limit as h tends to zero (Akinfenwa *et al., 2017*). Thus, as $h \to 0$, the method (24) tends to the difference system

$$A^{(1)} Y_w - A^{(0)} Y_{w-1} = 0 \tag{25}$$

whose first characteristic polynomial $\rho(\lambda)$ is given by

$$\rho(\lambda) = \left| \lambda A^{(1)} - A^0 \right| \tag{26}$$

Definition (Zero-stability): The block method (22) is said to be zero stable if the roots of the first characteristic polynomial $\rho(\lambda)$ satisfies $|\lambda_j| \le 1,\ j = 1, 2, 3, \ldots$ and for those roots with $|\lambda_j| = 1$, the multiplicity must not exceed 1 (Fatunla, 1991). Therefore, the characteristic polynomials of the methods OSDBM5 and OSDBM6 are respectively given as:

$$\rho(\lambda) = \lambda^5 (\lambda - 1) = 0$$
$$\lambda = \{0, 0, 0, 0, 0, 1\}$$

and

$$\rho(\lambda) = \lambda^6 (\lambda - 1) = 0$$
$$\lambda = \{0, 0, 0, 0, 0, 0, 1\}$$

Therefore, our methods are zero stable since they both satisfy $|\lambda_j| \le 1$.

4.3. Convergence: The necessary and sufficient conditions for one-step second derivative methods OSDBM5 and OSDBM6to be convergent are that they must be consistent and zero stable (Lambert, 1973). Following this theorem, OSDBM5 and OSDBM6 are convergent.

4.3. Region of Absolute stability

The region of absolute stability is determined by obtaining the stability polynomial of the form:

$$\sigma(z)=\left(A^{(1)}-zB^{(1)}-z^2C^{(1)}\right)^{-1}\left(A^{(0)}+zB^{(0)}+z^2C^{(0)}\right) \quad (27)$$

where $z=\lambda h$

The matrix $\sigma(z)$ has eigenvalues $\{0,0,0,...,\lambda_k\}$, and the dominant eigenvalue λ_k :□ →□ is a rational function with real coefficient given by

$$\lambda_k=\frac{P(z)}{P(-z)} \quad (28)$$

For our methods, the stability functions are given in the Appendix 1. It is clear from the stability functions that for $\mathrm{Re}(z)<0$, $|\lambda_k|\le 1$.The two methods are A-stable since their regions of absolute stability contains the left half-plane□⁻.

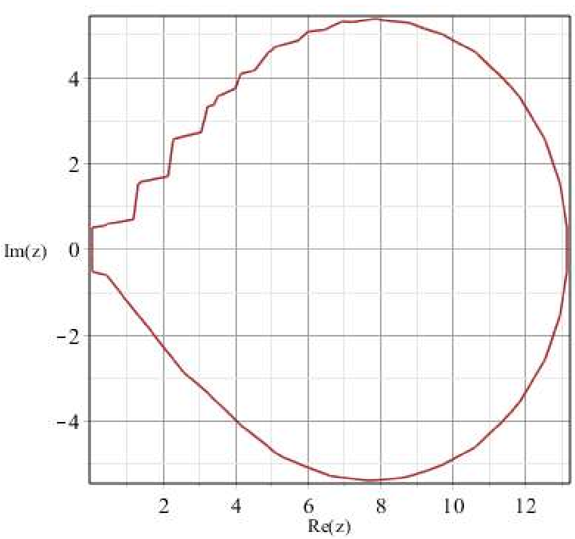

Figure 1. Stability Region for OSDBM5

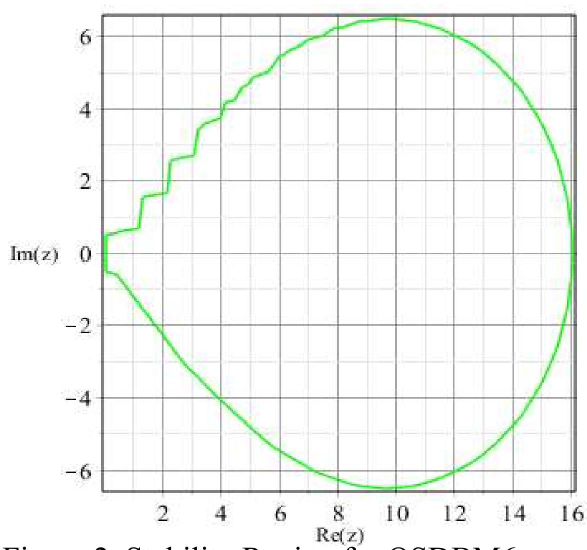

Figure 2. Stability Region for OSDBM6

5 Numerical experiments

In this section, we test the performance of the one-step second derivative block methods on some systems of initial value problems particularly on stiff problems.We find the absolute errors of the approximate solution on the partition πN as $|y(x)-y(x_n)|$ and also make comparisons with some existing methods in the literature. For the purpose of comparative analysis of performance of the new methods on the various numerical examples, we use the following notations:

- OSDBM5: The new One step second derivative block method with 5 intra-points
- OSDBM6: The new One step second derivative block method with 6 intra-points
- CEA: The method of Abhulimen(2014) - a 3-step third derivative of order p=5
- EOS: The method of Ehieie *et al.* (2013) – a 2-step method of order p=3.
- AUK: The method of Abhulimen and Ukpebor(2019) – a 4 step method of order p= 6
- AAK: The method of Akinfenwa(2014) – a 2-step method of order p=8.
- NJT: The method of Ngwane and Jator(2012) – a one-step method of order p= 6
- AJY: The method of Akinfenwa *et al.*(2014) – a 6-step method of order p=6
- NOK: The method of Nwachukwu and Okor(2018) – a 4-step method of order p=5
- ABG: The method of Akintububo (2019) – a 4-step method of order p= 8.

Problem 1. We considered the following IVP on the range $0 \le x \le 1$. Source: Ngwane and Jator (2012).

$$y' = -y + 95z,\ \ y(0) = 1$$
$$z' = -y - 97z,\ \ z(0) = 1$$

Exact solution:

$$y(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}$$
$$z(x) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}$$

Table 3: Comparative analysis of errors for Problem 1 at $x=1$.

h	Methods	$y(1)\|error\|$	$z(1)\|error\|$
0.03125	CEA	1.91×10^{-9}	2.01×10^{-9}
	EOS	3.40×10^{-9}	3.60×10^{-9}
	A&U	3.0×10^{-8}	1.4×10^{-9}
	NJT	5.0×10^{-14}	5.0×10^{-14}
	AAK	5.0×10^{-19}	5.0×10^{-20}
	OSDBM5	**4.40*10^(-36)**	**3.01*10^(-34)**
	OSDBM6	**3.00*10^(-37)**	**2.80*10^(-36)**
0.0625	CEA	1.91×10^{-9}	2.01×10^{-9}
	SOC	2.73×10^{-1}	2.49×10^{-1}
	AUK	3.0×10^{-8}	1.4×10^{-9}
	NJT	3.0×10^{-12}	3.0×10^{-12}
	AAK	1.0×10^{-16}	1.0×10^{-17}
	OSDBM5	**2.86*10^(-32)**	**3.01*10^(-34)**
	OSDBM6	**2.50*10^(-34)**	**2.80*10^(-36)**

Problem 2. We considered the following nonlinear IVP. Source: Akinfenwa *et al.* [3].

$$y' = -1002y + 1000z^2, \quad y(0) = 1$$
$$z' = y - z(1+z), \quad y(0) = 1$$

Exact solution: $y(x) = e^{-2x}, \quad z(x) = e^{-x}$

Table 4: Comparative analysis of errors for Problem 2 at $x = 1$ and $x = 10$.

x	h	N	Methods	$y\|error\|$	$z\|error\|$
1	0.02	50	AYJ k=6	9.11×10^{-13}	1.25×10^{-12}
	0.1	10	NJT	5.68×10^{-13}	6.57×10^{-13}
	0.008	125	NOK	1.80×10^{-15}	6.11×10^{-16}
	0.1	**10**	**OSDBM5**	**$2.07*10^{-32}$**	**$1.80*10^{-32}$**
	0.1	**10**	**OSDBM6**	**$7.00*10^{-35}$**	**$1.00*10^{-35}$**
10	0.02	500	AJY k=6	2.20×10^{-20}	1.25×10^{-12}
	0.01	1000	NJT	7.10×10^{-22}	7.82×10^{-18}
	0.02	500	ABG	2.12×10^{-21}	7.98×10^{-17}
	0.2	**50**	**OSDBM5**	**$9.28*10^{-36}$**	**$7.18*10^{-32}$**
	0.2	**50**	**OSDBM6**	**$3.12*10^{-40}$**	**$2.47*10^{-36}$**

Problem 3. We considered the following system of three linear equations. Source: Akinfenwa *et al.* (2011).

$$\begin{aligned}
&y' = -21y + 19z - 20w,\ y(0) = 1,\\
&z' = 19y - 20z + 20w,\ z(0) = 0,\\
&w' = 40y - 40z - 20w,\ w(0) = -1\\
&x \in [0,1]
\end{aligned}$$

Exact solution:

$$\begin{aligned}
y(x) &= \frac{1}{2}e^{-2x} + \frac{1}{2}e^{-40x}\sin(40x) + \frac{1}{2}e^{-40x}\cos(40x)\\
z(x) &= \frac{1}{2}e^{-2x} - \frac{1}{2}e^{-40x}\sin(40x) - \frac{1}{2}e^{-40x}\cos(40x)\\
w(x) &= e^{-40x}\sin(40x) - e^{-40x}\cos(40x)
\end{aligned}$$

Table 5: Comparative analysis of errors for Problem 3 at $x=3$.

x	h	N	Error in NJT	**Error in OSDBM5**	**Error in OSDBM6**
3	0.02	150	9.34×10^{-7}	8.80×10^{-36}	2.40×10^{-36}
	0.01	300	1.40×10^{-8}	9.00×10^{-37}	1.51×10^{-35}
	0.005	600	2.31×10^{-10}	4.36×10^{-35}	6.57×10^{-35}

Problem 4. We consider a singularly perturbed problem. Source: Nwanchukwu and Okor (2018).

$$y' = -\left(2+10^4\right)y + 10^4 z^2, \ \ y(0)=1$$
$$z' = y - z - z^2, \ \ z(0)=1$$

Analytical solution: $y(x)=e^{-2x}, \ \ z(x)=e^{-x}$

Table 6: Comparative analysis of errorsfor Problem 4.

x	Error in NOK h=0.01 y_n z_n	OSDBM5 h=0.1 y_n z_n	OSDBM6 h=0.1 y_n z_n
5	1.2×10^{-14} 9.29×10^{-13}	2.08810^{-36} 1.42×10^{-34}	6.90×10^{-38} 4.60×10^{-36}
10	6.07×10^{-19} 6.68×10^{-15}	1.88×10^{-40} 1.59×10^{-36}	6.40×10^{-42} 6.80×10^{-38}

Concluding Remarks

In this paper, we derived a modified multi-step method to overcome the Dahquist barrier theorem by imposing varieties of countable intra-step points for one-step methods from the Bhaskara cosine approximation formula, and incorporating higher derivatives in the derivation process of our algorithms for direct solution systems of first order initial value problems of ordinary differential equations. Analysis of basic properties of numerical methods was carried out and findings show that the methods are convergent and are A-stable of higher order. The effectiveness of the derived methods is demonstrated by considering two test problems for stiff systems. The desirable property of a numerical solution is to behave like that of the exact solution of the problem which can be seen in the figures presented.

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Appendix 1

Stability function for OSDBM5

$$-(1950105600 z^{12} + 279322951680 z^{11} + 20897182896768 z^{10} + 1015352172233984 z^9 + 35101727342868783 z^8 + 903878100820094306 z^7 + 17724686412082932100 z^6 + 266407560800727978240 z^5 + 3047378336600274270720 z^4 + 25923149620627061698560 z^3 + 156382175616219435294720 z^2 + 609147730495666755993600 z + 1218295460991333511987200) / (1950105600 z^{12} - 279322951680 z^{11} + 20897182896768 z^{10} - 1015352172233984 z^9 + 35101727342868783 z^8 - 903878100820094306 z^7 + 17724686412082932100 z^6 - 266407560800727978240 z^5 + 3047378336600274270720 z^4 - 25923149620627061698560 z^3 + 156382175616219435294720 z^2 - 609147730495666755993600 z + 1218295460991333511987200)$$

Stability function for OSDBM6

$$-(11356240665375 z^{14} + 2234688100339725 z^{13} + 233060693487675605 z^{12} + 16051110304222521080 z^{11} + 804715337510257806156 z^{10} + 30909829439677166074152 z^9 + 935272197554552490818640 z^8 + 22595458445517006370298880 z^7 + 437133164547728581867468800 z^6 + 6727170489686829437444259840 z^5 + 80933323572727023984099655680 z^4 + 737007271342534619008598016000 z^3 + 4795348300509213521550508032000 z^2 + 19928002546944865701199872000000 z + 39856005093889731402399744000000) / (11356240665375 z^{14} - 2234688100339725 z^{13} + 233060693487675605 z^{12} - 16051110304222521080 z^{11} + 804715337510257806156 z^{10} - 30909829439677166074152 z^9 + 935272197554552490818640 z^8 - 22595458445517006370298880 z^7 + 437133164547728581867468800 z^6 - 6727170489686829437444259840 z^5 + 80933323572727023984099655680 z^4 - 737007271342534619008598016000 z^3 + 4795348300509213521550508032000 z^2 - 19928002546944865701199872000000 z + 39856005093889731402399744000000)$$