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Direct Solution of $y''(x) = f(x, y, y')$ Using Four Points Block-Hybrid Linear Multistep Method of Order Seven with Applications

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ABSTRACT

The study aims to construct an implicit block hybrid method with four points to tackle general second order initial value problems of ordinary differential equations (ODEs) directly. Power series is used as the basis function to obtain the proposed method which involved the first and second derivatives of $f(x, y, y')$. From the investigation done, it was found that the proposed method is consistent and zero-stable, hence it is convergent. The proposed method's efficiency was obtained and a comparison was made in terms of accuracy to some existing methods with similar order and the ones higher than it. The new proposed method is able to solve linear, nonlinear and systems of equations of general second order Initial Value Problems and outperformed existing methods with impressive results. Applications of the proposed method to a real-life problem is discussed.

***Keywords*:** Second Order IVPs, Hybrid Block Method, System of equations implicit LMM, Collocation, power series, off-step, Zero Stability

1. INTRODUCTION

In this paper, we addressed an approximate solution of general second order initial value problem of the form (1);

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$$y''(x) = f(x, y, y'); \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \tag{1}$$

where f is continuously differentiable on the given interval $[a, b]$. We assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness if solution (1) [1].

Ordinary Differential Equations are often used to describe physical systems in science, engineering such as in ship dynamics, deflection of beams, problems in combustion theory, control theory, heat and mass transfer, which are modeled as second order Ordinary Differential Equations (ODEs) [2]. This is why the numerical solution of (1) is of great interest to researchers [3] and only few can be solved analytically. The discrete solution and continuous solution of (1) by Linear Multistep Method (LMM) has been studied by authors like [4,5]. Hence, the need to study numerical methods and their solution [6].

Moreover, recent studies show that several studies have been conducted to implement derivative methods in solving ODEs, but these methods have only the first derivative of $f(x, y, y')$; [7], which revealed that adding more derivatives might lead to more accurate numerical schemes. Conventionally, we often reduce (1) to system of first order ordinary differential equations and then use appropriate numerical methods such as Euler method to solve the resultant system [8]. The reduction process and the setbacks of this approach has been discussed by numerous authors among them is [9] which includes lengthy computation time, complicated computational work and wastage of human time. Each block contains r-point approximation values of $y_{n+1}, y_{n+2}, \dots, y_{n+r}$, at each iteration. The block method was presented by [10] and later extended by [11] and [12]. Hybrid methods were initially introduced to overcome zero stability barrier occurred in block methods mentioned by [13].

A direct solution can be obtained by using the collocation and interpolation techniques [14]. To determine the coefficients of the method via collocation and interpolation approach, the points must be collocated and interpolated which result in a system of linear equations. We were motivated to develop an order (k+3) block hybrid method which can be easily executed directly for solving both linear and nonlinear problems in the form (1) problem more accurately and efficiently.

In this paper, a hybrid block LMM utilizing both step and off-step points at collocation will be developed for direct solution of general second order IVPs with power series as the basis function. The basic properties of this method will be discussed and numerical experiment will be carried out on linear problem, system of equations and a real-life problem to validate the method.

2. DERIVATION OF THE METHOD

In the four-point block method, the interval $[a, b]$ is divided into a series of blocks that generate four approximate values $y_{n+1}, y_{n+2}, y_{n+3}$ *and* y_{n+4}, at each block, concurrently using one earlier block. Consider power series of a single variable as an approximate solution to (1) to be;

$$y(x) = \sum_{j=0}^{r+s-1} \alpha_j x^j \tag{2}$$

where $\alpha_j \in R$ are unknown parameters to be determined and $r+s$ is the sum of the number of interpolation and number of collocation points. The first and second derivatives of (2) are;

$$y'(x) = \sum_{j=1}^{r+s-1} j\,\alpha_j x^{j-1} \tag{3}$$

$$y''(x) = \sum_{j=2}^{r+s-1} j\,(j-1)\alpha_j x^{j-2} \tag{4}$$

Comparing (4) and (1), we have;

$$f(x,y,y') = \sum_{j=2}^{r+s-1} j\,(j-1)\alpha_j x^{j-2} \tag{5}$$

Interpolating (2) at $x_{n+j}, j = 1, \frac{5}{3}$ and Collocating (5) at $x_{n+j}, j = 0, \frac{2}{3}, 1, \frac{5}{3}, 2, 3{,}4$ gives rise to a system of nonlinear equation $Ax = b$;

$$\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+\frac{5}{3}} & x_{n+\frac{5}{3}}^2 & x_{n+\frac{5}{3}}^3 & x_{n+\frac{5}{3}}^4 & x_{n+\frac{5}{3}}^5 & x_{n+\frac{5}{3}}^6 & x_{n+\frac{5}{3}}^7 & x_{n+\frac{5}{3}}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x_{n+\frac{2}{3}}^2 & 20x_{n+\frac{2}{3}}^3 & 30x_{n+\frac{2}{3}}^4 & 42x_{n+\frac{2}{3}}^5 & 56x_{n+\frac{2}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{3}} & 12x_{n+\frac{5}{3}}^2 & 20x_{n+\frac{5}{3}}^3 & 30x_{n+\frac{5}{3}}^4 & 42x_{n+\frac{5}{3}}^5 & 56x_{n+\frac{5}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ y_{n+\frac{5}{3}} \\ f_n \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{6}$$

Solving (6) for $\alpha_j, j = 0(1)8$ by inversion method and performing necessary manipulation using Maple 17 software, we obtained coefficient of $\alpha_j$and β_j to produce continuous Hybrid-Block linear multistep method of the form;

$$y(x) = \alpha_1 y_{n+1} + \alpha_{\frac{5}{3}} y_{n+\frac{5}{3}} + \beta_0 f_n + \beta_{\frac{2}{3}} f_{n+\frac{2}{3}} + \beta_1 f_{n+1} + \beta_{\frac{5}{3}} f_{n+\frac{5}{3}} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4} \quad (7)$$

where $y(x)$ is the approximate solution of the initial value problem and $v = \frac{2}{3}, \frac{5}{3}$. α_j and β_j are coefficients that are continuously differentiable. Since (7) is continuous and differentiable, then α_0 and β_0 are not both zero.

We evaluated (7) at non-interpolating points $x_n, x_{n+\frac{2}{3}}, x_{n+2}, x_{n+3,} x_{n+4}$ that produced the discrete schemes;

$$y_n + \frac{3}{2} y_{n+\frac{5}{3}} = \frac{1}{1524096} h^2 \left(45801 f_n + 320180 f_{n+1} - 160293 f_{n+2} + 8154 f_{n+3} - 569 f_{n+4} + 654165 f_{n+\frac{2}{3}} + 402642 f_{n+\frac{5}{3}} \right) + \frac{5}{2} y_{n+1} \quad (8)$$

$$y_{n+\frac{2}{3}} + \frac{1}{2} y_{n+\frac{5}{3}} = \frac{1}{68584320} h^2 \left(26089 f_n + 9132452 f_{n+1} - 944013 f_{n+2} + 34802 f_{n+3} - 2153 f_{n+4} - 54675 f_{n+\frac{2}{3}} + 3238218 f_{n+\frac{5}{3}} \right) + \frac{3}{2} y_{n+1} \quad (9)$$

$$y_{n+2} - \frac{3}{2} y_{n+\frac{5}{3}} = \frac{1}{7620480} h^2 \left(3465 f_n + 317828 f_{n+1} - 29757 f_{n+2} + 3618 f_{n+3} - 233 f_{n+4} - 82179 f_{n+\frac{2}{3}} + 1057338 f_{n+\frac{5}{3}} \right) - \frac{1}{2} y_{n+1} \quad (10)$$

$$y_{n+3} - 3 y_{n+\frac{5}{3}} = \frac{1}{59535} h^2 \left(630 f_n + 27398 f_{n+1} + 60018 f_{n+2} + 4833 f_{n+3} - 134 f_{n+4} - 10818 f_{n+\frac{2}{3}} + 2547 f_{n+\frac{5}{3}} \right) - 2 y_{n+1} \quad (11)$$

$$y_{n+4} - \frac{9}{2} y_{n+\frac{5}{3}} = -\frac{1}{362880} h^2 \left(7119 f_n + 73612 f_{n+1} - 353283 f_{n+2} - 415098 f_{n+3} - 21343 f_{n+4} - 76365 f_{n+\frac{2}{3}} - 484722 f_{n+\frac{5}{3}} \right) - \frac{7}{2} y_{n+1} \quad (12)$$

The continuous scheme (7) is differentiated with respect to x to produce the first derivative that was evaluated at both the interpolation points $x_{n+1}, x_{n+\frac{5}{3}}$ and collocation points $x_n, x_{n+\frac{2}{3}}, x_{n+2}, x_{n+3}, x_{n+4}$ which gives;

$$h y'_n = -\frac{1}{3810240} h^2 \left[724689 f_n - 1348480 f_{n+1} - 1100127 f_{n+2} + 62856 f_{n+3} - 4541 f_{n+4} + 4407435 f_{n+\frac{2}{3}} + 2338488 f_{n+\frac{5}{3}} \right] - \frac{3}{2} y_{n+1} + \frac{3}{2} y_{n+\frac{5}{3}} \quad (13)$$

$$h y'_{n+\frac{2}{3}} = -\frac{1}{34292160} h^2 \left[9737 f_n + 16336768 f_{n+1} - 1127175 f_{n+2} + 37192 f_{n+3} - 2197 f_{n+4} + 3367251 f_{n+\frac{2}{3}} + 4239864 f_{n+\frac{5}{3}} \right] - \frac{3}{2} y_{n+1} + \frac{3}{2} y_{n+\frac{5}{3}} \quad (14)$$

$$hy'_{n+1} = -\frac{1}{238140}h^2\left[441f_n + 63518f_{n+1} - 11739f_{n+2} + 459f_{n+3} - 29f_{n+4} - 11169f_{n+\frac{2}{3}} + 37899f_{n+\frac{5}{3}}\right] - \frac{3}{2}y_{n+1} + \frac{3}{2}y_{n+\frac{5}{3}} \tag{15}$$

$$hy'_{n+\frac{5}{3}} = \frac{1}{2143260}h^2\left[3514f_n + 294770f_{n+1} - 125202f_{n+2} + 4331f_{n+3} - 266f_{n+4} - 80190f_{n+\frac{2}{3}} + 617463f_{n+\frac{5}{3}}\right] - \frac{3}{2}y_{n+1} + \frac{3}{2}y_{n+\frac{5}{3}} \tag{16}$$

$$hy'_{n+2} = \frac{1}{3810240}h^2\left[4095f_n + 429184f_{n+1} + 347151f_{n+2} + 2808f_{n+3} - 211f_{n+4} - 103419f_{n+\frac{2}{3}} + 1860552f_{n+\frac{5}{3}}\right] - \frac{3}{2}y_{n+1} + \frac{3}{2}y_{n+\frac{5}{3}} \tag{17}$$

$$hy'_{n+3} = \frac{1}{238140}h^2\left[5418f_n + 196546f_{n+1} + 440958f_{n+2} + 77355f_{n+3} - 1402f_{n+4} - 87246f_{n+\frac{2}{3}} - 234729f_{n+\frac{5}{3}}\right] - \frac{3}{2}y_{n+1} + \frac{3}{2}y_{n+\frac{5}{3}} \tag{18}$$

$$hy'_{n+4} = -\frac{1}{3810240}h^2\left[410193f_n + 10650752f_{n+1} + 9931425f_{n+2} - 6600312f_{n+3} - 1065533f_{n+4} - 5670261f_{n+\frac{2}{3}} - 17816904f_{n+\frac{5}{3}}\right] - \frac{3}{2}y_{n+1} + \frac{3}{2}y_{n+\frac{5}{3}} \tag{19}$$

We transformed (8) – (19) and solved the resulting equations linearly using the Gaussian Elimination Method with the help of Maple17 Software to obtain the continuous schemes;

$$y_{n+\frac{2}{3}} = \frac{1}{1071630}h^2\left(104083f_n - 335272f_{n+1} - 108318f_{n+2} + 6596f_{n+3} - 485f_{n+4} + 365580f_{n+\frac{2}{3}} + 205956f_{n+\frac{5}{3}}\right) + \frac{2}{3}hy'_n + y_n$$

$$y_{n+1} = \frac{1}{282240}h^2\left(45199f_n - 159180f_{n+1} - 51807f_{n+2} + 3146f_{n+3} - 231f_{n+4} + 205335f_{n+\frac{2}{3}} + 98658f_{n+\frac{5}{3}}\right) + hy'_n + y_n$$

$$y_{n+\frac{5}{3}} = \frac{25}{4572288}h^2\left(52479f_n - 146300f_{n+1} - 68775f_{n+2} + 4050f_{n+3} - 295f_{n+4} + 274095f_{n+\frac{2}{3}} + 138762f_{n+\frac{5}{3}}\right) + \frac{5}{3}hy'_n + y_n$$

$$y_{n+2} = \frac{1}{4410}h^2\left(1547f_n - 3864f_{n+1} - 2100f_{n+2} + 124f_{n+3} - 9f_{n+4} + 8262f_{n+\frac{2}{3}} + 4860f_{n+\frac{5}{3}}\right) + 2hy'_n + y_n$$

$$y_{n+3} = \frac{3}{31360}h^2\left(5761f_n - 8484f_{n+1} + 2583f_{n+2} + 1310f_{n+3} - 57f_{n+4} + 29889f_{n+\frac{2}{3}} + 16038f_{n+\frac{5}{3}}\right) + 3hy'_n + y_n$$

$$y_{n+4} = \frac{8}{2205}h^2\left(196f_n - 504f_{n+1} - 21f_{n+2} + 332f_{n+3} + 15f_{n+4} + 1215f_{n+\frac{2}{3}} + 972f_{n+\frac{5}{3}}\right) + 4hy'_n + y_n$$

$$y'_{n+\frac{2}{3}} = \frac{1}{2143260}h\left(407029f_n - 1779568f_{n+1} - 548373f_{n+2} + 33032f_{n+3} - 2417f_{n+4} + 2268729f_{n+\frac{2}{3}} + 1050408f_{n+\frac{5}{3}}\right) + y'_n$$

$$y'_{n+1} = \frac{1}{141120} h\left(26579 f_n - 87584 f_{n+1} - 33789 f_{n+2} + 2056 f_{n+3} - 151 f_{n+4} + 169857 f_{n+\frac{2}{3}} + 64152 f_{n+\frac{5}{3}}\right) + y'_n$$

$$y'_{n+\frac{5}{3}} = \frac{5}{6858432} h\left(263137 f_n - 296800 f_{n+1} - 476175 f_{n+2} + 25400 f_{n+3} - 1805 f_{n+4} + 1535355 f_{n+\frac{2}{3}} + 1237032 f_{n+\frac{5}{3}}\right) + y'_n$$

$$y'_{n+2} = \frac{1}{8820} h\left(1687 f_n - 2128 f_{n+1} - 1743 f_{n+2} + 152 f_{n+3} - 11 f_{n+4} + 9963 f_{n+\frac{2}{3}} + 9720 f_{n+\frac{5}{3}}\right) + y'_n$$

$$y'_{n+3} = \frac{3}{15680} h\left(1113 f_n + 2464 f_{n+1} + 8169 f_{n+2} + 1784 f_{n+3} - 37 f_{n+4} + 4131 f_{n+\frac{2}{3}} - 1944 f_{n+\frac{5}{3}}\right) + y'_n$$

$$y'_{n+4} = \frac{2}{2205} h\left(91 f_n - 3472 f_{n+1} - 3192 f_{n+2} + 1928 f_{n+3} + 307 f_{n+4} + 2916 f_{n+\frac{2}{3}} + 5832 f_{n+\frac{5}{3}}\right) + y'_n \qquad (20)$$

3. PROPERTIES OF THE BLOCK METHOD

3. 1. Order and Error constant

Let $\mathcal{L}$ be the Linear Difference Operator defined by;

$$\mathcal{L}[y(x);h] = \sum_{i=0}^{k} [\alpha_i \; y(x+ih) - h^2 \beta_i \, y''(x+ih)] \qquad (21)$$

where the function $y(x)$ is an arbitrary function that is continuously differentiable on $[a,b]$.

We used Taylor Series to expand (21) about $y(x)$ and we collected the coefficients of power of h as;

$$\mathcal{L}[y(x);h] = c_0 \, y(x) + c_1 \, hy'(x) + c_2 \, h^2 y''(x) + \cdots + c_q \, h^q y^q(x) + 0(h^{q+1}) \qquad (22)$$

where c_k are constants.

$$c_0 = \sum_{i=0}^{k} \alpha_i = \alpha_0 + \alpha_1 + \alpha_2 + \dots\dots + \alpha_k$$

$$c_1 = \sum_{i=0}^{k} i\alpha_i = (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_2 = \sum_{i=0}^{k} \frac{1}{2!} i^2 \alpha_i - \sum_{i=0}^{k} \beta_i = \begin{Bmatrix} \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + \cdots + k^2\alpha_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \cdots + k\beta_k) \end{Bmatrix}$$

Generally,

$$c_q = \sum_{i=0}^{k}\{\frac{1}{q!}i^q\alpha_i - \frac{1}{(q-2)!}i^{q-2}\beta_i\} \ \forall\ q = 0,1,2,\dots,p+2.$$

The difference operator (21) is said to be of order p if and only if in (22) $c_0 = c_1 = c_2 = \cdots c_{p+1} = 0$ and $c_{p+2} \neq 0$ *and* c_{p+2} is the error constant. This implies that the local truncation error is given as $T_{n+k} = c_{p+2}h^{p+2}y^{p+2}(x) + 0(h^{p+3})$.

We considered (9) and obtained its order and error constants as;

$$c_0 = -1 + \frac{3}{2} - \frac{1}{2} = 0$$

$$c_1 = -\frac{2}{3} + \frac{3}{2} - \frac{5}{6} = 0$$

$$c_2 = -\frac{2}{9} + \frac{3}{4} - \frac{25}{36} + \frac{3727}{9797760} - \frac{5}{6272} + \frac{326159}{2449440} + \frac{2221}{47040} - \frac{44953}{3265920} + \frac{17401}{34292160} - \frac{2153}{68584320} = 0$$

$$c_3 = -\frac{4}{81} + \frac{1}{4} - \frac{125}{324} - \frac{5}{9408} + \frac{326159}{2449440} + \frac{2221}{28224} - \frac{44953}{1632960} + \frac{17401}{11430720} - \frac{2153}{17146080} = 0$$

$$c_4 = -\frac{2}{243} + \frac{1}{16} - \frac{625}{3888} - \frac{5}{28224} + \frac{326159}{4898880} + \frac{11105}{169344} - \frac{44953}{1632960} + \frac{17401}{7610480} - \frac{2153}{8573040} = 0$$

$$c_5 = -\frac{4}{3645} + \frac{1}{80} - \frac{625}{11664} - \frac{5}{127008} + \frac{326159}{14696640} + \frac{55525}{1524096} - \frac{44953}{2449440} + \frac{17401}{7620480} - \frac{2153}{6429780} = 0$$

$$c_6 = -\frac{4}{32805} + \frac{1}{480} - \frac{3125}{209952} - \frac{5}{762048} + \frac{326159}{58786560} + \frac{277625}{18289152} - \frac{44953}{4898880} + \frac{17401}{10160640} - \frac{2153}{6429780} = 0$$

$$c_7 = -\frac{8}{688905} + \frac{1}{3360} - \frac{15625}{4408992} - \frac{1}{1143072} + \frac{326159}{293932800} + \frac{277625}{54867456} - \frac{44953}{12247200} + \frac{17401}{16934400} - \frac{2153}{8037225} = 0$$

$$c_8 = -\frac{2}{2066715} + \frac{1}{26880} - \frac{78125}{105815808} - \frac{1}{10287648} + \frac{326159}{1763596800} + \frac{1388125}{987614208} - \frac{44953}{36741600} + \frac{17401}{33868800} - \frac{4306}{24111675} = 0$$

$$c_9 = \frac{4}{55801305} + \frac{1}{241920} - \frac{390625}{2857026816} - \frac{1}{108020304} + \frac{326159}{12345177600} + \frac{6940625}{20739898368} - \frac{44953}{128595600} + \frac{17401}{79027200} - \frac{17224}{168781725} = -\frac{50473}{16665989760}$$

Hence, the method is of order 7 and $c_{p+2} \neq 0$; $c_9 = -\frac{50473}{16665989760}$. We repeated the same procedure for (10) – (12) which shows that the order of the block is $p = 7$ with the error constants; $C_{p+2} = \left[-\frac{50473}{16665989760}, -\frac{10369}{3333197952}, -\frac{3340}{26040609}, \frac{432493}{793618560}\right]^T$

3. 2. Consistency

A linear Multistep method is said to be consistent if order $p \geq 1$ and obeys the following axioms;

i. $\sum_{i=0}^{k} \alpha_i = 0$

ii. $\rho(r) = \rho'(r) = 0$

iii. $\rho''(r) = 2!\,\sigma(r)$

The first and second characteristics polynomial of our method are $\rho(r)$ and $\sigma(r)$ respectively.

According to [12], the sufficient condition for associated block method to be consistent is that $p \geq 1$. Since the proposed method is of order $p = 7$. Hence the proposed method is consistent.

3. 3. Zero Stability

Given block method as a single block r-point multistep method of the form;

$$A^{(0)}Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h^2 \sum_{i=0}^{k} B^i F_{m-i} \tag{23}$$

Applying the block in (20) we have;

$$\left|\det[\Omega I - A_1^{(1)}]\right| = \left\| \begin{bmatrix} \Omega & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\| = 0$$

$\Omega^5(\Omega - 1) = 0 \quad \rightarrow \quad \Omega_1 = 0, \Omega_2 = 0, \Omega_3 = 0, \Omega_4 = 0, \Omega_5 = 0, \Omega_6 = 1$

Since the roots of the polynomial satisfies $|\Omega| \leq 1$, and $|\Omega| = 1$ has multiplicity not greater than the order of the differential equation. This implies zero-stability, the proposed method is zero stable.

3. 4. Convergence

According to Fatunla 1971, A Linear Multistep Method is said to be convergent if the necessary and sufficient condition are satisfied i.e., consistent and zero stable. Hence the proposed method is convergent because it satisfies both consistency and zero-stability.

4. IMPLEMENTATION OF METHOD

In this section, we investigate the efficiency of our method on a linear problem, real life problem and system of equations of second order IVPs. We used Mathematica 11.3 software and Maple17 to compute the solution and absolute error of the approximate solutions were compared with the literature.

4. 1. Numerical problems

4. 1. 1. Cooling of a Body

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$, or the otherwise, find y in terms of t given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6In\ 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute.
Formulation of the Problem:

$$y'' = \frac{-y'}{3}, y(0) = 60, y'(0) = -\frac{80}{9}, h = 0.1 \qquad (24)$$

Exact Solution: $y(x) = \frac{80}{3}e^{-\frac{1}{3}x} + \frac{100}{3}$

4. 1. 2. System of equations

Consider the Stiefel and Bettis Problem:

$$y_1'' + y_1 = 0.001\cos(x), y_1(0) = 1, \ y_1'(0) = 0 \qquad h = \frac{1}{320}$$

$$y_2'' + y_2 = 0.001\sin(x), y_2(0) = 1, \qquad y_2'(0) = 0.9995 \qquad (25)$$

Exact solutions are given as;

$$y_1(x) = \cos(x) + 0.0005(x)\sin(x), \qquad y_2(x) = \sin(x) - 0.0005(x)\cos(x).$$

4. 1. 3. Slightly Stiff linear problem

$y'' = y'$ with the initial conditions $y(0) = 0, y'(0) = -1$, $h = 0.1$ (26)

Exact solution: $y(x) = 1 - e^x$

Table 1. Exact and Numerical Results of Problem 1

n	x_n	Exact solution	Computed solution	Error
0	0.0	1.000000000000000000	1.000000000000000000	0.00
1	0.1	59.125762679520157388	59.125762679520157532	1.44E-16
2	0.2	58.280186267509806339	58.280186267509806686	3.47E-16
3	0.3	57.462331147625588618	57.462331147625589314	6.96E-16
4	0.4	56.671288507811932107	56.671288507811932127	2.00E-17
5	0.5	55.906179330416375308	55.906179330416372921	2.39E-15
6	0.6	55.166153415412849564	55.166153415412844904	4.66E-15
7	0.7	54.450388435647511050	54.450388435647504326	6.72E-15
8	0.8	53.758089023057298472	53.758089023057288864	9.61E-15
9	0.9	53.088485884845809762	53.088485884845795829	1.39E-14
10	1.0	52.440834948634380011	52.440834948634361944	1.80E-14

Table 2a. Exact and Numerical Results of Problem 2

n	x_n	Exact solution	Computed solution	Error
0	0.0	1.000000000000000000	1.000000000000000000	0.00
1	0.003125	0.99999512207427819441	0.99999512207427819441	5.66851E-22
2	0.006250	0.99998048834470104865	0.99998048834470104865	9.22820E-22
3	0.009375	0.99995609895403291149	0.99995609895403291149	2.05284E-22
4	0.012500	0.99992195414021281668	0.99992195414021281668	3.01769E-21
5	0.015625	0.99987805423635216164	0.99987805423635216164	2.63888E-21

6	0.018750	0.99982439967073145770	0.99982439967073145770	1.88479E-22
7	0.021875	0.99976099096679615186	0.99976099096679615186	4.59462E-21
8	0.025000	0.99968782874315152015	0.99968782874315152015	3.80667E-21
9	0.028125	0.99960491371355663261	0.99960491371355663261	3.92052E-21
10	0.031250	0.99951224668691738996	0.99951224668691738996	1.20963E-21

Table 2b. Exact and Numerical Results for Problem 2

n	x_n	**Exact solution**	**Computed solution**	**Error**
0	0.0	1.000000000000000000	1.000000000000000000	0.00
1	0.003125	0.00312343242136885101	0.003123432421368851054	1.029194E-23
2	0.006250	0.00624683437101026369	0.006246834371010263687 2	2.885345E-23
3	0.009375	0.00937017537749407687	0.009370175377494076871 1	4.672538E-23
4	0.012500	0.01249342496998468092	0.01249342496998468092 0	3.866963E-22
5	0.015625	0.01561655267853828619	0.01561655267853828618 5	3.006767E-22
6	0.018750	0.01873952803440182810	0.01873952803440018281 1	4.670196E-22
7	0.021875	0.02186232057030198893	0.02186232057030198893 3	1.492241E-22
8	0.025000	0.02498489982075888438	0.02498489982075888438 0	1.541913E-22
9	0.028125	0.02810723532236682696	0.02810723532236682696 4	1.336198E-22
10	0.031250	0.0312292966140997484	0.03122929661409974848 4	1.882721E-22

Table 3. Exact and Numerical Results of Problem 3

n	x_n	**Exact solution**	**Computed solution**	**Error**
0	0.0	1.00000000000000000	1.00000000000000000	0.00
1	0.1	0.10517091807564762 48	0.10517091807550580842	1.418E-13

2	0.2	0.2214027581601698339	0.22140275815980660495	3.632E-13
3	0.3	0.3498588075760031040	0.34985880757524529372	7.578E-13
4	0.4	0.4918246976412703178	0.49182469764104531871	2.250E-13
5	0.5	0.6487212707001281468	0.64872127070219967247	2.072E-12
6	0.6	0.8221188003905089749	0.82211880039502205978	4.513E-12
7	0.7	1.0137527074704765216	1.0137527074774643484	6.988E-12
8	0.8	1.2255409284924676046	1.2255409285036358594	1.117E-11
9	0.9	1.4596031111569496638	1.4596031111752855615	1.834E-11
10	1.0	1.7182818284590452354	1.7182818284851586527	2.611E-11

5. DISCUSSION OF RESULTS

Table 1 – 3 shows the results of the proposed method with step number four and order of accuracy seven. The accuracy of the method for each corresponding problem are discussed below;

From Table 2a It was observed that the maximum absolute error of the proposed method is 9.22820E-22 which is (smaller) and more accurate. Also, the accuracy comparison in table 2b shows that the proposed method is substantially more accurate. The proposed method performed excellently in terms of efficiency and accuracy.

6. CONCLUSION

We explored an approach for solving second order ordinary differential equations by proposing an accurate implicit Hybrid-Block method through collocation and interpolation method that yields approximate solutions at suitable points when applied to solve Initial Value Problems (IVPs). The analysis of the basic properties shows that the method is consistent, convergent ad zero stable. The proposed method performed efficiently when applied to solve second order Initial Value Problems as seen in the low error constant and hence better approximation. The new method performed favourably by producing accurate and convergent results. Hence, it can be used to solve all kinds of second order initial value problems.

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