A UNIFORM ORDER CONTINUOUS HYBRID INTEGRATION FOR SOLVING THIRD ORDER ORDINARY DIFFRENTIAL EQUATIONS

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Abstract

A three -step Continuous Block Hybrid Method (CBHM) with two non-step points of order (6,6,6,6,6) is proposed for direct solution of the special and general third order initial value problems (IVPs). The main method and additional methods are obtained from the same continuous schemes derived via interpolation and collocation procedures. The stability properties of the methods are discussed and the stability region shown. The methods are applied in block form as simultaneous numerical integrators over non-overlaping interval. The efficiency of the proposed method was tested and was found to compete favorable with the existing methods.

Keywords: Collocation, Interpolation, Power series approximant, Grid points, Off-grid points and Block methods

1. Introduction

This paper considers the solution of special and general third order initial value problem of the form:

$$y''' = f(x,y),\ \ y(a) = y_0\ ,\ y'(a) = \eta_0\ ,\ y''(a) = \eta_1 \qquad (1)$$

$$y''' = f(x,y,y',y''),\ \ y(a) = y_0\ ,\ y'(a) = \eta_0\ ,\ y''(a) = \eta_1 \qquad (2)$$

However, only a limited number of numerical methods are available for solving (1) and (2) directly without reducing to a first order system of initial value problems. Some authors have proposed solution to higher order initial value problems of ordinary differential equations using different approaches (Awoyemi (1991),Awoyemi (2000),Kayode (2005) and Adekunle et al. (2013)). These methods mentioned which were implemented in Predictor-Corrector mode, like linear multistep methods and other standard method are usually applied to initial value problem as a single formula but the setbacks of this method are: (1) They are not self-starting (2) they advance the numerical integration of ordinary differential equation in one step at a time which lead to overlapping of piecewise polynomial solution mode. See (Mohammed and Adeniyi (2014) and Mohammed and Adeniyi (2015)) for details. The advantages of continuous method are widely reported by Awoyemi (2003).

In order to correct the setback of the method of Predictor-Corrector method, Fatunla(1991), Olabode (2009), Olabode and Yusuph (2009) and Yahaya and Mohammed (2010) proposed block methods for the solution of higher order differential equations with limitation to special type of tODEs

In view of the above mention, we extended the work of Olabode and Yusuph (2009) into a modified linear multi-step method by considering one-three off step point at collocation to handle both special and general third ordinary differential equations. The three step block hybrid method proposed is zero stable, consistent and more accurate than the existing one. Experimental results confirm the superiority of the new schemes over the existing method.

2. Derivation of the method

In this section the objective is to derive Hybrid Linear Multi-step Method (HLMM) of the form

$$\sum_{j=0}^{r-1}\alpha_j y_{n+j} = h^3\sum_{j=0}^{s-1}\beta_j f_{n+j} + h^3\beta_\mu f_{n+\mu} + h^3\beta_v f_{n+v} \quad (2)$$

Where α_j , β_j and β_v are unknown constants and v_j is not an integer. We note that α_k =1, $\beta_j \neq 0$, α_0 and β_0 do not both vanish. In order to obtain (2), we proceed by seeking to approximate the exact solution Y(x) of the form

$$Y(x) = \sum_{j=0}^{r+s-1} a_j x^j \, , \quad (3)$$

Where $x \in [a,b]$, a_j are unknown coefficients to be determined and $1 \leq r < k$, $S > 0$ are the number of interpolation and collocation points respectively. We construct the continuous approximation by imposing the following conditions.

$$Y(x_{n+j}) = y_{n+j} \, , \quad j = 0,1.2,\ldots\ldots, r-1 \quad (4)$$

$$Y'''(x_{n+j}) = f_{n+j} \quad (5)$$

Equation (4) and (5) lead to a system of (r+s) equations which is solved by Cramer's rule to obtain a_j. The continuous approximation is constructed by substituting the values of a_j into equation (3). After simplification, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{r-1}\alpha_j(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j(x) f_{n+j} + h^3\beta_\mu(x) f_{n+\mu} + h^3\beta_v(x) f_{n+v} \quad (6)$$

where $\alpha_j(x)$, $\beta_j(x)$, $\beta_\mu(x)$ and $\beta_v(x)$ are continuous coefficients. We note that since

equation (1) involves first and second derivatives, the first and second derivative formula

$$Y'(x)=\frac{1}{h}\left(\sum_{j=0}^{r-1}\alpha'_j(x)y_{n+j}+h^3\sum_{j=0}^{s-1}\beta'_j(x)f_{n+j}+h^3\beta'_\mu(x)f_{n+\mu}+h^3\beta'_v(x)f_{n+v}\right)$$

$$Y''(x)=\frac{1}{h^2}\left(\sum_{j=0}^{r-1}\alpha''_j(x)y_{n+j}+h^3\sum_{j=0}^{s-1}\beta''_j(x)f_{n+j}+h^3\beta'_\mu(x)f_{n+\mu}+h^3\beta''_v(x)f_{n+v}\right) \quad (7)$$

Equation (7) is easily obtained from (6) and is used to provide the first and second derivatives for the methods by imposing the condition

$$Y'(x)=\delta(x),\ Y''(x)=\gamma(x) \quad (8)$$

$$Y'(a)=\delta_0,\ Y''(a)=\gamma_0 \quad (9)$$

3. Three Step Block Hybrid Method with two off Step Collocation Point

To derive this methods, we use Eq.(6) to obtained a continuous 3-step HLM method with the following specification : r=3,s=6,k=3, $v=\frac{8}{3},\mu=\frac{5}{2},\gamma_i(x)=x^i, i=0,1,\ldots,8$.

We also express $\alpha_j(x),\beta_j(x)$ and $\beta_v(x)$ as a functions of t, where $t=\frac{x-x_n}{h}$ to obtain the continuous form as follows:

$$\alpha_0=\left(1-\frac{3}{2}t+\frac{1}{2}t^2\right),\ \alpha_1=\left(2t-t^2\right),\ \alpha_2=\left(-\frac{1}{2}t+\frac{1}{2}t^2\right)$$

$$\beta_0(x)=\frac{1}{201600}\left(10078t-28615t^2+33600t^3-21910t^4+8638t^5-2044t^6+268t^7-15t^8\right)$$

$$\beta_1(x)=\frac{1}{25200}\left(10016t-14400t^2+8400t^4-5404t^5+1617t^6-244t^7+15t^8\right)$$

$$\beta_2(x)=\frac{1}{3360}\left(-2082t+3809t^2-4200t^4+3542t^5-1274t^6+220t^7-15t^8\right)$$

$$\beta_{\frac{5}{2}}(x)=\frac{1}{1575}\left(3244t-5910t^2+6720t^4-5936t^5+2268t^6-416t^7+30t^8\right)$$

$$\beta_{\frac{8}{3}}(x)=\frac{1}{22400}\left(-40662t+74115t^2-85050t^4+75978t^5-29484t^6+5508t^7-40t^8\right)$$

$$\beta_3(x)=\frac{1}{5040}\left(1316t-2402t^2+2800t^4-2548t^5+1015t^6-196t^7+15t^8\right) \quad (10)$$

The MFDMs are obtained by evaluating (10) at $x=\left\{x_{n+3}, x_{n+\frac{8}{3}}, x_{n+\frac{5}{2}}\right\}$ to obtain the following

$$y_{n+3}=y_n-3y_{n+1}+3y_{n+2}+\frac{h^3}{1200}\left[11f_n+536f_{n+1}+900f_{n+2}-896f_{n+\frac{5}{2}}+729f_{n+\frac{8}{3}}-80f_{n+3}\right] \quad (11)$$

$$y_{n+\frac{8}{3}}=\frac{5}{9}y_n-\frac{16}{9}y_{n+1}+\frac{20}{9}y_{n+2}+\frac{h^3}{52488}\left[271f_n+12928f_{n+1}+19716f_{n+2}-21760f_{n+\frac{5}{2}}+16605f_{n+\frac{8}{3}}-1840f_{n+3}\right] \quad (12)$$

$$y_{n+\frac{5}{2}}=\frac{3}{8}y_n-\frac{5}{4}y_{n+1}+\frac{15}{8}y_{n+2}+\frac{h^3}{491520}\left[1729f_n+81304f_{n+1}+115740f_{n+2}-135424f_{n+\frac{5}{2}}+101331f_{n+\frac{8}{3}}-11080f_{n+3}\right] \quad (13)$$

In particular, to start the initial value problem for n = 0, we obtain the following equations from (9):

$$h\delta_0=-\frac{3}{2}y_0+2y_1-\frac{1}{2}y_2+h^3\left[\frac{5039}{100800}f_0+\frac{626}{1575}f_1-\frac{347}{560}f_2+\frac{3244}{1575}f_{\frac{5}{2}}-\frac{20331}{11200}f_{\frac{8}{3}}+\frac{47}{180}f_3\right] \quad (14)$$

$$h^2\gamma_0=y_0-2y_1+y_2+h^3\left[-\frac{5723}{20160}f_0-\frac{8}{7}f_1+\frac{3809}{1680}f_2-\frac{788}{105}f_{\frac{5}{2}}+\frac{14823}{2240}f_{\frac{8}{3}}-\frac{1201}{1260}f_3\right] \quad (15)$$

The derivatives are derived by

$\delta(x_{n+\tau})=\delta_{n+\tau}$ *and* $\gamma(x_{n+\tau})=\gamma_{n+\tau}, \tau=1,2,\frac{5}{2},\frac{8}{3}$ *and* 3 as follows:

$$h\delta_{n+1}=-\frac{1}{2}y_n+\frac{1}{2}y_{n+2}-\frac{h^3}{20160}\left[131f_n+3272f_{n+1}-1332f_{n+2}+5632f_{n+\frac{5}{2}}-5103f_{n+\frac{8}{3}}+760f_{n+3}\right]$$

$$h\delta_{n+2}=\frac{1}{2}y_n-2y_{n+1}+\frac{3}{2}y_{n+2}+\frac{h^3}{100800}\left[49f_n+21632f_{n+1}+21130f_{n+2}-27392f_{n+\frac{5}{2}}+19683f_{n+\frac{8}{3}}-200f_{n+3}\right]$$

$$h\delta_{n+\frac{5}{2}}=y_n-3y_{n+1}+2y_{n+2}+\frac{h^3}{806400}\left[7387f_n+360262f_{n+1}+602370f_{n+2}-628672f_{n+\frac{5}{2}}+486243f_{n+\frac{8}{3}}-54790f_{n+3}\right]$$

$$h\delta_{n+\frac{8}{3}} = \frac{7}{6}y_n - \frac{10}{3}y_{n+1} + \frac{13}{6}y_{n+2} + h^3\left[\frac{31111}{2939328}f_n + \frac{120314}{229635}f_{n+1} + \frac{1144879}{1224720}f_{n+2} - \frac{29084}{32805}f_{n+\frac{5}{2}} + \frac{43651}{60480}f_{n+\frac{8}{3}} - \frac{15115}{183708}f_{n+3}\right]$$

$$h\delta_{n+3} = \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} + \frac{h^3}{20160}\left[271f_n + 13672f_{n+1} + 26460f_{n+2} - 22528f_{n+\frac{5}{2}} + 21141f_{n+\frac{8}{3}} - 2056f_{n+3}\right]$$

$$h^2\gamma_{n+1} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{100800}\left[1653f_n + 12088f_{n+1} - 52860f_{n+2} + 150272f_{n+\frac{5}{2}} - 129033f_{n+\frac{8}{3}} + 17880f_{n+3}\right]$$

$$h^2\gamma_{n+2} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{100800}\left[841f_n + 46976f_{n+1} + 94140f_{n+2} - 125696f_{n+\frac{5}{2}} + 95499f_{n+\frac{8}{3}} - 10960f_{n+3}\right]$$

$$h^2\gamma_{n+\frac{5}{2}} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{322560}\left[2757f_n + 149360f_{n+1} + 363828f_{n+2} - 225536f_{n+\frac{5}{2}} + 220887f_{n+\frac{8}{3}} - 27456f_{n+3}\right]$$

$$h^2\gamma_{n+\frac{8}{3}} = y_n - 2y_{n+1} + y_{n+2} + h^3\left[\frac{209179}{24494400}f_n + \frac{177256}{382725}f_{n+1} + \frac{459667}{408240}f_{n+2} - \frac{234436}{382725}f_{n+\frac{5}{2}} + \frac{77467}{100800}f_{n+\frac{8}{3}} - \frac{26627}{306108}f_{n+3}\right]$$

$$h^2\gamma_{n+3} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{100800}\left[869f_n + 46584f_{n+1} + 115140f_{n+2} - 79104f_{n+\frac{5}{2}} + 115911f_{n+\frac{8}{3}} + 2200f_{n+3}\right]$$

This proposed method is consistent since its order is 6, its also zero-stable; above all, it has moderate interval of absolute stability as can be seen in figure 1. The proposed three step method (11)-(15) have order 6 and error constants given by the vector $C_9 = \left(\frac{1}{201600}, -\frac{5167}{1785641760}, -\frac{571}{123863-40}, \frac{1141}{12}, \frac{283}{4}\right)^T$

4. Convergence

The convergence of the proposed block hybrid methods is determined using the approach of Fatunla (1991) for hybrid linear multistep method, where the block hybrid method are represented in a single block, r point multistep method of the form

$$A^{(0)}Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h\sum_{i=0}^{k} B^{(i)}F_{m-i}, \tag{16}$$

Where *h* is a fixed mesh size within a block, A^i,B^i, i=O(1)k are r by r matrix coefficients and

A^0 is r by r identity matrix.

Y_m, Y_{m-1} , F_m and F_{m-1} are vectors of numerical estimates

Definition 1. The block method is zero stable provided the root $R_{i,j} = 1(1)k$ of the first characteristic polynomial $p(R)$ specified as

$$\rho(R) = \det\left|\sum_{i=0}^{k} A^{(i)} R^{k-i}\right| = 0 \qquad (17)$$

Satisfies $|R_j| \leq 1$ and for those roots with $|R_j| \leq 1$, the multiplicity must not exceed 2.

We can put the five integrator represented by equations (11) - (15) into the matrix-equation form and for easy analysis the result was normalized to obtain;

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{8}{3}} \\ f_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$+h^3 \begin{pmatrix} \frac{274}{1575} & -\frac{1727}{3360} & -\frac{2666}{1575} & -\frac{4779}{3200} & \frac{181}{840} \\ \frac{2348}{1575} & -\frac{346}{105} & \frac{17152}{1575} & -\frac{6723}{700} & \frac{436}{315} \\ \frac{101125}{36864} & -\frac{911875}{172032} & \frac{145375}{8064} & -\frac{3655125}{229376} & \frac{198125}{86016} \\ \frac{3731456}{1148175} & -\frac{461824}{76545} & \frac{3407872}{164025} & -\frac{260416}{14175} & \frac{610304}{229635} \\ \frac{1539}{350} & -\frac{243}{32} & \frac{4698}{175} & -\frac{531441}{22400} & \frac{963}{280} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{8}{3}} \\ f_{n+3} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{6179}{67200} \\ 0 & 0 & 0 & 0 & \frac{421}{900} \\ 0 & 0 & 0 & 0 & \frac{175625}{229376} \\ 0 & 0 & 0 & 0 & \frac{1011776}{1148175} \\ 0 & 0 & 0 & 0 & \frac{25461}{22400} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

(18)

The first characteristic polynomial of the block hybrid method (18) is given by

$$\rho(R) = \det\left(RA^0 - A^1\right)$$

Substituting the value of A^0 and A^1 into the function above gives

$$\rho(R) = \det\left[R\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{pmatrix}$$

$$= \left[R^4(R-1)\right] \qquad (19)$$

Therefore, R=0, R=1. The block method is zero stable and consistent since the order of the method p=6>1, and by Henrici (1962), the block method is convergent.

5. Region of Absolute Stability

The absolute stability region of the newly constructed hybrid linear multi-step methods (11)-(15) is plotted using Chollom (2004) by reformulating the methods as general linear methods and is shown in Figure 1 below.

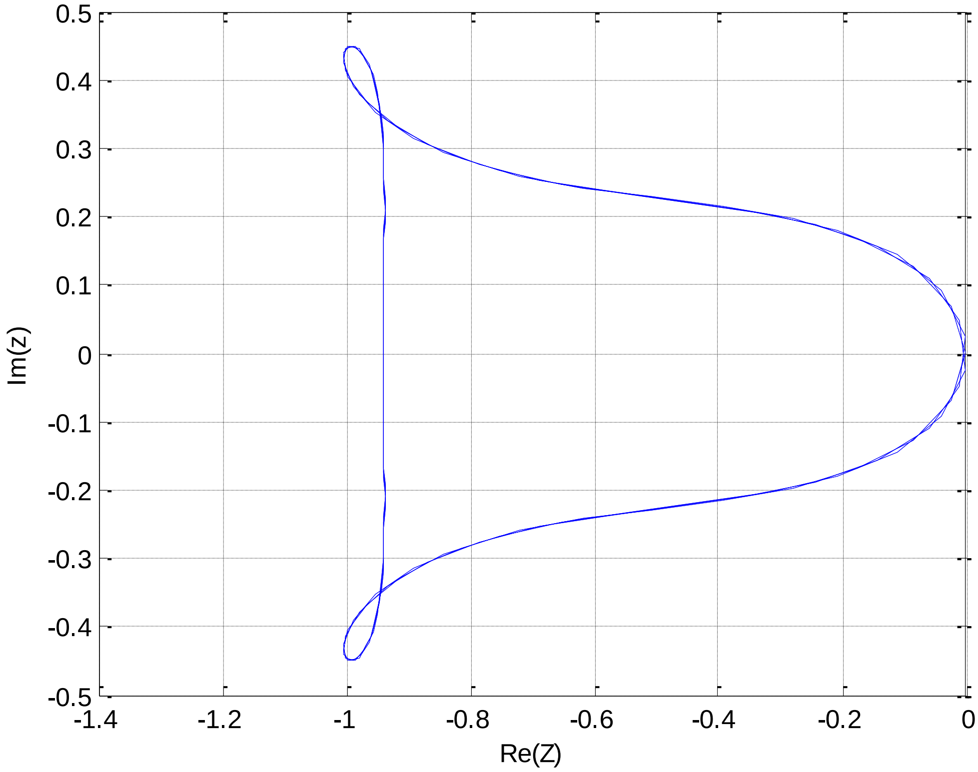

Fig. 1: Region of Absolute Stability Region of Hybrid Linear Multi-Step Method (HLMM)

6. Numerical Examples

We report here seven numerical examples taken from literature for the bases of comparison (see Tables 1-7).

Problem 1

$$y''' + e^{-y} - 3e^{-2y} + 2e^{-3y}$$

$$y(0) = \ln 2, \quad y'(0) = \tfrac{1}{2}, \quad y''(0) = \frac{1}{4}$$

Exact Solution is $y(x) = \ln\left(e^{x} + 1\right)$

Table 1: Showing Exact solutions and the computed results from the proposed methods for Problem 1

X	Exact Solution	Numerical Solution	Error	Error in R-K Method
0.1	0.7443966600	0.7443966601	1E-10	5.755E-09
0.2	0.7981388693	0.7981388694	1E-10	1.150E-07
0.3	0.8543552446	0.8543552445	1E-10	1.205E-07
0.4	0.9130152525	0.9130152524	1E-10	4.059E-08
0.5	0.9740769843	0.9740769842	1E-10	1.127E-07
0.6	1.037487950	1.037487951	1E-09	1.092E-07
0.7	1.103186049	1.103186049	0	7.339E-08
0.8	1.171100666	1.171100666	0	1.230E-07
0.9	1.241153875	1.241153875	0	9.167E-08
1.0	1.313261687	1.313261687	0	1.149E-07

Problem 2 $y''' = 3\sin x$ (Olabode and Yusuph (2009))

$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, h = 0.1$

Exact Solution is $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 2: Showing Exact solutions and the computed results from the proposed methods for Problem 2

X	Exact Solution	Numerical Solution	Error	Olabode and Yusuph (2009)
0.1	0.990012496	0.990012496	0	1.65922E-10
0.2	0.960199733	0.9601997335	5E-10	4.76275E-10
0.3	0.911009467	0.9110094673	3E-10	6.23182E-10
0.4	0.843182982	0.8431829819	1E-10	2.91345E-10
0.5	0.757747686	0.7577476855	5E-10	8.71118E-10
0.6	0.656006845	0.6560068445	5E-10	3.92904E-09
0.7	0.539526562	0.5395265615	5E-10	9.55347E-09
0.8	0.410120128	0.4101201276	4E-10	1.80415E-08
0.9	0.269829905	0.2698299042	8E-10	3.03120E-08
1.0	0.120906918	**0.1209069177**	3E-10	4.73044E-08

Problem 3 (Awoyemi et al (2014))

$$y''' + 4y' = x$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, h = 0.1$$

Exact Solution is $y(x) = -\frac{3}{16}\cos 2x + \frac{5}{16}$

Table 3: Showing Exact solutions and the computed results from the proposed methods for Problem 3

X	Exact Solution	Numerical Solution	Error	Error in Awoyemi et al (2014)
0.1	0.004987516700	0.004987516680	2E-11	1.1899E-11
0.2	0.01980106360	0.01980106378	1.8E-10	3.0422E-09
0.3	0.04399957220	0.04399957259	3.9E-10	7.7796E-08
0.4	0.07686749200	0.07686749244	4.4E-10	1.5559E-07
0.5	0.1174433176	0.1174433185	9E-10	3.0541E-07
0.6	0.1645579210	0.1645579226	1.6E-09	4.6102E-07
0.7	0.2168811607	0.2168811622	1.5E-09	3.138E-07
0.8	0.2729749104	0.2729749122	1.8E-09	7.0374E-07
0.9	0.3313503928	0.3313503951	2.3E-09	1.0177E-06
1.0	0.3905275319	0.3905275341	2.2E-09	1.6528E-06

Problem 4 (Sagir (2014))

$$y''' + 5y'' + 7y' + 3y = 0$$

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, h = 0.1$$

Exact Solution is $y(x) = e^{-x} + xe^{-x}$

Table 4: Showing Exact solutions and the computed results from the proposed methods for Problem 4

X	Exact Solution	Numerical Solution	Error	Error in Sagir (2014)
0.1	0.9953211598	0.9953211602	4E-10	6.4300E-08
0.2	0.9824769037	0.9824769045	8E-10	2.7200E-08
0.3	0.9630636869	0.9630636870	1E-10	3.0500E-08
0.4	0.9384480644	0.9384480648	4E-10	8.9800E-08
0.5	0.9097959895	0.9097959906	1.1E-09	4.4260E-07
0.6	0.8780986178	0.8780986179	1E-10	7.7260E-07
0.7	0.8441950165	0.8441950160	5E-10	1.9523E-06
0.8	0.8087921354	0.8087921343	1.1E-09	1.0274E-06
0.9	0.7724823534	0.7724823510	2.4E-09	1.3509E-06
1.0	0.7357588824	0. 7357588850	2.6E-09	1.3470E-05

Problem 5 Consider the linear singular IVP

$$y''' + \frac{\cos x}{\sin x} y'' = \sin x \cos x$$

$$y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0, h = 0.1$$

Exact Solution is $y(x) = 1 - 2x + \frac{x^2}{12} - \frac{\sin^2 x}{12}$

Table 5: Showing Exact solutions and the computed results from the proposed methods for Problem 5

X	Exact Solution	Numerical Solution	Error
0.1	0.8000027740	0.8000027741	1E-10
0.2	0.6000442080	0.6000442081	1E-10
0.3	0.4002223173	0.4002223172	1E-10
0.4	0.2006961129	0.2006961128	1E-10
0.5	0.00167926274	0.001679262669	7.1E-11
0.6	-0.1965684269	-0.1965684270	1E-10
0.7	-0.3937513691	-0.3937513692	1E-10
0.8	-0.5895499801		3E-10

		-0.5895499804	
0.9	-0.7836334206	-0.7836334211	5E-10
1.0	-0.9756727849	-0.9756727850	1E-10

Problem 6 Non-linear Blasius equations

$$2y''' + yy'' = 0$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, h = 0.1$$

The exact solution does not exist.

We compare our method with fourth order Runge-Kutta Method which shows an agreement with each other.

Table 6:Numerical methods for problem 6

x	Numerical Solution	R-K METHOD
0.1	0.0049999583	0.0049999552
0.2	0.0199986668	0.0199986591
0.3	0.0449898795	0.0449898741
0.4	0.0799573780	0.0799573773
0.4	0.1248700575	0.1248700476
0.6	0.1796771413	0.1796771264
0.7	0.2443036171	0.2443036129
0.8	0.3186460094	0.3186459795
0.9	0.4025686206	0.4025686062
1.0	0.4959003415	0.4959003376

Problem 7

Consider linear system

$$y''' = \frac{1}{68}(817y + 1393z + 448w)$$

$$z''' = -\frac{1}{68}(1141y + 2837z + 896w)$$

$$w''' = \frac{1}{68}(3059y + 4319z + 1592w)$$

With initial conditions

$$\begin{cases} y(0) = -2,\ z(0) = -2, w(0) = -12 \\ y'(0) = -12,\ z'(0) = 28, w'(0) = -33 \\ y''(0) = 20,\ z''(0) = -52,\ w''(0) = 5 \end{cases}$$

The analytical solution of the problem is given by

$$\begin{cases} y = e^{x} - 2e^{2x} + 3e^{-3x} \\ z = 3e^{x} + 2e^{2x} - 7e^{-3x} \\ w = -11e^{x} - 5e^{2x} + 4e^{-3x} \end{cases}$$

Table 13: Example 7 for k=3 with Two off-grid point at Collocation

X	Exact Solution			Numerical Solution			Error		
	y(x)	Z(x)	W(x)	Y(x)	Z(x)	W(x)	Y(x)	Z(x)	W(x)
0.1	0.884820064	0.572590725	-15.30062101	0.8848200328	0.5725907975	-15.30062105	3.12E-08	7.25E-08	4E-08
0.2	-0.115811730	2.806176217	-18.69930729	-0.115811924	2.806176663	18.69930753	1.93E-07	4.46E-07	2.4E-07
0.3	-1.074669813	4.847826406	-22.33276225	-1.074670301	4.847827514	-22.33276288	4.88E-07	1.11E-06	6.3E-07
0.4	-2.055674522	6.818196467	-26.33299947	-2.055675432	6.818198548	26.33300066	9.09E-07	2.08E-06	1.19E-06
0.5	-3.118451905	8.820816349	-30.83482248	-3.118453394	8.820819761	-30.83482447	1.48E-06	3.41E-06	1.99E-06
0.6	-4.322218381	10.94949803	-35.98269587	-4.322220576	10.94950306	-35.98269878	2.19E-06	5.03E-06	2.91E-06
0.7	-5.729277942	13.29446306	-41.93745391	-5.729280922	13.29446992	-41.93745794	2.98E-06	6.86E-06	4.03E-06
0.8	-7.408370060	15.94766196	-48.88324052	-7.408373873	15.94767075	-48.88324578	3.81E-06	8.79E-06	5.26E-06

0.9	-9.438075281	19.00766567	-57.03504949	-9.438079890	19.00767630	-57.03505590	4.61E-06	1.06E-05	6.41E-06
1.0	-11.91046916	22.58444820	-66.64723234	-11.9104655489	22.58446732	-66.647239321	3.61E-06	1.91E-05	6.98E-06

5. Conclusion

We have derived a three-step continuous Hybrid Linear Multi-step Method (HLMM) from which Multiple Finite Difference Methods (MFDMs) are obtained and applied to solve third order ordinary differential equations (ODE) without first adapting the ODE to an equivalent first order system. The MFDMs are applied as simultaneous numerical integrators over sub-intervals which do not overlap and hence they are more accurate than Single Finite Difference Methods (SFDMs) which are generally applied as single formulas over overlapping intervals. We have shown that the methods are convergent and have large intervals of absolute stability, which make them suitable candidates for computing solutions on wider intervals. In addition to providing additional methods and derivatives, the continuous HLMM can be used to obtain global error estimates. Our future research will be focused on adapting the MFDMs to solve third order partial differential equations.

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