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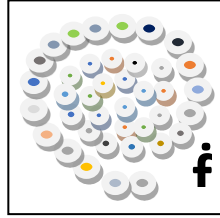
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# International Journal of Mathematical Analysis and Modelling

# Mathematical modelling of solid waste management

M.O. Ibrahim*and N.O. Abdurrahman†

### Abstract

Solid waste is anything that comes from domestic, commercial, or industrial sources that is no longer needed. It is deposited as undesired. Waste disposal did not become an issue when there were few habitations and a lot of open space. Waste disposal becomes a real concern in towns and cities when more individuals move there in pursuit of employment [6,12]. Using a set of ordinary differential equations, a mathematical model for managing solid waste is put forth in this study. The solution's existence and uniqueness are proven. In order to simulate the sensitive parameter for solid waste management, the next-generation matrix is used to identify the basic reproduction number $R_0$. It has been found that when waste production increases, so does the rate at which energy is produced from waste.

*Keywords*: biodegradable; incineration; reproductive number; waste; next generation matrix; decomposition

## 1    Introduction

Waste, according to [7], can be defined as materials that are not of any use, old, unwanted, or defective. It may however be reused by others or recycled for further use. Waste can be defined as by-products, materials, or things that are undesired or useless [15]. Waste disposal is the process or system whereby undesired stuff is disposed of through burying, burning, or throwing it into the ocean.

Waste management refers to the process of managing discarded waste materials that had served their purpose and are no longer useful. The practice of handling waste materials that have served their purpose and are no longer useful is referred to as waste management. According to [1] waste management is the collection, storage, treatment, and disposal of wastes in a manner that makes them harmless to human and animal life, the ecosystem, and the environment as a whole. Solid waste materials must be gathered, processed, recycled, and disposed of in the course of waste management. According to [11] the main reason for studying waste management is to understand the intricacies involved in managing waste and to ease waste management planning . Wastes are generated from several sources such as domestic, industrial, agricultural and commercial activities.

Domestic wastes are the waste materials produced by our households in our daily activities, it is usually referred to as refuse. A larger percentage of domestic waste is directly dumped on land and in water thereby increasing land or soil and water pollution [15]. Waste materials generated from industries include ashes, rubbish, build ing material wastes, toxic wastes, metal containers, plastic containers, paints, and other synthetic materials [15].

Modern techniques used in agriculture and the use of chemicals have contributed to the increase in the quantities of agricultural waste. Wastes are generated from commercial establishments such as restaurants, hotels, markets, offices, printing shops, automobile repair shops and hospitals. There are mainly two types of waste:- The waste which can be decomposed by the action of microbes is called biodegradable waste, examples include domestic waste, newspapers, and vegetable matter. The second

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type is the non-biodegradable which does not decompose or decay, examples include polythene bags, plastics, glass, aluminum cans, iron nails, and DDT [15].

Senzige and Makinde presented a mathematical model that examines how population dynamics affect solid waste generation and management. The model presupposed that each of the three age groups in the population would experience its own rate of natural mortality and waste generation [11]. It is projected that immigration and birth rates will increase the population. The results of the analytical and numerical work provide credence to the idea that increased population growth drives up the production of solid waste. A solid waste-free environment can be achieved with concerted treatment efforts, according to the sensitivity analysis, which showed that boosting treatment efforts for solid waste led to a significant drop in the accumulation of significant trash.

In order to stochastically examine the viability of waste-to-energy in metropolitan areas employing cutting-edge technologies like gasification, pyrolysis, and refuse-derived fuel (RDF), [13] used a quantitative Waste to Energy Recovery Assessment (WERA) methodology. To determine whether garbage to Energy systems can be made an independent investment, future policy measures such as feed-in tariffs, rewards for averted pollution, and increased garbage collecting costs are employed. By highlighting the common elements found across a wide range of studies. [9] conducted research into what encourages and what inhibits participation in waste diversion and minimization practices. He then proposed ways that these strategies could be used by government agencies and other organizations seeking to increase the diversion of waste from disposal to recovery.

# 2 Model Formulation

The process of solid waste management with a focus on energy generation and wealth creation consists of Manufactured goods $A(t)$, goods for domestic/ municipal use $M(t)$ , goods for Industrial use $I(t)$, Collected waste $C(t)$, Recycled wastes $K(t)$, wastes converted to Energy $E(t)$, Wealth generated from Energy $W(t)$, non-recyclable waste for Disposal $D(t)$.

Manufactured products move from $A(t)$ to $M(t)$ at the rate $v$ while some move to $I(t)$ at the rate $1 - v$. Waste generated from $M(t)$ and $I(t)$ are collected at the rate $\alpha_1$ and $\alpha_2$ respectively. Waste from $C(t)$ are recycled and disposed at the rates $\rho_1$ and $\rho_2$ respectively. Recycled product generate energy at the rate $\eta$ the energy generated created wealth at the rate $\beta$. Each of the compartments have a waste natural decomposition rate of $\mu$. $\phi = \xi(M + \omega I)$ is the rate at which goods are manufactured where $\xi$ is the rate at which goods are used domestically and industrially. The wastes from the industries are controlled at the rate $\omega$.

In formulating the model equations, the following assumptions were made:

1. The model assumed that manufactured goods are used for domestic and industrial purposes only.

2. The wastes generated in the industries are controlled and minimized.

3. Collected wastes are either recycled or disposed.

4. It is assumed that the waste are properly disposed

The schematic diagram in Figure 1 and the set of equations(1) provided below can be used to explain the mathematical model that represents the process of solid waste management with an emphasis on energy production and wealth development:-
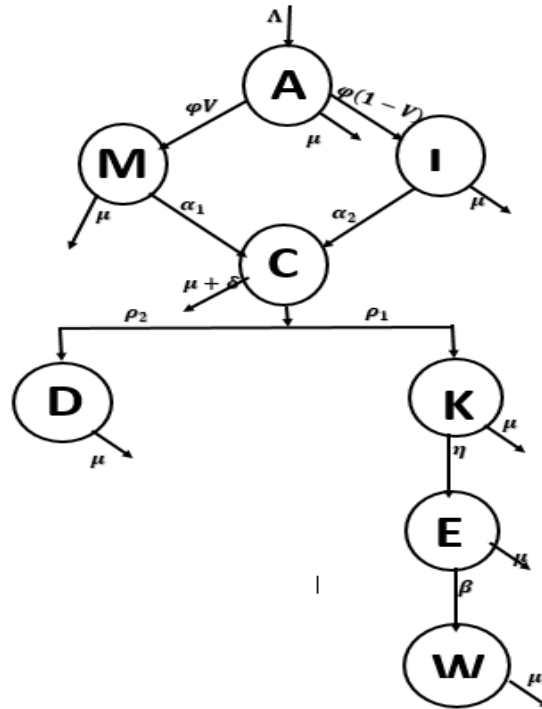
Figure 1: **Schematic Diagram**

$$
\left.\begin{aligned}
\frac{dA}{dt} &= \Lambda - \xi(M + \omega I)A - \mu A \\
\frac{dM}{dt} &= \xi v(M + \omega I)A - (\alpha_1 + \mu)M \\
\frac{dI}{dt} &= \xi(1 - v)(M + \omega I)A - (\alpha_2 + \mu)I \\
\frac{dC}{dt} &= \alpha_1 M + \alpha_2 I - (\rho_1 + \rho_2 + \mu + \delta)C \\
\frac{dK}{dt} &= \rho_1 C - (\omega + \mu + \delta)K \\
\frac{dE}{dt} &= \omega K - (\beta + \mu)E \\
\frac{dW}{dt} &= \beta E - \mu W \\
\frac{dD}{dt} &= \rho_2 C - \mu D
\end{aligned}\right\}
\tag{1}
$$

## 3 Basic properties of the Model

### 3.1 Existence and Uniqueness of the Solution

**Lipschitz Condition**

The Lipschitz condition establishes a limit on a function's modulus of continuity. A function $f : [a, b] \to \Re$ is said to satisfy the Lipschitz condition if there exists a constant $k$ known as the Lipschitz constant such that $|f(x_1) - f(x_2)| \le k|x_1 - x_2| \forall x_1, x_2 \in [a, b]$. The Lipschitz condition is used to prove that a differential equation's solution is unique. The model's equation system will be examined for the existence and uniqueness of a solution using Lipschitz criteria with Derrick and Grossman's

Table 1: Definition of Variables

| S/NO | variables | Description |
|---|---|---|
| 1 | A | Manufactured Goods |
| 2 | M | municipal / Goods used Domestic purpose |
| 3 | I | goods used in the Industries |
| 4 | C | Collected waste |
| 5 | K | Recycled waste |
| 6 | D | Disposed waste |
| 7 | E | Energy generated from Recycled waste |
| 8 | W | Wealth generated from Energy |

Table 2: Definition of Parameters

| s/no | Para | Description |
|---|---|---|
| 1 | $\Lambda$ | rate at which new products are produced |
| 2 | $\phi$ | rate at which manufactured goods are used |
| 3 | $v$ | Proportion of manufactured goods used domestically |
| 4 | $1-v$ | Proportion of manufactured goods used industrially |
| 5 | $\alpha_1$ | rate at which domestic waste are collated |
| 6 | $\alpha_2$ | rate at which industrial waste are collated |
| 7 | $\rho_1$ | Proportion of collated waste that can be recycled |
| 8 | $\rho_2$ | Proportion of collated waste that can decompose |
| 9 | $\omega$ | Rate at which recycled waste generate Energy. |
| 10 | $\beta$ | Rate at which wealth is created from Energy. |
| 11 | $\xi$ | Force of infection. |
| 12 | $\delta$ | Rate at which non-recyclable waste are disposed. |
| 13 | $\mu$ | Rate at which manufacture material deteriorates/decompose naturally. |

theorem in the sections that follow.[2]

**Theorem 1. [3]**

Derrick and Grossman theorem shall be applied to verify the existence and uniqueness of solution of the model. Let $D$ denotes the region $|t - t_0| \leq a, ||x - x_0|| \leq b, x = (x_1, x_2, ..., x_n)$, $x_0 = (x_10, x_20, ..., x_n0)$ and suppose $f(t, x)$ satisfies the Lipschitz condition $||f(t, x_1) - f(t, x_2)|| \leq k||x_1 - x_2||$ The pairs $(t, x_1)$ and $(t, x_2)$ belong to $D$ and k is a positive constant, hence there is a constant $\delta > 0$ such that there exists a unique continuous vector solution $x(t)$ of the system in the interval $t - t_0 \leq 0$. It is important to note thatthe condition is satisfied by the requirement that $\dfrac{\partial f_i}{\partial f_j}, i = 1, 2, ...,$ be continuous and bounded in $D$. We shall return to the model (2) and we consider the region $0 \leq a \leq \Re$. We look for a bounded solution in this region and whose partial derivatives satisfy $\delta \leq a \leq 0$, where $a$ and $\delta$ are constants.

**Theorem 2**

Let $D$ denotes the region $0 \leq a \leq \Re$. Then the model system (1) has a unique solution if it is established that $\dfrac{\partial f_i}{\partial f_j}, i = 1, 2, ..., 8.$ are continuous and bounded in $D$.

**Proof 1** By partially differentiating each compartment with respect to each of the state variables, it is necessary to demonstrate that the magnitude of each result is smaller than infinity and therefore prove the existence and uniqueness of the model's solution.

Let

$$
\left.
\begin{aligned}
f_1 &= \Lambda - \xi(M + \omega I)A - \mu A \\
f_2 &= \xi v(M + \omega I)A - (\alpha_1 + \mu)M \\
f_3 &= \xi(1 - v)(M + \omega I)A - (\alpha_2 + \mu)I \\
f_4 &= \alpha_1 M + \alpha_2 I - (\rho_1 + \rho_2 + \mu + \delta)C \\
f_5 &= \rho_1 C - (\omega + \mu + \delta)K \\
f_6 &= \omega K - (\beta + \mu)E \\
f_7 &= \beta E - \mu W \\
f_8 &= \rho_2 C - \mu D
\end{aligned}
\right\}
\tag{2}
$$

$$
\left.
\begin{aligned}
\frac{df_1}{dA} &= | -\xi(M + \omega I) - \mu| = |\xi(M + \omega I) - \mu| < \infty \\
\frac{df_1}{dM} &= | -(\xi A)| = |\xi A| < \infty \\
\frac{df_1}{dI} &= | -\xi \omega A| = |\xi \omega A| < \infty \\
\frac{df_1}{dC} = 0 < \infty; \quad \frac{df_1}{dK} &= 0 < \infty; \quad \frac{df_1}{dE} = 0 < \infty \\
\frac{df_1}{dW} &= 0 < \infty \quad \frac{df_1}{dD} = 0 < \infty
\end{aligned}
\right\}
\tag{3}
$$

$$
\left.
\begin{aligned}
\frac{df_2}{dA} = |v\xi(M + \omega I)| = | < \infty; \quad \frac{df_2}{dM} &= |(v\xi A - (\alpha_1 + \mu)) < \infty \\
\frac{df_2}{dI} = |(v\xi \omega A - (\alpha_2 + \mu))| < \infty; \quad \frac{df_2}{dC} = 0 < \infty; \quad \frac{df_2}{dK} &= 0 < \infty \\
\frac{df_2}{dE} = 0 < \infty; \quad \frac{df_2}{dW} = 0 < \infty; \quad \frac{df_2}{dD} &= 0 < \infty
\end{aligned}
\right\}
\tag{4}
$$

$$\left.\begin{array}{l}\dfrac{df_3}{dA} = |(1-v)\xi(M+\omega I)| < \infty; \dfrac{df_3}{dM} = |(1-v)\xi A| < \infty \\[2mm] \dfrac{df_3}{dI} = |(1-v)\xi\omega A - (\alpha_2+\mu)| < \infty; \dfrac{df_3}{dC} = 0 < \infty; \dfrac{df_3}{dK} = 0 < \infty \\[2mm] \dfrac{df_3}{dE} = 0 < \infty; \dfrac{df_3}{dW} = 0 < \infty; \dfrac{df_3}{dD} = 0 < \infty \end{array}\right\} \qquad (5)$$

$$\left.\begin{array}{l}\dfrac{df_4}{dA} = 0 < \infty; \dfrac{df_4}{dM} = |\alpha_1| < \infty; \dfrac{df_4}{dI} = |\alpha_2| < \infty \\[2mm] \dfrac{df_4}{dC} = |-(\rho_1+\rho_2+\mu+\delta)| = |(\rho_1+\rho_2+\mu+\delta)| < \infty; \dfrac{df_4}{dK} = 0 < \infty \\[2mm] \dfrac{df_4}{dE} = 0 < \infty; \dfrac{df_4}{dW} = 0 < \infty; \dfrac{df_4}{dD} = 0 < \infty \end{array}\right\} \qquad (6)$$

$$\left.\begin{array}{l}\dfrac{df_5}{dA} = 0 < \infty; \dfrac{df_5}{dM} = 0 < \infty; \dfrac{df_5}{dI} = 0 < \infty \\[2mm] \dfrac{df_5}{dC} = |\rho_1| < \infty; \dfrac{df_5}{dK} = |-(\eta+\mu)| = |\eta+\mu| < \infty \\[2mm] \dfrac{df_5}{dE} = 0 < \infty; \dfrac{df_5}{dW} = 0 < \infty; \dfrac{df_5}{dD} = 0 < \infty \end{array}\right\} \qquad (7)$$

$$\left.\begin{array}{l}\dfrac{df_6}{dA} = 0 < \infty; \dfrac{df_6}{dM} = 0 < \infty; \dfrac{df_6}{dI} = 0 < \infty \\[2mm] \dfrac{df_6}{dC} = 0 < \infty; \dfrac{df_6}{dK} = |\eta| < \infty; \dfrac{df_6}{dE} = |-(\beta+\mu)| = |(\beta+\mu)| < \infty \\[2mm] \dfrac{df_6}{dW} = 0 < \infty; \dfrac{df_6}{dD} = 0 < \infty \end{array}\right\} \qquad (8)$$

$$\left.\begin{array}{l}\dfrac{df_7}{dA} = 0 < \infty; \dfrac{df_7}{dM} = 0 < \infty; \dfrac{df_7}{dI} = 0 < \infty \\[2mm] \dfrac{df_7}{dC} = 0 < \infty; \dfrac{df_7}{dK} = 0 < \infty; \dfrac{df_7}{dE} = |\beta| < \infty \\[2mm] \dfrac{df_7}{dW} = |-\mu| = |\mu| < \infty; \dfrac{df_7}{dD} = 0 < \infty \end{array}\right\} \qquad (9)$$

$$\left.\begin{array}{l}\dfrac{df_8}{dA} = 0 < \infty; \dfrac{df_8}{dM} = 0 < \infty; \dfrac{df_8}{dI} = 0 < \infty \\[2mm] \dfrac{df_8}{dC} = |\rho_2| < \infty; \dfrac{df_8}{dK} = 0 < \infty; \dfrac{df_8}{dE} = 0 < \infty \\[2mm] \dfrac{df_8}{dW} = 0 < \infty; \dfrac{df_8}{dD} = |-\mu| = |\mu| < \infty \end{array}\right\} \qquad (10)$$

### 3.2 Positivity of Solution.

Theorem: Let $\Omega = (A, M, I, C, K, E, W, D) \in R_+^8 : A_0 > 0, M_0 > 0, I_0 > 0, C_0 > 0, K_0 > 0, E_0 > 0, W_0 > 0, D_0 >$
Then the solutions of A,M,I,C,K,E,W and D are positive for all $t \geq 0$
Proof: From the system of differential equation (2). The first equation is considered thus:

$$\frac{dA}{dt} = \Lambda - \xi(M+\omega I)A - \mu A$$

$$\left.\begin{aligned}
\frac{dA}{dt} &\geq (-\xi(M+\omega I)A - \mu A) \\
\int \frac{dA}{A(t)} &\geq \int (-(\xi(M+\omega I) - \mu)A)dt \\
\ln A(t) &\geq (-(\xi(M+\omega I) - \mu))t + b_1 \\
at\ t=0; A(0) = A_0 \ln A_0 = b_1 \\
\ln A(t) &\geq (-(\xi(M+\omega I) - \mu))t + \ln A_0 \\
\frac{\ln A(t)}{\ln A_0} &\geq -(\xi(M+\omega I) - \mu)t \\
\frac{A(t)}{A_0} &\geq \exp(-\xi(M+\omega I) - \mu)t \\
A(t) &= A_0 \exp(-\xi(M+\omega I) - \mu)t
\end{aligned}\right\} \quad (11)$$

The second equation is considered in the (12)

$$\frac{dM}{dt} = v\xi(M+\omega I)A - (\alpha_1 + \mu)M$$

$$\left.\begin{aligned}
\frac{dM}{dt} &\geq -(\alpha_1 + \mu)M \\
\int \frac{dM}{M(t)} &\geq \int -(\alpha_1 + \mu)dt \\
\ln M(t) &\geq -(\alpha_1 + \mu)t + b_2 \\
at\ t=0; M(0) = M_0 \ln M_0 = b_2 \\
\ln M(t) - \ln M_0 &\geq -(\alpha_1 + \mu)t \\
\ln \frac{M(t)}{M_0} &\geq -(\alpha_1 + \mu)t \\
\frac{M(t)}{M_0} &\geq \exp{-(\alpha_1 + \mu)t} \\
M(t) &= M_0 \exp{-(\alpha_1 + \mu)t}
\end{aligned}\right\} \quad (12)$$

The third equation is considered in (13)

$$\frac{dI}{dt} = \xi(1-v)(M+\omega I)A - (\alpha_2 + \mu)I$$

$$\left.\begin{aligned}
\frac{dI}{dt} &\geq -(\alpha_2 + \mu)I \\
\int \frac{dI}{I(t)} &\geq \int -(\alpha_2 + \mu)dt \\
\ln I(t) &\geq -(\alpha_2 + \mu)t + b_3 \\
at\ t=0; I(0) = I_0 \ln I_0 = b_3 \\
\ln I(t) - \ln I_0 &\geq -(\alpha_2 + \mu)t \\
\ln \frac{I(t)}{I_0} &\geq -(\alpha_2 + \mu)t \\
\frac{I(t)}{I_0} &\geq \exp{-(\alpha_2 + \mu)t} \\
I(t) &= I_0 \exp{-(\alpha_2 + \mu)t}
\end{aligned}\right\} \quad (13)$$

The fourth equation is considered in (14)

$$\frac{dC}{dt} = \alpha_1 M + \alpha_2 I - (\rho_1 + \rho_2 + \mu + \delta)C$$

$$\left.\begin{aligned}
\frac{dC}{dt} &\geq -(\rho_1 + \rho_2 + \mu + \delta)C \\
\int \frac{dC}{(t)} &\geq \int -(\rho_1 + \rho_2 + \mu + \delta)C dt \\
\ln C(t) &\geq -(\rho_1 + \rho_2 + \mu + \delta)t + b_4 \\
at\ t = 0; C(0) = C_0 \ln C_0 &= b_4 \\
\ln C(t) - \ln C_0 &\geq -(\rho_1 + \rho_2 + \mu + \delta)t \\
\ln \frac{C(t)}{C_0} &\geq -(\rho_1 + \rho_2 + \mu + \delta)t \\
\frac{C(t)}{C_0} &\geq \exp -(\rho_1 + \rho_2 + \mu + \delta)t \\
C(t) &= C_0 \exp -(\rho_1 + \rho_2 + \mu + \delta)t
\end{aligned}\right\} \quad (14)$$

The fifth equation is considered in (15)

$$\frac{dK}{dt} = \rho_1 C - (\omega + \mu + \delta)K$$

$$\left.\begin{aligned}
\frac{dK}{dt} &\geq -(\omega + \mu + \delta)K \\
\int \frac{dK}{K(t)} &\geq \int -(\omega + \mu + \delta)dt \\
\ln K(t) &\geq -(\omega + \mu + \delta)t + b_5 \\
at\ t = 0; K(0) = K_0 \ln K_0 = b_5 \ln K(t) - \ln K_0 &\geq -(\omega + \mu + \delta)t \\
\ln \frac{K(t)}{K_0} &\geq -(\omega + \mu + \delta)t \\
\frac{K(t)}{K_0} &\geq \exp -(\omega + \mu + \delta)t \\
K(t) &= K_0 \exp -(\omega + \mu + \delta)t
\end{aligned}\right\} \quad (15)$$

The sixth equation is considered in (16)
$$\frac{dE}{dt} = \omega K - (\beta + \mu)E$$

$$\left.\begin{aligned}
\frac{dE}{dt} &\geq -(\beta + \mu)E \\
\int \frac{dE}{E(t)} &\geq \int -(\beta + \mu)dt \\
\ln E(t) &\geq -(\beta + \mu)t + b_6 \\
at\ t = 0;\ E(0) = E_0 \ln E_0 &= b_6 \\
\ln E(t) - \ln E_0 &\geq -(\beta + \mu)t \\
\ln \frac{E(t)}{E_0} &\geq -(\beta + \mu)t \\
\frac{E(t)}{E_0} &\geq \exp -(\beta + \mu)t \\
E(t) &= E_0 \exp -(\beta + \mu)t
\end{aligned}\right\} \quad (16)$$

The seventh equation is considered in (17)

$$\frac{dW}{dt} = \beta E - \mu W$$

$$\left.\begin{array}{r}
\dfrac{dW}{dt} \geq -\mu W \\[2mm]
\displaystyle\int \dfrac{dW}{W(t)} \geq \int -\mu dt \\[2mm]
\ln W(t) \geq -\mu t + b_7 \\[2mm]
at\ t = 0;\ W(0) = W_0 \ln W_0 = b_7 \\[2mm]
\ln W(t) - \ln W_0 \geq -\mu t \\[2mm]
\ln \dfrac{W(t)}{W_0} \geq -\mu t \\[2mm]
\dfrac{W(t)}{W_0} \geq \exp -\mu t \\[2mm]
W(t) = W_0 \exp -\mu t
\end{array}\right\} \tag{17}$$

The eighth equation is considered in (18)

$$\frac{dD}{dt} = \rho_2 C - \mu D$$

$$\left.\begin{array}{r}
\dfrac{dD}{dt} \geq -\mu D \\[2mm]
\displaystyle\int \dfrac{dD}{D(t)} \geq \int -\mu dt \\[2mm]
\ln D(t) \geq -\mu + b_8 \\[2mm]
at\ t = 0;\ D(0) = D_0 \ln D_0 = b_8 \\[2mm]
\ln D(t) - \ln D_0 \geq -\mu t \\[2mm]
\ln \dfrac{D(t)}{D_0} \geq -\mu t \\[2mm]
\dfrac{D(t)}{D_0} \geq \exp -\mu t \\[2mm]
D(t) = D_0 \exp -\mu t
\end{array}\right\} \tag{18}$$

### 3.3 Existence of Waste-Free Equilibrium (WFE).

At the Waste-Free Equilibrium (WFE) point

$$\frac{dA}{dt} = \frac{dM}{dt} = \frac{dI}{dt} = \frac{dC}{dt} = \frac{dK}{dt} = \frac{dE}{dt} = \frac{dW}{dt} = \frac{dD}{dt} = 0$$

And Let

$$(A, M, I, C, K, E, W, D) = (A_0, M_0, I_0, C_0, K_0, E_0, W_0, D_0)$$

$$\left.\begin{array}{r}
\Lambda - \xi(M_0 + \omega I_0)A_0 - \mu A_0 = 0 \\[1mm]
\xi v(M_0 + \omega I_0)A_0 - (\alpha_1 + \mu)M_0 = 0 \\[1mm]
\xi(1 - v)(M_0 + \omega I_0)A_0 - (\alpha_2 + \mu)I_0 = 0 \\[1mm]
\alpha_1 M_0 + \alpha_2 I_0 - (\rho_1 + \rho_2 + \mu + \delta)C_0 = 0 \\[1mm]
\rho_1 C_0 - (\omega + \mu + \delta)K_0 = 0 \\[1mm]
\omega K_0 - (\beta + \mu)E_0 = 0 \\[1mm]
\beta E_0 - \mu W_0 = 0 \\[1mm]
\rho_2 C_0 - \mu D_0 = 0
\end{array}\right\} \tag{19}$$

Let

$$k_1 = (\alpha_1 + \mu), k_2 = (\alpha_2 + \mu), k_3 = (\rho_1 + \rho_2 + \mu + \delta), k_4 = (\omega + \mu + \delta), k_5 = (\beta + \mu)$$

$$\left.\begin{array}{l} \Lambda - \xi(M_0 + \omega I_0)A_0 - \mu A_0 = 0 \\ \xi v(M_0 + \omega I_0)A_0 - k_1 M_0 = 0 \\ \xi(1 - v)(M_0 + \omega I_0)A_0 - k_2 I_0 = 0 \\ \alpha_1 M_0 + \alpha_2 I_0 - k_3 C_0 = 0 \\ \rho_1 C_0 - k_4 K_0 = 0 \\ \omega K_0 - k_5 E_0 = 0 \\ \beta E_0 - \mu W_0 = 0 \\ \rho_2 C_0 - \mu D_0 = 0 \end{array}\right\} \tag{20}$$

multiply the second equation in(2) by $1 - v$ and the multiply the third equation in (2) by $v$

$$\xi v(1 - v)(M_0 + \omega I_0)A_0 - (1 - v)k_1 M_0 = 0$$

$$\xi v(1 - v)(M_0 + \omega I_0)A_0 - vk_2 I_0 = 0$$

subtracting gives

$$vk_2 I_0 - (1 - v)k_1 M_0 = 0$$

$$M_0 = \frac{vk_2 I_0}{(1 - v)k_1}$$

substituting for $M_0$ in the second equation of (2) gives

$$\xi v(\frac{vk_2 I_0}{(1 - v)k_1} + \omega I_0)A_0 - k_1 \frac{vk_2 I_0}{(1 - v)k_1} = 0$$

$$I_0 \frac{\xi v^2 A_0 k_2 + \xi v(1 - v)k_1 \omega - vk_1 k_2}{(1 - v)k_1} = 0$$

$$\implies I_0 = 0 \; or \; \xi v^2 A_0 k_2 + \xi v(1 - v)k_1 \omega - vk_1 k_2 = 0 \tag{21}$$

substituting for $I_0$ in first and second equation of (2) gives

$$M_0 = 0; \; A_0 = \frac{\Lambda}{\mu} \tag{22}$$

substituting (21) and (22) into equation (2) gives

$$C_0 = 0; \; D_0 = 0; \; E_0 = 0; \; K_0 = 0; \; W_0 = 0. \tag{23}$$

Therefore the Waste-Free Equilibrium (WFE)for the model equation (2) is given as

$$(A_0, M_0, I_0, C_0, K_0, E_0, W_0, D_0) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0) \tag{24}$$

### 3.4 Basic Reproduction Number $R_0$

The average number of secondary infections that an infected person causes over the course of their whole duration of infection is the basic reproduction number $R_0$. [4]

The reproduction number when applied to the model on waste management is considered as the parameter that governs the spread of waste in the system.

The value of $R_0$ determined whether the spread of waste dies out in the community or it spread so widely that it can be considered as an epidemic.

Whenever the value of the $R_0$ goes below unity $R_0 < 1$ it spread of the waste is insignificant and waste management becomes easier. when the reproduction number is above one $R_0 > 1$ it means that the spread of waste in the community becomes significant.

The basic reproduction is denoted by $R_0$ , that is $R_0 = \rho(FV^{-1})$ , where $\rho$ is the spectral radius (maximum eigenvalue) of $FV^{-1}$ [4,5,14] $f_i(x)$ is the rate at which novel diseases arise in compartment i, $v_i+$ is the rate at which people are moved into compartment i by methods other than epidemics, and $v_i-$ is the rate at which people are moved out of compartment i.

$$v_i = v_i + +v_i-$$

$FV^{-1}$ is called the "Next Generation Matrix" Equation(2) can be written as

$$\mathbf{f_i(x)} = \begin{pmatrix} vA_0\xi(M_0 + \omega I_0) \\ (1-v)A_0\xi(M_0 + \omega I_0) \\ 0 \end{pmatrix} \tag{25}$$

$$\mathbf{v_i(x)} = \begin{pmatrix} k_1 M_0 \\ k_2 I_0 \\ k_3 C_0 - \alpha_1 M_0 - \alpha_2 I_0 \end{pmatrix} \tag{26}$$

$$\mathbf{V(x)} = \begin{pmatrix} \dfrac{\partial v_1}{\partial M} & \dfrac{\partial v_1}{\partial I} & \dfrac{\partial v_1}{\partial C} \\ \dfrac{\partial v_2}{\partial M} & \dfrac{\partial v_2}{\partial I} & \dfrac{\partial v_2}{\partial C} \\ \dfrac{\partial v_3}{\partial M} & \dfrac{\partial v_3}{\partial I} & \dfrac{\partial v_3}{\partial C} \end{pmatrix} \tag{27}$$

$$\mathbf{F(x)} = \begin{pmatrix} vA_0\xi & vA_0\xi\omega & 0 \\ (1-v)A_0\xi & (1-v)A_0\xi\omega & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{28}$$

$$\mathbf{V(x)} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ -\alpha_1 & -\alpha_2 & k_3 \end{pmatrix} \tag{29}$$

$$\mathbf{V^{-1}} = \begin{pmatrix} \dfrac{k_2 k_3}{k_1 k_2 k_3} & 0 & 0 \\ 0 & \dfrac{k_1 k_3}{k_1 k_2 k_3} & 0 \\ \dfrac{-\alpha_1 k_2}{k_1 k_2 k_3} & \dfrac{-\alpha_2 k_1}{k_1 k_2 k_3} & \dfrac{k_1 k_2}{k_1 k_2 k_3} \end{pmatrix} \tag{30}$$

$$\mathbf{V^{-1}} = \begin{pmatrix} \dfrac{1}{k_1} & 0 & 0 \\ 0 & \dfrac{1}{k_2} & 0 \\ \dfrac{-\alpha_1}{k_1 k_3} & \dfrac{-\alpha_2}{k_2 k_3} & \dfrac{1}{k_3} \end{pmatrix} \tag{31}$$

$$\mathbf{FV^{-1}} = \begin{pmatrix} \dfrac{vA_0\xi}{k_1} & \dfrac{vA_0\xi\omega}{k_2} & 0 \\ \dfrac{(1-v)A_0\xi}{k_1} & \dfrac{(1-v)A_0\xi\omega}{k_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

$$|\mathbf{FV^{-1}} - \lambda| = \begin{vmatrix} \dfrac{vA_0\xi}{k_1} - \lambda & \dfrac{vA_0\xi\omega}{k_2} & 0 \\ \dfrac{(1-v)A_0\xi}{k_1} & \dfrac{(1-v)A_0\xi\omega}{k_2} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \tag{33}$$

$$\left(\frac{vA_0\xi}{k_1} - \lambda\right) - \lambda\left(\frac{(1-v)A_0\xi\omega}{k_2} - \lambda\right) - \frac{-vA_0\xi\omega}{k_2}\left(\frac{-\lambda(1-v)A_0\xi}{k_1}\right) + 0 = 0 \tag{34}$$

$$\frac{v(1-v)\lambda A_0^2\xi^2\omega}{k_1k_2} + \frac{vA_0\xi\lambda^2}{k_1} + \frac{(1-v)\lambda^2 A_0\xi\omega}{k_2} - \lambda^3 + \frac{v(1-v)\lambda A_0^2\xi^2\omega}{k_1k_2} = 0 \tag{35}$$

$$\frac{vA_0\xi\lambda^2}{k_1} + \frac{(1-v)\lambda^2 A_0\xi\omega}{k_2} - \lambda^3 = 0 \tag{36}$$

$$\frac{\lambda^2(vA_0\xi k_2 + (1-v)A_0\xi\omega k_1)}{k_1k_2} - \lambda^3 = 0 \tag{37}$$

$$\lambda^2\left(\frac{(vA_0\xi k_2 + (1-v)A_0\xi\omega k_1)}{k_1k_2} - \lambda\right) = 0 \tag{38}$$

$$\lambda_1 = 0, \lambda_2 = 0 \quad and \quad \lambda_3 = \frac{(vA_0\xi k_2 + (1-v)A_0\xi\omega k_1)}{k_1k_2} \tag{39}$$

Substituting $A_0 = \frac{\Lambda}{\mu}, k_1 = \alpha_1 + \mu, k_2 = \alpha_2 + \mu$

$$R_0 = \frac{\Lambda\xi(\mu\omega\, v + \omega v\alpha_1 - \mu\omega - v\mu - \omega\alpha_1 - v\alpha_2)}{\mu(\alpha_2 + \mu)(\alpha_1 + \mu)} \tag{40}$$
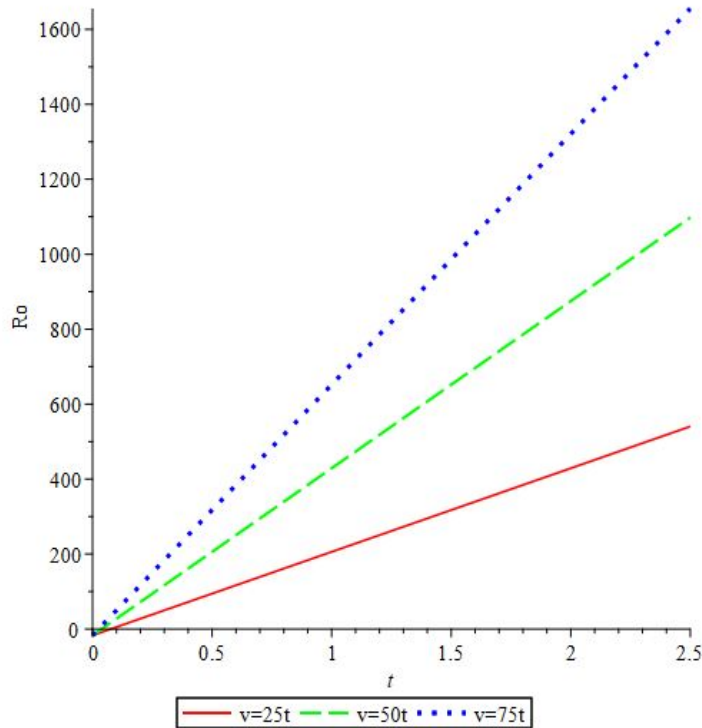
Figure 2: **graph of $R_0$ against $v$**

## 3.5 Local Stability Analysis of Waste Free Equilibrium Point(WFE).

The Jacobian matrix of the system of equations (2) can be used to analyze the local Stability Analysis of the Waste Free Equilibrium Point. The signs of the Jacobian matrix's eigenvalues are then used to calculate the stability.

**Theorem**: When all of the eigenvalues of the Jacobian matrix at the Waste Free Equilibrium Point are negative, the WFE point is considered to be locally asymptotically stable; otherwise, it is unstable.

**Proof**:The Jacobian matrix of equation (24) is given as

$$
\mathbf{J} = \begin{pmatrix}
\frac{\partial f_1}{\partial A} & \frac{\partial f_1}{\partial M} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial C} & \frac{\partial f_1}{\partial K} & \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial W} & \frac{\partial f_1}{\partial D} \\
\frac{\partial f_2}{\partial A} & \frac{\partial f_1}{\partial M} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial C} & \frac{\partial f_2}{\partial K} & \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial W} & \frac{\partial f_2}{\partial D} \\
\frac{\partial f_3}{\partial A} & \frac{\partial f_3}{\partial M} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial C} & \frac{\partial f_3}{\partial K} & \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial W} & \frac{\partial f_3}{\partial D} \\
\frac{\partial f_4}{\partial A} & \frac{\partial f_4}{\partial M} & \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial C} & \frac{\partial f_4}{\partial K} & \frac{\partial f_4}{\partial E} & \frac{\partial f_4}{\partial W} & \frac{\partial f_4}{\partial D} \\
\frac{\partial f_5}{\partial A} & \frac{\partial f_5}{\partial M} & \frac{\partial f_5}{\partial I} & \frac{\partial f_5}{\partial C} & \frac{\partial f_5}{\partial K} & \frac{\partial f_5}{\partial E} & \frac{\partial f_5}{\partial W} & \frac{\partial f_5}{\partial D} \\
\frac{\partial f_6}{\partial A} & \frac{\partial f_6}{\partial M} & \frac{\partial f_6}{\partial I} & \frac{\partial f_6}{\partial C} & \frac{\partial f_6}{\partial K} & \frac{\partial f_6}{\partial E} & \frac{\partial f_6}{\partial W} & \frac{\partial f_6}{\partial D} \\
\frac{\partial f_7}{\partial A} & \frac{\partial f_7}{\partial M} & \frac{\partial f_7}{\partial I} & \frac{\partial f_7}{\partial C} & \frac{\partial f_7}{\partial K} & \frac{\partial f_7}{\partial E} & \frac{\partial f_7}{\partial W} & \frac{\partial f_7}{\partial D} \\
\frac{\partial f_8}{\partial A} & \frac{\partial f_8}{\partial M} & \frac{\partial f_8}{\partial I} & \frac{\partial f_8}{\partial C} & \frac{\partial f_8}{\partial K} & \frac{\partial f_8}{\partial E} & \frac{\partial f_8}{\partial W} & \frac{\partial f_8}{\partial D}
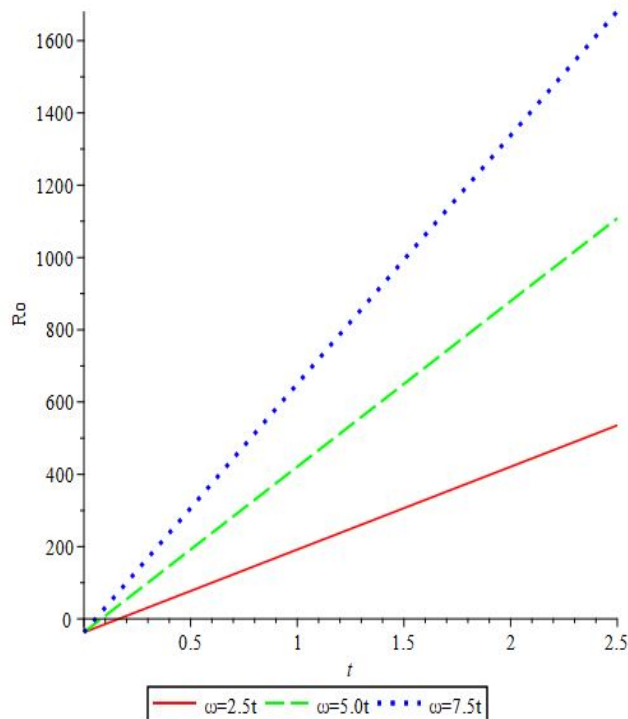\end{pmatrix}
\tag{41}
$$

Figure 3: **graph of** $R_0$ **against** $\omega$

$$\mathbf{J(E_0)} = \begin{pmatrix} -\xi(M_0 + \omega I_0) - \mu & \xi A_0 & \xi\omega A_0 & 0 & 0 & 0 & 0 & 0 \\ v\xi(M_0 + \omega I_0) & (v)\xi A_0 - k_1 & (v\xi\omega A_0) & 0 & 0 & 0 & 0 & 0 \\ (1-v)\xi(M_0 + \omega I_0) & ((1-v)\xi A_0) & ((1-v)\xi\omega A_0) - k_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & -k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_1 & -k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & -k_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & -\mu & 0 \\ 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & -\mu \end{pmatrix}$$

(42)

At WFE, $A_0 = \frac{\Lambda}{\mu}$; $M_0 = 0$; $I_0 = 0$

$$\mathbf{J(E_0)} = \begin{pmatrix} -\mu & \dfrac{\xi\Lambda}{\mu} & \dfrac{\xi\omega\Lambda}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{v\xi\Lambda - \mu k_1}{\mu} & \dfrac{v\xi\omega\Lambda}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{(1-v)\xi\Lambda}{\mu} & \dfrac{(1-v)\xi\omega\Lambda - \mu k_2}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\alpha_2\mu(v\xi\Lambda - \mu k_1) - \alpha_1\xi\omega v}{\mu(v\xi\Lambda - \mu k_1)} & -k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \end{pmatrix}$$

(43)

$$\mathbf{J(E_0)} = \begin{pmatrix} -\mu & \frac{\xi\Lambda}{\mu} & \frac{\xi\omega\Lambda}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v\xi\Lambda - \mu k_1}{\mu} & \frac{v\xi\omega\Lambda}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(v\xi\omega\Lambda k_1 - \xi\omega\Lambda k_1 - \xi v\Lambda k_2 + \mu k_2 k_1)}{v\xi\Lambda - \mu k_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \end{pmatrix} \tag{44}$$

$$= \begin{vmatrix} -\mu - \lambda & \frac{\xi\Lambda}{\mu} & \frac{\xi\omega\Lambda}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & (\frac{v\xi\Lambda - \mu k_1}{\mu}) - \lambda & \frac{v\xi\omega\Lambda}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\frac{v\xi\omega\Lambda k_1 - \xi\omega\Lambda k_1 - \xi v\Lambda k_2 + \mu k_2 k_1}{v\xi\Lambda - \mu k_1}) - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_3 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_4 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_5 - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu - \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu - \lambda \end{vmatrix} \tag{45}$$

$$|J(E_0) - \lambda| = ((-\mu - \lambda_1)(-\mu - \lambda_2)(-\mu - \lambda_3)(k_3 - \lambda_4)(k_4 - \lambda_5)(k_5 - \lambda_6)$$
$$(\frac{v\xi\omega\Lambda k_1 - \xi\omega\Lambda k_1 - \xi v\Lambda k_2 + \mu k_2 k_1}{v\xi\Lambda - \mu k_1}) - \lambda_7)((\frac{v\xi\Lambda - \mu k_1}{\mu}) - \lambda_8) \tag{46}$$

Let $k_6 = (\frac{v\xi\omega\Lambda k_1 - \xi\omega\Lambda k_1 - \xi v\Lambda k_2 + \mu k_2 k_1}{v\xi\Lambda - \mu k_1})$ ; and $k_7 = (\frac{v\xi\Lambda - \mu k_1}{\mu})$.

$$\lambda_1 = -\mu; \lambda_2 = -\mu; \lambda_3 = -\mu; \lambda_4 = k_3; \lambda_5 = k_4; \lambda_6 = k_5; \lambda_7 = k_6; \lambda_8 = k_7 \tag{47}$$

$R_0$ is asymptotically stable if $\lambda_7 < 0$ this implies that

$$k_7 < 0 \implies (v\xi\omega\Lambda k_1 - \xi\omega\Lambda k_1 - \xi v\Lambda k_2) < -\mu k_2 k_1 \tag{48}$$

multiply both sides by $\mu k_1 k_2$

$$\frac{v\xi\omega\Lambda k_1 - \xi\omega\Lambda k_1 - \xi v\Lambda k_2}{\mu k_2 k_1} < -1 \tag{49}$$

$\implies R_0 < 1$

Hence, $R_0$ is globally asymptotically stable

## 4  Result and Discussion

In this study, we discussed a mathematical model for managing solid waste with a focus on producing money and energy. We showed the presence and uniqueness of the solutions to establish the existence of the one and only solution to the model. To produce enough energy and riches, it is vital to manage the rate of waste production in both households and industries. Figures 1 and 2 are provided. The fundamental reproduction number $R_0$ was calculated using the next generation method. When $R_0 < 1$, we demonstrated that the Waste-Free equilibrium is globally asymptotically stable.

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