BLOCK METHOD APPROACH FOR COMPUTATION OF ERRORS OF SOME ADAMS CLASS OF METHODS.

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ABSTRACT

Traditionally, the error and order constant of block linear multistep methods were analyzed by examining each block members separately. This paper proposes a block-by-block analysis of the schemes as they appear for implementation. Specifically, cases when k= 2, 3, 4, and 5 for Adams Moulton (implicit) are reformulated as continuous schemes in order to generate a sufficient number of schemes required for the methods to be self-starting. The derivation was accomplished through the continuous collocation technique utilizing power series as the basis function, and the property of order and error constants is examined across the entire block for each case of the considered step number. The findings of the study generated error constants in block form for Adams Bashforth and Adams Moulton procedures at steps $k = 2,3,4,5$. Furthermore, the relevance of the study demonstrates that calculating all members' error constants at once, reduces the amount of time necessary to run the analysis. The new approach, for examining the order and error constants of a block linear multistep method, is highly recommended for application in solving real-world problems, modelled as ordinary and partial differential equations.

Keywords*:* Block linear multistep method, error computation.

INTRODUCTION

Typically, numerical methods for solving initial value problems (IVPs) of the first order are classified as either linear multistep methods (LMM) or Runge-Kutta methods (Akinfenwa et al., 2011). Methods can be separated into explicit and implicit categories. Adams – Bashforth methods and any Runge-Kutta method with a lower diagonal Butcher tableau are explicit linear multistep methods. Backward Differentiation Formula (BDF) and Adams-Moulton methods are examples of implicit linear multistep methods. In the integral range, linear multistep methods require less evaluation of the derivative function than one-step methods. For this reason, they have been immensely popular and are essential for numerically solving ordinary differential equations (Muhammad et al. 2014). However, these methods have limitations, including the overlap of solution models and the need for a starting value. Another limitation is that they produce discrete solution values, which renders them uneconomical for mass production. In this regard, a continuous formulation is desirable. The collocation method is likely the most essential numerical technique for the development of continuous methods – (Lie and Norsett, 1989; Onumanyi et al., 1994; 1999). The continuous method preserves the Runge-Kutta traditional advantage as it allows the generation of a necessary and sufficient number of schemes, which makes the method self-starting and is more accurate since it is implemented as a block method, as reported by Yahaya (2004). Block methods were first introduced by Milne (1953) for the purpose of obtaining starting values for predictor-corrector algorithms (Sarafyan, 1965). However, Rosser (1967), developed Milne's idea into algorithms for general use. Block methods have also been considered by Shampine and Watts (1969), Musa *et al.* (2012), Jator and Li, (2012), Akinfenwa *et al.*, 2013; Mohammed and Adeniyi, (2014), Badmus, *et al.* (2015), Omar and Adeyeye (2016), Akinfenwa *et al.*, (2017). Furthermore, error analysis of numerical methods is crucial; an acceptable linear multistep method (LMM) must be convergent. Consistency and zero stability are, however, the necessary and sufficient conditions for the convergence of a LMM. According to (Musa *et al.*, 2012), consistency controls the magnitude of the local truncation error while zero stability controls the manner in which the error is propagated at each step of the computation. A method which is not both consistent and zero stable is rejected outright and has no practical interest. In recent times, analysis of these properties has been carried out on the individual members of a block linear multistep method (Ibrahim *et al*., 2011; Muhammad *et al.* 2014), whose results may not be assumed for the entire block method. However, in this research paper, we reformulate the existing Adams Moulton methods (for cases when $k=2,3,4$ and 5) into continuous methods and generate the corresponding sufficient number of schemes to make each of them self-starting and carry out the analysis of their local truncation errors as one entity rather than the individual scheme in the block.

METHODOLOGY

Derivation of Continuous Forms of Adams Moulton Methods

A power series of a single variable x in the form:

$$p(x)=\sum_{j=0}^{\infty}a_j x^j \tag{1}$$

is used as the basis or trial function, to produce the approximate solution given as

$$y(x)=\sum_{j=0}^{r+s-1}a_j x^j \tag{2}$$

where $a_j \in \square$ $j=0,1,2,3\ldots r+s-1$ are unknown coefficients to be determined, r and s are the numbers of interpolation and collocation points respectively.

Differentiating (2),

$$y'(x)=f\left(x,y\right)=\sum_{j=1}^{r+s-1}ja_j x^{j-1} \tag{3}$$

Interpolating (2) and collocating (3) at specified points lead to a system of nonlinear equations of the form

$$AX=B \tag{4}$$

When the matrix inversion technique is used to solve equation (4), the values are returned to equation (2) and substituted in order to produce the following continuous Adams method scheme:

$$y\left(x\right)=\alpha_{k-1}\left(x\right)y_{n+k-1}+h\sum_{j=0}^{k}\beta_j\left(x\right)f_{n+j} \tag{5}$$

The continuous scheme is evaluated at the non-interpolating points in order to obtain the necessary number of equations for solving an ODE. Using k=2, 3, 4 and 5, we get the following block forms of the Adams Moulton methods in (6) through (9):

$$\left.\begin{aligned} y_{n+1} &= y_n + \tfrac{5}{12}hf_n + \tfrac{2}{3}hf_{n+1} - \tfrac{1}{12}hf_{n+2} \\ y_{n+2} &= y_{n+1} - \tfrac{1}{12}hf_n + \tfrac{2}{3}hf_{n+1} + \tfrac{5}{12}hf_{n+2} \end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned} y_{n+1} &= y_{n+2} + \tfrac{1}{24}hf_n - \tfrac{13}{24}hf_{n+1} - \tfrac{13}{24}hf_{n+2} + \tfrac{1}{24}hf_{n+3} \\ y_{n+2} &= y_n + \tfrac{1}{3}hf_n + \tfrac{4}{3}hf_{n+1} + \tfrac{1}{3}hf_{n+2} \\ y_{n+3} &= y_{n+2} + \tfrac{1}{24}hf_n - \tfrac{5}{24}hf_{n+1} + \tfrac{19}{24}hf_{n+2} + \tfrac{3}{8}hf_{n+3} \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} y_{n+1} &= y_{n+3} + \tfrac{1}{90}hf_n - \tfrac{17}{45}hf_{n+1} - \tfrac{19}{15}hf_{n+2} - \tfrac{17}{45}hf_{n+3} + \tfrac{1}{90}hf_{n+4} \\ y_{n+2} &= y_{n+3} - \tfrac{11}{720}hf_n + \tfrac{37}{360}hf_{n+1} - \tfrac{19}{30}hf_{n+2} - \tfrac{173}{360}hf_{n+3} + \tfrac{19}{720}hf_{n+4} \\ y_{n+3} &= y_n + \tfrac{27}{80}hf_n + \tfrac{51}{40}hf_{n+1} + \tfrac{9}{10}hf_{n+2} + \tfrac{21}{40}hf_{n+3} - \tfrac{3}{80}hf_{n+4} \\ y_{n+4} &= y_{n+3} - \tfrac{19}{720}hf_n + \tfrac{53}{360}hf_{n+1} - \tfrac{11}{30}hf_{n+2} + \tfrac{323}{360}hf_{n+3} + \tfrac{251}{720}hf_{n+4} \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} y_{n+1} &= y_{n+4} + \tfrac{3}{160}hf_n - \tfrac{69}{160}hf_{n+1} - \tfrac{87}{80}hf_{n+2} - \tfrac{87}{80}hf_{n+3} - \tfrac{69}{160}hf_{n+4} + \tfrac{3}{160}hf_{n+5} \\ y_{n+2} &= y_{n+4} + \tfrac{1}{90}hf_{n+1} - \tfrac{17}{45}hf_{n+2} - \tfrac{19}{15}hf_{n+3} - \tfrac{17}{45}hf_{n+4} + \tfrac{1}{90}hf_{n+5} \\ y_{n+3} &= y_{n+4} + \tfrac{11}{1440}hf_n - \tfrac{77}{1440}hf_{n+1} + \tfrac{43}{240}hf_{n+2} - \tfrac{511}{720}hf_{n+3} - \tfrac{637}{1440}hf_{n+4} + \tfrac{3}{160}hf_{n+5} \\ y_{n+4} &= y_n + \tfrac{14}{45}hf_n + \tfrac{64}{45}hf_{n+1} + \tfrac{8}{15}hf_{n+2} + \tfrac{64}{45}hf_{n+3} + \tfrac{14}{45}hf_{n+4} \\ y_{n+5} &= y_{n+4} + \tfrac{3}{160}hf_n - \tfrac{173}{1440}hf_{n+1} + \tfrac{241}{720}hf_{n+2} - \tfrac{133}{240}hf_{n+3} + \tfrac{1427}{1440}hf_{n+4} + \tfrac{95}{288}hf_{n+5} \end{aligned}\right\} \quad (9)$$

NUMERICAL EXPERIMENTS AND RESULTS

Block Error Analysis of Adams Methods

Following Nwachukwu and Okor (2018), the individual scheme of a linear multistep method can be written as:

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right) + h\left(\sum_{j=0}^{k}\beta_j y'(x+jh)\right) \quad (10)$$

Expanding (10) in Taylor series, the local truncation error associated with (5) is the linear difference operator

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right) - h\left(\sum_{j=0}^{k}\beta_j y'(x+jh)\right) \quad (11)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (3.34) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right] = c_0 y(x) + c_1 h y'(x) + \ldots + c_q h^q y^{(q)}(x) + \ldots \quad (12)$$

where the constant $c_q, q = 0,1,\ldots$ are given as follows

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k} \alpha_j \\ c_1 &= \sum_{j=1}^{k} j\alpha_j + \sum_{j=0}^{k} \beta_j \\ c_2 &= \frac{1}{2!}\sum_{j=1}^{k} (j)^2 \alpha_j + \sum_{j=1}^{k} j\beta_j \\ &\vdots \\ c_q &= \frac{1}{\mathrm{q}!}\sum_{j=1}^{k} (j)^q \alpha_j + \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j \end{aligned}\right\} \quad (13)$$

If the error constant of a linear multistep method is known, it is said to be of the order of accuracy (Akinfenwa et al., 2015). However, this method is typically used to determine the order of the block's members. In order to determine the order of the entire block, this method is extended further. Block linear multistep method is expressed in the following form to achieve this goal

$$\sum_{i=0}^{k} \alpha_{ij} y_{n+j} = h\sum_{j=0}^{k} \beta_{ij} f_{n+j} \quad (14)$$

Equation (14) is expanded to give the following system of equation.

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{21} & \dots & \alpha_{1k} \\ \alpha_{02} & \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2k} \\ \alpha_{03} & \alpha_{12} & \alpha_{23} & \dots & \alpha_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0k} & \alpha_{1k} & \alpha_{2k} & \cdots & \alpha_{kk} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+k} \end{pmatrix} = h \begin{pmatrix} \beta_{01} & \beta_{11} & \beta_{21} & \dots & \beta_{1k} \\ \beta_{02} & \beta_{12} & \beta_{22} & \cdots & \beta_{2k} \\ \beta_{03} & \beta_{12} & \beta_{23} & \dots & \beta_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{0k} & \beta_{1k} & \beta_{2k} & \cdots & \beta_{kk} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+k} \end{pmatrix} \quad (15)$$

where

$$\vec{\alpha}_0 = \begin{pmatrix} 01 \\ 02 \\ 03 \\ \vdots \\ 0k \end{pmatrix}, \vec{\alpha}_1 = \begin{pmatrix} 11 \\ 12 \\ 13 \\ \vdots \\ 1k \end{pmatrix} \ \vec{\alpha}_k = \begin{pmatrix} k1 \\ k2 \\ k3 \\ \vdots \\ kk \end{pmatrix} \text{ and } \vec{\beta}_0 = \begin{pmatrix} 01 \\ 02 \\ 03 \\ \vdots \\ 0k \end{pmatrix}, \vec{\beta}_1 = \begin{pmatrix} 11 \\ 12 \\ 13 \\ \vdots \\ 1k \end{pmatrix} \ \vec{\beta}_k = \begin{pmatrix} k1 \\ k2 \\ k3 \\ \vdots \\ kk \end{pmatrix}$$

Extending (12) to the vector form in (15),

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left[\vec{\alpha}_j y(x+jh) - h\vec{\beta}_j y'\left(x+jh, y(y+jh)\right)\right] \quad (16)$$

where, $y(x)$ exactly satisfies $y'(x) = f\left(x, y(x)\right)$. Using Taylor's series expansion of (16) as a starting point, concerning x yields

$$L\left[y(x);h\right]=\vec{C}_0 y(x)+\vec{C}_1 h y'(x)+\vec{C}_2 h^2 y''(x)+\ldots+\vec{C}_q h^q y^{(q)}(x) \qquad (17)$$

where

$$\vec{C}_0=\begin{pmatrix}c_{01}\\c_{02}\\c_{03}\\\vdots\\c_{0p}\end{pmatrix},\ \vec{C}_1=\begin{pmatrix}c_{11}\\c_{12}\\c_{13}\\\vdots\\c_{1p}\end{pmatrix}\ \vec{C}_p=\begin{pmatrix}c_{p1}\\c_{p2}\\c_{p3}\\\vdots\\c_{pp}\end{pmatrix} \qquad (18)$$

The block linear multistep technique is of order if and only if the local truncation error can be expressed as (Chollom et al., 2007).

Local Truncation Error of two-step Adams Moulton Block Method

A simple two-step method of Adams Moulton block analysis in (6) can be found by following the steps outlined above. The results are summarized as follows:

$$\vec{C}_0=\vec{\alpha}_0+\vec{\alpha}_1+\vec{\alpha}_2=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad (19)$$

$$\vec{C}_1=\vec{\alpha}_1+2\vec{\alpha}_2-(\beta_0+\beta_1+\beta_2)=\begin{pmatrix}1\\-1\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}-\left(\begin{pmatrix}\frac{5}{12}\\-\frac{1}{12}\end{pmatrix}+\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\end{pmatrix}+\begin{pmatrix}-\frac{1}{12}\\\frac{5}{12}\end{pmatrix}\right)=\begin{pmatrix}0\\0\end{pmatrix} \qquad (20)$$

$$\vec{C}_2=\frac{1}{2!}\left[\vec{\alpha}_1+2^2\vec{\alpha}_2\right]-(\beta_1+2\beta_2)=\frac{1}{2!}\left[\begin{pmatrix}1\\-1\end{pmatrix}+2^2\begin{pmatrix}0\\1\end{pmatrix}\right]-\left(\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\end{pmatrix}+2\begin{pmatrix}-\frac{1}{12}\\\frac{5}{12}\end{pmatrix}\right)=\begin{pmatrix}0\\0\end{pmatrix} \qquad (21)$$

$$\vec{C}_3=\frac{1}{3!}\left[\vec{\alpha}_1+2^3\vec{\alpha}_2\right]-\frac{1}{2!}(\beta_1+2\beta_2)=\frac{1}{3!}\left[\begin{pmatrix}1\\-1\end{pmatrix}+2^3\begin{pmatrix}0\\1\end{pmatrix}\right]-\frac{1}{2!}\left(\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\end{pmatrix}+2^2\begin{pmatrix}-\frac{1}{12}\\\frac{5}{12}\end{pmatrix}\right)=\begin{pmatrix}0\\0\end{pmatrix} \qquad (22)$$

$$\vec{C}_4=\frac{1}{4!}\left[\vec{\alpha}_1+2^4\vec{\alpha}_2\right]-\frac{1}{3!}(\beta_1+2^3\beta_2)=\frac{1}{4!}\left[\begin{pmatrix}1\\-1\end{pmatrix}+2^4\begin{pmatrix}0\\1\end{pmatrix}\right]-\frac{1}{3!}\left(\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\end{pmatrix}+2^3\begin{pmatrix}-\frac{1}{12}\\\frac{5}{12}\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{24}\\-\frac{1}{24}\end{pmatrix} \qquad (23)$$

Hence the two-step Adams Moulton block method is of order $(3,3)^T$ and the error constant is $\left(\frac{1}{24},-\frac{1}{24}\right)^T$

Local Truncation Error of three-step Adams Moulton Block Method

According to the Adams Moulton block method in (7), the order and error constant are as follows:

$$\vec{C}_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (24)$$

$$\vec{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{1}{24} \\ \frac{1}{3} \\ \frac{1}{24} \end{pmatrix} + \begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + \begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + \begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (25)$$

$$\vec{C}_2 = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (26)$$

$$\vec{C}_3 = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \frac{1}{2!}\left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3^2\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (27)$$

$$\vec{C}_4 = \frac{1}{4!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^4\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \frac{1}{3!}\left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2^3\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3^3\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (28)$$

$$\vec{C}_5 = \frac{1}{5!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^5\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \frac{1}{4!}\left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2^4\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3^4\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix} \right] = \begin{pmatrix} -\frac{11}{720} \\ -\frac{1}{90} \\ -\frac{19}{720} \end{pmatrix} \qquad (29)$$

As a result, the three-step Adams Moulton block method is ordered $(4,4,4)^T$, and the error constant is

$$\left(-\frac{11}{720},-\frac{1}{90},-\frac{10}{720}\right)^T$$

Local Truncation Error of four-step Adams Moulton Block Method

The order and error constant of the four-step Adams Moulton block method in (8) is presented as thus:

$$\vec{C}_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (30)$$

$$\vec{C}_1 = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right] - \left[\begin{pmatrix} \frac{1}{90} \\ -\frac{11}{720} \\ \frac{27}{80} \\ -\frac{19}{720} \end{pmatrix} + \begin{pmatrix} -\frac{17}{45} \\ \frac{37}{360} \\ \frac{51}{40} \\ \frac{53}{360} \end{pmatrix} + \begin{pmatrix} -\frac{19}{15} \\ -\frac{19}{30} \\ \frac{9}{10} \\ -\frac{11}{30} \end{pmatrix} + \begin{pmatrix} -\frac{17}{45} \\ -\frac{173}{360} \\ \frac{21}{40} \\ \frac{323}{360} \end{pmatrix} + \begin{pmatrix} \frac{1}{90} \\ \frac{19}{720} \\ -\frac{3}{80} \\ \frac{251}{720} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (31)$$

$$\vec{C}_2 = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3^2\begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right] - \left[\begin{pmatrix} -\frac{17}{45} \\ \frac{37}{360} \\ \frac{51}{40} \\ \frac{53}{360} \end{pmatrix} + 2\begin{pmatrix} -\frac{19}{15} \\ -\frac{19}{30} \\ \frac{9}{10} \\ -\frac{11}{30} \end{pmatrix} + 3\begin{pmatrix} -\frac{17}{45} \\ -\frac{173}{360} \\ \frac{21}{40} \\ \frac{323}{360} \end{pmatrix} + 4\begin{pmatrix} \frac{1}{90} \\ \frac{19}{720} \\ -\frac{3}{80} \\ \frac{251}{720} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (32)$$

$$\vec{C}_3 = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^3\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3^3\begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right] - \frac{1}{2!}\left[\begin{pmatrix} -\frac{17}{45} \\ \frac{37}{360} \\ \frac{51}{40} \\ \frac{53}{360} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{19}{15} \\ -\frac{19}{30} \\ \frac{9}{10} \\ -\frac{11}{30} \end{pmatrix} + 3^2\begin{pmatrix} -\frac{17}{45} \\ -\frac{173}{360} \\ \frac{21}{40} \\ \frac{323}{360} \end{pmatrix} + 4^2\begin{pmatrix} \frac{1}{90} \\ \frac{19}{720} \\ -\frac{3}{80} \\ \frac{251}{720} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (33)$$

$$\vec{C}_4 = \frac{1}{4!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^4\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3^4\begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right] - \frac{1}{3!}\left[\begin{pmatrix} -\frac{17}{45} \\ \frac{37}{360} \\ \frac{51}{40} \\ \frac{53}{360} \end{pmatrix} + 2^3\begin{pmatrix} -\frac{19}{15} \\ -\frac{19}{30} \\ \frac{9}{10} \\ -\frac{11}{30} \end{pmatrix} + 3^3\begin{pmatrix} -\frac{17}{45} \\ -\frac{173}{360} \\ \frac{21}{40} \\ \frac{323}{360} \end{pmatrix} + 4^3\begin{pmatrix} \frac{1}{90} \\ \frac{19}{720} \\ -\frac{3}{80} \\ \frac{251}{720} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (34)$$

$$\vec{C}_5=\frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^5\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^5\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{4!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^4\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^4\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^4\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad (35)$$

$$\vec{C}_6=\frac{1}{6!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^6\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^6\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{5!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^5\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^5\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^5\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\-\frac{11}{1440}\\\frac{3}{160}\\-\frac{3}{160}\end{pmatrix} \quad (36)$$

$$\vec{C}_7=\frac{1}{7!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^7\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^7\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{6!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^6\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^6\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^6\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}-\frac{1}{756}\\-\frac{241}{15120}\\\frac{19}{560}\\-\frac{641}{15120}\end{pmatrix} \quad (37)$$

As a result, the four-step Adams Moulton block method is of order $(6,5,5,5)^T$, and the error constant is constant.

$$\left(-\frac{1}{756},-\frac{11}{1440},\frac{3}{160},-\frac{3}{160}\right)^T$$

Local Truncation Error of five-step Adams Moulton Block Method

The order and error constant of the five-step Adams Moulton block method are presented in (9) as follows:

$$\vec{C}_0=\begin{pmatrix}0\\0\\0\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (38)$$

$$\vec{C}_1=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}-\left[\begin{pmatrix}\frac{3}{360}\\\frac{11}{1440}\\\frac{14}{45}\\\frac{3}{160}\\\frac{251}{720}\end{pmatrix}+\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

(39)

$$\vec{C}_2=\frac{1}{2!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^2\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^2\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^2\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (40)$$

$$\vec{C}_3=\frac{1}{3!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^3\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^3\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^3\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{2!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^2+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^2\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^2\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^2\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (41)$$

$$\vec{C}_4=\frac{1}{4!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^{4}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^{4}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^{4}\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^{4}\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{3!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^{3}+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^{3}\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^{3}\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^{3}\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (42)$$

$$\vec{C}_5=\frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^{5}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^{5}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^{5}\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^{5}\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{4!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^{4}+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^{4}\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^{4}\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^{4}\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (43)$$

$$\vec{C}_6=\frac{1}{6!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^{6}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^{6}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^{6}\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^{6}\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{5!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^{5}+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^{5}\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^{5}\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^{5}\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (44)$$

$$\vec{C}_7=\frac{1}{7!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^{7}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^{7}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^{7}\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^{7}\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{6!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^{6}+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^{6}\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^{6}\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^{6}\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}-\frac{13}{2240}\\-\frac{1}{756}\\-\frac{271}{60480}\\-\frac{8}{945}\\-\frac{863}{60480}\end{pmatrix} \quad (45)$$

As a result, the five-step Adams Moulton block method is of order $(6,6,6,6,6)^T$, and the error constant is

$$\left(-\frac{13}{2240},-\frac{1}{756},-\frac{271}{60480},-\frac{8}{945},-\frac{863}{60480}\right)^T$$

CONCLUSION

In this paper, a method is proposed for analyzing the order and error constants of linear multistep methods in a way that works well. For cases where k = 2, 3, 4, and 5, the continuous formulation of Adams-Moulton schemes is found using the collocation technique and a power series as the basis function. Instead of using the usual method of getting the order and error constants of each member in the block method, this research suggested using block analysis to get the error constants of all the members at once, which saved time on the computer.

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