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Application of open network queuing model in a banking system to reduce waiting time of customers

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Abstract

The need to use scientific techniques to determine optimal number of banking personnel (Servers) at different units of a banking system is the thrust of this work. In this paper, a network queuing model that determines optimal numbers of servers at different nodes of First City Monument Bank Minna Branch to reduce waiting time is presented. The relevant data were collected for a period of four (4) weeks, through direct observations and personal interview. The total expected waiting time of customers in the current system before modification was 52 minutes with total number of 11 servers in the system while the total new expected waiting time of the customers in the system after modification was reduced to 11 minutes with optimal number of 17 servers (personnel) in all the nodes. The study has determined optimal number of servers (personnel) at different nodes of the bank network system. Result from this study is important information to the management of the First City Monument Bank, Minna branch for efficient and better service delivery.

Keywords: FCMB; nodes; servers; customers; waiting time; network of queue.

1. Introduction

A Common situation that occurs in everyday life is that of queuing or waiting in line, when the demand for a service exceeds the capacity of the service, waiting is unsurprising and inevitable [14]. Queues or waiting lines are usually seen at hospitals, bus stops, supermarkets, traffic, airports, gas stations, bank counters and so on. Service delay is unavoidable as a system gets blocked [13]. When too much service is been provided, it's involves excessive cost and not providing enough service capacity causes the waiting line to become excessively long. The ultimate goal is to achieve an economic balance between the cost of service and the cost associated with the waiting for that service. Queuing systems theories have been used to study waiting time and predict the efficiency of services to be provided. In queuing theory, there are three basic components of a queuing process which are: - Arrivals patterns, the actual waiting line and service facilities. Customers arrive to the facility from an infinite calling population, with a random arrival pattern following

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poison process. Once customers arrive, they are served immediately if the server(s) is empty, or otherwise the customers wait in the queue for the next empty server. Mostly, the service is on a first come first serve (FCFS) basis although other methods like service at random order (SARO) can be used. Preference service depending on the level of risk, urgency or the social, economic or political standing of the customers and hold on line (HL) discipline, where important arriving customer takes the lead of the queue is rampant in many facilities. Customers who may feel to have waited for long in queue can balk or renege and seek alternative equivalent services elsewhere, however, the queue length and waiting time depends on the traffic intensity, which is the ratio of arrival and service rates. The service discipline follows an exponential pattern, with individual service time variation due to different nature of the problems to be handled [19]. Successful application of queuing models has been reported in the following works: ([1], [3], [11], [16]).

2. Materials and methods

The type of queuing system adopted by an organization solely dependent on the type of service being provided [11]. First City Monument Bank practice network type of queuing system. Queuing network is composed of several random queue systems, mostly limited and single queue systems. Diverse types of customers go by through the network in many ways and are served by the service nodes within the network system. A queuing network system has a set of nodes (i). Each node has a number of servers (s) and a single node can be regarded as a queuing system. customers can have access to the queuing network from any node. The arrival rate from the outside is λ and the arrival rate of Node i is λ_i. After the customer queues and gets the service at a node (the service rate of Node i is μ_i), he /she can leave the network system or go to another node, or even return to the former node [11].

2.1. Model formulation

The First City Monument Bank Minna branch consists of five main units, they are the Meter Greeter Unit, Customers Service Unit, Marketing Unit, Tellers Unit, and Customers Service Manager Unit. In this study, each department is regarded as a node of the network system. The data used in this research were collected from the five different departments of the Bank and they were collected based on the arrival and departure rate as well as time spent at each node. The method adopted for the data collection was direct observation and personal interview, for a period of one month. The collection of the data was for a total of six (6) hours at different time of the day, for each node. In a day, the number of arrivals and departures together with service time were taken at intervals of 5 minutes arrivals of customers into a node (λ), while the departure rate was obtained also by the average number of five (5) minutes departures of customers at that particular node.

2.2. Model Assumptions

The following are the model assumptions made for Network Queuing System of the First City Monument Bank (FCMB), Minna Branch.

i. The First City Monument Bank in the network queuing system is considered as an independent queuing system.
ii. Queuing discipline is usually first come first served in the bank.
iii. The arrival of the customers in the bank follows a Poisson arrival process.
iv. The service follows exponential distribution

v. The way customers enter the bank is not restricted.
vi. The banking personnel are regarded as servers
vii. All the banking personnel in the bank are working in full capacity
viii. Service rate is independent of line length.

We consider a banking network queuing system based on Jackson open network queuing model, the First City Monument Bank, Minna constitutes of five units. In this study, we assumed that customers who come in to bank for services will start by going first to the meter greeter unit and then move to the customer's service unit, then some customers proceed to tellers unit or customers service manager until all customers depart from the bank as captured by Figure 1 below,

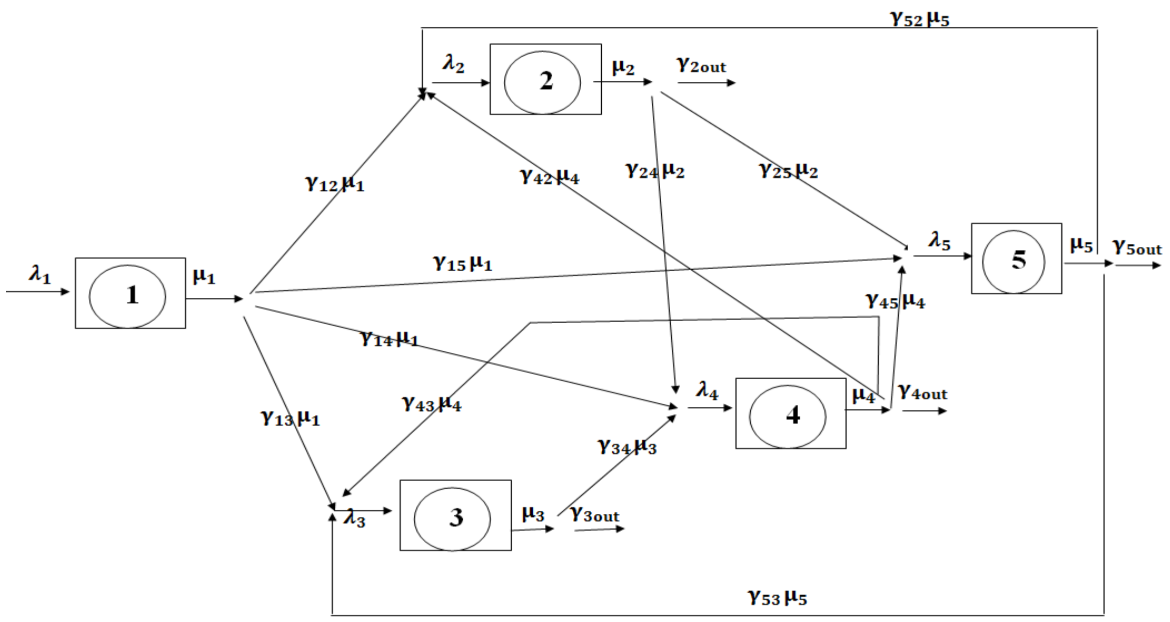

Figure 1 A Schematic diagram of the Bank (FCMB) Queuing Network

where: λ_i Is the arrival rate of the customer, for i = 1, 2,...5,
μ_i Is the departure rate out of the system, for i = 1, 2,…5,
γ_{ij} Are the weights of moving from server i to server j,
m_i for i= 1…,5, is the number servers at the various node points in the system.

The following are the nodes in the network queuing system of the bank where Node1, Node2, Node3, Node4, Node5 are defined as follow: Meter Greeter Unit denoted by node1; Customer Service Unit denoted by node2; Marketing Unit denoted by node3; Tellers Unit denoted by node4; Customer Service Manager Unit denoted by node5.

From Figure1, we obtained the following equations

$$\lambda_2 = \gamma_{12}\mu_1 + \gamma_{42}\mu_4 + \gamma_{52}\mu_5, \tag{1}$$

$$\lambda_3 = \gamma_{13}\mu_1 + \gamma_{43}\mu_4 + \gamma_{53}\mu_5, \tag{2}$$

$$\lambda_4 = \gamma_{14}\mu_1 + \gamma_{24}\mu_2 + \gamma_{34}\mu_3, \tag{3}$$

$$\lambda_5 = \gamma_{15}\mu_1 + \gamma_{25}\mu_2 + \gamma_{45}\mu_4. \tag{4}$$

Also,

$$\mu_1 = \gamma_{12}\mu_1 + \gamma_{13}\mu_1 + \gamma_{14}\mu_1 + \gamma_{15}\mu_1, \tag{5}$$

$$\mu_2 = \gamma_{24}\mu_2 + \gamma_{25}\mu_2 + \gamma_{2out}\mu_2, \tag{6}$$

$$\mu_3 = \gamma_{34}\mu_3 + \gamma_{3out}\mu_3, \tag{7}$$

$$\mu_4 = \gamma_{42}\mu_4 + \gamma_{43}\mu_4 + \gamma_{45}\mu_4 + \gamma_{4out}\mu_4, \tag{8}$$

$$\mu_5 = \gamma_{52}\mu_5 + \gamma_{53}\mu_5 + \gamma_{5out}\mu_5, \tag{9}$$

where,
$\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{24}, \gamma_{25}, \gamma_{2out}, \gamma_{34}, \gamma_{3out}, \gamma_{42}, \gamma_{43}, \gamma_{45}, \gamma_{4out}, \gamma_{52}, \gamma_{53}, \gamma_{5out}$
are to be determined.

Equation (1 – 9) can also be represented in the following forms, thus:

$$\begin{aligned}\lambda_2 = {} & \mu_1\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} \\ & + \mu_4\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + \mu_5\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out}\end{aligned} \tag{10}$$

$$\begin{aligned}\lambda_3 = {} & 0\gamma_{12} + \mu_1\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} \\ & + 0\gamma_{42} + \mu_4\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + \mu_5\gamma_{53} + 0\gamma_{5out}\end{aligned} \tag{11}$$

$$\begin{aligned}\lambda_4 = {} & 0\gamma_{12} + 0\gamma_{13} + \mu_1\gamma_{14} + 0\gamma_{15} + \mu_2\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + \mu_3\gamma_{34} + 0\gamma_{3out} \\ & + 0\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out}\end{aligned} \tag{12}$$

$$\begin{aligned}\lambda_5 = {} & 0\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + \mu_1\gamma_{15} + 0\gamma_{24} + \mu_2\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} \\ & + 0\gamma_{42} + 0\gamma_{43} + \mu_4\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out}\end{aligned} \tag{13}$$

$$\mu_1 = \mu_1\gamma_{12} + \mu_1\gamma_{13} + \mu_1\gamma_{14} + \mu_1\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out}$$

$$+0\gamma_{42}+0\gamma_{43}+0\gamma_{45}+0\gamma_{4out}+0\gamma_{52}+0\gamma_{53}+0\gamma_{5out}\,, \tag{14}$$

$$\mu_2 = 0\gamma_{12}+0\gamma_{13}+0\gamma_{14}+0\gamma_{15}+\mu_2\gamma_{24}+\mu_2\gamma_{25}+\mu_2\gamma_{2out}+0\gamma_{34}+0\gamma_{3out} \\ +0\gamma_{42}+0\gamma_{43}+0\gamma_{45}+0\gamma_{4out}+0\gamma_{52}+0\gamma_{53}+0\gamma_{5out}\,, \tag{15}$$

$$\mu_3 = 0\gamma_{12}+0\gamma_{13}+0\gamma_{14}+0\gamma_{15}+0\gamma_{24}+0\gamma_{25}+0\gamma_{2out}+\mu_3\gamma_{34}+\mu_3\gamma_{3out} \\ +0\gamma_{42}+0\gamma_{43}+0\gamma_{45}+0\gamma_{4out}+0\gamma_{52}+0\gamma_{53}+0\gamma_{5out}, \tag{16}$$

$$\mu_4 = 0\gamma_{12}+0\gamma_{13}+0\gamma_{14}+0\gamma_{15}+0\gamma_{24}+0\gamma_{25}+0\gamma_{2out}+0\gamma_{34}+0\gamma_{3out} \\ +\mu_4\gamma_{42}+\mu_4\gamma_{43}+\mu_4\gamma_{45}+\mu_4\gamma_{4out}+0\gamma_{52}+0\gamma_{53}+0\gamma_{5out}, \tag{17}$$

$$\mu_5 = 0\gamma_{12}+0\gamma_{13}+0\gamma_{14}+0\gamma_{15}+0\gamma_{24}+0\gamma_{25}+0\gamma_{2out}+0\gamma_{34}+0\gamma_{3out} \\ +0\gamma_{42}+0\gamma_{43}+0\gamma_{45}+0\gamma_{4out}+\mu_5\gamma_{52}+\mu_5\gamma_{53}+\mu_5\gamma_{5out}. \tag{18}$$

From model equation (10 - 18) can be represented in the matrix form as:

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 \\ 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & \mu_2 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & \mu_4 & \mu_4 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & \mu_5 & \mu_5 \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} \tag{19}$$

Equation (19) can be represented in the form

$$\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 \\ 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & \mu_2 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & \mu_4 & \mu_4 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & \mu_5 & \mu_5 \end{bmatrix}^{+} \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} \tag{20}$$

(Note: the " + " sign above the coefficient matrix indicates that is a Pseoudo Inverse. This is because is a non square matrix)

2.3. Mathematical Formulation for new Departure Rate

Reducing waiting time of the customers in the banking hall and increasing the efficiency of the bank is thrust of this research, hence we formulate new departure rate of each of the nodes in our network system. This is done using equation (1 – 4), thus, we have the following equations.

$$\lambda_2 = \gamma_{12}\mu_1 + 0\mu_2 + 0\mu_3 + \gamma_{42}\mu_4 + \gamma_{52}\mu_5, \tag{21}$$

$$\lambda_3 = \gamma_{13}\mu_1 + 0\mu_2 + 0\mu_3 + \gamma_{43}\mu_4 + \gamma_{53}\mu_5, \tag{22}$$

$$\lambda_4 = \gamma_{14}\mu_1 + \gamma_{24}\mu_2 + \gamma_{34}\mu_3 + 0\mu_4 + 0\mu_5, \tag{23}$$

$$\lambda_5 = \gamma_{15}\mu_1 + \gamma_{25}\mu_2 + 0\mu_3 + \gamma_{45}\mu_4 + 0\mu_5. \tag{24}$$

Model equation (21-24) can be transform to matrix as in equation (25):

$$\begin{bmatrix} \gamma_{12} & 0 & 0 & \gamma_{42} & \gamma_{52} \\ \gamma_{13} & 0 & 0 & \gamma_{43} & \gamma_{53} \\ \gamma_{14} & \gamma_{24} & \gamma_{34} & 0 & 0 \\ \gamma_{15} & \gamma_{25} & 0 & \gamma_{45} & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}, \tag{25}$$

where,

the arrival rate $\lambda_i = \frac{1}{mean\ number\ of\ arrival}$,for i = 1,2,...5, (26)

the departure rate $\mu_i = \frac{1}{mean\ number\ of\ departure}$,for i= 1,2,…5, (27)

and

$$\rho = \frac{\lambda_i}{\mu_i}, \quad for\ i = 1, 2, \cdots 5. \tag{28}$$

The expected number in the queue is given as

$$l_q = \frac{\rho}{m-\rho}, \tag{29}$$

where m stands for the number of servers at the node.

The expected waiting time in the queue is given as:

$$w_{qi} = \frac{l_q}{\lambda_i}, \tag{30}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho. \tag{31}$$

Finally, the expected waiting time in the system for node 1-5 is given as

$w_i = \frac{l_s}{\lambda_i}$, For i = 1,2,…5.

3. Results and discussion

We begin this section with computation of mean arrival and departure time for all the nodes in the network system. The results of the computation obtained from the raw data collected for four weeks is presented in Table 1 below.

	Node 1	Node 2	Node 3	Node 4	Node 5
Mean arrival	1.920	1.712	1.366	1.732	1.343
Mean departure	1.548	1.465	1.112	1.308	1.213

Table 1: Showing Data analysis for the Mean arrival as well as Mean departure

From Table 1 above, we obtained the Mean arrival and Mean departure for each node, the expected number of customers in the system, the expected number of customers in the queue and the expected waiting time in the system.

For node1, the arrival rate: $\lambda_1 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.920} = 0.52$ person per minute.
For node2, the arrival rate: $\lambda_2 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.712} = 0.58$ person per minute.
For node3, the arrival rate: $\lambda_3 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.366} = 0.73$ person per minute.
For node4, the arrival rate: $\lambda_4 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.732} = 0.58$ person per minute.
For node5, the arrival rate: $\lambda_2 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.343} = 0.74$ person per minute.

The departure rate for nodes is given as:

$\mu_1 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.548} = 0.646$
$\mu_2 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.465} = 0.683$
$\mu_3 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.112} = 0.899$
$\mu_4 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.308} = 0.765$
$\mu_5 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.213} = 0.824$.

Node 1:
$\rho = \frac{\lambda_1}{\mu_1} = \frac{0.52}{0.646} = 0.8$.

The expected number of customer in the queue is given as
$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{1-0.8} = 4.0$,
where m stands for the number of servers at the node1 (Meter Greeter Unit) .

The expected waiting time in the queue is given as
$w_{q1} = \frac{l_q}{\lambda_1} = \frac{4.0}{0.52} = 7.69$ minutes.

The expected number of customers in the system is given as
$l_s = l_q + \rho = 4.0 + 0.8 = 4.8$.

The expected waiting time in the system for node1 is given as
$w_1 = \frac{l_s}{\lambda_1} = \frac{4.8}{0.52} = 9.2$ minutes.

Node 2:
$\rho = \frac{\lambda_2}{\mu_2} = \frac{0.58}{0.683} = 0.8$.
The expected number of customers in the queue is given as
$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{2-0.8} = 0.7$,
where m stands for the number of servers at the node2 (Customer Service Unit).

The expected waiting time in the queue is given as
$w_{q2} = \frac{l_q}{\lambda_2} = \frac{0.7}{0.58} = 1.2$ minutes.
The expected number of customers in the system is given as
$l_s = l_q + \rho = 0.7 + 0.8 = 1.5$.
The expected waiting time in the system for node 2 is given as
$w_2 = \frac{l_s}{\lambda_2} = \frac{1.5}{0.58} = 26$ minutes.

Node 3:
$\rho = \frac{\lambda_3}{\mu_3} = \frac{0.73}{0.899} = 0.8$.

The expected number of customers in the queue is given as
$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{3-0.8} = 0.4$,
where m stands for the number of servers at the node3 (Marketing Unit).

The expected waiting time in the queue is given as
$w_{q3} = \frac{l_q}{\lambda_3} = \frac{0.4}{0.73} = 0.54$ Minute
The expected number of customers in the system is given as
$l_s = l_q + \rho = 0.4 + 0.8 = 1.2$
The expected waiting time in the system for node3 is given as
$w_3 = \frac{l_s}{\lambda_3} = \frac{1.2}{0.73} = 1.6 Minutes$.

Node 4:
.
$\rho = \frac{\lambda_4}{\mu_4} = \frac{0.58}{0.765} = 0.8$
The expected number of customers in the queue is given as
$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{4-0.8} = 0.3$,
where m stands for the number of servers at the node4 (Tellers Unit).

The expected waiting time in the queue is given as

$$w_{q4} = \frac{l_q}{\lambda_4} = \frac{0.3}{0.58} = 0.52 \text{ minutes.}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 0.3 + 0.8 = 1.1.$$

The expected waiting time in the system for node 4 is given as

$$w_4 = \frac{l_s}{\lambda_4} = \frac{1.1}{0.58} = 1.9 \text{ minutes.}$$

Node 5:

$$\rho = \frac{\lambda_5}{\mu_5} = \frac{0.74}{0.824} = 0.9.$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.9}{1-0.9} = 9.0,$$

where m stands for the number of servers at the node 5 (Customers Service Manager Unit).

The expected waiting time in the queue is given as

$$w_{q5} = \frac{l_q}{\lambda_5} = \frac{0.3}{0.74} = 0.41 \text{ minutes.}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 9.0 + 0.9 = 9.9.$$

The expected waiting time in the system for node5 is given as

$$w_5 = \frac{l_s}{\lambda_5} = \frac{9.9}{0.74} = 13.4 Minutes.$$

However, the total expected waiting time in the system before modification is:

$$W_t = w_1 + w_2 + w_3 + w_4 + w_5 = 9.2 + 26 + 1.6 + 1.9 + 13.4 = 52.1 \cong 52 \min utes$$

Now substituting the departure rates and arrival rates calculated above into equation (20), we have:

$$
\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix}
0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0 & 0 & 0 & 0.82 & 0 & 0 \\
0 & 0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0 & 0 & 0 & 0.82 & 0 \\
0 & 0 & 0.65 & 0 & 0.68 & 0 & 0 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.65 & 0 & 0.68 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0 & 0 & 0 & 0 \\
0.65 & 0.65 & 0.65 & 0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.68 & 0.68 & 0.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0.77 & 0.77 & 0.77 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.82 & 0.82 & 0.82
\end{bmatrix}^{+} \begin{bmatrix} 1.00 \\ 0.56 \\ 0.58 \\ 0.62 \\ 0.55 \\ 1.00 \\ 0.79 \\ 0.67 \\ 0.65 \end{bmatrix} \tag{32}
$$

The coefficient of the unknown in equation (20) is a rectangular matrix, to find the inverse of this Matrix, we use Moore-Penrose Generalized Inverse. See the following ([4], [5], [6], [7], [8], [9], [10], [12], [18], [20]).

Hence, we obtain equation (22)

$$
\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix}
0.3529 & -0.0322 & -0.1548 & -0.1134 & 0.3715 & 0.0894 & -0.0774 & -0.0518 & -0.1069 \\
-0.0322 & 0.3529 & -0.1548 & -0.1134 & 0.3715 & 0.0894 & 0.0774 & -0.0518 & -0.1069 \\
-0.2222 & -0.2222 & 0.3901 & -0.1626 & 0.4388 & -0.0758 & -0.195 & 0.1517 & 0.1481 \\
-0.0986 & -0.0986 & -0.0805 & 0.3893 & 0.3567 & -0.103 & 0.0402 & -0.048 & 0.0657 \\
0.0285 & 0.0285 & 0.4256 & -0.095 & -0.0969 & 0.38 & -0.2128 & 0.0095 & -0.019 \\
0.1577 & 0.1577 & -0.0666 & 0.4824 & -0.1828 & 0.3516 & 0.0333 & -0.1995 & -0.1051 \\
-0.1862 & -0.1862 & -0.359 & -0.3874 & 0.2797 & 0.739 & 0.1795 & 0.19 & 0.1241 \\
0.1405 & 0.1405 & 0.5135 & 0.1914 & -0.2465 & -0.235 & 0.305 & -0.1181 & -0.0937 \\
-0.1405 & -0.1405 & -0.5135 & -0.1914 & 0.2465 & 0.235 & 0.8185 & 0.1181 & 0.0937 \\
0.4789 & 0.0227 & 0.0388 & -0.0515 & -0.1222 & 0.0043 & -0.0194 & 0.2122 & -0.1672 \\
0.0227 & 0.4789 & 0.0388 & -0.0515 & -0.1222 & 0.0043 & -0.0194 & 0.2122 & -0.1672 \\
-0.056 & -0.056 & 0.1268 & 0.544 & -0.1397 & -0.2236 & -0.0634 & 0.2167 & 0.0374 \\
-0.4455 & -0.4455 & -0.2043 & -0.441 & 0.3841 & 0.2151 & 0.1022 & 0.6577 & 0.297 \\
0.4901 & 0.0042 & 0.0863 & 0.1382 & -0.1797 & -0.0748 & -0.0431 & -0.1581 & 0.2417 \\
0.0042 & 0.4901 & 0.0863 & 0.1382 & -0.1797 & -o.o748 & -0.0431 & -0.1581 & 0.2417 \\
-0.4943 & -0.4943 & -0.1726 & -0.2765 & 0.3594 & 0.1497 & 0.0863 & 0.3163 & 0.736
\end{bmatrix} \begin{bmatrix} 0.58 \\ 0.73 \\ 0.58 \\ 0.74 \\ 0.65 \\ 0.68 \\ 0.89 \\ 0.77 \\ 0.82 \end{bmatrix} \tag{33}
$$

Solving equation (33) using maple 17 software, we obtain equation (34) below:

$$\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} 0.251085 \\ 0.308850 \\ 0.113229 \\ 0.326733 \\ -0.211641 \\ 0.435043 \\ 0.353324 \\ 0.407175 \\ 0.592740 \\ 0.211245 \\ 0.279675 \\ 0.300992 \\ 0.208421 \\ 0.310075 \\ 0.382960 \\ 0.307033 \end{bmatrix} \tag{34}$$

To normalize the values of the probabilities in equation (34), since we do not have negative probabilities, we take the absolute value of the estimates and rescaled them so that the sum of each node sums up to 1. Thus, we have

$$\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} 0.251111 \\ 0.308882 \\ 0.113241 \\ 0.326767 \\ 0.211639 \\ 0.435040 \\ 0.353321 \\ 0.407210 \\ 0.592790 \\ 0.211175 \\ 0.279582 \\ 0.300892 \\ 0.208352 \\ 0.310054 \\ 0.382934 \\ 0.307012 \end{bmatrix} \tag{35}$$

From equation (35), the following deductions are made:

At node 1 (Meter Greeter Unit), the weights are $\gamma_{12}, \gamma_{13}, \gamma_{14}$ and γ_{15} with the values 0.251111, 0.308882, 0.113241 and 0.326767 respectively, shows that there is a high probability of a customer leaving node1 (Meter Greeter Unit) to join the queue for service at node5 (Customer Service Manager Unit) than any other node. The least probability is that a customer leaves node1 to node4 (Tellers Unit).

At node 2 (Customer Service Unit), the weights are γ_{24}, γ_{25} *and* γ_{2out} with values 0.211639, 0.435040 and 0.353321 respectively, shows that there is a high probability that a customer leaves node2 (Customer Service Unit) and goes directly to node5 (Customer Service Manager Unit).

At node 3 (Marketing Unit), the weights are γ_{34} *and* γ_{3out} with the values 0.407210 and 0.592790 respectively, which shows that there is a high probability that a customer leaves node3 (Marketing Unit) moves out of the system. The least probability is that a customer leaves node3 (Marketing Unit) to join the service at the node4 (Tellers Unit).

At node 4 (Tellers Unit), the weights are $\gamma_{42}, \gamma_{43}, \gamma_{45}$, *and* γ_{4out} with the values 0.211175, 0.279582, 0.300892 and 0.208352 respectively, which shows that, there is a high probability that a customer leaves node4 (Tellers Unit) to join the service at the node5 (Customers Service Manager Unit).

At node 5 (Customers Service Manager Unit) the weights are γ_{52}, γ_{53} *and* γ_{5out} with the values 0.310054, 0.382934 and 0.307012 respectively, which shows that there is a high probability that a customer leaves node5 (Customer Service Manager Unit) to join the queue for service either at node3 (Marketing Unit). The least probability here is that a customer leaves node5 (Customer Service Manager Unit) and goes out of the system.

3.1. Solution For New Departure Rate

To obtain the new departure rate for the network system, we solve equation (25), this is done by substituting the values of γ_{ij}'s and arrival rates obtained in the previous section into equation (25) then we obtain equation (36):

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 54.732 & 43.8693 & -0.1849 & 0.7306 \\ 6.2475 & -5.9549 & 0.2033 & 1.4693 \\ 17.8841 & 14.8199 & 2.3845 & -0.9486 \\ 69.3682 & 56.9901 & -0.0948 & 0.3747 \\ 6.0785 & -3.2277 & 0.2133 & -0.8430 \end{bmatrix} \begin{bmatrix} 0.58 \\ 0.73 \\ 0.58 \\ 0.74 \end{bmatrix} \tag{36}$$

Equation (36) becomes equation (37):

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 16.023 \\ 18.014 \\ 18.022 \\ 22.812 \\ 7.4242 \end{bmatrix} \tag{37}$$

Therefore, to serve Customer with the space interval of 5 minutes, we divide the values in equation (37) by 5. The recommended number of servers therefore for each node is given in equation (38):

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 3.2046 \\ 3.6028 \\ 3.6044 \\ 4.5624 \\ 1.4848 \end{bmatrix} \cong \begin{bmatrix} 3 \\ 4 \\ 4 \\ 5 \\ 1 \end{bmatrix} \tag{38}$$

The result in equation (38) shows that for the Bank to work optimally and to serve customers within the average time frame of 5 minutes, node1 will require a total of three servers, node2 will require a total of four servers, node3 will require a total of four servers, node4 will require a total of five servers, node five will require a total of one server. With the new estimated number of servers at each node given by equation (38). The arrival rates for each node are assumed to remain the same since we do not have control over it. A new departure rates and the expected waiting time for each node is estimated as follow.

The new departure rates for nodes are as follows:

$$\mu_1 = \textit{recommended departure rate per } 5 \min. = \tfrac{3}{5} = 0.6$$
$$\mu_2 = \textit{recommended departure rate per } 5 \min. = \tfrac{4}{5} = 0.8$$
$$\mu_3 = \textit{recommended departure rate per } 5 \min. = \tfrac{4}{5} = 0.8$$
$$\mu_4 = \textit{recommended departure rate per } 5 \min. = \tfrac{5}{5} = 1.0$$
$$\mu_5 = \textit{recommended departure rate per } 5 \min. = \tfrac{1}{5} = 0.2.$$

Finding the New Expected Waiting Time in the System.

Node 1:

$\rho = \frac{\lambda_1}{\mu_1} = \frac{0.52}{0.6} = 0.9$

The expected number of customers in the queue is given as

$, l_q = \frac{\rho}{m-\rho} = \frac{0.9}{3-0.9} = 0.4$

where m stands for the number of servers at the node1 (Meter Greeter Unit).

The expected waiting time in the queue is given as

$w_{q1} = \frac{l_q}{\lambda_1} = \frac{0.4}{0.52} = 0.8$ minutes

The expected number of customers in the system is given as

$l_s = l_q + \rho = 0.4 + 0.9 = 1.3$

The expected waiting time in the system for node1 is given as

$w_1 = \frac{l_s}{\lambda_1} = \frac{1.3}{0.52} = 2.5$ minutes.

Node 2:

$\rho = \frac{\lambda_2}{\mu_2} = \frac{0.58}{0.8} = 0.7$.

The expected number of customers in the queue is given as

$l_q = \frac{\rho}{m-\rho} = \frac{0.7}{4-0.7} = 0.2$,

where m stands for the number of servers at the node2 (Customer Service Unit).

The expected waiting time in the queue is given as

$w_{q2} = \frac{l_q}{\lambda_2} = \frac{0.2}{0.58} = 0.34$ minutes.

The expected waiting time in the system for node 2 is given as

$w_2 = \frac{l_s}{\lambda_2} = \frac{0.9}{0.58} = 1.6$ minutes.

Node 3:

$\rho = \frac{\lambda_3}{\mu_3} = \frac{0.73}{0.8} = 0.9$

The expected number of in the queue is given as

$l_q = \frac{\rho}{m-\rho} = \frac{0.9}{4-0.9} = 0.3$,

where m stands for the number of servers at the node3 (Marketing Unit).

The expected waiting time in the queue is given as

$w_{q3} = \frac{l_q}{\lambda_3} = \frac{0.3}{0.73} = 0.41$ minutes.

The expected number of customers in the system is given as

$l_s = l_q + \rho = 0.2 + 0.9 = 1.1$.

The expected waiting time in the system for node3 is given as

$w_3 = \frac{l_s}{\lambda_3} = \frac{1.1}{0.73} = 1.5$ minutes.

Node 4:

$\rho = \frac{\lambda_4}{\mu_4} = \frac{0.58}{1.0} = 0.6$

The expected number of customers in the queue is given as

$l_q = \frac{\rho}{m-\rho} = \frac{0.6}{5-0.6} = 0.1$,

where m stands for the number of servers at the node 4 (Tellers Unit).

The expected waiting time in the queue is given as

$w_{q4} = \frac{l_q}{\lambda_4} = \frac{0.1}{0.58} = 0.17$ minutes

The expected number of customers in the system is given as
$l_s = l_q + \rho = 0.1 + 0.6 = 0.7$
The expected waiting time in the system for node1 is given as
$w_4 = \frac{l_s}{\lambda_4} = \frac{0.7}{0.58} = 1.2$ minutes.

Node 5:
$\rho = \frac{\lambda_5}{\mu_5} = \frac{0.74}{1.0} = 0.7$.
The expected number of customers in the queue is given as
$l_q = \frac{\rho}{m-\rho} = \frac{0.7}{1-0.7} = 2.3$,
where m stands for the number of servers at the node5 (Customer Service Manager Unit).

The expected waiting time in the queue is given as
$w_{q5} = \frac{l_q}{\lambda_5} = \frac{2.3}{0.74} = 3.10$ minutes.

The expected number of customers in the system is given as
$l_s = l_q + \rho = 2.3 + 0.7 = 3.0$
The expected waiting time in the system for node1 is given as
$w_5 = \frac{l_s}{\lambda_5} = \frac{3.0}{0.74} = 4.1$minutes.

However, the total expected waiting time in the system after modification is
$W_t = w_1 + w_2 + w_3 + w_4 + w_5 = 2.5 + 1.6 + 1.5 + 1.2 + 4.1 = 10.9 \cong 11\, minutes$.

The summary of the computed performance measure for determination of optimal number of servers at network queuing nodes to reduce waiting time at the First City Monument Bank, Minna branch, is given in Table 2.

Nodes i	Number of Servers (m_i)	Probabilities (α_{ij})	ϱ_i	Lq	Ls	W_q	W_S
1	1	0.2511111. 0.308882 0.113241 0.326767	0.8	4.0	4.8	7.69	9.2
2	2	0.211639. 0.435040. 0.353321	0.8	0.7	1.5	1.20	26
3	3	0.407210. 0.592790.	0.8	0.4	1.6	0.54	1.6
4	4	0.211175. 0.279582. 0.300892 0.208352	0.8	0.3	1.1	0.52	1.9
5	1	0.310054. 0.382934. 0.307012	0.9	9.0	9.9	0.41	13.4
Total	11		4.1	14.4	18.9	10.36	52.1

Table 2: Showing all the results obtained before modification.

Nodes i	Number of Servers (m_i)	ϱ_i	Lq	Ls	W_q	W_S
1	3	0.9	0.4	1.3	0.8	2.5
2	4	0.7	0.2	0.9	0.34	1.6
3	4	0.9	0.3	1.1	0.41	1.5
4	5	0.6	0.1	0.7	0.17	1.2
5	1	0.7	2.3	3.0	3.10	4.1
Total	17	3.8	3.3	7	4.8	10.9

Table 3: Showing all the results obtained after modification

Nodes	Current number of servers	Optimal number of servers obtained
1	1	3
2	2	4
3	3	4
4	4	5
5	1	1
Total	11	17

Table 4: Showing the comparison between current number of servers and optimal number of servers obtained

4. Conclusion

The network queuing system of First City Monument Bank, Minna branch has been effectively investigated and studied. The study has determined optimal number of banking personnel at different nodes of the bank network system to reduce waiting time of the customers. Results from the study is a vital information to the management of the bank for proper planning and efficient service delivery. The network model could also be applied to other banks with little modifications.

References

[1] Adaji I. (2018). A determination of optimal performance for the queuing system. Unpublished Thesis of Master Technology in Department of Mathematics, Federal University of Technology, Minna.

[2] Adaji I., Lawal A., Abdullahi A., and Abdulkadir A. (2021). Performance evaluation of outpatient department waiting line system in a city. J. Appl. Sci. Environ. Manage., 25(1): 65-70.

[3] Adamu L., Adaji I., and Abdulkadir A. (2019) A study of waiting and service cost of a mult-server queuing system at national health insurance scheme (NHIS) Unit of the General Hospital Minna. Transactions of the Nigerian Association of Mathematical Physics, 10: 177-184

[4] Asmaa M. K., Asma A. E., and Afaf A. (2019). Applications of the Moore-Penrose generalized inverse to linear systems of algebraic equations. American Journal of Applied Mathematics, 7(6): 152-156.

[5] Bahadur-Thapa G., Lam-Estrada P., and Lo`pez-Bonilla J. (2018). On the Moore-Penrose generalized inverse matrix. World Scientific News, 5(9): 100- 110.

[6] Ben-Israel A. (1980). Generalized Inverses of Matrices and Their Applications. Springer, 154-186.

[7] Campbell S. L. and Meyer C. D. (1979). Generalized Inverses of Linear Transformations. London: Pitman Publishers.

[8] Golub G. H. and Kahan W. (1965). Calculating the singular values and pseudo-inverse of a matrix. SIAM J. Numer. Anal., 2 (3): 421 – 433.

[9] Greville T. N. (1960). The Pseudoinverse of a rectangular singular matrix and its application to the solution of systems of linear equations. SIAM Rev., 3(1): 38-43.

[10] Hearon J. Z. (1968). Generalized inverses and solutions of linear systems. Journal of research of National Bureau of standards-B Mathematical Sciences, 72: 303-308.

[11] Ismaila A. and Lawal A. (2020). Determination of optimal number of servers at network queuing node to reduce waiting time. International Journal of Optimal Model, 3(1): 34 – 49.

[12] Kanan A. M. (2017). Solving the systems of linear equations using the Moore-Penrose generalized inverse. Journal Massarat Elmeya, 1(2): 3-8.

[13] Kandenmir C. and Cavas I. (2007). An application of queuing theory to the relationship between insulin level and number of insulin receptors. Turk. J Biochem., 32(27): 32-38.

[14] Kembe M. M, Onah E. S., and Lorkegh S. (2012). A study of waiting and service cost of a multi-server queuing model in a specialist hospital. International Journal of Scientific and Technology Research, 1(8): 19-23.

[15] Narayanamoorthy S. and Ramya L. (2017). Multi server fuzzy queuing model using dsw algorithm with hexagonal fuzzy number. International Journal of Pure and Applied Mathematics, 11: 117 - 212.

[16] Owoloko E., Ayoku A., Adeleke J., and Edeki S. (2015). On the application of the open Jackson queuing network. Global International Journal of Applied Mathematics, 11(4): 2299-2313.

[17] Penrose R. (1955). A generalized inverse for matrices. Proc. Camb. Phil. Soc., 4(51): 406-413.
[18] Rao C. R. and Mitra S. K (1971). Generalized inverse of a matrix and its applications. New York: Wiley, 601-620.
[19] Rotich T. (2016). Utility analysis of an emergency medical service model using queuing theory. British Journal of Mathematics and Computer Science, 19(1), 1-18.
[20] Zuhair N. M. (1976). Generalized inverses and applications. New York: Academic Press.