COMPARISON OF REFINEMENT ACCELERATED RELAXATION ITERATIVE TECHNIQUES AND CONJUGATE GRADIENT TECHNIQUE FOR LINEAR SYSTEMS

Khadeejah James Audu

Department of Mathematics, Federal University of Technology, Minna, Nigeria.
Corresponding email: k.james@futminna.edu.ng

Abstract

Iterative methods use consecutive approximations to get more accurate results. A comparison of three iterative approaches to solving linear systems of this type $My = B$ is provided in this paper. We surveyed the Refinement Accelerated Relaxation technique, Refinement Extended Accelerated Relaxation technique, and Conjugate Gradient technique, and demonstrated algorithms for each of these approaches in order to get to the solutions more quickly. The algorithms are then transformed into the Python language and used as iterative methods to solve these linear systems. Some numerical investigations were carried out to examine and compare their convergence speeds. Based on performance metrics such as convergence time, number of iterations required to converge, storage, and accuracy, the research demonstrates that the conjugate gradient method is superior to other approaches, and it is important to highlight that the conjugate gradient technique is not stationary. These methods can help in situations that are similar to finite differences, finite element methods for solving partial differential equations, circuit and structural analysis. Based on the results of this study, iteration techniques will be used to help analysts understand systems of linear algebraic equations.

Keywords: linear systems, Accelerated Relaxation technique, Refinement techniques, conjugate gradient, convergence

1. Introduction

Linear systems are related to numerous difficulties in engineering and science as well as with applications of mathematics to social sciences and quantitative study of business, statistics, and economic problems. After the creation of access to computers, it is conceivable and relatively easy for us to solve a vast set of simultaneous linear algebraic equations. To grasp the decision of physical issues, it is sometimes appropriate to employ algorithms that converge to quickly solve these situations. The most challenging task is to solve square linear systems. Identifying solutions to the system of linear equations recognized as;

$$My = B \tag{1}$$

where M is a matrix (square) whose unique solution has the following form.

$$y = M^{-1}B \tag{2}$$

Assume M have non zero diagonals, then it is possible to decompose M into

$$M \equiv D - W - X \tag{3}$$

where D is the diagonal unit, W is strict lower and X is strict upper units of M. Again, further decomposing of M in the format

$$M = J - K \tag{4}$$

And substituting it (4) into (1), results into:

$$\left.\begin{aligned} (J-K)y &= B \\ Jy &= Ky + B \\ Jy^{(i+1)} &= Ky^{(i)} + B \\ y^{(i+1)} &= Ny^{(i)} + z \end{aligned}\right\} \tag{5}$$

where $N = J^{-1}K$ designates the iteration matrix and $e = J^{-1}B$ designates the iteration approach. By relating $D^{-1}M = \mathrm{I} - \mathrm{L} - \mathrm{U}$ to (3), we achieve $L = D^{-1}J$, $U = D^{-1}K$, $z = D^{-1}B$ and $I = D^{-1}D$. The convergence rate of any stationary iteration technique is evaluated through the spectral radius $\rho(N)$, which is represented as part of the iteration matrix. Typically, the rate of convergence of stationary iterative techniques rises when the spectral radius is small and reduces when the spectral radius is high. Several authors have come up with different iterative schemes based on the idea in (5), such as Audu *et al*. (2021a), Vatti *et al*. (2020a), Audu *et al*. (2021c) and Vatti *et al*. (2020b).

2. Literature Review

In order to improve the accuracy of numerical estimations for systems of linear equations (1), refinement of iterative techniques was introduced in the fifteenth century. It's for this reason that some researchers are looking into improving the convergence rates of various iterative techniques for solving linear systems by researching their refinement. Some authors have developed different refinement iterative techniques in this regard. Authors like Assefa and Teklehaymanot (2021) Audu *et al*. (2021b), Vatti *et al*. (2018), Dessalew *et al*. (2021) and Tesfaye *et al*. (2020).

A method that can also be used iteratively to solve large sparse linear systems is known as the conjugate gradient method. It is a useful tool in the process of approximating solutions to linearized partial differential equations since it is a technique that combines linear algebra and matrix manipulation. Abdelwasi and Shiker (2020) proposed an approach to solve large-scale nonlinear monotone equation systems. The recommended method can tackle large-scale problems without storing Jacobian matrix data or matrices at each iteration. With standard conditions, they established global convergence for the novel method, and numerical experiments show that it is promising and efficient. Wakili and Sadiq (2021) compared some stationary iterative methods and non-stationary methods for rapid solutions to linear equations. They contrast three iterative methods (Jacobi, Gauss-Seidel, and Conjugate Gradient) for solving linear systems of equations. Shareef and Sulaiman (2021) improved on the conjugate gradient technique and examine its performance on system of linear equations.

3. Methods

3.1. Refinement Accelerated Relaxation Technique (RART)

From the scheme in (5), the refinement of Accelerated Relaxation Technique (RART) is given as $y^{(i+1)} = N_{0,\Gamma,w}y^{(i)} + Pz$ or in the format;

$$y^{(i+1)} = \left((I-\Gamma\mathcal{L})^{-1}\left((1-w)I + [w-\Gamma]L + w\mathrm{U}\right)\right)^2 y^{(i)} +$$
$$\left(I + (I-\Gamma\mathcal{L})^{-1}\left((1-w)I + [w-\Gamma]L + w\mathrm{U}\right)\right)(I-\Gamma L)^{-1}wz \quad (6)$$

with iteration matrix as $RN_{0,\Gamma,w} = \left((I-\Gamma\mathcal{L})^{-1}\left((1-w)I + [w-\Gamma]L + w\mathrm{U}\right)\right)^2$ and its spectral radius is represented as $\rho R\left(N_{0,\Gamma,w}\right)$. The RART technique is convergent for symmetric, positive definite, Hermitian, M, L as well as H matrices.

3.1.1. Algorithm for RART Technique

1. Enter the entries of matrix M, select an initial guess y^0, maximum iteration number tolerance (ε) and $0, \Gamma, w \in (0,1)$
2. Get the diagonal and triangular matrices U, D and L from matrix M and $D^{-1}M$
3. Get the inverse of matrix $[I - \Gamma\mathrm{L}]$
4. Establish $A = (I-\Gamma L)^{-1}\left((1-w)I + [w-\Gamma]L + w\mathrm{U}\right)$
5. Establish $A1 = \left((I-\Gamma L)^{-1}\left((1-w)I + [w-\Gamma]L + w\mathrm{U}\right)\right)^2$
6. Establish $X = (I + A1)(I-\Gamma L)^{-1}wz$
7. Calculate $y^{(i+1)} = A1y^{(i)} + X$
8. End if $\left\|y^{(i+1)} - y^{(i)}\right\|_\infty < \varepsilon$

3.2. Refinement Extended Accelerated Relaxation Technique (REART)

Also utilizing the iterative approach (5), the refinement of Extended Accelerated Relaxation Technique (REART) is given as $y^{(i+1)} = N_{\alpha,\Gamma,w} y^{(i)} + Pz$ or explicitly as;

$$y^{(i+1)} = \left((I-(\alpha+\Gamma)L)^{-1}\left((1-w)I+[w-(\alpha+\Gamma)]L+w\mathrm{U}\right)\right)^2 y^{(i)} + \left(\mathcal{I}+(I-(\alpha+\Gamma)L)^{-1}\left((1-w)I+[w-(\alpha+\Gamma)]L+w\mathrm{U}\right)\right)(I-(\alpha+\Gamma)\mathrm{L})^{-1}wz \quad (7)$$

Its iteration matrix is $RN_{\alpha,\Gamma,w} = \left((I-(\alpha+\Gamma)L)^{-1}\left((1-w)I+[w-(\alpha+\Gamma)]L+w\mathrm{U}\right)\right)^2$ and the spectral radius is designated as $\rho R\left(N_{0,\Gamma,w}\right)$. The REART technique is convergent for symmetric, symmetric positive definite, M, L as well as H matrices

3.2.1. Algorithm for REART Technique

1. Enter the entries of matrix M, choose an initial estimate y^0, maximum iteration number tolerance (ε) and $\alpha, \Gamma, w \in (0,1)$
2. Get the diagonal and triangular matrices U, D and L from matrix M and $D^{-1}M$
3. Get the inverse of matrix $[\mathcal{I} - \alpha L - \Gamma L]$
4. Establish $C = (\mathcal{I}-(\beta+\mathscr{r})\mathcal{L})^{-1}\left((1-w)\mathcal{I}+[w-(\beta+\mathscr{r})]\mathcal{L}+w\mathcal{U}\right)$
5. Establish $C1 = \left((I-(\alpha+\Gamma)L)^{-1}\left((1-w)I+[w-(\alpha+\Gamma)]L+w\mathrm{U}\right)\right)^2$
6. Establish $F = (I+1)(I-(\alpha+\Gamma)\mathrm{L})^{-1}wz$
7. Calculate $y^{(i+1)} = C1y^{(i)} + F$
8. End if $\left\|y^{(i+1)} - y^{(i)}\right\|_\infty < \varepsilon$

3.3. Conjugate Gradient Technique

The conjugate gradient method is a mathematical algorithm for solving specific systems of linear equations, namely those with positive-definite matrices. The conjugate gradient method is frequently implemented as an iterative algorithm for sparse systems that are too large for direct implementation or other direct methods like the Cholesky decomposition. When numerically solving partial differential equations or optimization problems, large sparse systems frequently emerge. Unconstrained optimization problems, such as energy minimization, can also be solved using the conjugate gradient method. The conjugate gradient technique iteratively solves sparse linear systems. It approximates solutions to linearized partial differential equations using linear algebra and matrix manipulation. The conjugate gradient technique is derived from fundamental concepts. The conjugate gradient algorithm is presented below.

3.3.1. Algorithm for Conjugate gradient technique

1. Input initial estimate $y_0 = 0$
2. Set $r_0 = B - My_0$, $q_0 = r_0$ and $f = 0$
3. While r_f is not equal zero, construct the step size $\alpha_f = \dfrac{r_f^T r_f}{q_f^T q_f}$
4. Construct next iterate by stepping in direction q_f ; $y_{f+1} = y_f + \alpha_f q_f$
5. Construct new residual $r_{f+1} = r_f - \alpha_f Mq_f$
6. Construct scalar for linear combination for next direction $\beta_f = \dfrac{r_{f+1}^T r_{f+1}}{r_f^T r_f}$
7. Construct next conjugate vector $q_{f+1} = r_f + 1 + \beta_f q_f$
8. Increment $f = f + 1$
9. End the iteration and return y_{f+1} as the result

4. Investigation and Discussion of Results

This section performs some numerical experiments and presents the numerical results of the investigation. Two problems were solved using the refinement accelerated relaxation, extended refinement accelerated relaxation and conjugate gradient techniques. Python software was utilized for computing the solutions and the results are presented in Tables 1 - 3.

Problem 1: We consider a 5 x 5 linear system of (1), expressed in the format $My = B$;

$$\begin{pmatrix} 8.00 & -2.00 & -1.00 & 0.00 & 0.00 \\ -2.00 & 9.00 & -4.00 & -1.00 & 0.00 \\ -1.00 & -3.00 & 7.00 & -1.00 & -2.00 \\ 0.00 & -4.00 & -2.00 & 12 & -5.00 \\ 0.00 & 0.00 & -7.00 & -3.00 & 15 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 5.00 \\ 2.00 \\ 1.00 \\ 1.00 \\ 5.00 \end{pmatrix} \tag{8}$$

The exact solution is given as

$$y = M^{-1}B = \begin{pmatrix} 1.08250351535643 \\ 1.17591184318148 \\ 1.30820443651700 \\ 1.18538181190214 \\ 1.18090509942105 \end{pmatrix} \tag{9}$$

Problem 2: We consider the system of $9 \ x \ 9$ linear equations in the form $My = B$;

$$\begin{pmatrix} 1 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 7.5 \\ 2.5 \\ 2.5 \end{pmatrix} \tag{10}$$

The exact solution is given as

$$y = M^{-1}B = \begin{pmatrix} 7.468762595727535 \\ 9.875050382910125 \\ 4.38129786376461 \\ 13.7831519548569 \\ 7.65014107214833 \\ 3.314792422410320 \\ 13.2261185006046.5 \\ 9.12132204756147.5 \\ 5.60902861749295 \end{pmatrix} \tag{11}$$

Problem 3: We consider an applied problem of a metallic plate relating to heat transfer in Figure 1 (Source; Audu (2021)).

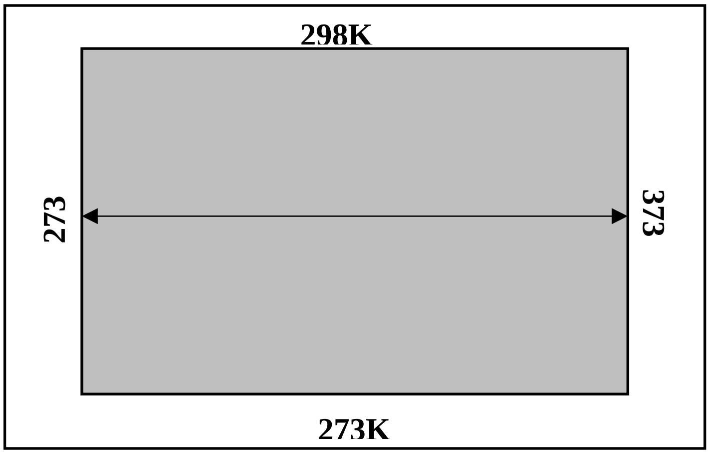

Figure 1: A Metallic plate with fixed Temperature

The metallic plate was discretized using central finite difference approach to obtain a sparse linear system $(My = B)$ of 64 equations and 64 unknowns. The linear equations are represented in equation (10);

$$\begin{pmatrix}
4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 4
\end{pmatrix} \times \begin{pmatrix} y_{1,1} \\ y_{2,1} \\ y_{3,1} \\ y_{4,1} \\ y_{5,1} \\ y_{6,1} \\ y_{7,1} \\ y_{8,1} \\ y_{1,2} \\ y_{2,2} \\ y_{3,2} \\ \vdots \\ y_{5,8} \\ y_{6,8} \\ y_{7,8} \\ y_{8,8} \end{pmatrix} = \begin{pmatrix} 546 \\ 273 \\ 273 \\ 273 \\ 273 \\ 273 \\ 273 \\ 646 \\ 273 \\ 0 \\ 0 \\ \vdots \\ 298 \\ 298 \\ 298 \\ 671 \end{pmatrix} \tag{12}$$

The exact solution is given as

$$y = \begin{bmatrix} 274.64, & 276.39, & 278.41, & 280.93, & 284.36, & 289.53, & 298.58, & 318.20, & 276.16, & 279.52, & 283.33, \\ 153287.96, & 293.97, & 302.17, & 313.58, & 328.23, & 277.49, & 282.20, & 287.42, & 293.61, & 301.40, & 311.61, \\ 325.34, 154344.23, & 278.60, & 284.34, & 290.53, & 297.67, & 306.40, & 317.53, & 331.92, & 350.36, & 279.57, \\ 286.05, & 292.71, 155300.12, & 309.01, & 320.19, & 334.47, & 352.30, & 280.63, & 287.56, & 294.14, & 301.10, \\ 309.32, & 319.75, & 333.47, 156351.33, & 282.39, & 289.43, & 295.19, & 300.84, & 307.40, & 316.03, & 328.31, \\ 346.58, & 286.49, & 292.57, & 296.35, 157299.65, & 303.43, & 308.66, & 317.16, & 333.69 \end{bmatrix}^T \tag{13}$$

Table 1: Convergence Result for Problem 1

Iteration Techniques	Number of iterations	CPU time (seconds)
RART	43	0.420
REART	31	0.360
Conjugate Gradient	5	0.050

Table 2: Convergence Result for Problem 2

Iteration Techniques	Number of iterations	CPU time (seconds)
RART	32	0.360
REART	21	0.280
Conjugate Gradient	8	0.070

Table 3: Convergence Result for Problem 3

Iteration Techniques	Number of iterations	CPU time (seconds)
RART	226	0.500
REART	150	0.301
Conjugate Gradient	63	0.150

According to Tables 1-3, the conjugate gradient technique outperforms the other two techniques and is the fastest in terms of system convergence. As for the refinement techniques, the refinement extended accelerated relaxation technique converges faster than the refinement accelerated relaxation technique. The conjugate gradient method converges faster and is more accurate than the refinement techniques, due to the fact that it takes a limited number of iterations to arrive at the final solution. It is observed that the refinement extended accelerated relaxation technique requires fewer iterations to reach its final solution than the refinement accelerated relaxation technique.

5. Conclusion

Three iterative techniques for solving linear systems, namely: refinement extended accelerated relaxation, conjugate gradient, and refinement accelerated relaxation, were examined and analysed in this study. The results obtained indicate that the conjugate gradient method converges faster than the refinement extended accelerated. The conjugate gradient uses a small number of iterations to reach its final solutions compared to the refinement relaxation techniques employed in the study. As a result of this, the conjugate gradient technique can be considered more reliable and accurate when finding solutions to linear systems than the compared methods. The three techniques are capable of solving linear equations effectively. However, the conjugate technique is considered the best among them.

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