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Graphical approach to the study of fixed point results involving hybrid contractions

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A B S T R A C T

In this work, a new class of general contractive mappings, with the name Jaggi-Suzuki-type hybrid (K-α-ϕ)-contractive mapping is discussed in metric space equipped with a graph and new criteria for which the mapping is a Picard operator are studied. The superiority of this type of contractive mapping lies in the fact that its contractive inequality can be fixed in different ways, depending on the specified constants. Substantial illustrations are furnished to validate the axioms of our obtained ideas and to show their difference from the existing concepts. Supplementarily, some corollaries which collapse our obtained notion to recently propounded results in the literature are brought out and analysed.

1. Introduction and preliminaries

The known Banach Contraction Principle (BCP) has brought to the limelight, the foundation of fixed point (FP) theory in metric spaces (MS). The usefulness of FP spans several arms of science and engineering. Several extensions of BCP have been obtained over the years by either generalizing the contractive conditions, introducing additional algebraic structures or altering the metric structures of the underlying space (see e.g [1–4]). In this direction, Jaggi [5] proposed an extension of BCP in the following manner.

Definition 1.1 (*[5]*)**.** On a MS (ζ, δ), a self-map $\digamma$ of ζ is called a Jaggi contractive mapping if for any $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$,

$$\delta(\digamma\hbar, \digamma r) \leq \lambda_1 \frac{\delta(\hbar, \digamma\hbar) + \delta(r, \digamma r)}{\delta(\hbar, r)} + \lambda_2 \delta(\hbar, r),$$

for all distinct points $\hbar, r \in \zeta$.

In similar manner, Suzuki [6] introduced a new version of Edelstein's result in the context of compact MS as given below.

Definition 1.2 (*[6]*)**.** On a MS (ζ, δ), a self-map $\digamma$ of ζ is called a Suzuki contractive mapping if for all distinct points $\hbar, r \in \zeta$,

$$\frac{1}{2}\delta(\hbar, \digamma\hbar) < \delta(\hbar, r) \Rightarrow \delta(\digamma\hbar, \digamma r) < \delta(\hbar, r).$$

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Following these developments, investigators in FP theory have examined a lot of new ideas in metric spaces and obtained more than a handful of interesting results (see e.g [7–11]).

In consistence with Shatanawi [12], let Φ be the set of all functions ϕ such that $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-decreasing function with $\lim_{j\to+\infty} \phi^j(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-map.

Let $\phi \in \Phi$ be a Φ-map such that there exist $j_0 \in \mathbb{N}$, $k \in (0,1)$ and a convergent series of non-negative terms $\sum_{j=1}^{\infty} v_j$ satisfying $\phi^{j+1}(t) \leq k\phi^j(t) + v_j$, for $j \geq j_0$ and any $t > 0$. Then ϕ is called a (c)-comparison function.

We present below, some definitions of admissible mappings on a non-empty set ζ.

Definition 1.3. Let $\alpha : \zeta \times \zeta \longrightarrow \mathbb{R}_+$ be a mapping.

(i) (Samet [13]) A self-map $\digamma$ of ζ is said to satisfy α-admissibility condition if for all $\hbar, r \in \zeta$,

$$\alpha(\hbar, r) \geq 1 \Rightarrow \alpha(\digamma\hbar, \digamma r) \geq 1.$$

(ii) (Karapınar [14]) A self-map $\digamma$ of ζ is said to satisfy triangular α-admissibility condition if for all $\hbar, r, t \in \zeta$,

1. $\digamma$ is α-admissible and
2. $\alpha(\hbar, r) \geq 1$ and $\alpha(r, t) \geq 1 \Rightarrow \alpha(\hbar, t) \geq 1$.

(iii) (Popescu [15]) A self-map $\digamma$ of ζ is said to satisfy α-orbital admissibility condition if for all $\hbar \in \zeta$,

$$\alpha(\hbar, \digamma\hbar) \geq 1 \Rightarrow \alpha(\digamma\hbar, \digamma^2\hbar) \geq 1.$$

(iv) (Popescu [15]) A self-map $\digamma$ of ζ is said to satisfy triangular α-orbital admissibility condition if $\digamma$ satisfies α-orbital admissibility condition and for all $\hbar, r \in \zeta$,

$$\alpha(\hbar, r) \geq 1 \text{ and } \alpha(r, \digamma r) \geq 1 \Rightarrow \alpha(\hbar, \digamma r) \geq 1.$$

Lemma 1.4 (*[16]*). *Let $\digamma$ be a self-map of ζ which satisfies the triangular α-orbital admissibility condition. If we can find $\hbar_0 \in \zeta$ such that $\alpha(\hbar_0, \digamma\hbar_0) \geq 1$, then*

$$\alpha(\hbar_j, \hbar_m) \geq 1 \quad \textit{for all } j, m \in \mathbb{N},$$

where the sequence $\{\hbar_j\}_{j\in\mathbb{N}}$ is given by $\hbar_{j+1} = \digamma\hbar_j$, $j \in \mathbb{N}$.

Not long ago, Noorwali and Yeşilkaya [17] launched a new notion of contractive mapping which is a refinement of some important ones in MS in the following manner.

Definition 1.5 (*[17]*). On a MS (ζ, δ), a self-map $\digamma$ of ζ is named a Jaggi-Suzuki-type hybrid contractive mapping, if we can find $\phi \in \Phi$ and $\alpha : \zeta \times \zeta \longrightarrow \mathbb{R}_+$ such that

$$\frac{1}{2}\delta(\hbar, \digamma\hbar) \leq \delta(\hbar, r) \Rightarrow \alpha(\hbar, r)\delta(\digamma\hbar, \digamma r) \leq \phi(\mathcal{M}(\hbar, r)), \tag{1.1}$$

for all $\hbar, r \in \zeta$, where

$$\mathcal{M}(\hbar, r) = \begin{cases} \left[\lambda_1 \left(\frac{\delta(\hbar,\digamma\hbar)\cdot\delta(r,\digamma r)}{\delta(\hbar,r)}\right)^\rho + \lambda_2\delta(\hbar, r)^\rho\right]^{\frac{1}{\rho}}, & \text{for } \rho > 0,\ \hbar \neq r; \\ \delta(\hbar, \digamma\hbar)^{\lambda_1} \cdot \delta(r, \digamma r)^{\lambda_2}, & for\ \rho = 0,\ \hbar, r \in \zeta \backslash Fix(\digamma), \end{cases}$$

$\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 < \frac{1}{2}$ and $Fix(\digamma) = \{\hbar \in \zeta : \digamma\hbar = \hbar\}$.

Hybrid contractive mappings are peculiar in the sense that they allow for the presentation of contractive conditions involving multiple terms, some of which involve self-composition of the mapping and, at the same time, admit several parameters, allowing for extensions in various ways depending on the parameters chosen. For some extensions of the idea of hybrid contractive mappings in FP theory, we refer to [16,18–26] and the references therein.

According to Petruşel and Rus [27], a Picard operator is a mapping $\digamma$ of a MS (ζ, δ) that has a unique FP $\hbar^*$ and $\lim_{j\to\infty} \digamma^j\hbar = \hbar^*$ for all $\hbar \in \zeta$. $\digamma$ is termed a weakly Picard operator if $\{\digamma^j\hbar\}_{j\in\mathbb{N}}$ is convergent for every $\hbar \in \zeta$ and the limit is a FP of $\digamma$.

The idea of graphic contractive mapping in MS was introduced by Jachymski [28]. Let (ζ, δ) be a MS and Δ, the diagonal of the Cartesian product $\zeta \times \zeta$. Take a directed graph K in the sense that its vertices set $U(K)$ is given by ζ, and its edges set, $D(K)$. The loops set, Δ is a subset of $D(K)$. Suppose that K does not contain parallel edges. Then K can be represented as $(U(K), D(K))$. In addition, K can be seen to be a weighted graph if we assign to each edge the distance between its vertices (see [[29], p. 376]). A graph K^{-1} is obtained from K if the direction of edges of K are reversed. In that case, $D(K^{-1}) = \{(\hbar, r) \in \zeta \times \zeta | (r, \hbar) \in D(K)\}$. Similarly, when the direction of edges of K are ignored or when the set of edges its is symmetric, the undirected graph $\tilde{K}$ is obtained. In that case, $D(\tilde{K}) = D(K) \cup D(K^{-1})$. A subgraph of a graph K is a pair (U', D') if $U(K) \supseteq U'$, $D(K) \supseteq D'$ and for each $(\hbar, r) \in D'$, $\hbar, r \in U'$. Let $\hbar, r \in U$. A sequence $\{\hbar_i\}_{i=0}^N$ of $N+1$ vertices satisfying $\hbar_0 = \hbar$, $\hbar_N = r$ and $(\hbar_{j-1}, \hbar_j) \in D(K)$ for all $i = 1, 2, \ldots, N$ defines a path in K from $\hbar$ to r of length $N \in \mathbb{N}$. If there is a path for any $\hbar, r \in U$, then K is a connected graph. If $\tilde{K}$ is connected, then K is said to be weakly connected.

Following these developments, FP results in MS equipped with graph have been discussed by many investigators (see, e.g. [23,30–35]). Particularly, Bojor [31] presented the results given below.

Definition 1.6 (*[31]*). On a MS (ζ, δ) equipped with a graph K, a self-map $_F$ of ζ is named a $(K\text{-}\phi)$-contractive mapping if:

(i) $_F$ preserves the edges of K, i.e., $(\hbar, r) \in D(K) \Rightarrow ({_F}\hbar, {_F}r) \in D(K)\ \forall \hbar, r \in \zeta$;
(ii) $\exists \phi \in \Phi$ which verifies

$$\delta({_F}\hbar, {_F}r) \leq \phi(\delta(\hbar, r))\ \forall (\hbar, r) \in D(K). \tag{1.2}$$

Definition 1.7 (*[31]*). A self-map $_F$ of ζ is said to satisfy orbital continuity condition if for all $\hbar \in \zeta$ and any sequence $\{k_j\}_{j\in\mathbb{N}}$, ${_F}^{k_j}\hbar \longrightarrow r \in \zeta$ implies that ${_F}({_F}^{k_j}\hbar) \longrightarrow {_F}r$ as $j \to \infty$.

Definition 1.8 (*[31]*). A self-map $_F$ of ζ is said to satisfy orbital K-continuity condition if for all $\hbar \in \zeta$ and a sequence $\{\hbar_j\}_{j\in\mathbb{N}}$, $\hbar_j \longrightarrow \hbar$ and $(\hbar_j, \hbar_{j+1}) \in D(K)$ imply that ${_F}\hbar_j \longrightarrow {_F}\hbar$ as $j \to \infty$.

Theorem 1.9 (*[31]*). *On a complete MS (ζ, δ) equipped with a graph K, and a $(K\text{-}\phi)$-contractive mapping $_F$, if we suppose in addition that:*

(i) K is weakly connected;
(ii) every sequence $\{\hbar_j\}_{j\in\mathbb{N}}$ in ζ with $\delta(\hbar_j, \hbar_{j+1}) \longrightarrow 0$ is such that we can find $k, j_0 \in \mathbb{N}$ satisfying $(\hbar_{kj}, \hbar_{km}) \in D(K)$ for all $j, m \in \mathbb{N}$ with $j, m \geq j_0$;
$(iii)_a$ $_F$ satisfies orbital continuity condition or;
$(iii)_b$ $_F$ satisfies orbital K-continuity condition and there is a subsequence $\{{_F}^{j_k}\hbar_0\}_{k\in\mathbb{N}}$ of $\{{_F}^{j}\hbar_0\}_{j\in\mathbb{N}}$ such that $({_F}^{j_k}\hbar_0, \hbar^*) \in D(K)$ for each $k \in \mathbb{N}$ and some $\hbar_0, \hbar^* \in \zeta$.

Then $_F$ is a Picard operator.

Motivated by the fact that hybrid FP results allow for the unification of both linear and nonlinear contractive mappings, research in the FP theory of hybrid contractive mappings is quickly expanding. Also, as evidenced by the striking applications of graph theory to such fields as cryptography and coding theory, on the other hand, FP results in MS equipped with a graph can be projected to play an increasingly significant role in applied mathematics, engineering, computing and communications. However, based on the existing results, we realized that hybrid FP notions in MS equipped with a graph have received insufficient attention. Therefore, inspired by the ideas in [17,28,31], we initiate a new concept of Jaggi-Suzuki-type hybrid $(K\text{-}\alpha\text{-}\phi)$-contractive mapping in MS equipped with graph and investigate the conditions for which this new contractive mapping is a Picard operator. Comparative examples are constructed to demonstrate that our obtained results are valid and distinct from the existing results in the literature. In addition, some corollaries are highlighted to show that the concept proposed in this manuscript complements and subsumes some well-known results in the literature.

Here and below, we consider ζ as non-empty. The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ denote the sets of natural numbers, real numbers, and non-negative real numbers respectively. $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$.

2. Main results

Presented below is the new idea of Jaggi-Suzuki-type hybrid $(K\text{-}\alpha\text{-}\phi)$-contractive mapping in MS equipped with a graph K.

Definition 2.1. On a MS (ζ, δ) equipped with a graph K, a self-map $_F$ of ζ is named a Jaggi-Suzuki-type hybrid $(K\text{-}\alpha\text{-}\phi)$-contractive mapping if:

(i) $_F$ preserves the edges of K;
(ii) $\exists \phi \in \Phi$ and $\alpha : \zeta \times \zeta \longrightarrow \mathbb{R}_+$ which verify

$$\frac{1}{2}\delta(\hbar, {_F}\hbar) \leq \delta(\hbar, r) \Rightarrow \alpha(\hbar, r)\delta({_F}\hbar, {_F}r) \leq \phi(\mathcal{M}(\hbar, r)), \tag{2.1}$$

for all $(\hbar, r) \in D(K)$, where

$$\mathcal{M}(\hbar, r) = \begin{cases} \left[\lambda_1 \left(\frac{\delta(\hbar, {_F}\hbar)\cdot\delta(r, {_F}r)}{\delta(\hbar, r)}\right)^\rho + \lambda_2 \delta(\hbar, r)^\rho\right]^{\frac{1}{\rho}}, & for\ \rho > 0, \quad \hbar \neq r; \\ \delta(\hbar, {_F}\hbar)^{\lambda_1} \cdot \delta(r, {_F}r)^{\lambda_2}, & fo\hbar \quad \rho = 0, \quad \hbar \neq {_F}\hbar, \end{cases}$$

and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$.

Example 2.2. Let $\zeta = \{\hbar | \hbar \leq 4, \hbar \in \mathbb{N}_0\}$ together with the metric $\delta(\hbar, r) = |\hbar - r|\ \forall \hbar, r \in \zeta$. Let a self-map $_F$ on ζ be defined by

$${_F}\hbar = \begin{cases} 2\hbar, & \text{if } \hbar \leq 1; \\ 1, & \text{if } 2 \leq \hbar \leq 4 \end{cases}$$

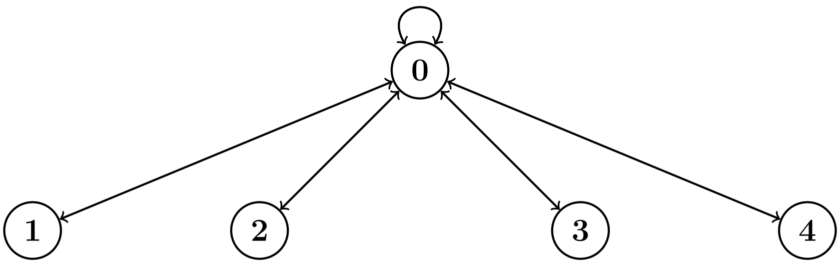

Fig. 1. Symmetric graph $\tilde{K}$ given in Example 2.2.

and $\alpha : \zeta \times \zeta \longrightarrow \mathbb{R}_+$ by

$$\alpha(\hbar, r) = \begin{cases} 0, & \text{if } \hbar, r \in \{0,1\},\ \hbar \neq r; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\digamma$ is a Jaggi-Suzuki-type hybrid (K-α-ϕ)-contractive mapping with $\phi(t) = \frac{4t}{5}$, $\lambda_1 = \frac{2}{5}$ and $\lambda_2 = \frac{3}{5}$ for $\rho = 0,5$, where the symmetric graph $\tilde{K}$ is given by $U(\tilde{K}) = \zeta$ and

$$D(\tilde{K}) = \{(0,0),(0,1),(0,2),(0,3),(0,4)\},$$

but $\digamma$ is not a Jaggi-Suzuki-type hybrid contractive mapping given in [17], since $\frac{1}{2}\delta(1, \digamma 1) = \frac{1}{2} < 1 = \delta(1,2)$, but $\alpha(1,2)\delta(\digamma 1, \digamma 2) = 1$ while $\phi(\mathcal{M}(1,2)) = \frac{4}{5}$ (see Fig. 1).

Our main result is given below.

Theorem 2.3. *On a complete MS (ζ, δ) equipped with a graph K, and a Jaggi-Suzuki-type hybrid (K-α-ϕ)-contractive mapping $\digamma$, if we suppose in addition that:*

(i) $\digamma$ satisfies triangular α-orbital admissibility condition;
(ii) we can find $\hbar_0 \in \zeta$ such that $\alpha(\hbar_0, \digamma\hbar_0) \geq 1$;
(iii) K is weakly connected;
(iv) every sequence $\{\hbar_j\}_{j\in\mathbb{N}}$ in ζ with $\delta(\hbar_j, \hbar_{j+1}) \longrightarrow 0$ is such that we can find $k, j_0 \in \mathbb{N}$ satisfying $(\hbar_{kj}, \hbar_{km}) \in D(K)$ for all $j, m \in \mathbb{N}$ with $j, m \geq j_0$;
$(v)_a$ $\digamma$ satisfies orbital continuity condition or;
$(v)_b$ $\digamma$ satisfies orbital K-continuity condition and there is a subsequence $\{\digamma^{j_k}\hbar_0\}_{k\in\mathbb{N}}$ of $\{\digamma^j\hbar_0\}_{j\in\mathbb{N}}$ such that $(\digamma^{j_k}\hbar_0, \hbar^*) \in D(K)$ for each $k \in \mathbb{N}$ and some $\hbar_0, \hbar^* \in \zeta$.

Then $\digamma$ is a Picard operator.

Proof. Let $\hbar_0 \in \zeta$ with $(\hbar_0, \digamma\hbar_0) \in D(K)$, $\alpha(\hbar_0, \digamma\hbar_0) \geq 1$, and define a sequence $\{\hbar_j\}_{j\in\mathbb{N}}$ by $\hbar_j = \digamma^j\hbar_0$ where $\hbar_j \neq \hbar_{j-1}$. Then a standard induction reveals that $(\digamma^j\hbar_0, \digamma^{j+1}\hbar_0) \in D(K)$. By (i) and Lemma 1.4, inequality (2.1) becomes

$$\begin{aligned} \frac{1}{2}\delta(\hbar_{j-1}, \digamma\hbar_{j-1}) = \frac{1}{2}\delta(\hbar_{j-1}, \hbar_j) \leq \delta(\hbar_{j-1}, \hbar_j) \\ \Rightarrow \alpha(\hbar_{j-1}, \hbar_j)\delta(\digamma\hbar_{j-1}, \digamma\hbar_j) \leq \phi(\mathcal{M}(\hbar_{j-1}, \hbar_j)). \end{aligned}$$

Now, we note that

$$\begin{aligned} \delta(\hbar_j, \hbar_{j+1}) = \delta(\digamma\hbar_{j-1}, \digamma\hbar_j) &\leq \alpha(\hbar_{j-1}, \hbar_j)\delta(\digamma\hbar_{j-1}, \digamma\hbar_j) \\ &\leq \phi(\mathcal{M}(\hbar_{j-1}, \hbar_j)). \end{aligned} \tag{2.2}$$

Using Case 1 of (2.1), we obtain

$$\begin{aligned} \mathcal{M}(\hbar_{j-1}, \hbar_j) &= \left[\lambda_1\left(\frac{\delta(\hbar_{j-1}, \digamma\hbar_{j-1}) \cdot \delta(\hbar_j, \digamma\hbar_j)}{\delta(\hbar_{j-1}, \hbar_j)}\right)^{\rho} + \lambda_2\delta(\hbar_{j-1}, \hbar_j)^{\rho}\right]^{\frac{1}{\rho}} \\ &= \left[\lambda_1\left(\frac{\delta(\hbar_{j-1}, \hbar_j) \cdot \delta(\hbar_j, \hbar_{j+1})}{\delta(\hbar_{j-1}, \hbar_j)}\right)^{\rho} + \lambda_2\delta(\hbar_{j-1}, \hbar_j)^{\rho}\right]^{\frac{1}{\rho}} \\ &= \left[\lambda_1\delta(\hbar_j, \hbar_{j+1})^{\rho} + \lambda_2\delta(\hbar_{j-1}, \hbar_j)^{\rho}\right]^{\frac{1}{\rho}}. \end{aligned}$$

Whence, (2.2) becomes

$$\delta(\hbar_j, \hbar_{j+1}) \leq \phi\left(\left[\lambda_1\delta(\hbar_j, \hbar_{j+1})^{\rho} + \lambda_2\delta(\hbar_{j-1}, \hbar_j)^{\rho}\right]^{\frac{1}{\rho}}\right).$$

Now, if $\delta(\hbar_{j-1},\hbar_j) \le \delta(\hbar_j,\hbar_{j+1})$, then we have

$$\begin{aligned}
\delta(\hbar_j,\hbar_{j+1}) \le\ & \phi\left(\left[\lambda_1\delta(\hbar_j,\hbar_{j+1})^\rho + \lambda_2\delta(\hbar_{j-1},\hbar_j)^\rho\right]^{\frac{1}{\rho}}\right)\\
\le\ & \phi\left(\left[\lambda_1\delta(\hbar_j,\hbar_{j+1})^\rho + \lambda_2\delta(\hbar_j,\hbar_{j+1})^\rho\right]^{\frac{1}{\rho}}\right)\\
\le\ & \phi\left(\left[(\lambda_1+\lambda_2)\delta(\hbar_j,\hbar_{j+1})^\rho\right]^{\frac{1}{\rho}}\right)\\
=\ & \phi\left(\delta(\hbar_j,\hbar_{j+1})\right)\\
<\ & \delta(\hbar_j,\hbar_{j+1}),
\end{aligned}$$

a contradiction. Therefore, $\delta(\hbar_j,\hbar_{j+1}) < \delta(\hbar_{j-1},\hbar_j)$, so that (2.2) becomes

$$\delta(\hbar_j,\hbar_{j+1}) \le \phi\left(\delta(\hbar_{j-1},\hbar_j)\right).$$

By inductive process, we see that

$$\delta(\hbar_j,\hbar_{j+1}) \le \phi^j\left(\delta(\hbar_0,\digamma\hbar_0)\right) \quad \text{for all } j\in\mathbb{N}.$$

Similar analysis of Case 2 yields

$$\begin{aligned}
\delta(\hbar_j,\hbar_{j+1}) &\le \phi\left(\delta(\hbar_{j-1},\digamma\hbar_{j-1})^{\lambda_1}\cdot\delta(\hbar_j,\digamma\hbar_j)^{\lambda_2}\right)\\
&< \delta(\hbar_{j-1},\digamma\hbar_{j-1})^{\lambda_1}\cdot\delta(\hbar_j,\digamma\hbar_j)^{\lambda_2}.
\end{aligned}$$

Since $\lambda_1+\lambda_2=1$, then the above inequality resolves into

$$\delta(\hbar_j,\hbar_{j+1}) < \delta(\hbar_{j-1},\hbar_j) \quad \text{for all } j\in\mathbb{N}.$$

Whence, inequality (2.2) becomes $\delta(\hbar_j,\hbar_{j+1}) \le \phi\left(\delta(\hbar_{j-1},\hbar_j)\right)$,
and by inductive process, we have

$$\delta(\hbar_j,\hbar_{j+1}) \le \phi^j\left(\delta(\hbar_0,\digamma\hbar_0)\right) \quad \text{for all } j\in\mathbb{N},$$

implying that $\delta(\digamma^j\hbar_0,\digamma^{j+1}\hbar_0) \le \phi^j\left(\delta(\hbar_0,\digamma\hbar_0)\right)$ for all $j\in\mathbb{N}$, $\rho\ge 0$. Hence, we have $\lim_{j\to\infty}\delta(\digamma^j\hbar_0,\digamma^{j+1}\hbar_0)=0$, and by (iv), we can find $k, j_0\in\mathbb{N}$ such that

$$(\digamma^{kj}\hbar_0,\digamma^{km}\hbar_0)\in D(K) \quad \text{for all } j,m\in\mathbb{N},\ j,m\ge j_0.$$

Since $\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+1)}\hbar_0)\longrightarrow 0$, then for any given $\epsilon>0$, we can choose $N\in\mathbb{N}$, $N\ge j_0$ such that

$$\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+1)}\hbar_0) < \epsilon-\phi(\epsilon) \quad \text{for any } j\in\mathbb{N}.$$

Since $(\digamma^{kj}\hbar_0,\digamma^{k(j+1)}\hbar_0)\in D(K)$, then for any $j\ge N$, we have

$$\begin{aligned}
\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+2)}\hbar_0) &\le \delta(\digamma^{kj}\hbar_0,\digamma^{k(j+1)}\hbar_0)+\delta(\digamma^{k(j+1)}\hbar_0,\digamma^{k(j+2)}\hbar_0)\\
&< \epsilon-\phi(\epsilon)+\phi^k(\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+1)}\hbar_0)) < \epsilon.
\end{aligned}$$

Similarly, since $(\digamma^{kj}\hbar_0,\digamma^{k(j+2)}\hbar_0)\in D(K)$, then for any $j\ge N$, we have

$$\begin{aligned}
\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+3)}\hbar_0) &\le \delta(\digamma^{kj}\hbar_0,\digamma^{k(j+1)}\hbar_0)+\delta(\digamma^{k(j+1)}\hbar_0,\digamma^{k(j+3)}\hbar_0)\\
&< \epsilon-\phi(\epsilon)+\phi^k(\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+2)}\hbar_0)) < \epsilon.
\end{aligned}$$

By inductive process, it can be seen that

$$\delta(\digamma^{kj}\hbar_0,\digamma^{k(j+m)}\hbar_0)\le\epsilon \quad \text{for any } j,m\in\mathbb{N},\ j\ge N.$$

Therefore, $\{\digamma^{kj}\hbar_0\}_{k\in\mathbb{N}}$ is a Cauchy sequence in (ζ,δ). Since (ζ,δ) is complete, then $\digamma^{kj}\hbar_0\longrightarrow\hbar^*$ as $j\to\infty$. Also, since $\delta(\digamma^j\hbar_0,\digamma^{j+1}\hbar_0)\longrightarrow 0$ as $j\to\infty$, then we have $\digamma^j\hbar_0\longrightarrow\hbar^*$ as $j\to\infty$.

Now for any arbitrary $\hbar\in\zeta$, we see that:

1. if $(\hbar,\hbar_0)\in D(K)$, then $(\digamma^j\hbar,\digamma^j\hbar_0)\in D(K)$ for all $j\in\mathbb{N}$. Therefore,

$$\delta(\digamma^j\hbar,\digamma^j\hbar_0)\le\phi^j(\delta(\hbar,\hbar_0)) \quad \text{for all } j\in\mathbb{N}.$$

   Letting $j\to\infty$, we have that $\digamma^j\hbar\longrightarrow\hbar^*$.
2. if $(\hbar,\hbar_0)\notin D(K)$, then by (iii), we can find a path in $\tilde{K}$, $\{\tilde{\hbar}_i\}_{i=0}^N$ from $\hbar_0$ to $\hbar$ such that $\tilde{\hbar}_0=\hbar_0$, $\tilde{\hbar}_N=\hbar$ with $(\tilde{\hbar}_{i-1},\tilde{\hbar}_i)\in D(\tilde{K})$ for all $i=1,2,\ldots,N$ such that by simple induction, we obtain

$$(\digamma^j\tilde{\hbar}_{i-1},\digamma^j\tilde{\hbar}_i)\in D(\tilde{K}) \quad \text{for } i=1,2,\ldots,N \text{ and}$$

$$\delta(\digamma^j\hbar_0,\digamma^j\hbar)\le\sum_{i=1}^N\phi^j(\delta(\tilde{\hbar}_{i-1},\tilde{\hbar}_i)),$$

   so that $\delta(\digamma^j\hbar_0,\digamma^j\hbar)\longrightarrow 0$, implying that $\digamma^j\hbar\longrightarrow\hbar^*$.

Table 1
Demonstration of contractive inequality (2.1).

Cases	$\hbar$	r	$\frac{1}{2}\delta(\hbar,\digamma\hbar)$	$\delta(\hbar,r)$	$\alpha(\hbar,r)\delta(\digamma\hbar,\digamma r)$	$\phi(\mathcal{M}(\hbar,r)),\ \rho=0$	$\phi(\mathcal{M}(\hbar,r)),\ \rho=3$
Case 1	2	4	0.5	2	1	1.32625	1.51591
	4	2	1	2	1	1.15456	1.51591
	4	6	1	2	2	2.23199	2.18633
	6	4	1.5	2	2	2.05813	2.18633
Case 2	1	1	0	0	0	0	–
Case 3	1	3	0	2	0	–	1.47600
	1	5	0	4	0	–	2.95201
	3	1	1	2	0	–	1.47600
	3	5	1	2	0	2.65250	2.73085
	5	1	2	4	0	–	2.95201
	5	3	2	2	0	2.30913	2.73085
Case 4	2	1	0.5	1	0	–	0.73800
	2	3	0.5	1	0	1.32625	1.36542
	2	5	0.5	3	0	2.01022	2.25638
	4	3	1	1	1	1.75	2.59881
	6	1	1.5	5	2	–	3.69001
Case 5	1	2	0	1	0	–	0.73800
	1	6	0	5	2	–	3.69001
	3	2	1	1	0	1.15456	1.36542
	3	4	1	1	1	1.75	2.59881
	5	2	2	3	0	1.52346	2.25638

Whence, for all $\hbar \in \zeta$, there is a unique point $\hbar^* \in \zeta$ such that

$$\lim_{j\to\infty} \digamma^j \hbar = \hbar^*.$$

To see that $\hbar^* \in Fix(\digamma)$, if $(\upsilon)_a$ holds, then clearly, $\hbar^* \in Fix(\digamma)$. Otherwise if $(\upsilon)_b$ holds, then since $\{\digamma^{j_k}\hbar_0\}_{k\in\mathbb{N}} \longrightarrow \hbar^*$ and $(\digamma^{j_k}\hbar_0, \hbar^*) \in D(K)$, then since $\digamma$ satisfies the orbital K-continuity condition, we have $\digamma^{j_k+1}\hbar_0 \longrightarrow \digamma\hbar^*$ as $k \to \infty$. Whence, $\digamma\hbar^* = \hbar^*$.

If there is some $r \in \zeta$ satisfying $\digamma r = r$, then from the above, we obtain $\digamma^j r \longrightarrow \hbar^*$, which means $r = \hbar^*$. Hence, $\digamma$ is a Picard operator. □

Example 2.4. Let $\zeta = \{\hbar | \hbar \le 6, \hbar \in \mathbb{N}\}$ be equipped with the Euclidean metric $\delta(\hbar,r) = |\hbar - r|$ for all $\hbar, r \in \zeta$. Then (ζ,δ) is a complete MS. Let $\digamma$ be a self-map on ζ defined by

$$\digamma\hbar = \begin{cases} \frac{\hbar}{2}, & \text{if } \hbar \in \{2i : i = \overline{1,3}\}; \\ 1, & \text{if } \hbar \in \{2i-1 : i = \overline{1,3}\} \end{cases}$$

and let $\alpha : \zeta \times \zeta \longrightarrow \mathbb{R}_+$ be defined by

$$\alpha(\hbar,r) = \begin{cases} 2, & \text{if } \hbar, r \in \{4,6\}; \\ 1, & \text{otherwise.} \end{cases}$$

Consider the symmetric graph $\tilde{K}$ given by $U(\tilde{K}) = \zeta$ and

$$D(\tilde{K}) = \{(1,1),(1,2),(1,3),(1,5),(1,6),(2,3),(2,4),(2,5),(3,4),(3,5),(4,6)\}.$$

It is obvious that $\digamma$ is edge preserving, $\digamma$ satisfies triangular α-orbital admissibility condition and K is weakly connected.

To demonstrate that $\digamma$ is a Jaggi-Suzuki-type hybrid (K-α-ϕ)-contractive mapping, let $\phi(t) = \frac{7t}{8}$ for all $t \ge 0$, $\lambda_1 = \frac{2}{5}$ and $\lambda_2 = \frac{3}{5}$ for $\rho = 0, 3$. The following instances are investigated:

Case 1: $\hbar, r \in \{2i : i = \overline{1,3}\}$, $\hbar \neq r$;
Case 2: $\hbar, r \in \{2i-1 : i = \overline{1,3}\}$, $\hbar = r$;
Case 3: $\hbar, r \in \{2i-1 : i = \overline{1,3}\}$, $\hbar \neq r$;
Case 4: $\hbar \in \{2i : i = \overline{1,3}\}$ and $r \in \{2i-1 : i = \overline{1,3}\}$;
Case 5: $\hbar \in \{2i-1 : i = \overline{1,3}\}$ and $r \in \{2i : i = \overline{1,3}\}$.

It will be demonstrated using the following Table 1 that contractive condition (2.1) is verified for each of the instances above.

It can be seen from Columns 4 and 5 of the above Table 1 that $\frac{1}{2}\delta(\hbar,\digamma\hbar) \le \delta(\hbar,r)$ for each of Cases $1-5$ and by implication, $\alpha(\hbar,r)\delta(\digamma\hbar,\digamma r) \le \phi(\mathcal{M}(\hbar,r))$ for all $(\hbar,r) \in \tilde{K}$ as shown by Columns 6, 7 and 8.

Fig. 2 below shows the symmetric graph $\tilde{K}$ for Example 2.4, while Figs. 3 and 4 further verify that contractive condition (2.1) is valid for Example 2.4.

For $\rho = 0$ and $\rho = 3$ respectively, Figs. 3 and 4 have shown that

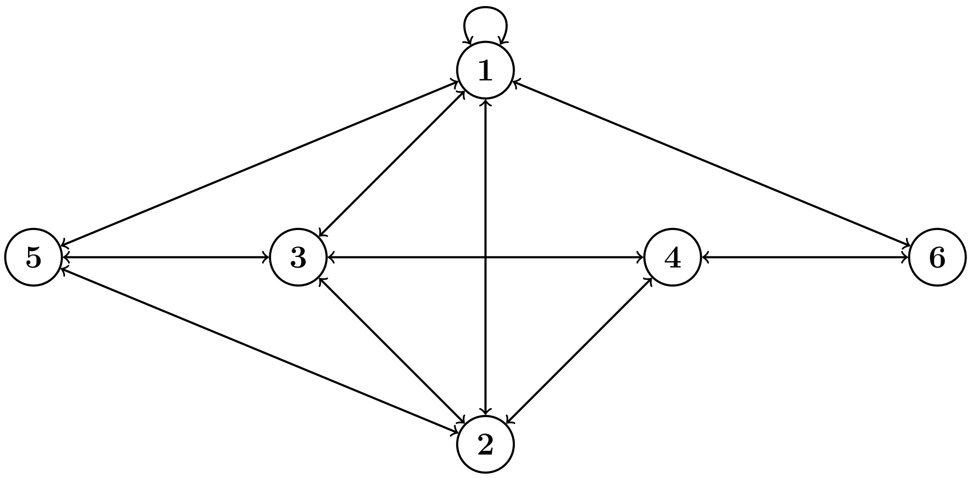

Fig. 2. Symmetric graph $\tilde{K}$ given in Example 2.4.

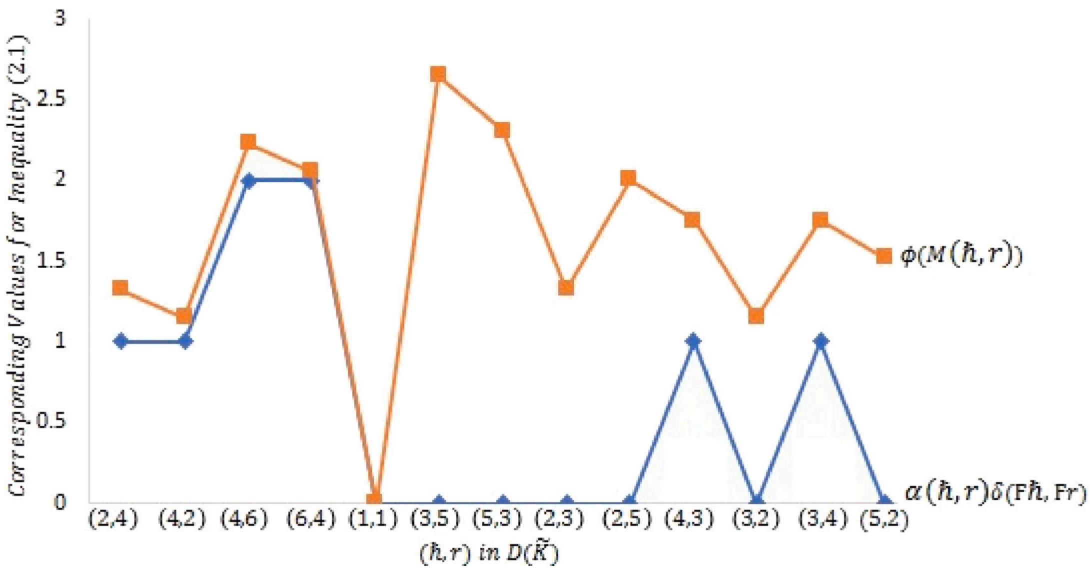

Fig. 3. Demonstration of contractive condition (2.1) for $\rho = 0$.

$\alpha(\hbar, r)\delta(_F\hbar, _Fr) \leq \phi(\mathcal{M}(\hbar, r))$ for all $(\hbar, r) \in \tilde{K}$ as defined in Example 2.4.

Hence, all the assumptions of Theorem 2.3 have been verified, there is a unique point $\hbar = 1$ such that $_F\hbar = \hbar$, and $\lim_{j\to\infty} {_F}^j\hbar = 1$ for all $\hbar \in \zeta$. Consequently, $_F$ is a Picard operator.

In what follows, we demonstrate that Theorem 3.2 obtained in [28] can be derived from our main results. In compliance with Jachymski [28], let $_F$ be a self-map of ζ. The set of all points $\hbar \in \zeta$ satisfying $(\hbar, _F\hbar) \in D(K)$ is denoted by $\zeta_{_F}$, that is,

$$\zeta_{_F} = \{\hbar \in \zeta : (\hbar, _F\hbar) \in D(K)\}.$$

Corollary 2.5 *(see [[28], Theorem 3.2]). On a complete MS (ζ, δ) equipped with a graph K, and a K-contractive mapping $_F$, if we suppose in addition that:*

(i) $\zeta_{_F} \neq \emptyset$ and K is weakly connected;
(ii) for any sequence $\{\hbar_j\}_{j\in\mathbb{N}}$ in ζ, if $\hbar_j \longrightarrow \hbar$ and $(\hbar_j, \hbar_{j+1}) \in D(K)$ for $j \in \mathbb{N}$, then there is a subsequence $\{\hbar_{kj}\}_{k\in\mathbb{N}}$ with $(\hbar_{kj}, \hbar) \in D(K)$ for $k \in \mathbb{N}$.

Then $_F$ is a Picard operator.

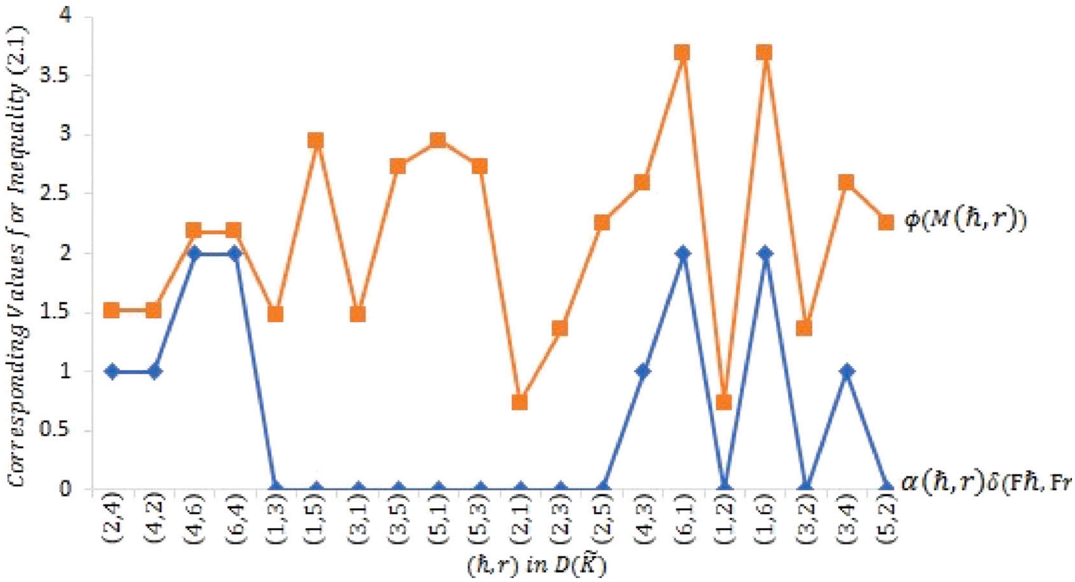

Fig. 4. Demonstration of contractive condition (2.1) for $\rho = 3$.

Proof. We can assume in Definition 2.1 that $\frac{1}{2}\delta(\hbar, {}_F\hbar) \leq \delta(\hbar, r)$ for all $\hbar, r \in \zeta$ and let $\alpha(\hbar, r) = 1$ for all $\hbar, r \in \zeta$, $\phi(t) = \mu t$ for all $t \geq 0$, $\mu \in (0, 1)$, $\rho > 0$, $\lambda_1 = 0$ and $\lambda_2 = 1$. In that case, Jaggi-Suzuki-type hybrid $(K\text{-}\alpha\text{-}\phi)$-contractive mapping becomes K-contractive mapping defined by Jachymski [28]. Hence, the proof is immediate from Theorem 3.2 of Jachymski [28]. □

The following Corollary 2.6 demonstrates that Theorem 1.9 of Bojor [31] is deducible from our main results.

Corollary 2.6 *(see [Theorem 1.9]). On a complete MS (ζ, δ) equipped with a graph K, and a $(K\text{-}\phi)$-contractive mapping ${}_F$, if we suppose in addition that:*

(i) K is weakly connected;
(ii) every sequence $\{\hbar_j\}_{j\in\mathbb{N}}$ in ζ with $\delta(\hbar_j, \hbar_{j+1}) \longrightarrow 0$ is such that we can find $k, j_0 \in \mathbb{N}$ satisfying $(\hbar_{kj}, \hbar_{km}) \in D(K)$ for all $j, m \in \mathbb{N}$ with $j, m \geq j_0$;
$(iii)_a$ ${}_F$ satisfies orbital continuity condition or;
$(iii)_b$ ${}_F$ satisfies orbital K-continuity condition and there is a subsequence $\{{}_F^{j_k}\hbar_0\}_{k\in\mathbb{N}}$ of $\{{}_F^{j}\hbar_0\}_{j\in\mathbb{N}}$ such that $({}_F^{j_k}\hbar_0, \hbar^*) \in D(K)$ for each $k \in \mathbb{N}$ and some $\hbar_0, \hbar^* \in \zeta$.

Then ${}_F$ is a Picard operator.

Proof. Consider Definition 2.1 and let $\frac{1}{2}\delta(\hbar, {}_F\hbar) \leq \delta(\hbar, r)$ for all $\hbar, r \in \zeta$, $\alpha(\hbar, r) = 1$ for all $\hbar, r \in \zeta$, $\rho > 0$, $\lambda_1 = 0$ and $\lambda_2 = 1$. Then Jaggi-Suzuki-type hybrid $(K\text{-}\alpha\text{-}\phi)$-contractive mapping reduces to $(K\text{-}\phi)$-contractive mapping given by Bojor [31] (see Definition 1.6). The proof follows similarly. □

Remark 2.7. It is obvious that we can obtain more consequences of our results by particularizing the values of the mappings $\alpha(\hbar, r)$, $\phi(t)$ and specializing the constants ρ and λ_i $(i = 1, 2)$.

3. Conclusion

The concept of Jaggi-Suzuki-type hybrid $(K\text{-}\alpha\text{-}\phi)$-contractive mapping in MS equipped with a graph is introduced in this paper (Definition 2.1). Sufficient criteria for the newly defined mapping to be a Picard operator are investigated (Theorem 2.3). Contrasting examples (Examples 2.2 and 2.4) with graphical depictions are built to validate the assumptions of our obtained results. Corollaries 2.5 and 2.6 are provided to show that the approach described herein is a generalization and improvement on some related results in the literature. The ideas in this paper are motivated by and compared to [17,28,31].

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Mohammed Shehu Shagari reports administratve support was provided by Ahmadu Bello University. Mohammed Shehu

Shagari reports a relationship with Ahmadu Bello University that includes: employment. Mohammed Shehu Shagari has patent pending to Not applicable. Not applicable If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to infuence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Boyd DW, Wong JS. On nonlinear contractions. Proc Amer Math Soc 1969;20(2):458–64.
[2] Chatterjea SK. Fixed point theorems for a sequence of mappings with contractive iterates. Publ l'Inst Math 1972;14(34):15–8.
[3] Ćirić LB. A generalization of Banach's contraction principle. Proc Amer Math Soc 1974;45(2):267–73.
[4] Maayah B, Moussaoui A, Bushnaq S, Abu Arqub O. The multistep Laplace optimized decomposition method for solving fractional-order coronavirus disease model (COVID-19) via the Caputo fractional approach. Demonstratio Math 2022;55(1):963–77.
[5] Jaggi DS. Some unique fixed point theorems. Indian J Pure Appl Math 1977;8(2):223–30.
[6] Suzuki T. A new type of fixed point theorem in metric spaces. Nonlinear Anal TMA 2009;71(11):5313–7.
[7] Alqahtani O, Karapınar E. A bilateral contraction via simulation function. Filomat 2019;33(15):4837–43.
[8] Abodayeh K, Karapınar E, Pitea A, Shatanawi W. Hybrid contractions on Branciari type distance spaces. Mathematics 2019;7(10):994.
[9] Alqahtani B, Aydi H, Karapınar E, Rakočević V. A solution for Volterra fractional integral equations by hybrid contractions. Mathematics 2019;7(8):694.
[10] Chifu C, Karapınar E. Admissible hybrid Z-contractions in b-metric spaces. Axioms 2019;9(2):17.
[11] Karapınar E, Agarwal RP, Yeşilkaya SS, Wang C. Fixed-point results for Meir–Keeler type contractions in partial metric spaces: A survey. Mathematics 2022;10(17):3109.
[12] Shatanawi W. Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces. Fixed Point Theory Appl. 2010;2010:1–9.
[13] Samet B, Vetro C, Vetro P. Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal Theory Methods Appl 2012;75:2154–65.
[14] Karapınar E, Kumam P, Salimi P. On α-ψ-Meir-Keeler contractive mappings. Fixed Point Theory Appl 2013;2013(94).
[15] Popescu O. Some new fixed point theorems for α-geraghty contraction type maps in metric spaces. Fixed Point Theory Appl 2014;2014(190).
[16] Karapınar E, Fulga A. An admissible hybrid contraction with an Ulam type stability. Demonstratio Math 2019;52:428–36.
[17] Noorwali M, Yeşilkaya SS. On Jaggi-Suzuki-type hybrid contraction mappings. J Funct Spaces 2021;2021(7). http://dx.doi.org/10.1155/2021/6721296, Article ID 6721296.
[18] Karapınar E, Fulga A. A hybrid contraction that involves Jaggi type. Symmetry 2019;11(715).
[19] Karapınar E, Fulga A, Shahzad N, Roldán López de Hierro AF. Solving integral equations by means of fixed point theory. J Funct Spaces 2022;2022:16, Article ID 7667499.
[20] Karapınar E. A survey on interpolative and hybrid contractions. In: Mathematical analysis in interdisciplinary research. Cham: Springer International Publishing; 2022, p. 431–75.
[21] Alansari M, Mohammed SS, Azam A, Hussain N. On multivalued hybrid contractions with applications. J Funct Spaces 2020;2020. Article ID 8401403.
[22] Jiddah JA, Alansari M, Mohamed OSK, Shagari MS, Bakery AA. Fixed point results of Jaggi-type hybrid contraction in generalized metric space. J Funct Spaces 2022;2022:9. http://dx.doi.org/10.1155/2022/2205423, Article ID 2205423.
[23] Jiddah JA, Shagari MS, Abdulazeez I, Imam AT. Fixed point of interpolative contraction on metric space endowed with graph. Int J Math Sci Optim Theory Appl 2022;8(2):58–67.
[24] Fulga A, Karapınar E, Petruşel G. On hybrid contractions in the context of Quasi-metric spaces. Mathematics 2020;8(5):675.
[25] Ogbumba RO, Shagari MS, Alansari M, Khalid TA, Mohamed EA, Bakery AA. Advancements in hybrid fixed point results and F-contractive operators. Symmetry 2023;15(6):1253.
[26] Ogbumba RO, Shagari MS, Azam A, Ali F, Alotaibi T. Existence results of certain nonlinear polynomial and integral equations via $\mathcal{F}$-contractive operators. AIMS Math 2023;8(12):28646–69.
[27] Petruşel A, Rus I. Fixed point theorems in ordered $\mathcal{L}$-spaces. Proc Amer Math Soc 2006;134(2):411–8.
[28] Jachymski J. The contraction principle for mappings on a metric space with a graph. Proc Amer Math Soc 2008;1(136):1359–73.
[29] Johnsonbaugh R. Discrete mathematics. 8th ed. London: Pearson Education Inc.; 2018.
[30] Gwóźdź-Łukawska G, Jachymski J. IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem. J Math Anal Appl 2009;356(2):453–63.
[31] Bojor F. Fixed point of φ-contraction in metric spaces with a graph. An Univ Craiova Math Comput Sci Ser 2010;37(4):85–92.
[32] Bojor F. Fixed point theorems for reich type contractions on metric spaces with a graph. J Nonlinear Anal 2012;75(2012):3895–901.
[33] Bojor F. Fixed points of Kannan mappings in metric spaces endowed with a graph. Math J Ovidius Univ Constantza 2012;20(1):31–40.
[34] Chifu C, Petruşel G. Generalized contractions in metric spaces endowed with a graph. Fixed Point Theory Appl 2012;1(161):9.
[35] Younis M, Singh D, Radenović S, Imdad M. Convergence theorems for generalized contractions and applications. Filomat 2020;34(3):945–64.