Boundary Value Technique for the Solution of Special Third Order Boundary Value Problems in Ordinary Differential Equations (ODEs)

Umaru Mohammed[1] Habibah Abdullah[2] Aliyu Alhaji Abdullahi[3] and Mikhail Semenov[4]
[1,2]Department of Mathematics, Federal University of Technology, Minna, Nigeria
[3]Department of Mechanical Engineering, Federal University of Technology, Minna, Nigeria
[4]School of Nuclear Science & Engineering, Tomsk Polytechnic University
[4]sme@tpu.ru
[3]aliuaabdullah@futminna.edu.ng
[2]bibalmaas@gmail.com
[1]umaru.mohd@futminna.edu.ng

Abstract

We develop a class of continuous modified multistep methods (CMMMs) which were use as boundary value methods for the numerical integration of special third order boundary value problems in ordinary differential equations. We investigate the basic properties of the methods and numerical experiments are given to show the performance of the approach.

Keywords: boundary value methods, continuous modified multistep methods

1. INTRODUCTION

Differential equations are important tools in solving real-world problems. These equations arise in several branches of sciences, engineering and technology, ranging from models that describe acoustic wave propagation in relaxing media, draining and coating flow problems to the deflection of a curved beam that has a constant or varying cross section. Boundary value problems also arise in these areas and as such numerical methods that are faster and accurate in solving them are of importance

Boundary value problems of third order have been discussed in many papers in recent years. Examples of such papers are (Abdullah *et al*2013; 2013) had developed a fifth order block method using constant step size with shooting technique to solve third order non-linear boundary value problems anddeveloped a fourth order two-point block method for solving non-linear third order boundary value problems.Khan and Aziz (2003) presented a forth order method that was based on quantic splines which was used to solve third order linear and non-linear boundary value problems. Collocation approximation was applied in deriving schemes that were applied as a block method to solve special third order initial value problems in Olabode (2009). Srivastava and Kumar (2011) andSahi*et al* (2013) had all worked in solving third order ordinary differential equations.Jator (2008) used a continuous linear multistep method to generate multiple finite difference methods that were assembled into a single block matrix that was used to generate third order BVPs. A family of three step hybrid methods independent of first and second derivative components using Taylor approach were proposed to solve special third order ODEs in Jikantoro*et al* (2018), These were all done without reducing the ODEs to equivalent systems of first order ODEs.

Ahmed (2017) used the variational iteration method to get numerical solutions to third order ordinary boundary value problems after reducing them to a system of first order ODEs

In this paper, the considered special third order ordinary differential equation is of the form:

$$y''' = f(x,y), \qquad y(a) = y_0, y'(a) = \delta_0, y''(a) = \beta_0, y(b) = y_M \tag{1}$$

$$y''' = f(x,y), \qquad y(a) = y_0, y'(a) = \delta_0, y'(b) = y_M, y''(b) = y_N \tag{2}$$

2. METHODOLOGY

In this section, the construction of the continuous linear multistep methods via the interpolation and collocation approach is discussed, which will be used to produce several discreteschemes for solving (1) and (2)

Algorithm

Step 1: Construct the continuous LMM (CLMM) with continuous coefficients as:

$$U(x) = \alpha_v(x)y_{n+v} + \alpha_{v-1}(x)y_{n+v-1} + \alpha_0(x)y_n + h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j} + h^3\beta_\mu(x)f_{n+\mu}, \tag{3}$$

Where $v = \begin{cases} \dfrac{k}{2} & for\ \ even\ k \\ \dfrac{k+1}{2} & for\ \ odd\ \ k \end{cases}$

Step 2: Obtain the main and additional methods by evaluating (3) in step 1 at x_{n+j} where

$j = 1(1)2v, j \neq v-1, v$

$$y_{n+j} + \alpha_v y_{n+v} + \alpha_{v-1}y_{n+v-1} + \alpha_0 y_n = h^3\sum_{i=o}^{k}\beta_i f_{n+i} + h^3\beta_w f_{n+w} \tag{4}$$

$j = 1,\ldots,v-1, v+2,\ldots,2v$

Step 3: Obtain the first and second derivative formulae which are used to obtain the additional methods by evaluating $U'(x)$ and $U''(x)$ at $x = x_{n+j}$, $j = 0(1)k$ as

$$U'(x) = \frac{1}{h}\left(\alpha_v'(x)y_{n+v} + \alpha_{v-1}'(x)y_{n+v-1} + \alpha_0'(x)y_n + h^3\sum_{i=0}^{k}\beta_i'(x)f_{n+i} + h^3\beta_\mu'(x)f_{n+\mu},\right)$$

$$U''(x) = \frac{1}{h^2}\left(\alpha_v''(x)y_{n+v} + \alpha_{v-1}''(x)y_{n+v-1} + \alpha_0''(x)y_n + h^3\sum_{i=0}^{k}\beta_i''(x)f_{n+i} + h^3\beta_\mu''(x)f_{n+\mu},\right)$$

by imposing that $U'(a) = y'_0, U'(b) = y'_N$ and $U''(a) = y''_0, U''(b) = y''_N$

Step 4: Combine the schemes obtained in steps 2 and 3 above to form a system of equations with form equivalent to $Ax = B$ where

$x = (M_0, M_1, M_2, \ldots, M_{N-1})^T$ and

$M_0 = (y_0, y_1, y_2, y_3)^T$, $M_1 = (y'_0, y'_1, y'_2, y'_3)^T$, $M_2 = (y''_0, y''_1, y''_2, y''_3)$

Step 5: Adopt matrix inversion algorithm to the system of equations in step 4 to obtain the values of the unknowns in the expected block method.

*Theorem 2.1***:***Let* $P_j(x) = 2xT_n(x) - T_{n-1}(x), n = 0(1)(k+3)$ *be the Chebyshev Polynomial used as basis function and W a vector given by* $W = (y_n, y_{n+v-1}, y_{n+v}, f_n, f_{n+1}, \ldots, f_k)^T$ *where T is the transpose. Consider the matrix V defined as*

$$V = \begin{pmatrix} P_0(x_n) & P_1(x_n) & \ldots & P_{k+3}(x_n) \\ P_0(x_{n+v-1}) & P_1(x_{n+v-1}) & \ldots & P_{k+3}(x_{n+v-1}) \\ P_0(x_{n+v}) & P_1(x_{n+v}) & \ldots & P_{k+3}(x_{n+v}) \\ P'''_0(x_n) & P'''_1(x_n) & \ldots & P'''_{k+3}(x_n) \\ P'''_0(x_{n+1}) & P'''_1(x_{n+1}) & \ldots & P'''_{k+3}(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots \\ P'''_0(x_{n+\mu}) & P'''_1(x_{n+\mu}) & \ldots & P'''_{k+3}(x_{n+\mu}) \\ P'''_0(x_{n+k}) & P'''_1(x_{n+k}) & \ldots & P'''_{k+3}(x_{n+k}) \end{pmatrix}$$

and V_j *obtained by replacing the jth column of V by the vector W and let (3) satisfy the following conditions*

$$U(x_{n+j}) = y_{n+j} \quad j = 0, v-1, v$$

$$U'''(x_{n+j}) = f_{n+j} \qquad j = 0(1)k \tag{4}$$

then the continuous representation (3) is equivalent to

$$U(x) = \sum_{j=0}^{k+4} \frac{\det(V_j)}{\det(V)} P_j(x) \tag{5}$$

Assuming the basis functions as

$$\begin{cases} \alpha_j(x) = \sum_{i=0}^{k+4} \alpha_{i+1,j} P_i(x), \\ h^3\beta_j(x) = \sum_{i=0}^{k+4} h^3 \beta_{i+1.j} P_i(x), \\ h^3\beta_\mu(x) == \sum_{i=0}^{k+4} h^3 \beta_{i+1.\mu} P_i(x), \end{cases} \quad (6)$$

where $\alpha_{i+1,j}$, $\beta_{i+1.j}$, $\beta_{i+1.\mu}$, are coefficients to be determined.

Substituting (6) into (3) yields

$$U(x) = \sum_{i=0}^{k+4} \alpha_{i+1,v} P_i(x) y_{n+v} + \sum_{i=0}^{k+4} \alpha_{i+1,v-1} P_i(x) y_{n+v-1} + \sum_{i=0}^{k+4} \alpha_{i+1,0} P_i(x) y_n +$$
$$h^3 \sum_{j=0}^{k} \sum_{i=0}^{k+4} \beta_{i+1,j} P_i(x) f_{n+j} + h^3 \sum_{i=1}^{k+4} \beta_{i+1,\mu} P_i(x) f_{n+\mu},$$

which is simplified as

$$U(x) = \sum_{i=0}^{k+4} \left\{ \alpha_{i+1,v} y_{n+v} + \alpha_{i+1,v-1} y_{n+v-1} + \alpha_{i+1,0} y_n + \sum_{j=0}^{k} h^3 \beta_{i+1,j} f_{n+j} + h^3 \beta_{i+1,w} f_{n+w} \right\} P_i(x)$$
$$\approx U(x) = \sum_{i=0}^{k+4} \Gamma_i P_i(x) \quad (7)$$

where $\Gamma_i = \alpha_{i+1}, 0, y_n, \ldots$

imposing conditions (4) on (7), we obtain a system of (k+5) equations which can be expressed as $VH = W$ where $H = (\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_{k+4})^T$ is a vector of (k+5) undetermined coefficients.

We proceed to find the elements of H using Cramer's rule, thus;

$$\Gamma_i = \frac{\det(V_j)}{\det(V)}, \; j = 0(1)(k+3)$$

replacing the jth column of V by W gives the value of V_j

$$\Rightarrow U(x) = \sum_{i=0}^{k+4} \frac{\det(V_j)}{\det(V)} P_i(x)$$

3. SPECIFICATION OF THE METHODS

Evaluating the CMMM (3) at $x_{n+i,} i = 1, \ldots, v-2, v+1, \ldots, k$ and using it to obtain the first derivative formulae given by

$$U'(x)=\frac{1}{h}\left(\alpha'_v(x)y_{n+v}+\alpha'_{v-1}(x)y_{n+v-1}+\alpha'_0(x)y_n+h^3\sum_{j=0}^{k}\beta'_j(x)f_{n+j}+h^3\beta'_w(x)f_{n+w},\right) \tag{8}$$

effectively applied by imposing

$$U'(a)=y'_0, U'(b)=y'_N \tag{9}$$

to produce derivative formula of the form (8). The second derivative formula is also obtained from (3), this is given by

$$U''(x)=\frac{1}{h^2}\left(\alpha''_v(x)y_{n+v}+\alpha''_{v-1}(x)y_{n+v-1}+\alpha''_0(x)y_n+h^3\sum_{j=0}^{k}\beta''_j(x)f_{n+j}+h^3\beta''_w(x)f_{n+w},\right) \tag{10}$$

effectively imposed by applying

$$U''(a)=y''_0, U''(b)=y''_N \tag{11}$$

to generate the formulae in (10)

4. CONVERGENCE OF THE METHOD

Here the convergence of the method is established. The equation (3) is evaluated at x_{n+1}, $x_{n+2}, \ldots,$ x_{n+v-2}, $x_{n+v+1}, \ldots,$ $x_{n+\omega}$, x_{n+2v} to give

$$y_{n+1}+\alpha_v^{(1)}y_{n+v}+\alpha_{v-1}^{(1)}y_{n+v-1}+\alpha_0^{(1)}y_0=h^3\sum_{i=0}^{k}\beta_i^{(1)}f_{n+i}+h^3\beta_\omega^{(1)}f_{n+\omega}$$

$$y_{n+2}+\alpha_v^{(2)}y_{n+v}+\alpha_{v-1}^{(2)}y_{n+v-1}+\alpha_0^{(2)}y_0=h^3\sum_{i=0}^{k}\beta_i^{(2)}f_{n+i}+h^3\beta_\omega^{(2)}f_{n+\omega}$$

$$\vdots$$

$$y_{n+v-2}+\alpha_v^{(v-2)}y_{n+v}+\alpha_{v-1}^{(v-2)}y_{n+v-1}+\alpha_0^{(v-2)}y_0=h^3\sum_{i=0}^{k}\beta_i^{(v-2)}f_{n+i}+h^3\beta_\omega^{(v-2)}f_{n+\omega} \tag{12}$$

$$y_{n+v+1}+\alpha_v^{(v+1)}y_{n+v}+\alpha_{v-1}^{(v+1)}y_{n+v-1}+\alpha_0^{(v+1)}y_0=h^3\sum_{i=0}^{k}\beta_i^{(v+1)}f_{n+i}+h^3\beta_\omega^{(v+1)}f_{n+\omega}$$

$$\vdots$$

$$y_{n+\omega}+\alpha_v^{(\omega)}y_{n+v}+\alpha_{v-1}^{(\omega)}y_{n+v-1}+\alpha_0^{(\omega)}y_0=h^3\sum_{i=0}^{k}\beta_i^{(\omega)}f_{n+i}+h^3\beta_\omega^{(\omega)}f_{n+\omega}$$

$$y_{n+k}+\alpha_v^{(k)}y_{n+v}+\alpha_{v-1}^{(k)}y_{n+v-1}+\alpha_0^{(k)}y_0=h^3\sum_{i=0}^{k}\beta_i^{(k)}f_{n+i}+h^3\beta_\omega^{(k)}f_{n+\omega}$$

$U'(x)$ is evaluated at x_{n+j} $j=0(1)k$ and $x_{n+\omega}$ to give

$$hy'_n+\alpha'^{(0)}_v y_{n+v}+\alpha'^{(0)}_{v-1}y_{n+v-1}+\alpha'^{(0)}_0 y_n=h^3\sum_{i=0}^{k}\beta'^{(0)}_i f_{n+i}+h^3\beta'^{(0)}_\omega f_{n+\omega}$$

$$hy'_n+\alpha'^{(1)}_v y_{n+v}+\alpha'^{(1)}_{v-1}y_{n+v-1}+\alpha'^{(1)}_0 y_n=h^3\sum_{i=0}^{k}\beta'^{(1)}_i f_{n+i}+h^3\beta'^{(1)}_\omega f_{n+\omega}$$

$$\vdots \qquad (13)$$

$$hy'_n+\alpha'^{(\omega)}_v y_{n+v}+\alpha'^{(\omega)}_{v-1}y_{n+v-1}+\alpha'^{(\omega)}_0 y_n=h^3\sum_{i=0}^{k}\beta'^{(\omega)}_i f_{n+i}+h^3\beta'^{(\omega)}_\omega f_{n+\omega}$$

$$hy'_n+\alpha'^{(k)}_v y_{n+v}+\alpha'^{(k)}_{v-1}y_{n+v-1}+\alpha'^{(k)}_0 y_n=h^3\sum_{i=0}^{k}\beta'^{(k)}_i f_{n+i}+h^3\beta'^{(k)}_\omega f_{n+\omega}$$

And also evaluate $U''(x)$ to give

$$hy''_n+\alpha''^{(0)}_v y_{n+v}+\alpha''^{(0)}_{v-1}y_{n+v-1}+\alpha''^{(0)}_0 y_n=h^3\sum_{i=0}^{k}\beta''^{(0)}_i f_{n+i}+h^3\beta''^{(0)}_\omega f_{n+\omega}$$

$$hy''_n+\alpha''^{(1)}_v y_{n+v}+\alpha''^{(1)}_{v-1}y_{n+v-1}+\alpha''^{(1)}_0 y_n=h^3\sum_{i=0}^{k}\beta''^{(1)}_i f_{n+i}+h^3\beta''^{(1)}_\omega f_{n+\omega}$$

$$\vdots \qquad (14)$$

$$hy''_n+\alpha''^{(\omega)}_v y_{n+v}+\alpha''^{(\omega)}_{v-1}y_{n+v-1}+\alpha''^{(\omega)}_0 y_n=h^3\sum_{i=0}^{k}\beta''^{(\omega)}_i f_{n+i}+h^3\beta''^{(\omega)}_\omega f_{n+\omega}$$

$$hy''_n+\alpha''^{(k)}_v y_{n+v}+\alpha''^{(k)}_{v-1}y_{n+v-1}+\alpha''^{(k)}_0 y_n=h^3\sum_{i=0}^{k}\beta''^{(k)}_i f_{n+i}+h^3\beta''^{(k)}_\omega f_{n+\omega}$$

All the equations in (12) to (14) are of order $O(h^{k+})$ and can be compactly written in matrix form by introducing the following notations. Let P be a $3N\times 3N$ matrix defined by

$P=\begin{pmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{pmatrix}$ where P_{ij} are $N\times N$ matrices given as

$$P_{11}=\begin{pmatrix} & \alpha'^{(0)}_{v-1} & \alpha'^{(0)}_{v} & & \alpha'^{(0)}_{0} & & & \\ & \alpha''^{(0)}_{v-1} & \alpha''^{(0)}_{v} & & \alpha''^{(0)}_{0} & & & \\ 1 & \alpha^{(1)}_{v-1} & \alpha'^{(1)}_{v} & & \alpha^{(1)}_{0} & & & \\ 1 & \alpha^{(2)}_{v-1} & \alpha'^{(2)}_{v} & & \alpha^{(2)}_{0} & & & \\ & \vdots & \vdots & & \vdots & & & \\ 1 & \alpha^{(v-1)}_{v-1} & \alpha^{(v-1)}_{v} & & \alpha^{(v-1)}_{0} & & & \\ 1 & \alpha^{(v)}_{v-1} & \alpha^{(v)}_{v} & & \alpha^{(v)}_{0} & & & \\ & \vdots & \vdots & & \vdots & & & \\ 1 & \alpha^{(k)}_{v-1} & \alpha^{(k)}_{v} & & \alpha^{(k)}_{0} & & & \\ & & & & \alpha'^{(0)}_{v-1} & \alpha'^{(0)}_{v} & \alpha'^{(0)}_{0} \\ & & & 1 & \alpha^{(1)}_{v-1} & \alpha^{(1)}_{v} & \alpha^{(1)}_{0} \\ & & & & & \vdots & \vdots \\ & & & 1 & \alpha^{(v-1)}_{v-1} & \alpha^{(v-1)}_{v} & \alpha^{(0)}_{0} \\ & & & 1 & \alpha^{(v)}_{v-1} & \alpha^{(v)}_{v} & \alpha^{(v)}_{0} \\ & & & & & \ddots & \ddots \\ & & & 1 & \alpha^{(k)}_{v-1} & \alpha^{(k)}_{v} & \alpha^{(k)}_{0} \end{pmatrix}$$

$$P_{21}=\begin{pmatrix} \alpha'^{(1)}_{v-1} & \alpha'^{(1)}_{v} & \alpha'^{(1)}_{0} & & & \\ \alpha'^{(2)}_{v} & \alpha'^{(2)}_{v} & \alpha'^{(2)}_{0} & & & \\ \vdots & \vdots & \vdots & & & \\ \alpha'^{(k)}_{v-1} & \alpha'^{(k)}_{v} & \alpha'^{(k)}_{0} & & & \\ & & & \alpha'^{(1)}_{v-1} & \alpha'^{(1)}_{v} & \alpha'^{(1)}_{0} \\ & & & \ddots & \ddots & \ddots \\ & & & \alpha'^{(k)}_{v-1} & \alpha'^{(k)}_{v} & \alpha'^{(2v)}_{0} \end{pmatrix}$$

$$P_{31}=\begin{pmatrix} \alpha''^{(1)}_{v-1} & \alpha''^{(1)}_{v} & \alpha''^{(1)}_{0} & & & \\ \alpha''^{(2)}_{v} & \alpha''^{(2)}_{v} & \alpha''^{(2)}_{0} & & & \\ \vdots & \vdots & \vdots & & & \\ \alpha''^{(k)}_{v-1} & \alpha''^{(k)}_{v} & \alpha''^{(k)}_{0} & & & \\ & & & \alpha''^{(1)}_{v-1} & \alpha''^{(1)}_{v} & \alpha''^{(1)}_{0} \\ & & & \ddots & \ddots & \ddots \\ & & & \alpha''^{(k)}_{v-1} & \alpha''^{(k)}_{v} & \alpha''^{(2v)}_{0} \end{pmatrix}$$

P_{12}, P_{13}, P_{23}, P_{32}, are $N \times N$ null matrices and P_{22}, P_{33} are $N \times N$ identity matrices. Similarly, another matrix Q which is a $3N \times 3N$ matrix defined as

$$Q=\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}$$

Where Q_{ij} are $N \times N$ matrices given as

$$Q_{11} = \begin{pmatrix} \begin{matrix} \beta_1'^{(0)} & \beta_2'^{(0)} & \cdots & \beta_k'^{()} \\ \beta_1''^{()} & \beta_2''^{()} & \cdots & \beta_k''^{()} \\ \beta_1^{(1)} & \beta_2^{(1)} & \cdots & \beta_k^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_1^{(v-2)} & \beta_2^{(v-2)} & \cdots & \beta_k^{(v-2)} \\ \beta_1^{(v+1)} & \beta_2^{(v+1)} & \cdots & \beta_k^{(v+1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_1^{(k)} & \beta_2^{(k)} & \cdots & \beta_k^{(k)} \end{matrix} & & \\ & \begin{matrix} \beta_0'^{(0)} & \beta_1'^{(0)} & \cdots & \beta_k'^{(0)} \\ \beta_0'^{(0)} & \beta_1'^{(1)} & \cdots & \beta_k'^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_0^{(v-2)} & \beta_1^{(v-2)} & \cdots & \beta_k^{(v-2)} \\ \beta_0^{(v+1)} & \beta_1^{(v+1)} & \cdots & \beta_k^{(v+1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_0^{(k)} & \beta_1^{(k)} & \cdots & \beta_k^{(k)} \end{matrix} & \\ & \ddots & \\ & & \begin{matrix} \beta_0^{(k)} & \beta_1^{(k)} & \cdots & \beta_k^{(k)} \end{matrix} \end{pmatrix}$$

$$Q_{21} = \begin{pmatrix} \begin{matrix} \beta_1'^{(1)} & \beta_2'^{(1)} & \cdots & \beta_k'^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_1'^{(k)} & \beta_2'^{(k)} & \cdots & \beta_k'^{(1)} \end{matrix} & & \\ & \begin{matrix} \beta_0'^{(1)} & \beta_1'^{(1)} & \cdots & \beta_k'^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_0'^{(k)} & \beta_1'^{(k)} & \cdots & \beta_k'^{(k)} \end{matrix} & \\ & \ddots & \\ & & \begin{matrix} \beta_0'^{(k)} & \beta_1'^{(k)} & \cdots & \beta_k'^{(k)} \end{matrix} \end{pmatrix}$$

$$Q_{31} = \begin{pmatrix} \begin{matrix} \beta_1''^{(1)} & \beta_2''^{(1)} & \cdots & \beta_k''^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_1''^{(k)} & \beta_2''^{(k)} & \cdots & \beta_k''^{(1)} \end{matrix} & & \\ & \begin{matrix} \beta_0''^{(1)} & \beta_1''^{(1)} & \cdots & \beta_k''^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_0''^{(k)} & \beta_1''^{(k)} & \cdots & \beta_k''^{(k)} \end{matrix} & \\ & \ddots & \\ & & \begin{matrix} \beta_0''^{(k)} & \beta_1''^{(k)} & \cdots & \beta_k''^{(k)} \end{matrix} \end{pmatrix}$$

Q_{12}, Q_{13}, Q_{22}, Q_{23}, Q_{32}, Q_{33} are $N \times N$ null matrices

And then the following vectors are defined

$$\bar{Y} = (y_{n+1}, \ldots, y_{n+k}, hy'_{n+1}, \ldots, hy'_{n+k}, h^2 y''_{n+1}, \ldots, h^2 y''_{n+k})^T$$

$$Y = (y(x_{n+1}), \ldots, y(x_{n+k}), hy'(x_{n+1}), \ldots, hy'(x_{n+k}), h^2 y''(x_{n+1}), \ldots, h^2 y''(x_{n+k}))^T$$

$$F = (f_{n+1}, \ldots, f_{n+2v}, hf'_{n+1}, \ldots, hf'_{n+k}, h^2 f''_{n+1}, \ldots, h^2 f''_{n+k})^T$$

$$L(h) = (l_1, \ldots, l_N, l'_1, \ldots, l'_N, l''_1, \ldots, l''_N)^T$$

$$C = (\beta_0'^{(0)} h^3 f_0 - hy'_0, \beta_0''^{(0)} h^3 f_0 - hy''_0, \beta_0^{(0)} h^3 f_0 - y_0, \beta_0^{(1)} h^3 f_0, \ldots, \beta_0^{(v-2)} h^3 f_0, \beta_0^{(v+1)} h^3 f_0, \ldots,$$
$$\beta_0^{(k)} h^3 f_0, 0, \ldots, 0, \beta_0'^{(1)} h^3 f_0 - \alpha_0'^{(1)} y_0, \beta_0'^{(k)} h^3 f_0 - \alpha_0'^{(k)} y_0, 0, \ldots 0, \beta_0''^{(1)} h^3 f_0 - \alpha_0''^{(1)} y_0,$$
$$\beta_0''^{(k)} h^3 f_0 - \alpha_0''^{(k)} y_0, 0, \ldots 0)^T$$

With $L(h)$ representing the local truncation error vector at the point x_n of the methods (12) to (14).

Theorem 4.1: Let (y_i, y'_i, y''_i) be an approximation to the solution vector $(y(x_i), y'(x_i), y''(x_i))$ for the systems (1) and (2). If $e_i = |y(x_i) - y_i|, e'_i = |y'(x_i) - y'_i|, e''_i = |y''(x_i) - y''_i|$, where the exact solution given by the vector $(y(x), y'(x), y''(x))$ is several times differentiable and if $\|E\| = \|Y - \bar{Y}\|$, then the BVMs are said to be convergent of order $k+1$ which implies that

$\|E\| = O(h^{k+1})$, where k is the step number.

Proof: Consider the exact form of the system formed from (12) to (14) and given by

$$PY - h^3 QF(Y) + C + L(h) = 0 \qquad (15)$$

where $L(h)$ is the truncation error vector obtained from the formulas (12) to (14). The approximate form of the system is given by

$$P\bar{Y} - h^3 QF(\bar{Y}) + C = 0 \qquad (16)$$

where $\bar{Y}$ is the approximate solution of vector Y.

Subtracting (15) from (16) and letting $E = |\bar{Y} - Y| = (e_1, \ldots e_N, e'_1, \ldots e'_N, e''_1, \ldots e''_N)^T$ and using the mean value theorem, we have the error system

$$(P - h^3 QB)E = L(h) \qquad (17)$$

where B is the Jacobian matrix and its entries $B_{rs}, r, s = 1,2,3,$ are defined as

$$B_{rs} = \begin{pmatrix} \frac{\partial f_1^{(r-1)}}{\partial y_1^{(s-1)}} & \cdots & \frac{\partial f_1^{(r-1)}}{\partial f_N^{(s-1)}} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N^{(r-1)}}{\partial f_1^{(s-1)}} & \cdots & \frac{\partial f_N^{(r-1)}}{\partial f_N^{(s-1)}} \end{pmatrix}$$

From (16) and $L(h)$

$$E = (P - h^3 QB)^{-1} L(h)$$

$$E = SL(h)$$

$$\|E\| = \|SL(h)\|$$

$$= O(h^{-3})O(h^{k+4})$$

$$= O(h^{k+1})$$

Which show that the methods are convergent and the global errors are of order $O(h^{k+1})$

Numerical Examples

Here, two numerical examples are considered.

Problem 1: Consider the third order boundary value problem (Jator*et al*, 2018)

$$y'''(x) - xy(x) = (x^3 - 2x^2 - 5x - 3)e^x, 0 \prec x \prec 1$$

$$y(0) = y(1) = 0, y'(0) = 1$$

Exact: $y(x) = (x - x^2)e^x$

The problem above was solved using the proposed method and the behaviour of the method was observed. The results gotten are compared with the results in Jator (2018) and the results gotten from the Extended Trapezoidal Methods (ETRs). Maximum of absolute error was obtained within the interval of integration. It is observed from Table 1 that the proposed methods did better than the methods of Jator (2018) and the ETRs in terms of accuracy.

Table 1: Comparison of the Proposed Methods, Jator (2018) and ETRs

N	Proposed Method	Jator (2018)	ETRs
6	8.33E-07	1.525E-05	1.089E-02
12	2.06E-08	9.257E-07	7.666E-05

24	7.0E-10	5.873E-08	5.108E-06
48	6.52E-11	3.683E-09	3.300E-07
96	4.31E-12	2.305E-10	2.098E-08
192	2.13E-13	1.428E-11	1.323E-09

Problem 2: $y''' = -2e^{-3y} + \frac{4}{(1+x)^3}, y(0) = 0, y(1) = \ln 2, y'(0) = 1, 0 \le x \le 1$

Exact: $y(x) = \ln(1+x)$

The proposed method was applied to the problem above and the results were compared to the results in Mohammed (2016). The maximum error was also obtained within the interval of integration. Table 2 suggests that the proposed methods generate results that are at least of approximate accuracy with Mohammed (2016).

Table 2: Comparison of the Proposed Methods and Mohammed (2016).

N	Proposed Method	Mohammed (2016)
6	1.24E-06	1.079E-06
9	-	1.290E-07
12	3.93E-08	2.770E-08
15	-	8.990E-09
24	2.45E-10	-
48	9.32E-12	-
96	4.69E-14	-
192	8.38E-16	-

Note tha:t Mohammed proposed a three step method with one off grid point for the solution of third order ODEs

5. CONCLUSION

In this paper, CMMMs have been proposed using the boundary value technique to integrate special third order boundary value problems in ordinary differential equations. This has been done without reducing the differential equations to systems of first order ODEs. The convergence of this class of methods was carried out and numerical examples were given. The efficiency of the methods was given in the tables 1 and 2. A future research will be carried out based on applying this approach to higher order ODEs while increasing the number of off grid points.

REFERENCES

Abdullah, A. S., Majid, Z. A. andSenu, N. (2013) Solving Third Order Boundary Value Problemwith Fifth Order Block Method. *Mathematical Methods in Engineering and Economics*, ISBN: 978-1-61804-230-9

Abdullah, A. S., Majid, Z. A. andSenu, N. (2013). Solving Third Order Boundary Value Problem Using Fourth Order Block Method. *Applied Mathematical Sciences,*Vol. 7(53), 2629-2645

Ahmed, J. (2017). Numerical Solutions of Third-Order Boundary Value Problems Associated with Draining and Coating Flows. *Kyungpook Mathematical Journal*, Vol. 57,651-665

Brugnano, L. andTrigiante, D. (1998).*Solving Differential Problems by Multistep Initial and Boundary Value Problems*. Gordon and Breach Science Publishers, London

Khan, A. and Aziz, T. (2003). The Numerical Solution of Third-Order Boundary Value Problems Using Quintic Splines. *Applied Mathematics and Computation,*Vol. 137, 253-260

Jator, S. N. (2008). On the Numerical Integration of Third Order Boundary Value Problems by a Linear Multistep Method. *International Journal of Pure and Applied Mathematics*, Vol. 46(3), 375-388

Jator, S., Okunlola, T., Biala, T., andAdeniyi, R. (2018). Direct Integrators for the General Third-Order Ordinary Differential Equations with an Application to the Korteweg–de Vries Equation. *Int. J. Appl. Comput. Math*, Vol. 4, 110

Jikantoro, Y. D., Ismail, F., Senu, N. & Ibrahim, Z. B. (2018). A New Integrator for Special Third Order Differential Equations with Application to Thin Film Flow Problem. *Indian J. Pure Appl. Math.*, Vol. 49(1), 151-167

Mohammed, U. (2016). *A Class of Block Hybrid Linear Multistep Methods for Solution of Second and Third Order Ordinary Differential Equations*. A PhD thesis submitted to the University of Ilorin, Ilorin, Nigeria

Olabode, B. T. (2009). An Accurate Scheme by Block Method for Third Order Ordinary Differential Equations.The Pacific Journal of Science and Technology, Vol. 10(1), 136-142

Sahi, R. K., Jator, S. N. and Khan, N. A. (2013). Continuous Fourth Derivative Method for Third Order Boundary Value Problems. International Journal of Pure and Applied Mathematics, *International Journal of Pure and Applied Mathematics*, Vol. 85(5), 907-923

Srivastava, P. K. and Kumar, M. (2011). Numerical Treatment of Nonlinear Third Order Boundary Value Problem. *Applied Mathematics,* Vol. 2, 959-964