MODELLING TRANSIENT MAGNETOHYDRODYNAMIC FREE CONVECTION FLOW BETWEEN TWO LONG VERTICAL PARALLEL PLATES WITH VISCOUS ENERGY DISSIPATION

BY

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FEDERAL UNIVERSITY OF TECHNOLOGY

MINNA, NIGERIA

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTERS OF TECHNOLOGY (MTech) IN MATHEMATICS

NOVEMBER, 2023

DECLARATION

I hereby declare that this thesis titled: "**Modelling Transient Magnetohydrodynamic Free Convection Flow Between Two Long Vertical Parallel Plates With Viscous Energy Dissipation**"is a collection of my original research work and it has not been presented for any other qualification anywhere. Information from other sources (published or unpublished) has been duly acknowledged.

..…………………………
SIGNATURE/DATE

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CERTIFICATION

The thesis titled: "**Modelling Transient Magnetohydrodynamic Free Convection Flow Between Two Long Vertical Parallel Plates With Viscous Energy Dissipation"** by ADEOTI Lydia Olubunmi, (MTech/SPS/2019/10501) meets the regulations governing the award of the degree of Masters of Technology of the Federal University of Technology, Minna and it is approved for its contribution to scientific knowledge and literary presentation.

Prof. Y. A. Yahaya ----------------------------
SUPERVISOR Signature &Date

Prof. A.I. Enagi ----------------------------
HEAD OF DEPARTMENT Signature &Date

Prof. M. Jiya -----------------------------
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Engr. Prof. O. K. Abubakre -----------------------------
DEAN, POSTGRADUATE SCHOOL Signature &Date

DEDICATION

This thesis is dedicated to the Almighty God for His divine grace, protection, favour and inspiration throughout this programme and to my Immediate Family members.

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ABSTRACT

This thesis presents mathematical model for transient magnetohydrodynamic free convection flow between two long vertical parallel plates with viscous energy dissipation.The partial differential equations governing the phenomenon were non-dimensionalized using some dimensionless quantities. The dimensionless coupled non-linear partial differential equations were solved using harmonic solution technique. The results obtained were presented graphically and discussed. From the results obtained, it was observed that increase in Peclet number, Eckert number and Grashof's number lead to increase in the velocity profiles. Increase in Reynold number leads to reduction in the concentration profile. The Concentration profile also increases with time. It was also observed that increase in Reynold number leads to increase in the velocity profile at $E_c = 1$. The result from this research work is of importance to industries in meteorology, solar physics, geophysics, planetary magnetospheres, aeronautical plasma flows, chemical engineering and electronics, pumps and generators.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Convectional heat transfer fluids, including oil, water and ethylene glycol mixture are poor heat transfer fluids. Since the thermal conductivity of these fluids play an important role in determining the coefficient of heat transfer between the heat transfer medium and the heat transfer surface, numerous methods have been used to improve the thermal conductivity of these fluids by suspending nano/micrometer-sized particle materials in liquids (Chon *et al.*, 2006).

The experimental and theoretical works on Magnethohydrodynamic (MHD) flow with chemical reaction have been done extensively in various areas i.e sustain plasma confinement for controlled thermo nuclear fusion, liquid metal cooling of nuclear reactions and electromagnetic casting of metals keeping the above facts in many authors attracted in this field of study of heat and mass transfer through dusty fluids (Ugwu *et al.*, 2021).

Flow of conducting fluid in external magnetic field produce a variety of new effects, which are not realized in usual hydrodynamics. MHD analyzed these phenomena. It also studies the arising of a flow of conducting fluid due to the current passing through the fluid (so-called electrically induced vortex-type flows). MHD described the frontier area combining classical fluid mechanics and electrodynamics. It was a relatively young discipline in natural science and engineering, starting with the pioneering work of Hartmann (1937) on a liquid metal duet flow under the influence of a strong external magnetic field. Today MHD has developed into vast field of applied and fundamental research in engineering and physical sciences.

Electromagnetic methods of action on electrically conducting medium were widely used both in technical devices such as pump, flow meters, generators and industrial processes in metallurgy and material processing. Another common application of MHD in metallurgy was MHD separation that was used for electromagnetic removal of non-metallic inclusions from melts and metal extraction from Oxides and slag i.e. MHD was used for cleaning liquid metals of impurities as well as for the separation of multiphase systems into their components (Herman and Yeshajahu, 1993).

Nowadays, electromagnetic pumps and their modifications were widely used in metallurgy and materials processing in order to transport and dose (exact batching) melting metal (IvIev *et a.*, 1993). The advantages of MHD pumps over mechanical pumps are: the absence of moving and rotating parts (this increases their reliability), noiseless operations (better vibration and noise characteristics), relative simplicity of control, being completely hermetically sealed. They can be utilized even with chemically aggressive, reactive and very hot fluids. Therefore, they are used also in chemical industries (Al-Habahbeh *et al.*, 2016).

1.2 Statement of the Research Problem

The need for study of transient magnetohydrodynamics free convectional flow with viscous energy dissipation have rapidly increase in recent years as the efficiency of the devices used in industries and engineering depend on the particles suspended in the fluid under the effect of magnetic field. Therefore, investigation of MHD free convection flow between two long vertical parallel plates is important for improving the existing industrial processes and for developing new MHD devices.

1.3 Aim and Objectives of the Research

1.3.1 Aim

The aim of this work is to investigate the effect of viscous energy dissipation on transient MHD free convention flow between two long vertical parallel plates.

1.3.2 Objectives

The objectives of this study are to:

i. Formulate a model for the transient MHD flow with viscous energy dissipation.

ii. Obtain the analytical solution of the model using harmonic solution technique.

iii. Provide the graphical representation of the system responses.

1.4 Significance of the Study

Magnetohydrodynamics (MHD) finds its application in meteorology, solar physics, geophysics and motion of the earth core. MHD free convection flow have also significant applications in the field of stellar and planetary magnetospheres, aeronautical plasma flows, chemical engineering and electronics which arises to the use of Partial Differential Equations (PDE) to model these physical phenomena. Since analytical methods are most times restricted in handling of such PDE due to its coupled nature, harmonic method would go a long way in providing solutions to these physical problems.

1.5 Scope and Limitation

1.5.1 Scope

The scope of this work is to study the governing equations by providing analytical solution using harmonic solution method for the analysis of transient MHD free convection flow with viscous energy dissipation.

1.5.2 Limitation

This work is limited to the mathematical study of transient MHD free convection flow with viscous energy dissipation.

1.6 Definition of Terms

Convection: this is heat transfer by mass motion of fluid such as air or water when the heated fluid is caused to move away from the source of heat, carrying energy on it.

Fluid: A substance that has no fixed shape and yield easily to external pressure, either gas or liquid

Heat transfer: Is the exchange of thermal energy between physical systems. The rate of heat transfer is dependent on the temperature of the system and the properties of the intervening medium through which the heat is transferred.

Magnetic field: Is the region around a magnetic material or a moving electric charge within which the force of magnetism acts.

Magnetism: Is a physical process produced by the motion of electric charge, which results in attractive and repulsive forces between objects.

Mathematical modelling: A representation of a system, process or relationship in a mathematical form in which equations are used to simulate the behavior of the system or process under study.

Nano fluids: is a fluid containing nanometer sized particles called nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in base fluid. These particles used in nano fluids are typically made of metals, oxides, carbides, or carbon nanotubes.

Order: the order of a differential equation is the order of the highest derivative involved in the equation.

Ordinary Differential Equation (ODE): An equation containing a single independent variable.

Parallel plate: Is an arrangement of two metal plates connected in parallel separated from each other by some distance.

Partial Differential Equation (PDE): An equation containing two or more independent variables.

Particles: is a small localized objects to which can be ascribed several physical or chemical properties such as volume, density or mass. They vary greatly in size or quantity, from subatomic particles like electron, to microscopic particles like atoms and molecules, to macroscopic particles like powders and other granular materials.

Perturbation technique: is a class of analytical methods for determining approximate solutions of nonlinear equations.

Viscosity: Is the measure of fluids resistance to gradual deformation by shear stress or tensile stress. It is the friction between the molecules of fluids.

Viscous: Having a thick, sticky consistency between solid and liquid.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Related Literature

Transient free convection flows under the influence of a magnetic field have attracted the interest of many researchers in view of their applications in modern materials processing where magnetic fields were known to achieve excellent manipulation and control of electrically-conducting materials (Ibrahim and Shankar, 2014). Magnetohydrodynamic (MHD) convection flows find significant applications in renewable energy devices, including MHD power generators as well as nuclear reactor transport processes (Mukhopadhyay, 2011) wherein magnetic field was employed to regulate heat transfer rates. Therefore great effort have been made to analyze the transient MHD in the presence of viscous energy dissipation.

Uddin *et al.* (2014) studied analysis and computation of magneto-convective non-Newtonian nanofluid slip flow from a permeable stretching sheet. Rana *et al*. (2013) used a variational finite element method to simulate rotating magnetic nanofluid boundary layer flow, heat and mass transfer from an extruding sheet. Recently Rajesh and Chamkha (2014) presented a mathematical model for the unsteady free convective flow and heat transfer of a viscous fluid from a moving vertical cylinder in the presence of thermal radiation. Later Rajesh and Anwar (2014) numerically studied the effects of MHD on the transient free convection flow of a viscous, electrically conducting, and incompressible nanofluid past a moving semi-infinite vertical cylinder with temperature oscillation.

Nelson and Wood (1989) had presented numerical analysis of developing laminar flow between vertical parallel plates for combined heat and mass transfer natural convection with uniform wall temperature/concentration and uniform heat/mass flux boundary conditions. They also had presented an analytical solution for the fully developed combined heat and mass transfer natural convection between vertical parallel plates with asymmetric boundary conditions. Unsteady free convection couette flow between two vertical parallel plates has been studied by Singh (1998). Lee (1999) had studied a combined numerical and theoretical investigation of laminar natural convection heat and mass transfer in open vertical parallel plates with unheated entry and unheated exit for various thermal and concentration boundary conditions.

Singh *et al*. (1996) had studied the transient free convection flow of a viscous incompressible fluid in a vertical parallel plate channel, when the walls are heated asymmetrically. Jha (2001) had studied the combined effect of natural convection and uniform transverse magnetic field on the unsteady couette flow. Narahari *et al*. (2002) had studied the transient free convection flow between two infinite vertical parallel plates with constant heat flux at one boundary. Jha *et al*. (2003) have presented the transient free convection flow in a vertical channel as a result of symmetric heating of the channel walls.

Narahari (2008) had studied the transient free convection flow of a viscous incompressible fluid between two infinite vertical parallel plates in the presence of constant temperature and mass diffusion.

Mohammed *et al.* (2015) presented an analytical method to describe the heat and mass transfer in the flow of an incompressible viscous fluid past an infinite vertical plate with the governing equations accounting for viscous dissipation effect and mass transfer with chemical reaction of constant reaction rate.

Ahmad *et al.* (2019) considered the transient free convection flow of nanofluids between two vertical parallel plates in the presence of radiation and damped thermal flux. The generalized Fourier's law is considered in thermal flux constitutive equation with a weakly memory. The integral transform technique is used for finding the exact solutions of the fractional governing differential equations for fluid temperature and velocity field. The solutions are presented in the term of the time-fractional derivative of the Wright function and Robotnov and Hartley function.

Ostrach (1952) had studied laminar free convection flow of a viscous incompressible fluid between two vertical walls with constant wall temperature. Ostrach (1954) and Sparrow *et al.* (1959) had studied the combined effect of a steady free and forced convection laminar flow and heat transfer between two vertical parallel walls.

Megaraju *et al.* (2021) analyzed the transient MHD current of an exponentially accelerated isothermal vertical plate with Hall current and chemical reaction effects, and use the finite element method to solve the dimensionless equation under appropriate initial and boundary conditions. The Galerkin finite element method is used for numerical solution. It is a powerful and stable technology that provides excellent convergence and is suitable for processing coupled systems with partial differential equations. Their results are useful in fields related to heat and mass transfer research.

Pantokratoras (2006) had presented their results for a steady free convection flow between vertical parallel plates by considering different conditions on the wall temperature.

Susmay and Bidyasagar (2020) investigated unsteady magnetohydrodynamic free convective flow of an electrically conducting and incompressible fluid past a permeable and periodically moving infinite flat plate with slippage at the surface in the presence of Hall current, rotation, thermal radiation and internal heat generation/absorption. Their model equations are converted into non-dimensional form using suitable dimensionless variables and parameters. Exact analytical solutions, in closed form, for the velocity and temperature fields was obtained with the help of Laplace transform technique. They concluded that the Primary fluid velocity increases with the increase of rotation, Hall current, heat generation and thermal radiation whereas it decreases on increasing magnetic field, suction/injection, frequency oscillation and slip parameters and Secondary fluid velocity is accelerated with Hall current, heat generation and thermal radiation parameters whereas magnetic field, rotation, suction/injection, frequency of oscillation and slip parameters have the tendency to decelerate the flow in secondary direction.

Kettleborought (1972) had described numerically the transient laminar two-dimensional motion of a viscous incompressible fluid between two heated vertical plates in which the motion is generated by a temperature gradient perpendicular to the direction of the body force.

Olanrewaju *et al*. (2012) studied internal heat generation effect on thermal boundary layer with convective surface boundary conditions. They showed that an increase in the internal heat generation prevent the rapid flow of heat from the lower surface to the upper surface of the plate. Kabir *et al*. (2013) examined effects of viscous dissipation on magnetohydrodynamics natural convection flow along a vertical wavy surface with heat generation. They discovered that velocity, temperature and skin friction coefficient enhance higher values of internal heat generation parameter but the same reason the rate of heat transfer reduces.

Ajibade *et al.* (2021) investigated the effects of dynamic viscosity and nonlinear thermal radiation on free convective flow through a vertical porous channel. The study is aimed at finding the possible effects of changing viscosity and nonlinear thermal radiation on the flow characteristics of the fluid. A semi-analytic method of solution popularly called Adomian decomposition method (ADM) is used to split the equations into series after which computer simulation is deployed for the final solution of the equations.

Daniel *et al*. (2013) investigated an unsteady forced and free convection flow past an infinite permeable vertical plate. They observed that thermal boundary layers increases towards the plate with injection and reduced towards the plate with suction and also seen that temperature is higher near the plate with injection while velocity is enhanced near the plate with suction and injection. Mostafa (2015) studied variable fluid properties effects on Hydromagnetic fluid flow over an exponentially stretching sheet. They observed that the local velocity Nusselt number increased with the increase of suction parameter, and heat generation parameter.

Krishna and Chamkha (2020) investigated the Hall and ion slip effects on the MHD convective flow of elastico-viscous fluid through porous medium between two rigidly rotating parallel plates with time fluctuating sinusoidal pressure gradient. Analytical solutions for the velocity, temperature and concentration were evaluated and discussed computationally with the help of graphical profiles. For engineering interest, they obtained skin friction, Nusselt number, Sherwood number and volumetric flow rate and discussed numerically. Elasticity and magnetic field resist the fluid motion gets thinner boundary layer. Lesser frequency of oscillating pressure gradient frightens the reverse flow.

Sajid and Hayat (2008) studied the influence of thermal radiation on the boundary layer flow due to an exponentially stretching sheet using homotopy analysis method. The study of MHD has important applications, and may be used to deal with problems such as cooling of nuclear reactors by liquid sodium and induction flow metre, which depends on the potential difference of the fluid in the direction perpendicular to the motion and to the magnetic field.

Isah *et al*. (2019) discussed thermal radiation and variable pressure effects on natural convective heat and mass transfer fluid flow in porous medium. They obtain solutions for time dependent energy, concentration and momentum equations by the perturbation series method after transforming into ordinary differential equations.

Olayiwola (2016) presented an analytical method for studying chemically reacting flow in a laminar premixed flame of carbon monoxide/oxygen mixture in the region of the stagnation point his result showed that the velocity increased as prandtl number increased, Biot number decreased the fluid velocity and enhanced the species concentration and flam temperature.

Attia and Aboul-Hassan (2002) studied the flow of a conducting, viscoelastic fluid between two horizontal porous plates in the presence of transverse magnetic field. The plates were assumed to be non-conducting and maintained at two fixed points but at different temperatures. The fluid viscosity was assumed to be temperature dependent and the fluid was subjected to a uniform suction from above and injective from below. The motion of the fluid was produced by a uniform horizontal pressure gradient. The equation of motion and energy were solved numerically to yield the velocity and temperature distributions.

Rajput and Sahu (2011) studied transient free convection MHD flow between two long vertical parallel plates with constant temperature and variable mass diffusion which neglects viscous energy dissipation. Their model equations are:

$$\frac{\partial u'}{\partial t'} = g\beta\left(T' - T_d'\right) + g\beta^*\left(C' - C_d'\right) + \upsilon\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_0^2}{\rho}u' \tag{2.1}$$

$$\frac{\partial T'}{\partial t'} = \frac{k}{\rho C_p}\frac{\partial^2 T'}{\partial y'^2} \tag{2.2}$$

$$\frac{\partial C'}{\partial t'} = D\frac{\partial^2 C'}{\partial y'^2} \tag{2.3}$$

With initial and boundary conditions as follows,

$$\left.\begin{aligned} &u'(y',0) = 0, \quad u'(0,t') = 0, \quad u'(d,t') = 0 \\ &T'(y',0) = T_d', \quad T'(0,t') = T_w', \quad T'(d,t') = T_d' \\ &C'(y',0) = C_d', \quad C'(0,t') = C_d' + \left(C_w' - C_d'\right)\frac{t'\upsilon}{d^2}, \quad C'(d,t') = C_d' \end{aligned}\right\} \tag{2.4}$$

Where,

u' is the velocity of the fluid,

g is the acceleration due to gravity,

β as the volumetric coefficient of thermal espansion,

t' as the time,

d is the distance between two vertical plates,

T' - the temperature of the fluid,

T_d' - the temperature of the plate at $y' = d$

β^* is the volumetric coefficient of concentration expansion,

C' - species concentration of the fluid,

C_d' - species concentration at the plate $y' = d$,

υ is the kinematic viscosity,

y' - the coordinate axis normal to the plates,

ρ is the density,

C_p - the specific heat at constant pressure,

k - the thermal conductivity of the fluid,

D is the mass diffusion coefficient,

T_w' is the temperature of the plate at $y' = 0$,

C_w' - species concentration at the plate $y' = 0$,

B_0 is the uniform magnetic field,

σ is the electrical conductivity.

2.1 Summary of Review and Gap to fill

In reviewing the above literature, it has been discovered that several works had been carried out on transient free convection flow of a viscous incompressible fluid. Some authors considered transient magnetohydrodynamic free convection flow between two long vertical parallel plates without considering viscous energy dissipation. In view of the above, this research work seeks to consider transient magnetohydrodynamic free convection flow between two long vertical parallel plates with viscous energy dissipation, thereby extended the work of Rajput and Sahu (2011) by incorporating viscous energy dissipation to the energy equation.

CHAPTER THREE

3.0 MATERIALS AND METHOD

3.1 Mathematical Formulation

In this problem, the x' - axis is considered along one of the vertical plates and y' - axis is taken normal to the plates. Initially, at time $t' \leq 0$ the temperature of the fluid and the plates are same as $T_d{}'$ and concentration of the fluid is $C_d{}'$. At $t' > 0$, the temperature of the fluid near the plate (at $y' = 0$) is raised to $T_w{}'$ and the concentration of the fluid near the plate (at $y' = 0$) is raised linearly with time t, causing the flow of free convection currents. The governing equations under the usual Boussinesq's approximation are as follows:

$$\frac{\partial u'}{\partial t'} = \upsilon \frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_0^2}{\rho} u' + g\beta\left(T' - T_d{}'\right) + g\beta^*\left(C' - C_d{}'\right) \tag{3.1}$$

$$\frac{\partial T'}{\partial t'} = \frac{k}{\rho C_p}\frac{\partial^2 T'}{\partial y'^2} + \frac{\mu}{\rho C_p}\left(\frac{\partial u'}{\partial y'}\right)^2 \tag{3.2}$$

$$\frac{\partial C'}{\partial t'} = D\frac{\partial^2 C'}{\partial y'^2} \tag{3.3}$$

With initial and boundary conditions as;

$$\left.\begin{array}{l} u'(y',0)=0, \quad u'(0,t')=0, \quad u'(d,t')=0 \\ T'(y',0)=T_d{}', \quad T'(0,t')=T_w{}', \quad T'(d,t')=T_d{}' \\ C'(y',0)=C_d{}', \quad C'(0,t')=C_d{}'+\left(C_w{}'-C_d{}'\right)\dfrac{\upsilon}{d^2}t', \quad C'(d,t')=C_d{}' \end{array}\right\} \qquad (3.4)$$

Where,

u' is the velocity of the fluid,

g is the acceleration due to gravity,

β as the volumetric coefficient of thermal espansion,

t' as the time,

d is the distance between two vertical plates,

T' - the temperature of the fluid,

$T_d{}'$ - the temperature of the plate at $y' = d$

β^* is the volumetric coefficient of concentration expansion,

C' - species concentration of the fluid,

$C_d{}'$ - species concentration at the plate $y' = d$,

υ is the kinematic viscosity,

μ is the dynamic viscosity

y' - the coordinate axis normal to the plates,

ρ is the density,

C_p - the specific heat at constant pressure,

k - the thermal conductivity of the fluid,

D is the mass diffusion coefficient,

$T_w{}'$ is the temperature of the plate at $y' = 0$,

$C_w{}'$ - species concentration at the plate $y' = 0$,

B_0 is the uniform magnetic field,

σ is the electrical conductivity.

3.2 Non-dimensionalisation

Equation (3.1), (3.2), (3.3) and (3.4) were non-dimensionalised using the following dimensionless variables

$$\left.\begin{array}{l} y = \dfrac{y'}{d},\quad t = \dfrac{Ut'}{d},\quad u = \dfrac{u'}{U},\quad \omega = \dfrac{d\omega'}{U} \\ \theta = \dfrac{T' - T_d{}'}{T_w{}' - T_d{}'},\quad \phi = \dfrac{C' - C_d{}'}{C_w{}' - C_d{}'} \end{array}\right\} \tag{3.5}$$

From equation (3.5), equation (3.6) was obtained

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t'} = \dfrac{U}{d}\dfrac{\partial}{\partial t},\quad \dfrac{\partial}{\partial y'} = \dfrac{1}{d}\dfrac{\partial}{\partial y},\quad \dfrac{\partial^2}{\partial y'^2} = \dfrac{1}{d^2}\dfrac{\partial^2}{\partial y^2} \\ T' = \left(T_w{}' - T_d{}'\right)\theta + T_d{}',\quad C' = \left(C_w{}' - C_d{}'\right)\phi + C_d{}' \end{array}\right\} \tag{3.6}$$

Equation (3.1) becomes,

$$\frac{U^2}{d}\frac{\partial u}{\partial t}=\frac{\upsilon U}{d^2}\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2 U}{\rho}u+g\beta\left(T_w'-T_d'\right)\theta+g\beta^*\left(C_w'-C_d'\right)\phi \tag{3.7}$$

Multiply through by $\frac{d}{U^2}$ to have,

$$\frac{\partial}{\partial t}=\frac{\upsilon}{dU}\frac{\partial^2}{\partial y^2}-\frac{\sigma B_0^2 d}{\rho U}u+\frac{g\beta d\left(T_w'-T_d'\right)}{U^2}\theta+\frac{g\beta^* d\left(C_w'-C_d'\right)}{U^2}\phi \tag{3.8}$$

Equation (3.2) becomes,

$$\frac{U\left(T_w'-T_d'\right)}{d}\frac{\partial\theta}{\partial t}=\frac{k\left(T_w'-T_d'\right)}{\rho C_p d^2}\frac{\partial^2\theta}{\partial y^2}+\frac{\mu U^2}{\rho C_p d^2}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.9}$$

Multiply equation (3.9) through by $\frac{d}{U\left(T_w'-T_d'\right)}$ to have,

$$\frac{\partial\theta}{\partial t}=\frac{k}{\rho C_p dU}\frac{\partial^2\theta}{\partial y^2}+\frac{\mu U}{\rho C_p d\left(T_w'-T_d'\right)}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.10}$$

And equation (3.3) becomes,

$$\frac{U\left(C_w'-C_d'\right)}{d}\frac{\partial\phi}{\partial t}=\frac{D\left(C_w'-C_d'\right)}{d^2}\frac{\partial^2\phi}{\partial y^2} \tag{3.11}$$

Multiply (3.11) through by $\frac{d}{U\left(C_w'-C_d'\right)}$ to have,

$$\frac{\partial\phi}{\partial t}=\frac{D}{dU}\frac{\partial^2\phi}{\partial y^2} \tag{3.12}$$

Equation (3.4) also becomes,

$$\left.\begin{aligned}
&u(y,0)=\frac{u'(y',0)}{U}=0\ \ as\ u'(y',0)=0,\ \ u(0,t)=\frac{u'(y',0)}{U}=0\ \ as\ u'(y',0)=0,\\
&u(1,t)==\frac{u'(d,0)}{U}=0\ \ as\ u'(d,0)=0,\ \ \theta(y,0)=\frac{T'-T_d'}{T_w'-T_d'}=0\ as\ T'(y',0)=T_d',\\
&\theta(0,t)=\frac{T'-T_d'}{T_w'-T_d'}=1\ \ as\ \ T'(0,t')=T_w',\ \theta(1,t)=\frac{T'-T_d'}{T_w'-T_d'}=0\ as\ T'(d,0)=T_d',\\
&\phi(y,0)=\frac{C'-C_d'}{C_w'-C_d'}=0\quad as\quad C'(y',0)=C_d',\\
&\phi(0,t)=\frac{C'-C_d'}{C_w'-C_d'}=\frac{t}{R_e}\quad as\quad C'(0,t')=C_d'+\left(C_w'-C_d'\right)\frac{\upsilon}{d^2}t',\\
&\phi(0,t)=\frac{C'-C_d'}{C_w'-C_d'}=0\ \ as\ \ C'(d,t')=C_d'
\end{aligned}\right\} \tag{3.13}$$

The dimensionless equations becomes,

$$\frac{\partial u}{\partial t}=\frac{1}{R_e}\frac{\partial^2 u}{\partial y^2}-\frac{H_a^2}{R_e}u+\frac{G_{r\theta}}{R_e}\theta+\frac{G_{r\phi}}{R_e}\phi \tag{3.14}$$

$$\frac{\partial\theta}{\partial t}=\frac{1}{P_e}\frac{\partial^2\theta}{\partial y^2}+\frac{E_c}{R_e}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.15}$$

$$\frac{\partial\phi}{\partial t}=\frac{1}{P_{em}}\frac{\partial^2\phi}{\partial y^2} \tag{3.16}$$

With dimensionless initial and boundary conditions as

$$\left.\begin{aligned}&u(y,0)=0, \quad u(0,t)=0, \quad u(1,t)=0\\&\theta(y,0)=0, \quad \theta(0,t)=1, \quad \theta(1,t)=0\\&\phi(y,0)=0, \quad \phi(0,t)=\frac{t}{R_e}, \quad \phi(1,t)=0\end{aligned}\right\} \tag{3.17}$$

Where,

$\frac{dU}{\upsilon}=R_e$ = Reynolds number

$\frac{\sigma B_0^2 d}{\rho U}=N=\frac{H_a^2}{R_e}$ = Stuarts number

$\frac{g\beta d^2\left(T_w'-T_d'\right)}{\upsilon U}=G_{r\theta}$ = Thermal Grashof number

$\frac{g\beta^* d^2\left(C_w'-C_d'\right)}{\upsilon U}=G_{r\phi}$ = Solutel Grashof number

$\frac{\rho C_p dU}{k}=P_e$ = Peclet Energy number

$\frac{U^2}{C_p\left(T_w'-T_d'\right)}=E_c$ = Eckert number

$\frac{dU}{D}=P_{em}$ = Peclet mass number

$\frac{\sigma B_0^2 d^2}{\mu}=H_a^2$ = Hartman number (3.18)

3.3 Method of Solution

For a purely oscillating flow, we have

$$u(y,t)=u(y)e^{i\omega t},\quad \theta(y,t)=\theta(y)e^{2i\omega t},\quad \phi(y,t)=\phi(y)e^{i\omega t} \tag{3.19}$$

So,

$$\left.\begin{aligned}
&\frac{\partial u(y,t)}{\partial t}=i\omega e^{i\omega t}u(y),\quad \frac{\partial u(y,t)}{\partial y}=e^{i\omega t}\frac{du(y)}{dy},\quad \left(\frac{\partial u(y,t)}{\partial y}\right)^2=e^{2i\omega t}\left(\frac{du(y)}{dy}\right)^2,\\
&\frac{\partial^2 u(y,t)}{\partial y^2}=e^{i\omega t}\frac{d^2u(y)}{dy^2},\quad \frac{\partial \theta(y,t)}{\partial t}=2i\omega e^{2i\omega t}\theta(y),\quad \frac{\partial \theta(y,t)}{\partial y}=e^{2i\omega t}\frac{d\theta(y)}{dy},\\
&\frac{\partial^2 \theta(y,t)}{\partial y^2}=e^{2i\omega t}\frac{d^2\theta(y)}{dy^2},\quad \frac{\partial \phi(y,t)}{\partial t}=i\omega e^{i\omega t}\phi(y),\quad \frac{\partial \phi(y,t)}{\partial y}=e^{i\omega t}\frac{d\phi(y)}{dy},\\
&\frac{\partial^2 \phi(y,t)}{\partial y^2}=e^{i\omega t}\frac{d^2\phi(y)}{dy^2}
\end{aligned}\right\} \tag{3.20}$$

Put equation (3.20) in equations (3.14), (3.15) and (3.16) to have,

$$i\omega e^{i\omega t}u(y)=\frac{e^{i\omega t}}{R_e}\frac{d^2u(y)}{dy^2}-\frac{H_a^2e^{i\omega t}}{R_e}u(y)+\frac{G_{r\theta}e^{2i\omega t}}{R_e}\theta(y)+\frac{G_{r\phi}e^{i\omega t}}{R_e}\phi(y) \tag{3.21}$$

$$2i\omega e^{2i\omega t}\theta(y)=\frac{e^{2i\omega t}}{P_e}\frac{d^2\theta(y)}{dy^2}+\frac{E_ce^{2i\omega t}}{R_e}\left(\frac{du(y)}{dy}\right)^2 \tag{3.22}$$

$$i\omega e^{i\omega t}\phi(y)=\frac{e^{i\omega t}}{P_{em}}\frac{d^2\phi(y)}{dy^2} \tag{3.23}$$

Multiplying equation (3.21) by $\dfrac{R_e}{e^{i\omega t}}$

The following equations were obtained

$$i\omega R_e u = \frac{d^2u}{dy^2} - H_a^2 u + e^{i\omega t} G_{r\theta}\theta + G_{r\phi}\phi \tag{3.24}$$

$$\frac{d^2u}{dy^2} - \left(H_a^2 + i\omega R_e\right)u = -e^{i\omega t} G_{r\theta}\theta - G_{r\phi}\phi \tag{3.25}$$

So,

$$\frac{d^2u}{dy^2} - b_1^2 u = -e^{i\omega t} G_{r\theta}\theta - G_{r\phi}\phi \tag{3.26}$$

Where,

$$b_1 = \sqrt{H_a^2 + i\omega R_e} \tag{3.27}$$

Multiply equation (3.22) by $\frac{P_e}{e^{2i\omega t}}$ to have,

$$2i\omega P_e\theta = \frac{d^2\theta}{dy^2} + \frac{E_c P_e}{R_e}\left(\frac{du}{dy}\right)^2 \tag{3.28}$$

$$\frac{d^2\theta}{dy^2} - 2i\omega P_e\theta = -\frac{E_c P_e}{R_e}\left(\frac{du}{dy}\right)^2 \tag{3.29}$$

So,

$$\frac{d^2\theta}{dy^2} - b_2^2\theta = -\frac{E_c P_e}{R_e}\left(\frac{du}{dy}\right)^2 \tag{3.30}$$

Where,

$$b_2 = \sqrt{2i\omega P_e} \tag{3.31}$$

Multiply equation (3.23) by $\frac{P_{em}}{e^{i\omega t}}$ to have,

$$i\omega P_{em}\phi = \frac{d^2\phi}{dy^2} \tag{3.32}$$

$$\frac{d^2\phi}{dy^2} - i\omega P_{em}\phi = 0 \tag{3.33}$$

$$\frac{d^2\phi}{dy^2} - b_3^2\phi = 0 \tag{3.34}$$

Where,

$$b_3 = \sqrt{i\omega P_{em}} \tag{3.35}$$

The corresponding boundary conditions are,

$$\left.\begin{array}{l} u(0) = \frac{u(0,t)}{e^{i\omega t}} = 0 \ \ at\, u(0,t) = 0, \ \ u(1) = \frac{u(1,t)}{e^{i\omega t}} = 0 \ at \ \ u(0,t) = 0, \\ \theta(0) = \frac{\theta(0,t)}{e^{2i\omega t}} = e^{-2i\omega t} \ \ at \ \theta(0,t) = 1, \ \ \theta(1) = \frac{\theta(1,t)}{e^{2i\omega t}} = 0 \ at \ \theta(0,t) = 0, \\ \phi(0) = \frac{\phi(0,t)}{e^{i\omega t}} = \frac{te^{-i\omega t}}{R_e} \ \ at \ \ \phi(0,t) = \frac{t}{R_e}, \ \ \phi(1) = \frac{\phi(1,t)}{e^{i\omega t}} = 0 \ \ at \ \ \phi(1,t) = 0 \end{array}\right\} \tag{3.36}$$

That is,

$$\left.\begin{array}{l} u(0) = 0, \quad u(1) = 0, \\ \theta(0) = e^{-2i\omega t}, \quad \theta(1) = 0, \\ \phi(0) = \frac{te^{-i\omega t}}{R_e}, \ \phi(1) = 0 \end{array}\right\} \tag{3.37}$$

Let,

$0 < G_{r\theta} << 1$ to give,

$$\left.\begin{aligned} u &= u_0 + G_{r\theta}u_1 + G_{r\theta}^2 u_2 + \ldots \\ \theta &= \theta_0 + G_{r\theta}\theta_0 + G_{r\theta}^2\theta_0 + \ldots \\ \phi &= \phi_0 + G_{r\theta}\phi_1 + G_{r\theta}^2\phi_2 + \ldots \end{aligned}\right\} \tag{3.38}$$

Putting equation (3.38) in equations (3.26), (3.30) and (3.34), the following equations were obtained

$$\frac{d^2u_0}{dy^2} + G_{r\theta}\frac{d^2u_1}{dy^2} + \ldots - b_1^2\left(u_0 + G_{r\theta}u_1 + \ldots\right) = -e^{i\omega t}G_{r\theta}\left(\theta_0 + G_{r\theta}\theta_0 + \ldots\right) - G_{r\phi}\left(\phi_0 + G_{r\theta}\phi_1 + \ldots\right)$$

(3.39)

$$\frac{d^2\theta_0}{dy^2} + G_{r\theta}\frac{d^2\theta_1}{dy^2} + \ldots - b_2^2\left(\theta_0 + G_{r\theta}\theta_0 + \ldots\right) = -\frac{E_cP_e}{R_e}\left(\frac{du_0}{dy} + G_{r\theta}\frac{du_0}{dy} + \ldots\right)^2 \tag{3.40}$$

$$\frac{d^2\phi_0}{dy^2} + G_{r\theta}\frac{d^2\phi_1}{dy^2} + \ldots - b_3^2\left(\phi_0 + G_{r\theta}\phi_1 + \ldots\right) = 0 \tag{3.41}$$

For Order 0, That is $O\left(G_{r\theta}^0\right):1$ we have,

$$\left.\begin{aligned} &\frac{d^2u_0}{dy^2} - b_1^2u_o = -G_{r\phi}\phi_0 \\ &u_0(0) = 0, \quad u_0(1) = 0 \end{aligned}\right\} \tag{3.42}$$

$$\left.\begin{aligned} &\frac{d^2\theta_0}{dy^2} - b_2^2\theta_0 = -\frac{E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)^2 \\ &\theta_0(0) = e^{-2i\omega t}, \quad \theta_0(1) = 0 \end{aligned}\right\} \tag{3.43}$$

$$\left.\begin{aligned}&\frac{d^2\phi_0}{dy^2}-b_3^2\phi_0=0\\&\phi_0(0)=\frac{te^{-i\omega t}}{R_e},\quad \phi_0(1)=0\end{aligned}\right\}\tag{3.44}$$

For Order 1, That is $O\left(G_{r\theta}^{1}\right):G_{r\theta}$ becomes,

$$\left.\begin{aligned}&\frac{d^2u_1}{dy^2}-b_1^2u_1=-e^{i\omega t}\theta_0-G_{r\phi}\phi_1\\&u_1(0)=0,\quad u_1(1)=0\end{aligned}\right\}\tag{3.45}$$

$$\left.\begin{aligned}&\frac{d^2\theta_1}{dy^2}-b_2^2\theta_1=-\frac{2E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\\&\theta_1(0)=0,\quad \theta_1(1)=0\end{aligned}\right\}\tag{3.46}$$

$$\left.\begin{aligned}&\frac{d^2\phi_1}{dy^2}-b_3^2\phi_1=0\\&\phi_1(0)=0,\quad \phi_1(1)=0\end{aligned}\right\}\tag{3.47}$$

Solving equation (3.44) ,

Seeking

$$\phi_0=e^{my}\tag{3.48}$$

Differentiating equation (3.48) twice with respect to y to have

$$\frac{d^2\phi_0}{dy^2}=m^2e^{my}\tag{3.49}$$

Put equations (3.49) and (3.48) in equation (3.44) to have,

$$m^2e^{my} - b_3^2e^{my} = 0 \quad (3.50)$$

$$\left(m^2 - b_3^2\right)e^{my} = 0 \quad (3.51)$$

For $e^{my} \neq 0$,

$$\left(m^2 - b_3^2\right) = 0 \quad (3.52)$$

$$m^2 - b_3^2 = 0 \quad (3.53)$$

$$m^2 = b_3^2 \quad (3.54)$$

$$m = \pm b_3 \quad (3.55)$$

So assume a solution to be

$$\phi_0\left(y\right) = A_1e^{b_3y} + A_2e^{-b_3y} \quad (3.56)$$

Applying the boundary conditions,

$\phi_0\left(0\right) = \dfrac{te^{-i\omega t}}{R_e}$ to have,

$$A_1 + A_2 = \frac{te^{-i\omega t}}{R_e} \quad (3.57)$$

$$A_1 = \frac{te^{-i\omega t}}{R_e} - A_2 \quad (3.58)$$

Also for $\phi_0\left(1\right) = 0$,

$$A_1 e^{b_3} + A_2 e^{-b_3} = 0 \tag{3.59}$$

$$\left(\frac{te^{-i\omega t}}{R_e} - A_2\right)e^{b_3} + A_2 e^{-b_3} = 0 \tag{3.60}$$

$$-A_2 e^{b_3} + A_2 e^{-b_3} = -\frac{te^{-i\omega t}e^{b_3}}{R_e} \tag{3.61}$$

$$-A_2\left(e^{b_3} - e^{-b_3}\right) = -\frac{te^{-i\omega t}e^{b_3}}{R_e} \tag{3.62}$$

$$A_2 = \frac{te^{-i\omega t}e^{b_3}}{R_e\left(e^{b_3} - e^{-b_3}\right)} \tag{3.63}$$

Put equation (3.63) in equation (3.58) to have,

$$A_1 = \frac{te^{-i\omega t}}{R_e} - \frac{te^{-i\omega t}e^{b_3}}{R_e\left(e^{b_3} - e^{-b_3}\right)} \tag{3.64}$$

$$A_1 = \frac{te^{-i\omega t}\left(e^{b_3} - e^{-b_3}\right) - te^{-i\omega t}e^{b_3}}{R_e\left(e^{b_3} - e^{-b_3}\right)} \tag{3.65}$$

$$A_1 = \frac{-te^{-i\omega t}e^{-b_3}}{R_e\left(e^{b_3} - e^{-b_3}\right)} \tag{3.66}$$

$$A_1 = \frac{te^{-i\omega t}e^{-b_3}}{R_e\left(e^{-b_3} - e^{b_3}\right)} \tag{3.67}$$

So the solution to equation (3.44) is

$$\phi_0\left(y\right) = A_1 e^{b_3 y} + A_2 e^{-b_3 y} \tag{3.68}$$

Where,

$$\left.\begin{aligned} A_1 &= \frac{te^{-i\omega t}e^{-b_3}}{R_e\left(e^{-b_3}-e^{b_3}\right)} \\ A_2 &= \frac{te^{-i\omega t}e^{b_3}}{R_e\left(e^{b_3}-e^{-b_3}\right)} \end{aligned}\right\} \tag{3.69}$$

Solving equation (3.47)

$$\left.\begin{aligned} &\frac{d^2\phi_1}{dy^2}-b_3^2\phi_1=0 \\ &\phi_1(0)=0, \quad \phi_1(1)=0 \end{aligned}\right\}$$

Seeking,

$$\phi_1 = e^{my} \tag{3.70}$$

The second derivative of equation (3.70) with respect to y gives,

$$\frac{d^2\phi_1}{dy^2}=m^2e^{my} \tag{3.71}$$

Put equations (3.71) and (3.70) in equation (3.47) to give,

$$m^2e^{my}-b_3^2e^{my}=0 \tag{3.72}$$

$$\left(m^2-b_3^2\right)e^{my}=0 \tag{3.73}$$

For $e^{my}\neq 0$ to have,

$$\left(m^2-b_3^2\right)=0 \tag{3.74}$$

$$m^2 - b_3^2 = 0 \tag{3.75}$$

$$m^2 = b_3^2 \tag{3.76}$$

$$m = \pm b_3 \tag{3.77}$$

So assume a solution of,

$$\phi_1(y) = A_3 e^{b_3 y} + A_4 e^{-b_3 y} \tag{3.78}$$

Applying the boundary conditions, that is

For $\phi_1(0) = 0$,

$$A_3 + A_4 = 0 \tag{3.79}$$

$$A_3 = -A_4 \tag{3.80}$$

Also for $\phi_1(1) = 0$,

$$A_3 e^{b_3} + A_4 e^{-b_3} = 0 \tag{3.81}$$

Put equation (3.80) into equation (3.81) to have,

$$-A_4 e^{b_3} + A_4 e^{-b_3} = 0 \tag{3.82}$$

$$A_4\left(e^{-b_3} - e^{b_3}\right) = 0 \tag{3.83}$$

That is,

$$A_4 = 0 \tag{3.84}$$

Put equation (3.84) into equation (3.80) to have,

$$A_3 = 0 \tag{3.85}$$

Put equations (3.85) and (3.84) into equation (3.87), the solution to (3.47) becomes,

$$\phi_1(y) = 0 \tag{3.86}$$

Solving equation (3.42)

$$\left.\begin{aligned} &\frac{d^2u_0}{dy^2} - b_1^2 u_o = -G_{r\phi}\phi_0 \\ &u_0(0) = 0, \quad u_0(1) = 0 \end{aligned}\right\}$$

Put equation (3.68) into equation (3.42) to give,

$$\frac{d^2u_0}{dy^2} - b_1^2 u_o = -G_{r\phi}\left(A_1 e^{b_3 y} + A_2 e^{-b_3 y}\right) \tag{3.87}$$

Solving the homogeneous part,

$$\frac{d^2u_0}{dy^2} - b_1^2 u_o = 0 \tag{3.88}$$

Seeking,

$$u_0 = e^{my} \tag{3.89}$$

The second derivative of equation (3.89) with respect to y gives

$$\frac{d^2u_0}{dy^2} = m^2 e^{my} \tag{3.90}$$

Putting equations (3.90) and (3.89) in equation (3.88) gives,

$$m^2 e^{my} - b_1^2 e^{my} = 0 \tag{3.91}$$

$$\left(m^2 - b_1^2\right) e^{my} = 0 \tag{3.92}$$

For $e^{my} \neq 0$,

$$\left(m^2 - b_1^2\right) = 0 \tag{3.93}$$

$$m^2 - b_1^2 = 0 \tag{3.94}$$

$$m^2 = b_1^2 \tag{3.95}$$

So,

$$m = \pm b_1 \tag{3.96}$$

The complimentary solution is given by

$$u_{0c}\left(y\right) = A_5 e^{b_1 y} + A_6 e^{-b_1 y} \tag{3.97}$$

Assume a particular solution to be

$$u_{0p}\left(y\right) = A_7 e^{b_3 y} + A_8 e^{-b_3 y} \tag{3.98}$$

The second derivative of equation (3.98) with respect to y gives

$$\frac{d^2 u_{0p}}{dy^2} = A_7 b_3^2 e^{b_3 y} + A_8 b_3^2 e^{-b_3 y} \tag{3.99}$$

Putting equations (3.99) and (3.98) into equation (3.87) gives,

$$A_7 b_3^2 e^{b_3 y} + A_8 b_3^2 e^{-b_3 y} - b_1^2 \left(A_7 e^{b_3 y} + A_8 e^{-b_3 y} \right) = -G_{r\phi} \left(A_1 e^{b_3 y} + A_2 e^{-b_3 y} \right) \tag{3.100}$$

Comparing the variables in equation (3.100),

$$A_7 b_3^2 - b_1^2 A_7 = -G_{r\phi} A_1 \tag{3.101}$$

$$A_8 b_3^2 - b_1^2 A_8 = -G_{r\phi} A_2 \tag{3.102}$$

From equation (3.101),

$$A_7 = \frac{-G_{r\phi} A_1}{b_3^2 - b_1^2} \tag{3.103}$$

From equation (3.102),

$$A_8 = \frac{-G_{r\phi} A_2}{b_3^2 - b_1^2} \tag{3.104}$$

So,

$$u_{0p}(y) = A_7 e^{b_3 y} + A_8 e^{-b_3 y} \tag{3.105}$$

The solution to equation (3.42) is given by,

$$u_0(y) = u_{0c}(y) + u_{op}(y) \tag{3.106}$$

Put equations (3.97) and (3.105) into equation (3.106),

$$u_0(y) = A_5 e^{b_1 y} + A_6 e^{-b_1 y} + A_7 e^{b_3 y} + A_8 e^{-b_3 y} \tag{3.107}$$

Applying the boundary conditions,

$u_0(0)=0$ gives,

$$A_5 + A_6 + A_7 + A_8 = 0 \tag{3.108}$$

$$A_5 = -A_7 - A_8 - A_6 \tag{3.109}$$

Also $u_0(1)=0$ gives,

$$A_5 e^{b_1} + A_6 e^{-b_1} + A_7 e^{b_3} + A_8 e^{-b_3} = 0 \tag{3.110}$$

Put equation (3.109) into equation (3.110),

$$\left(-A_7 - A_8 - A_6\right)e^{b_1} + A_6 e^{-b_1} + A_7 e^{b_3} + A_8 e^{-b_3} = 0 \tag{3.111}$$

$$A_6\left(e^{-b_1} - e^{b_1}\right) = A_7 e^{b_1} + A_8 e^{b_1} - A_7 e^{b_3} - A_8 e^{-b_3} \tag{3.112}$$

So,

$$A_6 = \frac{A_7 e^{b_1} + A_8 e^{b_1} - A_7 e^{b_3} - A_8 e^{-b_3}}{\left(e^{-b_1} - e^{b_1}\right)} \tag{3.113}$$

Put equation (3.113) into equation (3.109),

$$A_5 = -A_7 - A_8 - \frac{A_7 e^{b_1} + A_8 e^{b_1} - A_7 e^{b_3} - A_8 e^{-b_3}}{\left(e^{-b_1} - e^{b_1}\right)} \tag{3.114}$$

$$A_5 = \frac{\left(-A_7 - A_8\right)\left(e^{-b_1} - e^{b_1}\right) - A_7 e^{b_1} - A_8 e^{b_1} + A_7 e^{b_3} + A_8 e^{-b_3}}{\left(e^{-b_1} - e^{b_1}\right)} \tag{3.115}$$

$$A_5 = \frac{-\left(A_7 + A_8\right)e^{-b_1} + A_7 e^{b_3} + A_8 e^{-b_3}}{\left(e^{-b_1} - e^{b_1}\right)} \tag{3.116}$$

So the solution to equation (3.42) is,

$$u_0\left(y\right) = A_5 e^{b_1 y} + A_6 e^{-b_1 y} + A_7 e^{b_3 y} + A_8 e^{-b_3 y} \tag{3.117}$$

Solving equation (3.43)

$$\left.\begin{array}{l} \dfrac{d^2\theta_0}{dy^2} - b_2^2\theta_0 = -\dfrac{E_c P_e}{R_e}\left(\dfrac{du_0}{dy}\right)^2 \\ \theta_0\left(0\right) = e^{-2i\omega t}, \quad \theta_0\left(1\right) = 0 \end{array}\right\} \tag{3.118}$$

Differentiate equation (3.117) with respect to y,

$$\frac{du_0}{dy} = A_5 b_1 e^{b_1 y} - A_6 b_1 e^{-b_1 y} + A_7 b_3 e^{b_3 y} - A_8 b_3 e^{-b_3 y} \tag{3.119}$$

Taking the square of both sides of equation (3.119) gives,

$$\left(\frac{du_0}{dy}\right)^2 = \left(A_5 b_1 e^{b_1 y} - A_6 b_1 e^{-b_1 y} + A_7 b_3 e^{b_3 y} - A_8 b_3 e^{-b_3 y}\right)\left(A_5 b_1 e^{b_1 y} - A_6 b_1 e^{-b_1 y} + A_7 b_3 e^{b_3 y} - A_8 b_3 e^{-b_3 y}\right) \tag{3.120}$$

$$\left(\frac{du_0}{dy}\right)^2 = \left(\begin{array}{l} A_5^2 b_1^2 e^{2b_1 y} - A_5 A_6 b_1^2 + A_5 A_7 b_1 b_3 e^{(b_1+b_3)y} - A_5 A_8 b_1 b_3 e^{(b_1-b_3)y} - A_5 A_6 b_1^2 + \\ A_6^2 b_1^2 e^{-2b_1 y} - A_6 A_7 b_1 b_3 e^{(b_3-b_1)y} + A_6 A_8 b_1 b_3 e^{-(b_1+b_3)y} + A_5 A_7 b_1 b_3 e^{(b_1+b_3)y} - \\ A_6 A_7 b_1 b_3 e^{(b_3-b_1)y} + A_7^2 b_3^2 e^{2b_3 y} - A_7 A_8 b_3^2 - A_5 A_8 b_1 b_3 e^{(b_1-b_3)y} + A_6 A_8 b_1 b_3 e^{-(b_1+b_3)y} \\ -A_7 A_8 b_3^2 + A_8^2 b_3^2 e^{-2b_3 y} \end{array}\right) \tag{3.121}$$

$$\left(\frac{du_0}{dy}\right)^2 = \left(\begin{array}{l} A_5^2 b_1^2 e^{2b_1 y} - 2\left(A_5 A_6 b_1^2 + A_7 A_8 b_3^2\right) + 2A_5 A_7 b_1 b_3 e^{(b_1+b_3)y} - 2A_5 A_8 b_1 b_3 e^{(b_1-b_3)y} \\ +A_6^2 b_1^2 e^{-2b_1 y} - 2A_6 A_7 b_1 b_3 e^{(b_3-b_1)y} + 2A_6 A_8 b_1 b_3 e^{-(b_1+b_3)y} + A_7^2 b_3^2 e^{2b_3 y} + A_8^2 b_3^2 e^{-2b_3 y} \end{array}\right) \tag{3.122}$$

Putting equation (3.122) into equation (3.43) gives,

$$\frac{d^2\theta_0}{dy^2} - b_2^2\theta_0 = -\frac{E_c P_e}{R_e}\begin{pmatrix} A_5^2 b_1^2 e^{2b_1 y} - 2\left(A_5 A_6 b_1^2 + A_7 A_8 b_3^2\right) + 2A_5 A_7 b_1 b_3 e^{(b_1+b_3)y} - \\ 2A_5 A_8 b_1 b_3 e^{(b_1-b_3)y} + A_6^2 b_1^2 e^{-2b_1 y} - 2A_6 A_7 b_1 b_3 e^{(b_3-b_1)y} + \\ 2A_6 A_8 b_1 b_3 e^{-(b_1+b_3)y} + A_7^2 b_3^2 e^{2b_3 y} + A_8^2 b_3^2 e^{-2b_3 y} \end{pmatrix} \tag{3.123}$$

Solving the homogeneous part, that is

$$\frac{d^2\theta_0}{dy^2} - b_2^2\theta_0 = 0 \tag{3.124}$$

We seek,

$$\theta_0 = e^{my} \tag{3.125}$$

The second derivative of equation (3.125) with respect to y is

$$\frac{d^2\theta_0}{dy^2} = m^2 e^{my} \tag{3.126}$$

Putting equations (3.126) and (3.125) into equation (3.124) gives,

$$m^2 e^{my} - b_2^2 e^{my} = 0 \tag{3.127}$$

$$\left(m^2 - b_2^2\right)e^{my} = 0 \tag{3.128}$$

For $e^{my} \neq 0$,

$$m^2 - b_2^2 = 0 \tag{3.129}$$

$$m^2 = b_2^2 \tag{3.130}$$

$$m = \pm b_2 \tag{3.131}$$

So,

$$\theta_{0c}\left(y\right) = A_9 e^{b_2 y} + A_{10} e^{-b_2 y} \tag{3.132}$$

Assume a particular solution to be,

$$\theta_{0p}\left(y\right) = \begin{pmatrix} A_{11}e^{2b_1 y} + A_{12} + A_{13}e^{(b_1+b_3)y} + A_{14}e^{(b_1-b_3)y} + A_{15}e^{-2b_1 y} + A_{16}e^{(b_3-b_1)y} + A_{17}e^{-(b_1+b_3)y} \\ +A_{18}e^{2b_3 y} + A_{19}e^{-2b_3 y} \end{pmatrix} \tag{3.133}$$

Differentiating equation (3.133) with respect to y ,

$$\frac{d\theta_{0p}}{dy} = \begin{pmatrix} 2A_{11}b_1 e^{2b_1 y} + A_{13}\left(b_1+b_3\right)e^{(b_1+b_3)y} + A_{14}\left(b_1-b_3\right)e^{(b_1-b_3)y} - 2A_{15}b_1 e^{-2b_1 y} + \\ A_{16}\left(b_3-b_1\right)e^{(b_3-b_1)y} - A_{17}\left(b_1+b_3\right)e^{-(b_1+b_3)y} + 2A_{18}b_3 e^{2b_3 y} - 2A_{19}b_3 e^{-2b_3 y} \end{pmatrix} \tag{3.134}$$

Differentiating (3.134) with respect to y gives,

$$\frac{d^2\theta_{0p}}{dy^2} = \begin{pmatrix} 4A_{11}b_1^2 e^{2b_1 y} + A_{13}\left(b_1+b_3\right)^2 e^{(b_1+b_3)y} + A_{14}\left(b_1-b_3\right)^2 e^{(b_1-b_3)y} + 4A_{15}b_1^2 e^{-2b_1 y} + \\ A_{16}\left(b_3-b_1\right)^2 e^{(b_3-b_1)y} + A_{17}\left(b_1+b_3\right)^2 e^{-(b_1+b_3)y} + 4A_{18}b_3^2 e^{2b_3 y} + 4A_{19}b_3^2 e^{-2b_3 y} \end{pmatrix} \tag{3.135}$$

Putting equations (3.135) and (3.133) into equation (3.123) gives,

$$\begin{pmatrix} 4A_{11}b_1^2 e^{2b_1 y} + A_{13}\left(b_1+b_3\right)^2 e^{(b_1+b_3)y} \\ +A_{14}\left(b_1-b_3\right)^2 e^{(b_1-b_3)y} + 4A_{15}b_1^2 e^{-2b_1 y} + \\ A_{16}\left(b_3-b_1\right)^2 e^{(b_3-b_1)y} + A_{17}\left(b_1+b_3\right)^2 e^{-(b_1+b_3)y} \\ +4A_{18}b_3^2 e^{2b_3 y} + 4A_{19}b_3^2 e^{-2b_3 y} \end{pmatrix} - b_2^2 \begin{pmatrix} A_{11}e^{2b_1 y} + A_{12} + A_{13}e^{(b_1+b_3)y} + A_{14}e^{(b_1-b_3)y} \\ +A_{15}e^{-2b_1 y} + A_{16}e^{(b_3-b_1)y} + A_{17}e^{-(b_1+b_3)y} \\ +A_{18}e^{2b_3 y} + A_{19}e^{-2b_3 y} \end{pmatrix} =$$
$$-\frac{E_c P_e}{R_e}\begin{pmatrix} A_5^2 b_1^2 e^{2b_1 y} - 2\left(A_5 A_6 b_1^2 + A_7 A_8 b_3^2\right) + 2A_5 A_7 b_1 b_3 e^{(b_1+b_3)y} - 2A_5 A_8 b_1 b_3 e^{(b_1-b_3)y} \\ +A_6^2 b_1^2 e^{-2b_1 y} - 2A_6 A_7 b_1 b_3 e^{(b_3-b_1)y} + 2A_6 A_8 b_1 b_3 e^{-(b_1+b_3)y} + A_7^2 b_3^2 e^{2b_3 y} + A_8^2 b_3^2 e^{-2b_3 y} \end{pmatrix} \tag{3.136}$$

Comparing the variables in equation (3.136) gives,

$$4A_{11}b_1^2 - b_2^2 A_{11} = -\frac{E_c P_e A_5^2 b_1^2}{R_e} \tag{3.137}$$

$$-A_{12}b_2^2 = \frac{2E_c P_e \left(A_5 A_6 b_1^2 + A_7 A_8 b_3^2\right)}{R_e} \tag{3.138}$$

$$A_{13}\left(b_1 + b_3\right)^2 - b_2^2 A_{13} = -\frac{2E_c P_e A_5 A_7 b_1 b_3}{R_e} \tag{3.139}$$

$$A_{14}\left(b_1 - b_3\right)^2 - b_2^2 A_{14} = \frac{2E_c P_e A_5 A_8 b_1 b_3}{R_e} \tag{3.140}$$

$$4A_{15}b_1^2 - b_2^2 A_{15} = -\frac{E_c P_e A_6^2 b_1^2}{R_e} \tag{3.141}$$

$$A_{16}\left(b_3 - b_1\right)^2 - b_2^2 A_{16} = \frac{2E_c P_e A_6 A_7 b_1 b_3}{R_e} \tag{3.142}$$

$$A_{17}\left(b_1 + b_3\right)^2 - b_2^2 A_{17} = -\frac{2E_c P_e A_6 A_8 b_1 b_3}{R_e} \tag{3.143}$$

$$4A_{18}b_3^2 - b_2^2 A_{18} = -\frac{2E_c P_e A_7^2 b_3^2}{R_e} \tag{3.144}$$

$$4A_{19}b_3^2 - b_2^2 A_{19} = -\frac{E_c P_e A_8^2 b_3^2}{R_e} \tag{3.145}$$

From equation (3.137), equation (3.146) is obtained

$$A_{11} = -\frac{E_c P_e A_5^2 b_1^2}{R_e\left(4b_1^2 - b_2^2\right)} \tag{3.146}$$

From equation (3.138), equation (3.147) is obtained

$$A_{12} = -\frac{2E_c P_e \left(A_5 A_6 b_1^2 + A_7 A_8 b_3^2 \right)}{R_e b_2^2} \quad (3.147)$$

From equation (3.139), equation (3.148) is obtained

$$A_{13} = -\frac{2E_c P_e A_5 A_7 b_1 b_3}{R_e \left(\left(b_1 + b_3 \right)^2 - b_2^2 \right)} \quad (3.148)$$

From equation (3.140), equation (3.149) is obtained

$$A_{14} = \frac{2E_c P_e A_5 A_8 b_1 b_3}{R_e \left(\left(b_1 - b_3 \right)^2 - b_2^2 \right)} \quad (3.149)$$

From equation (3.141), equation (3.150) is obtained

$$A_{15} = -\frac{E_c P_e A_6^2 b_1^2}{R_e \left(4b_1^2 - b_2^2 \right)} \quad (3.150)$$

From equation (3.142), equation (3.151) is obtained

$$A_{16} = \frac{2E_c P_e A_6 A_7 b_1 b_3}{R_e \left(\left(b_3 - b_1 \right)^2 - b_2^2 \right)} \quad (3.151)$$

From equation (3.143), equation (3.152) is obtained

$$A_{17} = -\frac{2E_c P_e A_6 A_8 b_1 b_3}{R_e \left(\left(b_1 + b_3 \right)^2 - b_2^2 \right)} \quad (3.152)$$

From equation (3.144), equation (3.153) is obtained

$$A_{18} = -\frac{2E_c P_e A_7^2 b_3^2}{R_e\left(4b_3^2 - b_2^2\right)} \tag{3.153}$$

From equation (3.145), equation (3.154) is obtained

$$A_{19} = -\frac{E_c P_e A_8^2 b_3^2}{R_e\left(4b_3^2 - b_2^2\right)} \tag{3.154}$$

The general solution to equation (3.43) is given by,

$$\theta_0\left(y\right) = \theta_{0c}\left(y\right) + \theta_{0p}\left(y\right) \tag{3.155}$$

That is,

$$\begin{aligned}\theta_0\left(y\right) = {} & A_9 e^{b_2 y} + A_{10} e^{-b_2 y} + A_{11} e^{2b_1 y} + A_{12} + A_{13} e^{(b_1+b_3)y} + A_{14} e^{(b_1-b_3)y} + A_{15} e^{-2b_1 y} + \\ & A_{16} e^{(b_3-b_1)y} + A_{17} e^{-(b_1+b_3)y} + A_{18} e^{2b_3 y} + A_{19} e^{-2b_3 y}\end{aligned} \tag{3.156}$$

Applying the boundary conditions, that is

For $\theta_0\left(0\right) = e^{-2i\omega t}$,

$$A_9 + A_{10} + A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} = e^{-2i\omega t} \tag{3.157}$$

$$A_9 = -\left(A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} - e^{-2i\omega t}\right) - A_{10} \tag{3.158}$$

$$A_9 = -B_{34} - A_{10} \tag{3.159}$$

Where,

$$B_{34} = A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} - e^{-2i\omega t} \tag{3.160}$$

For $\theta_0(1)=0$,

$$A_9e^{b_2}+A_{10}e^{-b_2}+A_{11}e^{2b_1}+A_{12}+A_{13}e^{(b_1+b_3)}+A_{14}e^{(b_1-b_3)}+A_{15}e^{-2b_1}+A_{16}e^{(b_3-b_1)}+ A_{17}e^{-(b_1+b_3)}+A_{18}e^{2b_3}+A_{19}e^{-2b_3}=0 \tag{3.161}$$

Put equation (3.159) into equation (3.161) gives,

$$-B_{34}e^{b_2}-A_{10}e^{b_2}+A_{10}e^{-b_2}+A_{11}e^{2b_1}+A_{12}+A_{13}e^{(b_1+b_3)}+A_{14}e^{(b_1-b_3)}+A_{15}e^{-2b_1}+ A_{16}e^{(b_3-b_1)}+A_{17}e^{-(b_1+b_3)}+A_{18}e^{2b_3}+A_{19}e^{-2b_3}=0 \tag{3.162}$$

So,

$$A_{10}=\frac{\left(\begin{array}{l}B_{34}e^{b_2}-A_{11}e^{2b_1}-A_{12}-A_{13}e^{(b_1+b_3)}-A_{14}e^{(b_1-b_3)}-A_{15}e^{-2b_1}-A_{16}e^{(b_3-b_1)}-\\ A_{17}e^{-(b_1+b_3)}-A_{18}e^{2b_3}-A_{19}e^{-2b_3}\end{array}\right)}{e^{-b_2}-e^{b_2}} \tag{3.163}$$

Put equation (3.163) into equation (3.159) gives,

$$A_9=\frac{\left(\begin{array}{l}-B_{34}\left(e^{-b_2}-e^{b_2}\right)-B_{34}e^{b_2}+A_{11}e^{2b_1}+A_{12}+A_{13}e^{(b_1+b_3)}+A_{14}e^{(b_1-b_3)}+A_{15}e^{-2b_1}+\\ A_{16}e^{(b_3-b_1)}+A_{17}e^{-(b_1+b_3)}+A_{18}e^{2b_3}+A_{19}e^{-2b_3}\end{array}\right)}{e^{-b_2}-e^{b_2}} \tag{3.164}$$

$$A_9=\frac{\left(\begin{array}{l}-B_{34}e^{-b_2}+A_{11}e^{2b_1}+A_{12}+A_{13}e^{(b_1+b_3)}+A_{14}e^{(b_1-b_3)}+A_{15}e^{-2b_1}+A_{16}e^{(b_3-b_1)}+\\ A_{17}e^{-(b_1+b_3)}+A_{18}e^{2b_3}+A_{19}e^{-2b_3}\end{array}\right)}{e^{-b_2}-e^{b_2}} \tag{3.165}$$

Therefore,

$$\theta_0(y)=A_9e^{b_2y}+A_{10}e^{-b_2y}+A_{11}e^{2b_1y}+A_{12}+A_{13}e^{(b_1+b_3)y}+A_{14}e^{(b_1-b_3)y}+A_{15}e^{-2b_1y}+ A_{16}e^{(b_3-b_1)y}+A_{17}e^{-(b_1+b_3)y}+A_{18}e^{2b_3y}+A_{19}e^{-2b_3y} \tag{3.166}$$

Solving equation (3.45) that is,

$$\frac{d^2u_1}{dy^2} - b_1^2 u_1 = -e^{i\omega t}\theta_0 - G_{r\phi}\phi_1$$
$$u_1(0) = 0, \quad u_1(1) = 0 \tag{3.167}$$

Put equation (3.166) into equation (3.167) gives,

$$\frac{d^2u_1}{dy^2} - b_1^2 u_1 = -e^{i\omega t}\begin{pmatrix} A_9 e^{b_2 y} + A_{10}e^{-b_2 y} + A_{11}e^{2b_1 y} + A_{12} + A_{13}e^{(b_1+b_3)y} + A_{14}e^{(b_1-b_3)y} \\ + A_{15}e^{-2b_1 y} + A_{16}e^{(b_3-b_1)y} + A_{17}e^{-(b_1+b_3)y} + A_{18}e^{2b_3 y} + A_{19}e^{-2b_3 y} \end{pmatrix} \tag{3.168}$$

For the homogeneous solution,

$$\frac{d^2u_1}{dy^2} - b_1^2 u_1 = 0 \tag{3.169}$$

Seeking,

$$u_1 = e^{my} \tag{3.170}$$

The second derivative of equation (3.170) with respect to y is

$$\frac{d^2u_1}{dy^2} = m^2 e^{my} \tag{3.171}$$

Put equations (3.171) and (3.170) into equation (3.169) gives,

$$m^2 e^{my} - b_1^2 e^{my} = 0 \tag{3.172}$$

$$\left(m^2 - b_1^2\right)e^{my} = 0 \tag{3.173}$$

From $e^{my} \neq 0$ we have,

$$m^2 - b_1^2 = 0 \tag{3.174}$$

$$m^2 = b_1^2 \tag{3.175}$$

That is,

$$m = \pm b_1 \tag{3.176}$$

So,

$$u_{1c}\left(y\right) = A_{20}e^{b_1 y} + A_{21}e^{-b_1 y} \tag{3.177}$$

Assume a particular solution to be,

$$\begin{aligned} u_{1p}\left(y\right) = A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} + A_{25} + A_{26}e^{(b_1+b_3)y} + A_{27}e^{(b_1-b_3)y} + A_{28}e^{-2b_1 y} + \\ A_{29}e^{(b_3-b_1)y} + A_{30}e^{-(b_1+b_3)y} + A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{aligned} \tag{3.178}$$

The second derivative of equation (3.178) with respect to y is

$$\begin{aligned} \frac{d^2u_1}{dy^2} = A_{22}b_2^2e^{b_2 y} + A_{23}b_2^2e^{-b_2 y} + 4A_{24}b_1^2e^{2b_1 y} + A_{26}\left(b_1+b_3\right)^2 e^{(b_1+b_3)y} + A_{27}\left(b_1-b_3\right)^2 e^{(b_1-b_3)y} \\ +4A_{28}b_1^2e^{-2b_1 y} + A_{29}\left(b_3-b_1\right)^2 e^{(b_3-b_1)y} + A_{30}\left(b_1+b_3\right)^2 e^{-(b_1+b_3)y} + 4A_{31}b_3^2e^{2b_3 y} + 4A_{32}b_3^2e^{-2b_3 y} \end{aligned} \tag{3.179}$$

Put equations (3.179) and (3.178) into equation (3.168) gives,

$$\begin{aligned} &\begin{pmatrix} A_{22}b_2^2e^{b_2 y} + A_{23}b_2^2e^{-b_2 y} + 4A_{24}b_1^2e^{2b_1 y} + A_{26}\left(b_1+b_3\right)^2 e^{(b_1+b_3)y} + \\ A_{27}\left(b_1-b_3\right)^2 e^{(b_1-b_3)y} + 4A_{28}b_1^2e^{-2b_1 y} + A_{29}\left(b_3-b_1\right)^2 e^{(b_3-b_1)y} \\ +A_{30}\left(b_1+b_3\right)^2 e^{-(b_1+b_3)y} + 4A_{31}b_3^2e^{2b_3 y} + 4A_{32}b_3^2e^{-2b_3 y} \end{pmatrix} - b_1^2\begin{pmatrix} A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} \\ +A_{25} + A_{26}e^{(b_1+b_3)y} + A_{27}e^{(b_1-b_3)y} \\ +A_{28}e^{-2b_1 y} + A_{29}e^{(b_3-b_1)y} + A_{30}e^{-(b_1+b_3)y} \\ +A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{pmatrix} \\ &= -e^{i\omega t}\begin{pmatrix} A_9e^{b_2 y} + A_{10}e^{-b_2 y} + A_{11}e^{2b_1 y} + A_{12} + A_{13}e^{(b_1+b_3)y} + A_{14}e^{(b_1-b_3)y} + A_{15}e^{-2b_1 y} + \\ A_{16}e^{(b_3-b_1)y} + A_{17}e^{-(b_1+b_3)y} + A_{18}e^{2b_3 y} + A_{19}e^{-2b_3 y} \end{pmatrix} \end{aligned} \tag{3.180}$$

Comparing the variables in equation (3.180) gives,

$$A_{22}b_2^2 - b_1^2 A_{22} = -e^{i\omega t} A_9 \tag{3.181}$$

$$A_{23}b_2^2 - b_1^2 A_{23} = -e^{i\omega t} A_{10} \tag{3.182}$$

$$4A_{24}b_1^2 - b_1^2 A_{24} = -e^{i\omega t} A_{11} \tag{3.183}$$

$$-b_1^2 A_{25} = -e^{i\omega t} A_{12} \tag{3.184}$$

$$A_{26}\left(b_1 + b_3\right)^2 - b_1^2 A_{26} = -e^{i\omega t} A_{13} \tag{3.185}$$

$$A_{27}\left(b_1 - b_3\right)^2 - b_1^2 A_{27} = -e^{i\omega t} A_{14} \tag{3.186}$$

$$4A_{28}b_1^2 - b_1^2 A_{28} = -e^{i\omega t} A_{15} \tag{3.187}$$

$$A_{29}\left(b_3 - b_1\right)^2 - b_1^2 A_{29} = -e^{i\omega t} A_{16} \tag{3.188}$$

$$A_{30}\left(b_1 + b_3\right)^2 - b_1^2 A_{30} = -e^{i\omega t} A_{17} \tag{3.189}$$

$$4A_{31}b_3^2 - b_1^2 A_{31} = -e^{i\omega t} A_{18} \tag{3.190}$$

$$4A_{32}b_3^2 - b_1^2 A_{32} = -e^{i\omega t} A_{19} \tag{3.191}$$

From (3.181) we have,

$$A_{22} = -\frac{e^{i\omega t} A_9}{b_2^2 - b_1^2} \tag{3.192}$$

From equation (3.182), equation (3.193) is obtained

$$A_{23} = -\frac{e^{i\omega t} A_{10}}{b_2^2 - b_1^2} \tag{3.193}$$

From equation (3.183), equation (3.194) is obtained

$$A_{24} = -\frac{e^{i\omega t} A_{11}}{3b_1^2} \tag{3.194}$$

From equation (3.184), equation (3.195) is obtained

$$A_{25} = \frac{e^{i\omega t} A_{12}}{b_1^2} \tag{3.195}$$

From equation (3.185), equation (3.196) is obtained

$$A_{26} = -\frac{e^{i\omega t} A_{13}}{\left(b_1 + b_3\right)^2 - b_1^2} \tag{3.196}$$

From equation (3.186), equation (3.197) is obtained

$$A_{27} = -\frac{e^{i\omega t} A_{14}}{\left(b_1 - b_3\right)^2 - b_1^2} \tag{3.197}$$

From equation (3.187), equation (3.198) is obtained

$$A_{28} = -\frac{e^{i\omega t} A_{15}}{3b_1^2} \tag{3.198}$$

From equation (3.188), equation (3.199) is obtained

$$A_{29} = -\frac{e^{i\omega t} A_{16}}{\left(b_3 - b_1\right)^2 - b_1^2} \tag{3.199}$$

From equation (3.189), equation (3.200) is obtained

$$A_{30} = -\frac{e^{i\omega t}A_{17}}{\left(b_1 + b_3\right)^2 - b_1^2} \tag{3.200}$$

From equation (3.190), equation (3.201) is obtained

$$A_{31} = -\frac{e^{i\omega t}A_{18}}{4b_3^2 - b_1^2} \tag{3.201}$$

From equation (3.191), equation (3.202) is obtained

$$A_{32} = -\frac{e^{i\omega t}A_{19}}{4b_3^2 - b_1^2} \tag{3.202}$$

So,

$$\begin{aligned} u_{1p}\left(y\right) = {} & A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} + A_{25} + A_{26}e^{\left(b_1+b_3\right)y} + A_{27}e^{\left(b_1-b_3\right)y} + A_{28}e^{-2b_1 y} + \\ & A_{29}e^{\left(b_3-b_1\right)y} + A_{30}e^{-\left(b_1+b_3\right)y} + A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{aligned} \tag{3.203}$$

The general solution for equation (3.45) is given by

$$u_1\left(y\right) = u_{1c}\left(y\right) + u_{1p}\left(y\right) \tag{3.204}$$

That is,

$$\begin{aligned} u_1\left(y\right) = {} & A_{20}e^{b_1 y} + A_{21}e^{-b_1 y} + A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} + A_{25} + A_{26}e^{\left(b_1+b_3\right)y} + A_{27}e^{\left(b_1-b_3\right)y} + \\ & A_{28}e^{-2b_1 y} + A_{29}e^{\left(b_3-b_1\right)y} + A_{30}e^{-\left(b_1+b_3\right)y} + A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{aligned} \tag{3.205}$$

Applying the boundary conditions that is,

For $u_1\left(0\right) = 0$,

$$A_{20}+A_{21}+A_{22}+A_{23}+A_{24}+A_{25}+A_{26}+A_{27}+A_{28}+A_{29}+A_{30}+A_{31}+A_{32}=0 \tag{3.206}$$

$$A_{20}=-\left(A_{22}+A_{23}+A_{24}+A_{25}+A_{26}+A_{27}+A_{28}+A_{29}+A_{30}+A_{31}+A_{32}\right)-A_{21} \tag{3.207}$$

So,

$$A_{20}=-A_{33}-A_{21} \tag{3.208}$$

Where,

$$A_{33}=A_{22}+A_{23}+A_{24}+A_{25}+A_{26}+A_{27}+A_{28}+A_{29}+A_{30}+A_{31}+A_{32} \tag{3.209}$$

Also for $u_1(1)=0$,

$$\begin{aligned}&A_{20}e^{b_1}+A_{21}e^{-b_1}+A_{22}e^{b_2}+A_{23}e^{-b_2}+A_{24}e^{2b_1}+A_{25}+A_{26}e^{(b_1+b_3)}+A_{27}e^{(b_1-b_3)}+A_{28}e^{-2b_1}+\\&A_{29}e^{(b_3-b_1)}+A_{30}e^{-(b_1+b_3)}+A_{31}e^{2b_3}+A_{32}e^{-2b_3}=0\end{aligned} \tag{3.210}$$

From equation (3.208) into equation (3.210), equation (3.211) is obtained

$$\begin{aligned}&-A_{33}e^{b_1}-A_{21}e^{b_1}+A_{21}e^{-b_1}+A_{22}e^{b_2}+A_{23}e^{-b_2}+A_{24}e^{2b_1}+A_{25}+A_{26}e^{(b_1+b_3)}+A_{27}e^{(b_1-b_3)}+\\&A_{28}e^{-2b_1}+A_{29}e^{(b_3-b_1)}+A_{30}e^{-(b_1+b_3)}+A_{31}e^{2b_3}+A_{32}e^{-2b_3}=0\end{aligned} \tag{3.211}$$

So,

$$A_{21}=\frac{\begin{pmatrix}A_{33}e^{b_1}-A_{22}e^{b_2}-A_{23}e^{-b_2}-A_{24}e^{2b_1}-A_{25}-A_{26}e^{(b_1+b_3)}-A_{27}e^{(b_1-b_3)}-A_{28}e^{-2b_1}-\\A_{29}e^{(b_3-b_1)}-A_{30}e^{-(b_1+b_3)}-A_{31}e^{2b_3}-A_{32}e^{-2b_3}\end{pmatrix}}{e^{-b_1}-e^{b_1}} \tag{3.212}$$

From equation (3.212) into equation (3.208), equation (3.213) is obtained

$$A_{20} = -A_{33} - \frac{\begin{pmatrix} A_{33}e^{b_1} - A_{22}e^{b_2} - A_{23}e^{-b_2} - A_{24}e^{2b_1} - A_{25} - A_{26}e^{(b_1+b_3)} - A_{27}e^{(b_1-b_3)} - A_{28}e^{-2b_1} \\ -A_{29}e^{(b_3-b_1)} - A_{30}e^{-(b_1+b_3)} - A_{31}e^{2b_3} - A_{32}e^{-2b_3} \end{pmatrix}}{e^{-b_1} - e^{b_1}} \tag{3.213}$$

$$A_{20} = \frac{\begin{pmatrix} -A_{33}\left(e^{-b_1} - e^{b_1}\right) - A_{33}e^{b_1} + A_{22}e^{b_2} + A_{23}e^{-b_2} + A_{24}e^{2b_1} + A_{25} + A_{26}e^{(b_1+b_3)} + A_{27}e^{(b_1-b_3)} \\ +A_{28}e^{-2b_1} + A_{29}e^{(b_3-b_1)} + A_{30}e^{-(b_1+b_3)} + A_{31}e^{2b_3} + A_{32}e^{-2b_3} \end{pmatrix}}{e^{-b_1} - e^{b_1}} \tag{3.214}$$

$$A_{20} = \frac{\begin{pmatrix} -A_{33}e^{-b_1} + A_{22}e^{b_2} + A_{23}e^{-b_2} + A_{24}e^{2b_1} + A_{25} + A_{26}e^{(b_1+b_3)} + A_{27}e^{(b_1-b_3)} \\ +A_{28}e^{-2b_1} + A_{29}e^{(b_3-b_1)} + A_{30}e^{-(b_1+b_3)} + A_{31}e^{2b_3} + A_{32}e^{-2b_3} \end{pmatrix}}{e^{-b_1} - e^{b_1}} \tag{3.215}$$

So,

$$\begin{aligned} u_1(y) = {} & A_{20}e^{b_1 y} + A_{21}e^{-b_1 y} + A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} + A_{25} + A_{26}e^{(b_1+b_3)y} + A_{27}e^{(b_1-b_3)y} + \\ & A_{28}e^{-2b_1 y} + A_{29}e^{(b_3-b_1)y} + A_{30}e^{-(b_1+b_3)y} + A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{aligned} \tag{3.216}$$

Solving equation (3.46) that is,

$$\left.\begin{aligned} & \frac{d^2\theta_1}{dy^2} - b_2^2\theta_1 = -\frac{2E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) \\ & \theta_1(0) = 0, \quad \theta_1(1) = 0 \end{aligned}\right\} \tag{3.217}$$

Where,

$$u_0(y) = A_5e^{b_1 y} + A_6e^{-b_1 y} + A_7e^{b_3 y} + A_8e^{-b_3 y} \tag{3.218}$$

$$\begin{aligned} u_1(y) = {} & A_{20}e^{b_1 y} + A_{21}e^{-b_1 y} + A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} + A_{25} + A_{26}e^{(b_1+b_3)y} + A_{27}e^{(b_1-b_3)y} + \\ & A_{28}e^{-2b_1 y} + A_{29}e^{(b_3-b_1)y} + A_{30}e^{-(b_1+b_3)y} + A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{aligned} \tag{3.219}$$

Differentiate equations (3.218) and (3.219) with respect to y gives,

$$\frac{du_0}{dy} = A_5 b_1 e^{b_1 y} - A_6 b_1 e^{-b_1 y} + A_7 b_3 e^{b_3 y} - A_8 b_3 e^{-b_3 y} \tag{3.220}$$

$$\begin{aligned}\frac{du_1}{dy} &= A_{20} b_1 e^{b_1 y} - A_{21} b_1 e^{-b_1 y} + A_{22} b_2 e^{b_2 y} - A_{23} b_2 e^{-b_2 y} + 2A_{24} b_1 e^{2b_1 y} + A_{26}\left(b_1 + b_3\right) e^{\left(b_1+b_3\right)y} + \\ & A_{27}\left(b_1 - b_3\right) e^{\left(b_1-b_3\right)y} - 2A_{28} b_1 e^{-2b_1 y} + A_{29}\left(b_3 - b_1\right) e^{\left(b_3-b_1\right)y} - A_{30}\left(b_1 + b_3\right) e^{-\left(b_1+b_3\right)y} + 2A_{31} b_3 e^{2b_3 y} \\ & -2A_{32} b_3 e^{-2b_3 y}\end{aligned} \tag{3.221}$$

Multiplying equation (3.220) with equation (3.221) gives,

$$\begin{aligned}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) &= A_5 A_{20} b_1^2 e^{2b_1 y} - A_5 A_{21} b_1^2 + A_5 A_{22} b_1 b_2 e^{\left(b_1+b_2\right)y} - A_5 A_{23} b_1 b_2 e^{\left(b_1-b_2\right)y} \\ & +2A_5 A_{24} b_1^2 e^{3b_1 y} + A_5 A_{26} b_1\left(b_1 + b_3\right) e^{\left(2b_1+b_3\right)y} + A_5 A_{27} b_1\left(b_1 - b_3\right) e^{\left(2b_1-b_3\right)y} - \\ & 2A_5 A_{28} b_1^2 e^{-b_1 y} + A_5 A_{29} b_1\left(b_3 - b_1\right) e^{b_3 y} - A_5 A_{30} b_1\left(b_1 + b_3\right) e^{-b_3 y} + 2A_5 A_{31} b_1 b_3 e^{\left(b_1+2b_3\right)y} \\ & -2A_5 A_{32} b_1 b_3 e^{\left(b_1-2b_3\right)y} - A_6 A_{20} b_1^2 + A_6 A_{21} b_1^2 e^{-2b_1 y} - A_6 A_{22} b_1 b_2 e^{\left(b_2-b_1\right)y} + \\ & A_6 A_{23} b_1 b_2 e^{-\left(b_1+b_2\right)y} - 2A_6 A_{24} b_1^2 e^{b_1 y} - A_6 A_{26} b_1\left(b_1 + b_3\right) e^{b_3 y} - A_6 A_{27} b_1\left(b_1 - b_3\right) e^{-b_3 y} \\ & +2A_5 A_{28} b_1^2 e^{-3b_1 y} - A_6 A_{29} b_1\left(b_3 - b_1\right) e^{\left(b_3-2b_1\right)y} + A_6 A_{30} b_1\left(b_1 + b_3\right) e^{-\left(2b_1+b_3\right)y} - \\ & 2A_6 A_{31} b_1 b_3 e^{\left(b_1+2b_3\right)y} + 2A_6 A_{32} b_1 b_3 e^{-\left(b_1+2b_3\right)y} + A_7 A_{20} b_1 b_3 e^{\left(b_1+b_3\right)y} - A_7 A_{21} b_1 b_3 e^{\left(b_3-b_1\right)y} \\ & +A_7 A_{22} b_2 b_3 e^{\left(b_2+b_3\right)y} - A_7 A_{23} b_2 b_3 e^{\left(b_3-b_2\right)y} + 2A_7 A_{24} b_1 b_3 e^{\left(b_3-2b_1\right)y} + \\ & A_7 A_{26} b_3\left(b_1 + b_3\right) e^{\left(b_1+2b_3\right)y} + A_7 A_{27} b_3\left(b_1 - b_3\right) e^{b_1 y} - 2A_7 A_{28} b_1 b_3 e^{\left(b_3-2b_1\right)y} + \\ & A_7 A_{29} b_3\left(b_3 - b_1\right) e^{\left(2b_3-b_1\right)y} - A_7 A_{30} b_3\left(b_1 + b_3\right) e^{-b_1 y} + 2A_7 A_{31} b_3^2 e^{3b_3 y} - 2A_7 A_{32} b_3^2 e^{-b_3 y} \\ & -A_8 A_{20} b_1 b_3 e^{\left(b_1-b_3\right)y} + A_8 A_{21} b_1 b_3 e^{-\left(b_1+b_3\right)y} - A_8 A_{22} b_2 b_3 e^{\left(b_2-b_3\right)y} + A_8 A_{23} b_2 b_3 e^{-\left(b_2+b_3\right)y} \\ & -2A_8 A_{24} b_1 b_3 e^{\left(2b_1-b_3\right)y} - A_8 A_{26} b_3\left(b_1 + b_3\right) e^{b_1 y} - A_8 A_{27} b_3\left(b_1 - b_3\right) e^{\left(b_1-2b_3\right)y} + \\ & 2A_8 A_{28} b_1 b_3 e^{-\left(2b_1+b_3\right)y} - A_8 A_{29} b_3\left(b_3 - b_1\right) e^{-b_1 y} + A_8 A_{30} b_3\left(b_1 + b_3\right) e^{-\left(b_1+2b_3\right)y} \\ & -2A_8 A_{31} b_3^2 e^{b_3 y} + 2A_8 A_{32} b_3^2 e^{-3b_3 y}\end{aligned} \tag{3.222}$$

Factorizing equation (3.222) gives,

$$\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) = A_5A_{20}b_1^2e^{2b_1y} + A_5A_{22}b_1b_2e^{(b_1+b_2)y} - A_5A_{23}b_1b_2e^{(b_1-b_2)y} + 2A_5A_{24}b_1^2e^{3b_1y} +$$
$$\left(A_5A_{26}b_1\left(b_1+b_3\right)+2A_7A_{24}b_1b_3\right)e^{(2b_1+b_3)y} + \left(A_5A_{27}b_1\left(b_1-b_3\right)-2A_8A_{24}b_1b_3\right)e^{(2b_1-b_3)y} -$$
$$\left(2A_5A_{28}b_1^2 + A_7A_{30}b_3\left(b_1+b_3\right)+A_8A_{29}b_3\left(b_3-b_1\right)\right)e^{-b_1y} + \begin{pmatrix} A_5A_{29}b_1\left(b_3-b_1\right)- \\ A_6A_{26}b_1\left(b_1+b_3\right)- \\ 2A_8A_{31}b_3^2 \end{pmatrix} e^{b_3y}$$
$$-\left(A_5A_{30}b_1\left(b_1+b_3\right)+A_6A_{27}b_1\left(b_1-b_3\right)+2A_7A_{32}b_3^2\right)e^{-b_3y} + \begin{pmatrix} 2A_5A_{31}b_1b_3 - \\ 2A_6A_{31}b_1b_3 + \\ A_7A_{26}b_3\left(b_1+b_3\right) \end{pmatrix} e^{(b_1+2b_3)y}$$
$$-\left(2A_5A_{32}b_1b_3 + A_8A_{27}b_3\left(b_1-b_3\right)\right)e^{(b_1-2b_3)y} + A_6A_{21}b_1^2e^{-2b_1y} - A_6A_{22}b_1b_2e^{(b_2-b_1)y} +$$
$$A_6A_{23}b_1b_2e^{-(b_1+b_2)y} - \begin{pmatrix} 2A_6A_{24}b_1^2 - A_7A_{27}b_3\left(b_1-b_3\right)+ \\ A_8A_{26}b_3\left(b_1+b_3\right) \end{pmatrix} e^{b_1y} + 2A_6A_{28}b_1^2e^{-3b_1y} -$$
$$\left(A_6A_{29}b_1\left(b_3-b_1\right)+2A_7A_{28}b_1b_3\right)e^{(b_3-2b_1)y} + \left(A_6A_{30}b_1\left(b_1+b_3\right)+2A_8A_{28}b_1b_3\right)e^{-(2b_1+b_3)y}$$
$$+\left(2A_6A_{32}b_1b_3 + A_8A_{30}b_3\left(b_1+b_3\right)\right)e^{-(b_1+2b_3)y} + A_7A_{20}b_1b_3e^{(b_1+b_3)y} - A_7A_{21}b_1b_3e^{(b_3-b_1)y}$$
$$+A_7A_{22}b_2b_3e^{(b_2+b_3)y} - A_7A_{23}b_2b_3e^{(b_3-b_2)y} + A_7A_{29}b_3\left(b_3-b_1\right)e^{(2b_3-b_1)y} + 2A_7A_{31}b_3^2e^{3b_3y}$$
$$-A_8A_{20}b_1b_3e^{(b_1-b_3)y} + A_8A_{21}b_1b_3e^{-(b_1+b_3)y} - A_8A_{22}b_2b_3e^{(b_2-b_3)y} + A_8A_{23}b_2b_3e^{-(b_2+b_3)y} +$$
$$2A_8A_{32}b_3^2e^{-3b_3y} - \left(A_5A_{21}b_1^2 + A_6A_{20}b_1^2\right) \tag{3.223}$$

Putting equation (3.223) into equation (3.217) gives,

$$
\frac{d^2\theta_1}{dy^2}-b_2^2\theta_1=-\frac{2E_cP_e}{R_e}\left(\begin{array}{l}
A_5A_{20}b_1^2e^{2b_1y}+A_5A_{22}b_1b_2e^{(b_1+b_2)y}-A_5A_{23}b_1b_2e^{(b_1-b_2)y}+\\
2A_5A_{24}b_1^2e^{3b_1y}+\left(A_5A_{26}b_1\left(b_1+b_3\right)+2A_7A_{24}b_1b_3\right)e^{(2b_1+b_3)y}\\
+\begin{pmatrix}A_5A_{27}b_1\left(b_1-b_3\right)-\\ 2A_8A_{24}b_1b_3\end{pmatrix}e^{(2b_1-b_3)y}-\begin{pmatrix}2A_5A_{28}b_1^2+\\ A_7A_{30}b_3\left(b_1+b_3\right)+\\ A_8A_{29}b_3\left(b_3-b_1\right)\end{pmatrix}e^{-b_1y}\\
+\begin{pmatrix}A_5A_{29}b_1\left(b_3-b_1\right)\\ -A_6A_{26}b_1\left(b_1+b_3\right)-2A_8A_{31}b_3^2\end{pmatrix}e^{b_3y}-\begin{pmatrix}A_5A_{30}b_1\left(b_1+b_3\right)\\ +A_6A_{27}b_1\left(b_1-b_3\right)\\ +2A_7A_{32}b_3^2\end{pmatrix}e^{-b_3y}\\
+\begin{pmatrix}2A_5A_{31}b_1b_3-2A_6A_{31}b_1b_3\\ +A_7A_{26}b_3\left(b_1+b_3\right)\end{pmatrix}e^{(b_1+2b_3)y}-\begin{pmatrix}2A_5A_{32}b_1b_3+\\ A_8A_{27}b_3\left(b_1-b_3\right)\end{pmatrix}e^{(b_1-2b_3)y}\\
+A_6A_{21}b_1^2e^{-2b_1y}-A_6A_{22}b_1b_2e^{(b_2-b_1)y}+A_6A_{23}b_1b_2e^{-(b_1+b_2)y}-\\
\left(2A_6A_{24}b_1^2-A_7A_{27}b_3\left(b_1-b_3\right)+A_8A_{26}b_3\left(b_1+b_3\right)\right)e^{b_1y}+\\
2A_6A_{28}b_1^2e^{-3b_1y}-\begin{pmatrix}A_6A_{29}b_1\left(b_3-b_1\right)\\ +2A_7A_{28}b_1b_3\end{pmatrix}e^{(b_3-2b_1)y}+\\
\left(A_6A_{30}b_1\left(b_1+b_3\right)+2A_8A_{28}b_1b_3\right)e^{-(2b_1+b_3)y}+\begin{pmatrix}2A_6A_{32}b_1b_3+\\ A_8A_{30}b_3\\ \left(b_1+b_3\right)\end{pmatrix}e^{-(b_1+2b_3)y}\\
+A_7A_{20}b_1b_3e^{(b_1+b_3)y}-A_7A_{21}b_1b_3e^{(b_3-b_1)y}+A_7A_{22}b_2b_3e^{(b_2+b_3)y}-\\
A_7A_{23}b_2b_3e^{(b_3-b_2)y}+A_7A_{29}b_3\left(b_3-b_1\right)e^{(2b_3-b_1)y}+2A_7A_{31}b_3^2e^{3b_3y}\\
-A_8A_{20}b_1b_3e^{(b_1-b_3)y}+A_8A_{21}b_1b_3e^{-(b_1+b_3)y}-A_8A_{22}b_2b_3e^{(b_2-b_3)y}+\\
A_8A_{23}b_2b_3e^{-(b_2+b_3)y}+2A_8A_{32}b_3^2e^{-3b_3y}-\left(A_5A_{21}b_1^2+A_6A_{20}b_1^2\right)
\end{array}\right) \tag{3.224}
$$

For the homogeneous part that is,

$$
\frac{d^2\theta_1}{dy^2}-b_2^2\theta_1=0 \tag{3.225}
$$

Seeking,

$$\theta_1(y) = e^{my} \tag{3.226}$$

The second derivative of equation (3.226) with respect to y gives,

$$\frac{d^2\theta_1}{dy^2} = m^2 e^{my} \tag{3.227}$$

Putting equations (3.227) and (3.226) into equation (3.225) gives,

$$m^2 e^{my} - b_2^2 e^{my} = 0 \tag{3.228}$$

$$\left(m^2 - b_2^2\right) e^{my} = 0 \tag{3.229}$$

For $e^{my} \neq 0$ we have,

$$m^2 - b_2^2 = 0 \tag{3.230}$$

$$m^2 = b_2^2 \tag{3.231}$$

$$m = \pm b_2 \tag{3.232}$$

So,

$$\theta_{1c}(y) = A_{33} e^{b_2 y} + A_{34} e^{-b_2 y} \tag{3.233}$$

Assume a particular solution to be,

$$\theta_{1p}(y)=\begin{pmatrix} A_{35}e^{2b_1y}+A_{36}e^{(b_1+b_2)y}+A_{37}e^{(b_1-b_2)y}+A_{38}e^{3b_1y}+A_{39}e^{(2b_1+b_3)y}+A_{40}e^{(2b_1-b_3)y} \\ +A_{41}e^{-b_1y}+A_{42}e^{b_3y}+A_{43}e^{-b_3y}+A_{44}e^{(b_1+2b_3)y}+A_{45}e^{(b_1-2b_3)y}+A_{46}e^{-2b_1y}+ \\ A_{47}e^{(b_2-b_1)y}+A_{48}e^{-(b_1+b_2)y}+A_{49}e^{b_1y}+A_{50}e^{-3b_1y}+A_{51}e^{(b_3-2b_1)y}+A_{52}e^{-(2b_1+b_3)y}+ \\ A_{53}e^{-(b_1+2b_3)y}+A_{54}e^{(b_1+b_3)y}+A_{55}e^{(b_3-b_1)y}+A_{56}e^{(b_2+b_3)y}+A_{57}e^{(b_3-b_2)y}+A_{58}e^{(2b_3-b_1)y} \\ +A_{59}e^{3b_3y}+A_{60}e^{(b_1-b_3)y}+A_{61}e^{-(b_1+b_3)y}+A_{62}e^{(b_2-b_3)y}+A_{63}e^{-(b_2+b_3)y}+A_{64}e^{-3b_3y}+A_{65} \end{pmatrix} \quad (3.234)$$

The second derivative of equation (3.234) with respect to y gives,

$$\frac{d^2\theta_1}{dy^2}=\begin{pmatrix} 4A_{35}b_1^2e^{2b_1y}+A_{36}(b_1+b_2)^2e^{(b_1+b_2)y}+A_{37}(b_1-b_2)^2e^{(b_1-b_2)y}+9A_{38}b_1^2e^{3b_1y}+ \\ A_{39}(2b_1+b_3)^2e^{(2b_1+b_3)y}+A_{40}(2b_1-b_3)^2e^{(2b_1-b_3)y}+A_{41}b_1^2e^{-b_1y}+A_{42}b_3^2e^{b_3y} \\ +A_{43}b_3^2e^{-b_3y}+A_{44}(b_1+2b_3)^2e^{(b_1+2b_3)y}+A_{45}(b_1-2b_3)^2e^{(b_1-2b_3)y}+4A_{46}b_1^2e^{-2b_1y} \\ +A_{47}(b_2-b_1)^2e^{(b_2-b_1)y}+A_{48}(b_1+b_2)^2e^{-(b_1+b_2)y}+A_{49}b_1^2e^{b_1y}+9A_{50}b_1^2e^{-3b_1y}+ \\ A_{51}(b_3-2b_1)^2e^{(b_3-2b_1)y}+A_{52}(2b_1+b_3)^2e^{-(2b_1+b_3)y}+A_{53}(b_1+2b_3)^2e^{-(b_1+2b_3)y}+ \\ A_{54}(b_1+b_3)^2e^{(b_1+b_3)y}+A_{55}(b_3-b_1)^2e^{(b_3-b_1)y}+A_{56}(b_2+b_3)^2e^{(b_2+b_3)y}+ \\ A_{57}(b_3-b_2)^2e^{(b_3-b_2)y}+A_{58}(2b_3-b_1)^2e^{(2b_3-b_1)y}+9A_{59}b_3^2e^{3b_3y}+A_{60}(b_1-b_3)^2e^{(b_1-b_3)y} \\ +A_{61}(b_1+b_3)^2e^{-(b_1+b_3)y}+A_{62}(b_2-b_3)^2e^{(b_2-b_3)y}+A_{63}(b_2+b_3)^2e^{-(b_2+b_3)y}+9A_{64}b_3^2e^{-3b_3y} \end{pmatrix} \quad (3.235)$$

Put equations (3.235) and (3.234) into equation (3.224) gives,

$$
\left(\begin{array}{l}
4A_{35}b_1^2e^{2b_1y}+A_{36}\left(b_1+b_2\right)^2e^{(b_1+b_2)y}+A_{37}\left(b_1-b_2\right)^2e^{(b_1-b_2)y}+9A_{38}b_1^2e^{3b_1y}+A_{39}\left(2b_1+b_3\right)^2e^{(2b_1+b_3)y}\\
+A_{40}\left(2b_1-b_3\right)^2e^{(2b_1-b_3)y}+A_{41}b_1^2e^{-b_1y}+A_{42}b_3^2e^{b_3y}+A_{43}b_3^2e^{-b_3y}+A_{44}\left(b_1+2b_3\right)^2e^{(b_1+2b_3)y}+\\
A_{45}\left(b_1-2b_3\right)^2e^{(b_1-2b_3)y}+4A_{46}b_1^2e^{-2b_1y}+A_{47}\left(b_2-b_1\right)^2e^{(b_2-b_1)y}+A_{48}\left(b_1+b_2\right)^2e^{-(b_1+b_2)y}+A_{49}b_1^2e^{b_1y}+\\
9A_{50}b_1^2e^{-3b_1y}+A_{51}\left(b_3-2b_1\right)^2e^{(b_3-2b_1)y}+A_{52}\left(2b_1+b_3\right)^2e^{-(2b_1+b_3)y}+A_{53}\left(b_1+2b_3\right)^2e^{-(b_1+2b_3)y}+\\
A_{54}\left(b_1+b_3\right)^2e^{(b_1+b_3)y}+A_{55}\left(b_3-b_1\right)^2e^{(b_3-b_1)y}+A_{56}\left(b_2+b_3\right)^2e^{(b_2+b_3)y}+A_{57}\left(b_3-b_2\right)^2e^{(b_3-b_2)y}\\
+A_{58}\left(2b_3-b_1\right)^2e^{(2b_3-b_1)y}+9A_{59}b_3^2e^{3b_3y}+A_{60}\left(b_1-b_3\right)^2e^{(b_1-b_3)y}+A_{61}\left(b_1+b_3\right)^2e^{-(b_1+b_3)y}\\
+A_{62}\left(b_2-b_3\right)^2e^{(b_2-b_3)y}+A_{63}\left(b_2+b_3\right)^2e^{-(b_2+b_3)y}+9A_{64}b_3^2e^{-3b_3y}
\end{array}\right)
$$

$$
-b_2^2\left(\begin{array}{l}
A_{35}e^{2b_1y}+A_{36}e^{(b_1+b_2)y}+A_{37}e^{(b_1-b_2)y}+A_{38}e^{3b_1y}+A_{39}e^{(2b_1+b_3)y}+A_{40}e^{(2b_1-b_3)y}\\
+A_{41}e^{-b_1y}+A_{42}e^{b_3y}+A_{43}e^{-b_3y}+A_{44}e^{(b_1+2b_3)y}+A_{45}e^{(b_1-2b_3)y}+A_{46}e^{-2b_1y}+\\
A_{47}e^{(b_2-b_1)y}+A_{48}e^{-(b_1+b_2)y}+A_{49}e^{b_1y}+A_{50}e^{-3b_1y}+A_{51}e^{(b_3-2b_1)y}+A_{52}e^{-(2b_1+b_3)y}+\\
A_{53}e^{-(b_1+2b_3)y}+A_{54}e^{(b_1+b_3)y}+A_{55}e^{(b_3-b_1)y}+A_{56}e^{(b_2+b_3)y}+A_{57}e^{(b_3-b_2)y}+A_{58}e^{(2b_3-b_1)y}\\
+A_{59}e^{3b_3y}+A_{60}e^{(b_1-b_3)y}+A_{61}e^{-(b_1+b_3)y}+A_{62}e^{(b_2-b_3)y}+A_{63}e^{-(b_2+b_3)y}+A_{64}e^{-3b_3y}+A_{65}
\end{array}\right)
$$

$$
=-\frac{2E_cP_e}{R_e}\left(\begin{array}{l}
A_5A_{20}b_1^2e^{2b_1y}+A_5A_{22}b_1b_2e^{(b_1+b_2)y}-A_5A_{23}b_1b_2e^{(b_1-b_2)y}+2A_5A_{24}b_1^2e^{3b_1y}+\\
\left(A_5A_{26}b_1\left(b_1+b_3\right)+2A_7A_{24}b_1b_3\right)e^{(2b_1+b_3)y}+\left(A_5A_{27}b_1\left(b_1-b_3\right)-2A_8A_{24}b_1b_3\right)e^{(2b_1-b_3)y}\\
-\left(2A_5A_{28}b_1^2+A_7A_{30}b_3\left(b_1+b_3\right)+A_8A_{29}b_3\left(b_3-b_1\right)\right)e^{-b_1y}+\left(\begin{array}{l}A_5A_{29}b_1\left(b_3-b_1\right)-\\ A_6A_{26}b_1\left(b_1+b_3\right)-\\ 2A_8A_{31}b_3^2\end{array}\right)e^{b_3y}-\\
\left(A_5A_{30}b_1\left(b_1+b_3\right)+A_6A_{27}b_1\left(b_1-b_3\right)+2A_7A_{32}b_3^2\right)e^{-b_3y}+\left(\begin{array}{l}2A_5A_{31}b_1b_3-\\ 2A_6A_{31}b_1b_3\\ +A_7A_{26}b_3\left(b_1+b_3\right)\end{array}\right)e^{(b_1+2b_3)y}\\
-\left(2A_5A_{32}b_1b_3+A_8A_{27}b_3\left(b_1-b_3\right)\right)e^{(b_1-2b_3)y}+A_6A_{21}b_1^2e^{-2b_1y}-A_6A_{22}b_1b_2e^{(b_2-b_1)y}\\
+A_6A_{23}b_1b_2e^{-(b_1+b_2)y}-\left(\begin{array}{l}2A_6A_{24}b_1^2-A_7A_{27}b_3\left(b_1-b_3\right)\\ +A_8A_{26}b_3\left(b_1+b_3\right)\end{array}\right)e^{b_1y}+2A_6A_{28}b_1^2e^{-3b_1y}-\\
\left(A_6A_{29}b_1\left(b_3-b_1\right)+2A_7A_{28}b_1b_3\right)e^{(b_3-2b_1)y}+\left(A_6A_{30}b_1\left(b_1+b_3\right)+2A_8A_{28}b_1b_3\right)e^{-(2b_1+b_3)y}+\\
\left(2A_6A_{32}b_1b_3+A_8A_{30}b_3\left(b_1+b_3\right)\right)e^{-(b_1+2b_3)y}+A_7A_{20}b_1b_3e^{(b_1+b_3)y}-A_7A_{21}b_1b_3e^{(b_3-b_1)y}+\\
A_7A_{22}b_2b_3e^{(b_2+b_3)y}-A_7A_{23}b_2b_3e^{(b_3-b_2)y}+A_7A_{29}b_3\left(b_3-b_1\right)e^{(2b_3-b_1)y}+2A_7A_{31}b_3^2e^{3b_3y}-\\
A_8A_{20}b_1b_3e^{(b_1-b_3)y}+A_8A_{21}b_1b_3e^{-(b_1+b_3)y}-A_8A_{22}b_2b_3e^{(b_2-b_3)y}+A_8A_{23}b_2b_3e^{-(b_2+b_3)y}+\\
2A_8A_{32}b_3^2e^{-3b_3y}-\left(A_5A_{21}b_1^2+A_6A_{20}b_1^2\right)
\end{array}\right) \qquad (3.236)
$$

Comparing the variables in equation (3.236) gives,

$$4A_{35}b_1^2 - b_2^2 A_{35} = -\frac{2E_c P_e A_5 A_{20} b_1^2}{R_e} \tag{3.237}$$

$$A_{36}\left(b_1 + b_2\right)^2 - b_2^2 A_{36} = -\frac{2E_c P_e A_5 A_{22} b_1 b_2}{R_e} \tag{3.238}$$

$$A_{37}\left(b_1 - b_2\right)^2 - b_2^2 A_{37} = \frac{2E_c P_e A_5 A_{23} b_1 b_2}{R_e} \tag{3.249}$$

$$9A_{38}b_1^2 - b_2^2 A_{38} = -\frac{4E_c P_e A_5 A_{24} b_1^2}{R_e} \tag{3.240}$$

$$A_{39}\left(2b_1 + b_3\right)^2 - b_2^2 A_{39} = -\frac{2E_c P_e \left(A_5 A_{26} b_1 \left(b_1 + b_3\right) + 2A_7 A_{24} b_1 b_3\right)}{R_e} \tag{3.241}$$

$$A_{40}\left(2b_1 - b_3\right)^2 - b_2^2 A_{40} = -\frac{2E_c P_e \left(A_5 A_{27} b_1 \left(b_1 - b_3\right) - 2A_8 A_{24} b_1 b_3\right)}{R_e} \tag{3.242}$$

$$A_{41}b_1^2 - b_2^2 A_{41} = \frac{2E_c P_e \left(2A_5 A_{28} b_1^2 + A_7 A_{30} b_3 \left(b_1 + b_3\right) + A_8 A_{29} b_3 \left(b_3 - b_1\right)\right)}{R_e} \tag{3.243}$$

$$A_{42}b_3^2 - b_2^2 A_{42} = -\frac{2E_c P_e \left(A_5 A_{29} b_1 \left(b_3 - b_1\right) - A_6 A_{26} b_1 \left(b_1 + b_3\right) - 2A_8 A_{31} b_3^2\right)}{R_e} \tag{3.244}$$

$$A_{43}b_3^2 - b_2^2 A_{43} = -\frac{2E_c P_e \left(A_5 A_{30} b_1 \left(b_1 + b_3\right) + A_6 A_{27} b_1 \left(b_1 - b_3\right) + 2A_7 A_{32} b_3^2\right)}{R_e} \tag{3.245}$$

$$A_{44}\left(b_1 + 2b_3\right)^2 - b_2^2 A_{44} = -\frac{2E_c P_e \left(2A_5 A_{31} b_1 b_3 - 2A_6 A_{31} b_1 b_3 + A_7 A_{26} b_3 \left(b_1 + b_3\right)\right)}{R_e} \tag{3.246}$$

$$A_{45}\left(b_1-2b_3\right)^2-b_2^2A_{45}=\frac{2E_cP_e\left(2A_5A_{32}b_1b_3+A_8A_{27}b_3\left(b_1-b_3\right)\right)}{R_e} \tag{3.247}$$

$$4A_{46}b_1^2-b_2^2A_{46}=-\frac{2E_cP_eA_6A_{21}b_1^2}{R_e} \tag{3.248}$$

$$A_{47}\left(b_2-b_1\right)^2-b_2^2A_{47}=\frac{2E_cP_eA_6A_{22}b_1b_2}{R_e} \tag{3.249}$$

$$A_{48}\left(b_1+b_2\right)^2-b_2^2A_{48}=-\frac{2E_cP_e}{R_e}A_6A_{23}b_1b_2 \tag{3.250}$$

$$A_{49}b_1^2-b_2^2A_{49}=-\frac{2E_cP_e\left(2A_6A_{24}b_1^2-A_7A_{27}b_3\left(b_1-b_3\right)+A_8A_{26}b_3\left(b_1+b_3\right)\right)}{R_e} \tag{3.251}$$

$$9A_{50}b_1^2-b_2^2A_{50}=-\frac{4E_cP_eA_6A_{28}b_1^2}{R_e} \tag{3.252}$$

$$A_{51}\left(b_3-2b_1\right)^2-b_2^2A_{51}=-\frac{2E_cP_e\left(A_6A_{29}b_1\left(b_3-b_1\right)+2A_7A_{28}b_1b_3\right)}{R_e} \tag{3.253}$$

$$A_{52}\left(2b_1+b_3\right)^2-b_2^2A_{52}=-\frac{2E_cP_e\left(A_6A_{30}b_1\left(b_1+b_3\right)+2A_8A_{28}b_1b_3\right)}{R_e} \tag{3.254}$$

$$A_{53}\left(b_1+2b_3\right)^2-b_2^2A_{53}=-\frac{2E_cP_e\left(2A_6A_{32}b_1b_3+A_8A_{30}b_3\left(b_1+b_3\right)\right)}{R_e} \tag{3.255}$$

$$A_{54}\left(b_1+b_3\right)^2-b_2^2A_{54}=-\frac{2E_cP_e}{R_e}A_7A_{20}b_1b_3 \tag{3.256}$$

$$A_{55}\left(b_3-b_1\right)^2-b_2^2A_{55}=\frac{2E_cP_eA_7A_{21}b_1b_3}{R_e} \tag{3.257}$$

$$A_{56}\left(b_2+b_3\right)^2-b_2^2A_{56}=-\frac{2E_cP_eA_7A_{22}b_2b_3}{R_e} \tag{3.258}$$

$$A_{57}\left(b_3-b_2\right)^2-b_2^2A_{57}=\frac{2E_cP_e}{R_e}A_7A_{23}b_2b_3 \tag{3.259}$$

$$A_{58}\left(2b_3-b_1\right)^2-b_2^2A_{58}=-\frac{2E_cP_eA_7A_{29}b_3\left(b_3-b_1\right)}{R_e} \tag{3.260}$$

$$9A_{59}b_3^2-b_2^2A_{59}=-\frac{4E_cP_eA_7A_{31}b_3^2}{R_e} \tag{3.261}$$

$$A_{60}\left(b_1-b_3\right)^2-b_2^2A_{60}=\frac{2E_cP_eA_8A_{20}b_1b_3}{R_e} \tag{3.262}$$

$$A_{61}\left(b_1+b_3\right)^2-b_2^2A_{61}=-\frac{2E_cP_e}{R_e}A_8A_{21}b_1b_3 \tag{3.263}$$

$$A_{62}\left(b_2-b_3\right)^2-b_2^2A_{62}=\frac{2E_cP_eA_8A_{22}b_2b_3}{R_e} \tag{3.264}$$

$$A_{63}\left(b_2+b_3\right)^2-b_2^2A_{63}=-\frac{2E_cP_eA_8A_{23}b_2b_3}{R_e} \tag{3.265}$$

$$9A_{64}b_3^2-b_2^2A_{64}=-\frac{4E_cP_eA_8A_{32}b_3^2}{R_e} \tag{3.266}$$

$$-b_2^2A_{65}=\frac{2E_cP_e\left(A_5A_{21}b_1^2+A_6A_{20}b_1^2\right)}{R_e} \tag{3.267}$$

From equation (3.237), equation (3.268) is obtained

$$A_{35} = -\frac{2E_c P_e A_5 A_{20} b_1^2}{R_e\left(4b_1^2 - b_2^2\right)} \tag{3.268}$$

From equation (3.238), equation (3.269) is obtained

$$A_{36} = -\frac{2E_c P_e A_5 A_{22} b_1 b_2}{R_e\left(\left(b_1 + b_2\right)^2 - b_2^2\right)} \tag{3.269}$$

From equation (3.239), equation (3.270) is obtained

$$A_{37} = \frac{2E_c P_e A_5 A_{23} b_1 b_2}{R_e\left(\left(b_1 - b_2\right)^2 - b_2^2\right)} \tag{3.270}$$

From equation (3.240), equation (3.271) is obtained

$$A_{38} = -\frac{4E_c P_e A_5 A_{24} b_1^2}{R_e\left(9b_1^2 - b_2^2\right)} \tag{3.271}$$

From equation (3.241), equation (3.272) is obtained

$$A_{39} = -\frac{2E_c P_e\left(A_5 A_{26} b_1\left(b_1 + b_3\right) + 2A_7 A_{24} b_1 b_3\right)}{R_e\left(\left(2b_1 + b_3\right)^2 - b_2^2\right)} \tag{3.272}$$

From equation (3.242), equation (3.273) is obtained

$$A_{40} = -\frac{2E_c P_e\left(A_5 A_{27} b_1\left(b_1 - b_3\right) - 2A_8 A_{24} b_1 b_3\right)}{R_e\left(\left(2b_1 - b_3\right)^2 - b_2^2\right)} \tag{3.273}$$

From equation (3.243), equation (3.274) is obtained

$$A_{41} = \frac{2E_c P_e \left(2A_5 A_{28} b_1^2 + A_7 A_{30} b_3 \left(b_1 + b_3\right) + A_8 A_{29} b_3 \left(b_3 - b_1\right)\right)}{R_e \left(b_1^2 - b_2^2\right)} \tag{3.274}$$

From equation (3.244), equation (3.275) is obtained

$$A_{42} = -\frac{2E_c P_e \left(A_5 A_{29} b_1 \left(b_3 - b_1\right) - A_6 A_{26} b_1 \left(b_1 + b_3\right) - 2A_8 A_{31} b_3^2\right)}{R_e \left(b_3^2 - b_2^2\right)} \tag{3.275}$$

From equation (3.245), equation (3.276) is obtained

$$A_{43} = -\frac{2E_c P_e \left(A_5 A_{30} b_1 \left(b_1 + b_3\right) + A_6 A_{27} b_1 \left(b_1 - b_3\right) + 2A_7 A_{32} b_3^2\right)}{R_e \left(b_3^2 - b_2^2\right)} \tag{3.276}$$

From equation (3.246), equation (3.277) is obtained

$$A_{44} = -\frac{2E_c P_e \left(2A_5 A_{31} b_1 b_3 - 2A_6 A_{31} b_1 b_3 + A_7 A_{26} b_3 \left(b_1 + b_3\right)\right)}{R_e \left(\left(b_1 + 2b_3\right)^2 - b_2^2\right)} \tag{3.277}$$

From equation (3.247), equation (3.278) is obtained

$$A_{45} = \frac{2E_c P_e \left(2A_5 A_{32} b_1 b_3 + A_8 A_{27} b_3 \left(b_1 - b_3\right)\right)}{R_e \left(\left(b_1 - 2b_3\right)^2 - b_2^2\right)} \tag{3.278}$$

From equation (3.248), equation (3.279) is obtained

$$A_{46} = -\frac{2E_c P_e A_6 A_{21} b_1^2}{R_e \left(4b_1^2 - b_2^2\right)} \tag{3.279}$$

From equation (3.249), equation (3.280) is obtained

$$A_{47} = \frac{2E_c P_e A_6 A_{22} b_1 b_2}{R_e\left(\left(b_2 - b_1\right)^2 - b_2^2\right)} \tag{3.280}$$

From equation (3.250), equation (3.281) is obtained

$$A_{48} = -\frac{2E_c P_e A_6 A_{23} b_1 b_2}{R_e\left(\left(b_1 + b_2\right)^2 - b_2^2\right)} \tag{3.281}$$

From equation (3.251), equation (3.282) is obtained

$$A_{49} = -\frac{2E_c P_e\left(2A_6 A_{24} b_1^2 - A_7 A_{27} b_3\left(b_1 - b_3\right) + A_8 A_{26} b_3\left(b_1 + b_3\right)\right)}{R_e\left(b_1^2 - b_2^2\right)} \tag{3.282}$$

From equation (3.252), equation (3.283) is obtained

$$A_{50} = -\frac{4E_c P_e A_6 A_{28} b_1^2}{R_e\left(9b_1^2 - b_2^2\right)} \tag{3.283}$$

From equation (3.253), equation (3.284) is obtained

$$A_{51} = \frac{2E_c P_e\left(A_6 A_{29} b_1\left(b_3 - b_1\right) + 2A_7 A_{28} b_1 b_3\right)}{R_e\left(\left(b_3 - 2b_1\right)^2 - b_2^2\right)} \tag{3.284}$$

From equation (3.254), equation (3.285) is obtained

$$A_{52} = -\frac{2E_c P_e\left(A_6 A_{30} b_1\left(b_1 + b_3\right) + 2A_8 A_{28} b_1 b_3\right)}{R_e\left(\left(2b_1 + b_3\right)^2 - b_2^2 A\right)} \tag{3.285}$$

From equation (3.255), equation (3.286) is obtained

$$A_{53} = -\frac{2E_c P_e \left(2A_6 A_{32} b_1 b_3 + A_8 A_{30} b_3 \left(b_1 + b_3\right)\right)}{R_e \left(\left(b_1 + 2b_3\right)^2 - b_2^2\right)} \tag{3.286}$$

From equation (3.256), equation (3.287) is obtained

$$A_{54} = -\frac{2E_c P_e A_7 A_{20} b_1 b_3}{R_e \left(\left(b_1 + b_3\right)^2 - b_2^2\right)} \tag{3.287}$$

From equation (3.257), equation (3.288) is obtained

$$A_{55} = \frac{2E_c P_e A_7 A_{21} b_1 b_3}{R_e \left(\left(b_3 - b_1\right)^2 - b_2^2\right)} \tag{3.288}$$

From equation (3.258), equation (3.289) is obtained

$$A_{56} = -\frac{2E_c P_e A_7 A_{22} b_2 b_3}{R_e \left(\left(b_2 + b_3\right)^2 - b_2^2\right)} \tag{3.289}$$

From equation (3.259), equation (3.290) is obtained

$$A_{57} = \frac{2E_c P_e A_7 A_{23} b_2 b_3}{R_e \left(\left(b_3 - b_2\right)^2 - b_2^2\right)} \tag{3.290}$$

From equation (3.260), equation (3.291) is obtained

$$A_{58} = -\frac{2E_c P_e A_7 A_{29} b_3 \left(b_3 - b_1\right)}{R_e \left(\left(2b_3 - b_1\right)^2 - b_2^2\right)} \tag{3.291}$$

From equation (3.261), equation (3.292) is obtained

$$A_{59} = -\frac{4E_c P_e A_7 A_{31} b_3^2}{R_e \left(9b_3^2 - b_2^2\right)} \tag{3.292}$$

From equation (3.262), equation (3.293) is obtained

$$A_{60} = \frac{2E_c P_e A_8 A_{20} b_1 b_3}{R_e \left(\left(b_1 - b_3\right)^2 - b_2^2\right)} \tag{3.293}$$

From equation (3.263), equation (3.294) is obtained

$$A_{61} = -\frac{2E_c P_e A_8 A_{21} b_1 b_3}{R_e \left(\left(b_1 + b_3\right)^2 - b_2^2\right)} \tag{3.294}$$

From equation (3.264), equation (3.295) is obtained

$$A_{62} = \frac{2E_c P_e A_8 A_{22} b_2 b_3}{R_e \left(\left(b_2 - b_3\right)^2 - b_2^2\right)} \tag{3.295}$$

From equation (3.265), equation (3.296) is obtained

$$A_{63} = -\frac{2E_c P_e A_8 A_{23} b_2 b_3}{R_e \left(\left(b_2 + b_3\right)^2 - b_2^2\right)} \tag{3.296}$$

From equation (3.266), equation (3.297) is obtained

$$A_{64} = -\frac{4E_c P_e A_8 A_{32} b_3^2}{R_e \left(9b_3^2 - b_2^2\right)} \tag{3.297}$$

From equation (3.267), equation (3.298) is obtained

$$A_{65} = -\frac{2E_c P_e \left(A_5 A_{21} b_1^2 + A_6 A_{20} b_1^2\right)}{R_e b_2^2} \tag{3.298}$$

The general solution to equation (3.46) is given by

$$\theta_1(y) = \theta_{1c}(y) + \theta_{1p}(y) \tag{3.299}$$

So,

$$\begin{aligned}\theta_1(y) = {} & A_{33}e^{b_2 y} + A_{34}e^{-b_2 y} + A_{35}e^{2b_1 y} + A_{36}e^{(b_1+b_2)y} + A_{37}e^{(b_1-b_2)y} + A_{38}e^{3b_1 y} + A_{39}e^{(2b_1+b_3)y} \\ & + A_{40}e^{(2b_1-b_3)y} + A_{41}e^{-b_1 y} + A_{42}e^{b_3 y} + A_{43}e^{-b_3 y} + A_{44}e^{(b_1+2b_3)y} + A_{45}e^{(b_1-2b_3)y} + A_{46}e^{-2b_1 y} + \\ & A_{47}e^{(b_2-b_1)y} + A_{48}e^{-(b_1+b_2)y} + A_{49}e^{b_1 y} + A_{50}e^{-3b_1 y} + A_{51}e^{(b_3-2b_1)y} + A_{52}e^{-(2b_1+b_3)y} + \\ & A_{53}e^{-(b_1+2b_3)y} + A_{54}e^{(b_1+b_3)y} + A_{55}e^{(b_3-b_1)y} + A_{56}e^{(b_2+b_3)y} + A_{57}e^{(b_3-b_2)y} + A_{58}e^{(2b_3-b_1)y} \\ & + A_{59}e^{3b_3 y} + A_{60}e^{(b_1-b_3)y} + A_{61}e^{-(b_1+b_3)y} + A_{62}e^{(b_2-b_3)y} + A_{63}e^{-(b_2+b_3)y} + A_{64}e^{-3b_3 y} + A_{65}\end{aligned} \tag{3.300}$$

Applying the boundary conditions that is,

For $\theta_1(0) = 0$,

$$\begin{aligned}& A_{33} + A_{34} + A_{35} + A_{36} + A_{37} + A_{38} + A_{39} + A_{40} + A_{41} + A_{42} + A_{43} + A_{44} + A_{45} + A_{46} + \\ & A_{47} + A_{48} + A_{49} + A_{50} + A_{51} + A_{52} + A_{53} + A_{54} + A_{55} + A_{56} + A_{57} + A_{58} + A_{59} + A_{60} + \\ & A_{61} + A_{62} + A_{63} + A_{64} + A_{65} = 0\end{aligned} \tag{3.301}$$

$$A_{33} = -\begin{pmatrix} A_{35} + A_{36} + A_{37} + A_{38} + A_{39} + A_{40} + A_{41} + A_{42} + A_{43} + A_{44} + A_{45} + A_{46} + A_{47} \\ + A_{48} + A_{49} + A_{50} + A_{51} + A_{52} + A_{53} + A_{54} + A_{55} + A_{56} + A_{57} + A_{58} + A_{59} + A_{60} \\ + A_{61} + A_{62} + A_{63} + A_{64} + A_{65} \end{pmatrix} - A_{34} \tag{3.302}$$

So,

$$A_{33} = -A_{68} - A_{34} \tag{3.303}$$

Where,

$$A_{68}=\begin{pmatrix} A_{35}+A_{36}+A_{37}+A_{38}+A_{39}+A_{40}+A_{41}+A_{42}+A_{43}+A_{44}+A_{45}+A_{46}+A_{47} \\ +A_{48}+A_{49}+A_{50}+A_{51}+A_{52}+A_{53}+A_{54}+A_{55}+A_{56}+A_{57}+A_{58}+A_{59}+A_{60} \\ +A_{61}+A_{62}+A_{63}+A_{64}+A_{65} \end{pmatrix} \qquad (3.304)$$

Also for $\theta_1(1)=0$,

$$\begin{aligned} &A_{33}e^{b_2}+A_{34}e^{-b_2}+A_{35}e^{2b_1}+A_{36}e^{(b_1+b_2)}+A_{37}e^{(b_1-b_2)}+A_{38}e^{3b_1}+A_{39}e^{(2b_1+b_3)}+A_{40}e^{(2b_1-b_3)} \\ &+A_{41}e^{-b_1}+A_{42}e^{b_3}+A_{43}e^{-b_3}+A_{44}e^{(b_1+2b_3)}+A_{45}e^{(b_1-2b_3)}+A_{46}e^{-2b_1}+A_{47}e^{(b_2-b_1)}+ \\ &A_{48}e^{-(b_1+b_2)}+A_{49}e^{b_1}+A_{50}e^{-3b_1}+A_{51}e^{(b_3-2b_1)}+A_{52}e^{-(2b_1+b_3)}+A_{53}e^{-(b_1+2b_3)}+A_{54}e^{(b_1+b_3)}+ \\ &A_{55}e^{(b_3-b_1)}+A_{56}e^{(b_2+b_3)}+A_{57}e^{(b_3-b_2)}+A_{58}e^{(2b_3-b_1)}+A_{59}e^{3b_3}+A_{60}e^{(b_1-b_3)}+A_{61}e^{-(b_1+b_3)} \\ &+A_{62}e^{(b_2-b_3)}+A_{63}e^{-(b_2+b_3)}+A_{64}e^{-3b_3}+A_{65}=0 \end{aligned} \qquad (3.305)$$

Putting equation (3.303) into equation (3.305) gives,

$$\begin{aligned} &-A_{68}e^{b_2}-A_{34}e^{b_2}+A_{34}e^{-b_2}+A_{35}e^{2b_1}+A_{36}e^{(b_1+b_2)}+A_{37}e^{(b_1-b_2)}+A_{38}e^{3b_1}+A_{39}e^{(2b_1+b_3)}+A_{40}e^{(2b_1-b_3)} \\ &+A_{41}e^{-b_1}+A_{42}e^{b_3}+A_{43}e^{-b_3}+A_{44}e^{(b_1+2b_3)}+A_{45}e^{(b_1-2b_3)}+A_{46}e^{-2b_1}+A_{47}e^{(b_2-b_1)}+ \\ &A_{48}e^{-(b_1+b_2)}+A_{49}e^{b_1}+A_{50}e^{-3b_1}+A_{51}e^{(b_3-2b_1)}+A_{52}e^{-(2b_1+b_3)}+A_{53}e^{-(b_1+2b_3)}+A_{54}e^{(b_1+b_3)}+ \\ &A_{55}e^{(b_3-b_1)}+A_{56}e^{(b_2+b_3)}+A_{57}e^{(b_3-b_2)}+A_{58}e^{(2b_3-b_1)}+A_{59}e^{3b_3}+A_{60}e^{(b_1-b_3)}+A_{61}e^{-(b_1+b_3)} \\ &+A_{62}e^{(b_2-b_3)}+A_{63}e^{-(b_2+b_3)}+A_{64}e^{-3b_3}+A_{65}=0 \end{aligned} \qquad (3.306)$$

So equation (3.306) becomes,

$$A_{34}=\frac{\begin{pmatrix} A_{68}e^{b_2}-A_{35}e^{2b_1}-A_{36}e^{(b_1+b_2)}-A_{37}e^{(b_1-b_2)}-A_{38}e^{3b_1}-A_{39}e^{(2b_1+b_3)}-A_{40}e^{(2b_1-b_3)} \\ -A_{41}e^{-b_1}-A_{42}e^{b_3}-A_{43}e^{-b_3}-A_{44}e^{(b_1+2b_3)}-A_{45}e^{(b_1-2b_3)}-A_{46}e^{-2b_1}-A_{47}e^{(b_2-b_1)}- \\ A_{48}e^{-(b_1+b_2)}-A_{49}e^{b_1}-A_{50}e^{-3b_1}-A_{51}e^{(b_3-2b_1)}-A_{52}e^{-(2b_1+b_3)}-A_{53}e^{-(b_1+2b_3)}-A_{54}e^{(b_1+b_3)}- \\ A_{55}e^{(b_3-b_1)}-A_{56}e^{(b_2+b_3)}-A_{57}e^{(b_3-b_2)}-A_{58}e^{(2b_3-b_1)}-A_{59}e^{3b_3}-A_{60}e^{(b_1-b_3)}-A_{61}e^{-(b_1+b_3)} \\ -A_{62}e^{(b_2-b_3)}-A_{63}e^{-(b_2+b_3)}-A_{64}e^{-3b_3}-A_{65} \end{pmatrix}}{e^{-b_2}-e^{b_2}} \qquad (3.307)$$

Putting equation (3.307) into equation (3.303) gives,

$$A_{33} = -A_{68} - \frac{\begin{pmatrix} A_{68}e^{b_2} - A_{35}e^{2b_1} - A_{36}e^{(b_1+b_2)} - A_{37}e^{(b_1-b_2)} - A_{38}e^{3b_1} - A_{39}e^{(2b_1+b_3)} - A_{40}e^{(2b_1-b_3)} \\ -A_{41}e^{-b_1} - A_{42}e^{b_3} - A_{43}e^{-b_3} - A_{44}e^{(b_1+2b_3)} - A_{45}e^{(b_1-2b_3)} - A_{46}e^{-2b_1} - A_{47}e^{(b_2-b_1)} - \\ A_{48}e^{-(b_1+b_2)} - A_{49}e^{b_1} - A_{50}e^{-3b_1} - A_{51}e^{(b_3-2b_1)} - A_{52}e^{-(2b_1+b_3)} - A_{53}e^{-(b_1+2b_3)} - \\ A_{54}e^{(b_1+b_3)} - A_{55}e^{(b_3-b_1)} - A_{56}e^{(b_2+b_3)} - A_{57}e^{(b_3-b_2)} - A_{58}e^{(2b_3-b_1)} - A_{59}e^{3b_3} - \\ A_{60}e^{(b_1-b_3)} - A_{61}e^{-(b_1+b_3)} - A_{62}e^{(b_2-b_3)} - A_{63}e^{-(b_2+b_3)} - A_{64}e^{-3b_3} - A_{65} \end{pmatrix}}{e^{-b_2} - e^{b_2}} \quad (3.308)$$

$$A_{33} = -\frac{\begin{pmatrix} -A_{68}\left(e^{-b_2} - e^{b_2}\right) - A_{68}e^{b_2} + A_{35}e^{2b_1} + A_{36}e^{(b_1+b_2)} + A_{37}e^{(b_1-b_2)} + A_{38}e^{3b_1} + A_{39}e^{(2b_1+b_3)} + \\ A_{40}e^{(2b_1-b_3)} + A_{41}e^{-b_1} + A_{42}e^{b_3} + A_{43}e^{-b_3} + A_{44}e^{(b_1+2b_3)} + A_{45}e^{(b_1-2b_3)} + A_{46}e^{-2b_1} + \\ A_{47}e^{(b_2-b_1)} + A_{48}e^{-(b_1+b_2)} + A_{49}e^{b_1} + A_{50}e^{-3b_1} + A_{51}e^{(b_3-2b_1)} + A_{52}e^{-(2b_1+b_3)} + A_{53}e^{-(b_1+2b_3)} + \\ A_{54}e^{(b_1+b_3)} + A_{55}e^{(b_3-b_1)} + A_{56}e^{(b_2+b_3)} + A_{57}e^{(b_3-b_2)} + A_{58}e^{(2b_3-b_1)} + A_{59}e^{3b_3} + A_{60}e^{(b_1-b_3)} + \\ A_{61}e^{-(b_1+b_3)} + A_{62}e^{(b_2-b_3)} + A_{63}e^{-(b_2+b_3)} + A_{64}e^{-3b_3} + A_{65} \end{pmatrix}}{e^{-b_2} - e^{b_2}} \quad (3.309)$$

So equation (3.309) becomes,

$$A_{33} = -\frac{\begin{pmatrix} -A_{68}e^{-b_2} + A_{35}e^{2b_1} + A_{36}e^{(b_1+b_2)} + A_{37}e^{(b_1-b_2)} + A_{38}e^{3b_1} + A_{39}e^{(2b_1+b_3)} + A_{40}e^{(2b_1-b_3)} \\ +A_{41}e^{-b_1} + A_{42}e^{b_3} + A_{43}e^{-b_3} + A_{44}e^{(b_1+2b_3)} + A_{45}e^{(b_1-2b_3)} + A_{46}e^{-2b_1} + A_{47}e^{(b_2-b_1)} + \\ A_{48}e^{-(b_1+b_2)} + A_{49}e^{b_1} + A_{50}e^{-3b_1} + A_{51}e^{(b_3-2b_1)} + A_{52}e^{-(2b_1+b_3)} + A_{53}e^{-(b_1+2b_3)} + A_{54}e^{(b_1+b_3)} + \\ A_{55}e^{(b_3-b_1)} + A_{56}e^{(b_2+b_3)} + A_{57}e^{(b_3-b_2)} + A_{58}e^{(2b_3-b_1)} + A_{59}e^{3b_3} + A_{60}e^{(b_1-b_3)} + A_{61}e^{-(b_1+b_3)} \\ +A_{62}e^{(b_2-b_3)} + A_{63}e^{-(b_2+b_3)} + A_{64}e^{-3b_3} + A_{65} \end{pmatrix}}{e^{-b_2} - e^{b_2}} \quad (3.310)$$

Therefore,

$$\theta_1(y) = A_{33}e^{b_2 y} + A_{34}e^{-b_2 y} + A_{35}e^{2b_1 y} + A_{36}e^{(b_1+b_2)y} + A_{37}e^{(b_1-b_2)y} + A_{38}e^{3b_1 y} + A_{39}e^{(2b_1+b_3)y} + A_{40}e^{(2b_1-b_3)y} + A_{41}e^{-b_1 y} + A_{42}e^{b_3 y} + A_{43}e^{-b_3 y} + A_{44}e^{(b_1+2b_3)y} + A_{45}e^{(b_1-2b_3)y} + A_{46}e^{-2b_1 y} + A_{47}e^{(b_2-b_1)y} + A_{48}e^{-(b_1+b_2)y} + A_{49}e^{b_1 y} + A_{50}e^{-3b_1 y} + A_{51}e^{(b_3-2b_1)y} + A_{52}e^{-(2b_1+b_3)y} + A_{53}e^{-(b_1+2b_3)y} + A_{54}e^{(b_1+b_3)y} + A_{55}e^{(b_3-b_1)y} + A_{56}e^{(b_2+b_3)y} + A_{57}e^{(b_3-b_2)y} + A_{58}e^{(2b_3-b_1)y} + A_{59}e^{3b_3 y} + A_{60}e^{(b_1-b_3)y} + A_{61}e^{-(b_1+b_3)y} + A_{62}e^{(b_2-b_3)y} + A_{63}e^{-(b_2+b_3)y} + A_{64}e^{-3b_3 y} + A_{65} \tag{3.311}$$

Putting equations (3.117), (3.216), (3.166), (3.311), (3.68) and (3.86) in equation (3.38) gives,

$$u(y) = u_0(y) + G_{r\theta}u_1(y) \tag{3.312}$$

$$\theta(y) = \theta_0(y) + G_{r\theta}\theta_1(y) \tag{3.313}$$

$$\phi(y) = \phi_0(y) \tag{3.314}$$

The transformed solution becomes,

$$u(y) = \begin{pmatrix} A_5e^{b_1 y} + A_6e^{-b_1 y} + \\ A_7e^{b_3 y} + A_8e^{-b_3 y} \end{pmatrix} + G_{r\theta}\begin{pmatrix} A_{20}e^{b_1 y} + A_{21}e^{-b_1 y} + A_{22}e^{b_2 y} + A_{23}e^{-b_2 y} + A_{24}e^{2b_1 y} + \\ A_{25} + A_{26}e^{(b_1+b_3)y} + A_{27}e^{(b_1-b_3)y} + A_{28}e^{-2b_1 y} + A_{29}e^{(b_3-b_1)y} \\ +A_{30}e^{-(b_1+b_3)y} + A_{31}e^{2b_3 y} + A_{32}e^{-2b_3 y} \end{pmatrix} \tag{3.315}$$

$$\theta(y) = \begin{pmatrix} A_9e^{b_2 y} + A_{10}e^{-b_2 y} + A_{11}e^{2b_1 y} + \\ A_{12} + A_{13}e^{(b_1+b_3)y} + A_{14}e^{(b_1-b_3)y} \\ +A_{15}e^{-2b_1 y} + A_{16}e^{(b_3-b_1)y} + \\ A_{17}e^{-(b_1+b_3)y} + A_{18}e^{2b_3 y} + A_{19}e^{-2b_3 y} \end{pmatrix} + G_{r\theta}\begin{pmatrix} A_{33}e^{b_2 y} + A_{34}e^{-b_2 y} + A_{35}e^{2b_1 y} + A_{36}e^{(b_1+b_2)y} + \\ A_{37}e^{(b_1-b_2)y} + A_{38}e^{3b_1 y} + A_{39}e^{(2b_1+b_3)y} + A_{40}e^{(2b_1-b_3)y} \\ +A_{41}e^{-b_1 y} + A_{42}e^{b_3 y} + A_{43}e^{-b_3 y} + A_{44}e^{(b_1+2b_3)y} + \\ A_{45}e^{(b_1-2b_3)y} + A_{46}e^{-2b_1 y} + A_{47}e^{(b_2-b_1)y} + A_{48}e^{-(b_1+b_2)y} \\ +A_{49}e^{b_1 y} + A_{50}e^{-3b_1 y} + A_{51}e^{(b_3-2b_1)y} + A_{52}e^{-(2b_1+b_3)y} + \\ A_{53}e^{-(b_1+2b_3)y} + A_{54}e^{(b_1+b_3)y} + A_{55}e^{(b_3-b_1)y} + A_{56}e^{(b_2+b_3)y} \\ +A_{57}e^{(b_3-b_2)y} + A_{58}e^{(2b_3-b_1)y} + A_{59}e^{3b_3 y} + A_{60}e^{(b_1-b_3)y} + \\ A_{61}e^{-(b_1+b_3)y} + A_{62}e^{(b_2-b_3)y} + A_{63}e^{-(b_2+b_3)y} + A_{64}e^{-3b_3 y} + A_{65} \end{pmatrix} \tag{3.316}$$

$$\phi(y) = A_1e^{b_3 y} + A_2e^{-b_3 y} \tag{3.317}$$

Recall equation (3.19) on the general solutions

$$u(y,t)=u(y)e^{i\omega t},\quad \theta(y,t)=\theta(y)e^{2i\omega t},\quad \phi(y,t)=\phi(y)e^{i\omega t}$$

3.4: Skin-friction of the Fluid Velocity $u(y,t)$.

The dimensionless stress tensor in terms of the skin-friction coefficient at the plate $y=0$ is given by

$$CF_0=\left(\frac{\partial u(y,t)}{\partial y}\right)_{y=0}=\left(\frac{du}{dy}\right)_{y=0}\left(e^{i\omega t}\right)$$

$$CF_0=\left(\left(A_5-A_6\right)b_1+\left(A_7-A_8\right)b_3+G_{r\theta}\begin{pmatrix}\left(A_{20}-A_{21}+2A_{24}+2A_{28}\right)b_1+\\ \left(A_{22}-A_{23}\right)b_2+\left(A_{26}-A_{30}\right)\left(b_1+b_3\right)+\\ A_{27}\left(b_1-b_3\right)+A_{29}\left(b_3-b_1\right)+2\left(A_{31}-A_{32}\right)b_3\end{pmatrix}\right)e^{i\omega t} \quad (3.318)$$

3.5: Nusselt-number of the Temperature of the Fluid $\theta(y,t)$.

The dimensionless rate of heat transfer in terms of the Nusselt number at the plate $y=0$ is given by

$$Nu_0=-\left(\frac{\partial\theta(y,t)}{\partial y}\right)_{y=0}=\left(\frac{d\theta}{dy}\right)_{y=0}\left(e^{2i\omega t}\right)$$

$$Nu_0=-\left(\begin{pmatrix}\left(A_9-A_{10}\right)b_2+2\left(A_{11}-A_{15}\right)b_1\\ +A_{13}\left(b_1+b_3\right)+A_{14}\left(b_1-b_3\right)+\\ A_{16}\left(b_3-b_1\right)-A_{17}\left(b_1+b_3\right)+\\ 2\left(A_{18}-A_{19}\right)b_3\end{pmatrix}+G_{r\theta}\begin{pmatrix}\left(A_{33}-A_{34}\right)b_2+\left(2A_{35}-A_{41}+3A_{38}-2A_{46}+A_{49}-3A_{50}\right)b_1\\ +\left(A_{36}-A_{48}\right)\left(b_1+b_2\right)+A_{37}\left(b_1-b_2\right)+\left(A_{39}-A_{52}\right)\left(2b_1+b_3\right)\\ +A_{40}\left(2b_1-b_3\right)+\left(A_{42}-A_{43}+3A_{59}-3A_{64}\right)b_3+\\ \left(A_{44}-A_{53}\right)\left(b_1+2b_3\right)+A_{45}\left(b_1-2b_3\right)+A_{47}\left(b_2-b_1\right)\\ +A_{51}\left(b_3-2b_1\right)+\left(A_{54}-A_{61}\right)\left(b_1+b_3\right)+A_{55}\left(b_3-b_1\right)\\ +\left(A_{56}-A_{63}\right)\left(b_2+b_3\right)+A_{57}\left(b_3-b_2\right)+A_{58}\left(2b_3-b_1\right)\\ +A_{60}\left(b_1-b_3\right)+A_{62}\left(b_2-b_3\right)\end{pmatrix}\right)e^{2i\omega t} \quad (3.319)$$

3.6: Sherwood number of the Concentration of the Fluid $\phi(y,t)$.

The dimensionless rate of species concentration in terms of the Sherwood number at the plate $y=0$ is given by

$$S_h=-\left(\frac{\partial\phi(y,t)}{\partial y}\right)_{y=0}=\left(\frac{d\phi}{dy}\right)_{y=0}\left(e^{i\omega t}\right) \quad (3.320)$$

$$S_h=-\left(A_1b_3-A_2b_3\right)e^{i\omega t}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Results

In this analysis, the effect of Peclet number (P_e), Hartman number (H_a), Eckert number (E_c), Peclet mass number (P_{em}), Reynold number (R_e), Solutal Grashof number $(G_{r\phi})$, Grashof thermal number $(G_{r\theta})$, time (t), on the velocity $u(y,t)$ of the fluid, concentration of the fluid $\phi(y,t)$ and temperature of the fluid $\theta(y,t)$ were examined. The results obtained from the solutions are shown in Figure 4.1 through 4.25. The effect of Peclet number (P_e) on velocity $u(y,t)$ against distance is depicted in figure 4.1. The effect of Peclet number (P_e) on temperature of the fluid $\theta(y,t)$ against distance is depicted in figure 4.2. The effect of Peclet number (P_e) on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.3. The effect of Hartman number (H_a) on velocity $u(y,t)$ against distance is depicted in figure 4.4. The effect of Hartman number (H_a) on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.5. The effect of Eckert number (E_c) on the velocity $u(y,t)$ against distance is depicted in figure 4.6. The effect of Eckert number (E_c) on temperature of the fluid $\theta(y,t)$ against distance is depicted in figure 4.7. The effect of Eckert number (E_c) on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.8. The effect of Peclet mass number (P_{em}) on the velocity $u(y,t)$ against distance is depicted in figure 4.9. The effect of Peclet mass number (P_{em}) on temperature of the fluid $\theta(y,t)$ against distance is depicted in figure 4.10. The effect of Peclet mass number (P_{em}) on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.11. The effect of Reynold number (R_e) on velocity of the fluid $u(y,t)$ against distance is depicted in figure 4.12. The effect of Reynold number (R_e) on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.13. The effect of Grashof Thermal number $(G_{r\theta})$ on velocity of the fluid $u(y,t)$ against distance is depicted in figure 4.14. The effect of Grashof Thermal number $(G_{r\theta})$ on temperature of the fluid $\theta(y,t)$ against time is depicted in figure 4.15. The effect of Grashof Thermal number $(G_{r\theta})$ on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.16. The effect of Solutal Grashof number $(G_{r\phi})$ on velocity of the fluid $u(y,t)$ against distance is depicted in figure 4.17. The effect of Solutal Grashof number $(G_{r\phi})$ on temperature $\theta(y,t)$ against time is depicted in figure 4.18. The effect of Solutal Grashof number $(G_{r\phi})$ on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.19. The effect of time (t) on velocity of the fluid $u(y,t)$ against distance is depicted in figure 4.20. The effect of time (t) on temperature $\theta(y,t)$ against time is depicted in figure 4.21. The effect of time (t) on concentration of the fluid $\phi(y,t)$ against distance is depicted in figure 4.22.

Skin friction of the fluid velocity, Nusset number of the fluid temperature and Sherwood number of the mass transfer of the fluid given by equations (3.285), (3.286) and (3.287) respectively were computed using MAPLE 17 and presented in Tables 4.1, 4.2 and 4.3.

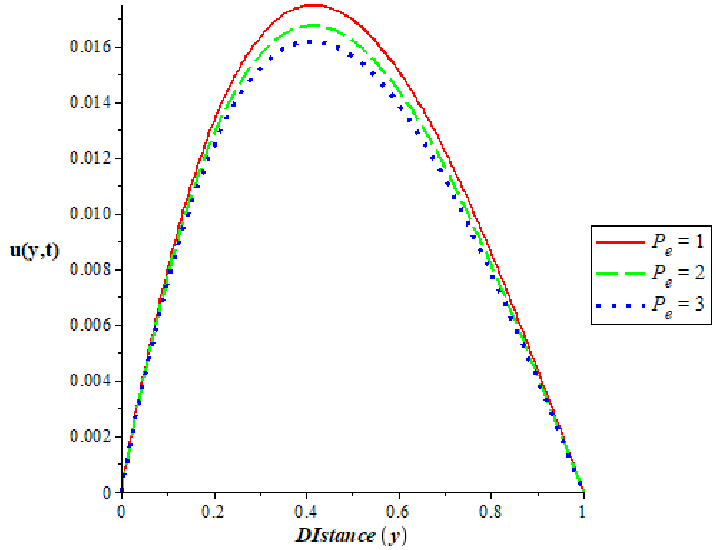

Figure 4.1: Effect of Peclet Number (P_e) on Velocity of The Fluid $u(y,t)$ Along Distance

It is observed that velocity of the fluid reduces with an increase in the Peclet number (P_e) at steady time.

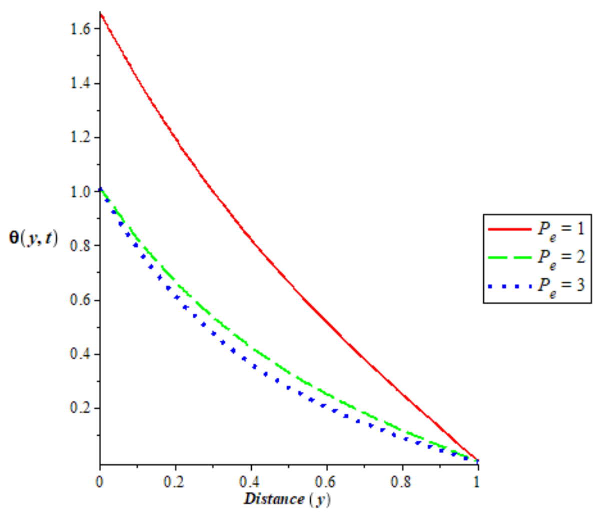

Figure 4.2: Effect of Peclet Number (P_e) on Temperature Profile $\theta(y,t)$ Along Distance

It is observed that the temperature of the fluid $\theta(y,t)$ reduces with increase in the Peclet number (P_e) at steady time.

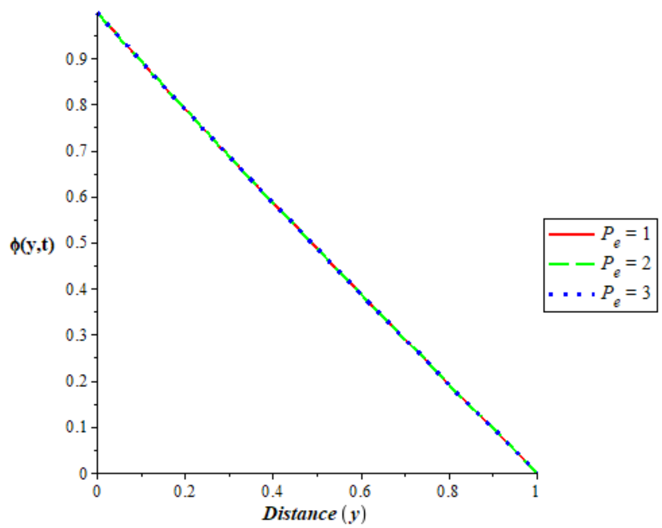

Figure 4.3: Effect of Peclet Number (P_e) on Concentration Profile $\phi(y,t)$

It is observed that the concentration of the fluid $\phi(y,t)$ does not change much with an increase in Peclet number (P_e) at steady time.

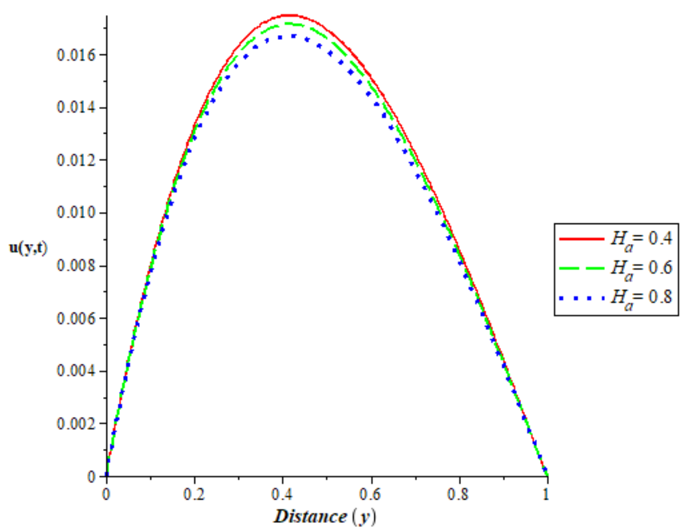

Figure 4.4: Effect of Hartman Number (H_a) on Velocity of The Fluid $u(y,t)$ Along Distance.

It is observed that velocity of the fluid reduces with an increase in the Hartman number (H_a) at steady time.

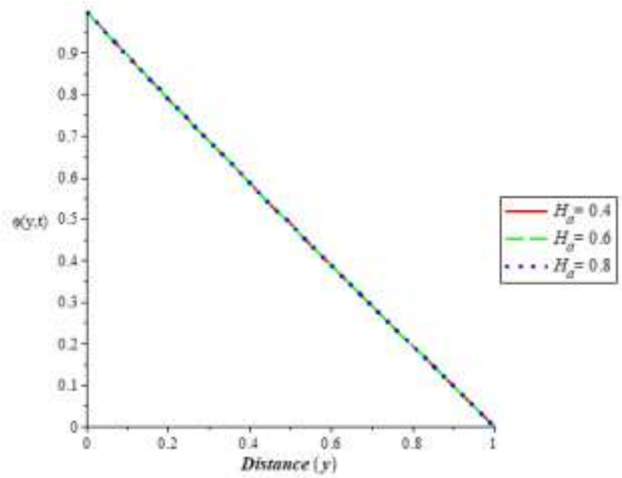

Figure 4.5: Effect of Hartman Number (Ha) on Concentration Profile $\phi(y,t)$ Along Distance

It is observed that the concentration of the fluid $\phi(y,t)$ does not change much with an increase in the Hartman number (H_a) at steady time.

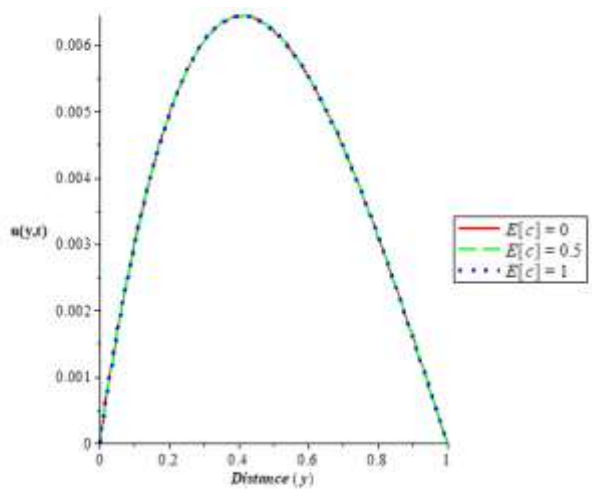

Figure 4.6: Effect of Eckert Number (E_c) on Velocity of the Fluid $u(y,t)$ Along Distance

It is observed that velocity of the fluid $u(y,t)$ does not change much with an increase in the Eckert number (E_c) at steady time.

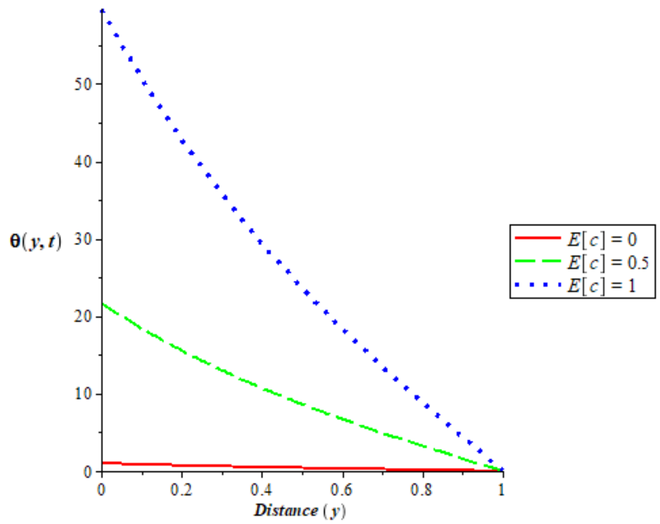

Figure 4.7: Effect of Eckert Number (E_c) on Temperature Profile $\theta(y,t)$ along Distance

It is observed that the temperature of the fluid $\theta(y,t)$ increases with increase in Eckert number (E_c) at steady time.

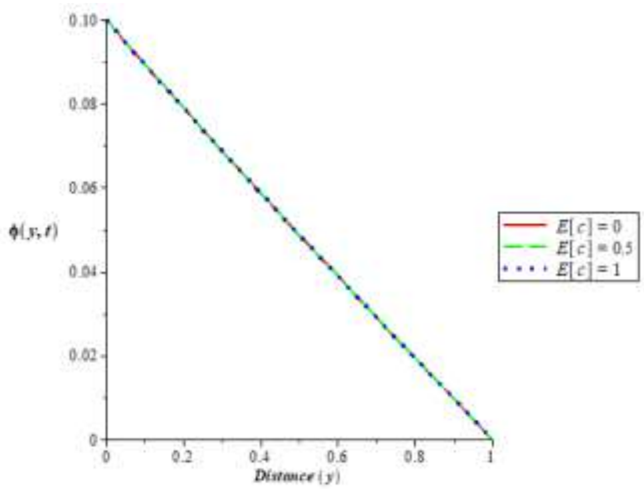

Figure 4.8: Effect of Eckert Number (E_c) on Concentration Profile $\phi(y,t)$

It is observed that the concentration of the fluid $\phi(y,t)$ does not change much with an increase in the Eckert number (E_c) at steady time.

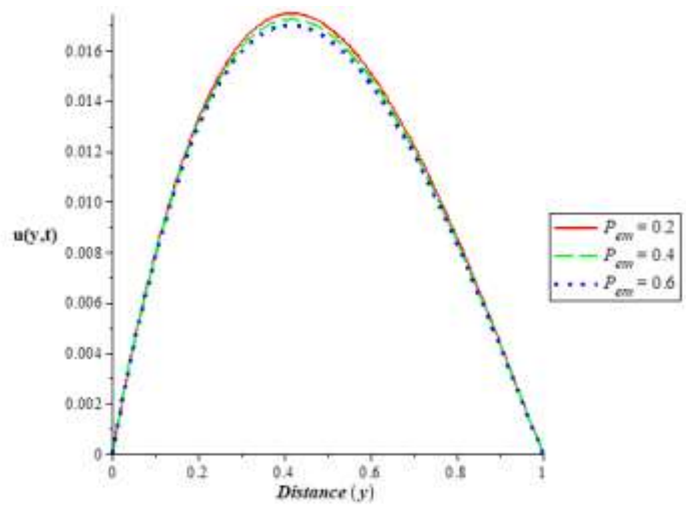

Figure 4.9: Effect of Peclet Mass Number (P_{em}) on Velocity of The Fluid $u(y,t)$ Along Distance

It is observed that the velocity of the $u(y,t)$ reduces with an increase in the Peclet mass number (P_{em}) at steady time.

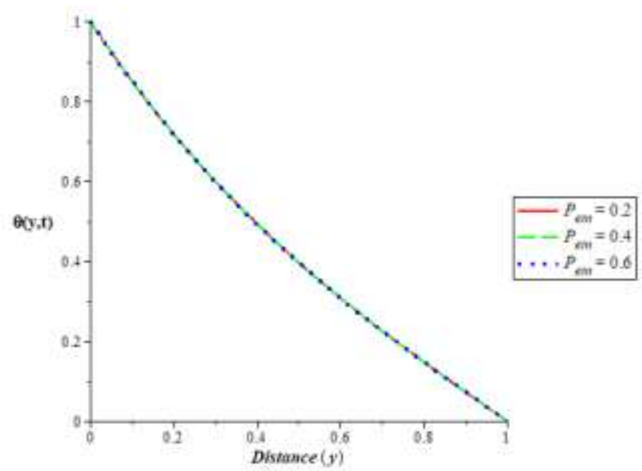

Figure 4.10: Effect of Peclet Mass Number (P_{em}) on Temperature Profile $\theta(y,t)$ Along Distance

It is observed that temperature of the fluid $\theta(y,t)$ does not change much with an increase in the Peclet mass number (P_{em}) at steady time.

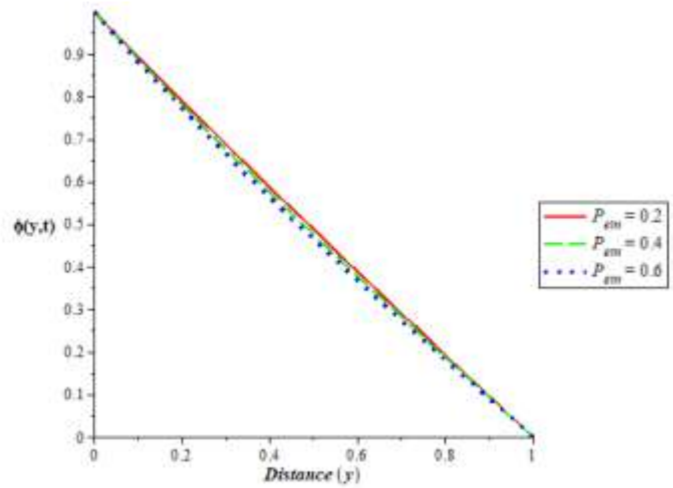

Figure 4.11 Effect of Peclet Mass Number (P_{em}) on Concentration Profile $\phi(y,t)$ Along Distance

It is observed that concentration of the fluid $\phi(y,t)$ reduces with an increase in the Peclet mass number (P_{em}) at steady time.

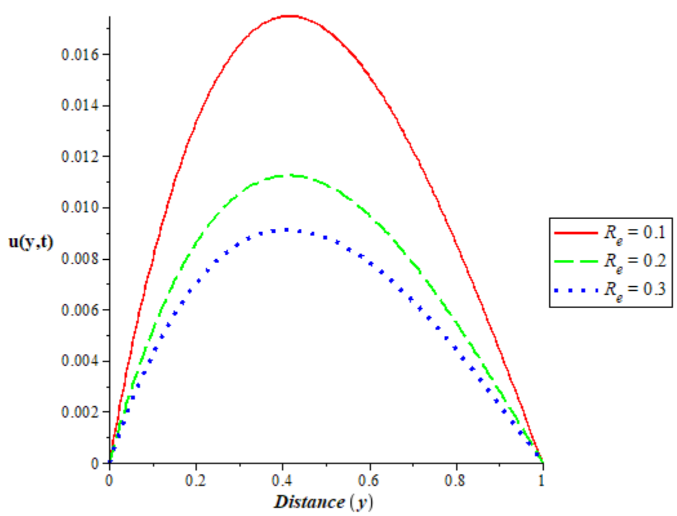

Figure 4.12: Effect of Reynold Number (R_e) on Velocity of the Fluid $u(y,t)$ Along Distance

It is observed that velocity of the fluid reduces with an increase in Reynold number (R_e) at steady time.

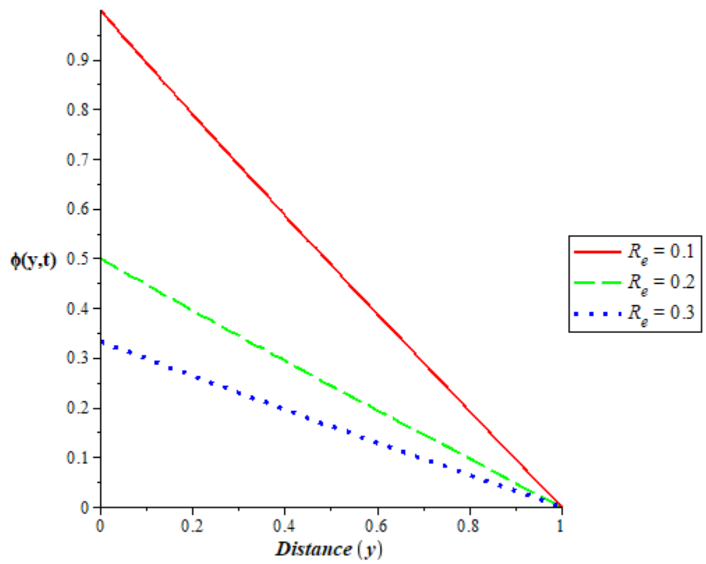

Figure 4.13: Effect of Reynold Number (R_e) on Concentration Profile $\phi(y,t)$ Along Distance

It is observed that the concentration of the fluid reduces with an increase in Reynold number (R_e) at steady time.

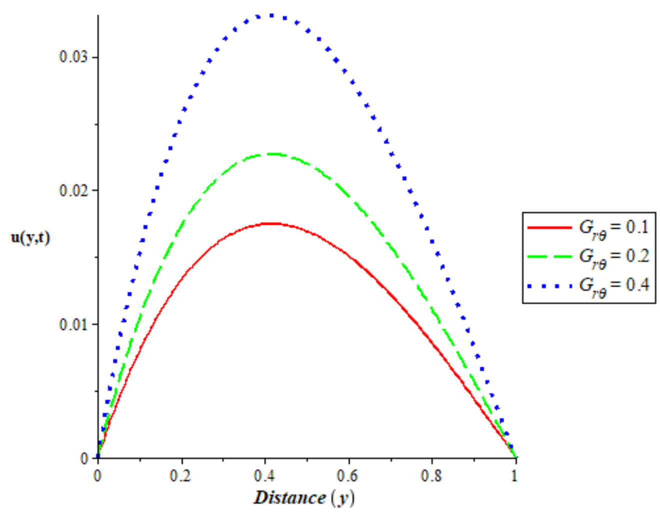

Figure 4.14: Effect of Thermal Grashof Number $(G_{r\theta})$ on Velocity of the Fluid $u(y,t)$ Along Distance

It is observed that velocity of the fluid increases with an increase in Thermal Grashof number $(G_{r\theta})$ at steady time.

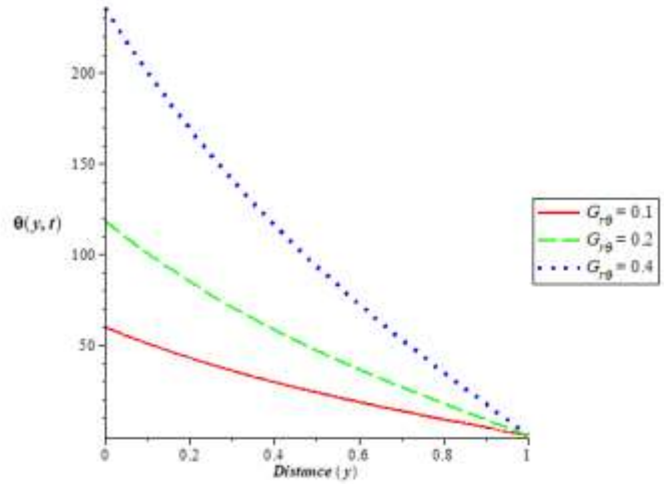

Figure 4.15: Effect of Thermal Grashof Number $\left(G_{r\theta}\right)$ on Temperature Profile $\theta\left(y,t\right)$ Along Distance

It is observed that the temperature of the fluid increases with an increase in Thermal Grashof number $\left(G_{r\theta}\right)$ at steady time.

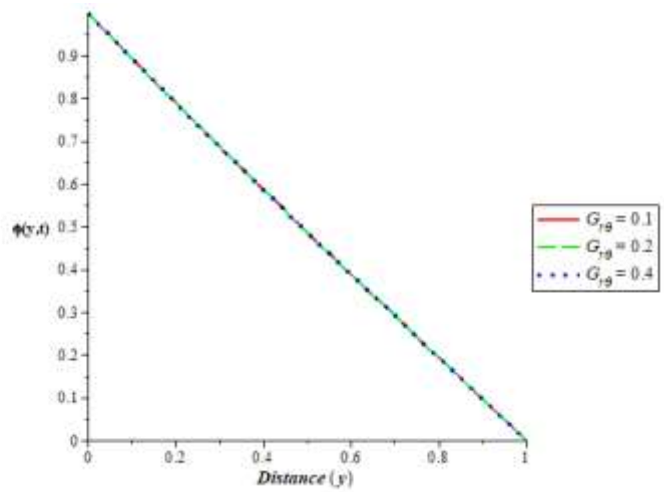

Figure 4.16: Effect of Thermal Grashof Number $\left(G_{r\theta}\right)$ on Concentration Profile $\phi\left(y,t\right)$ Along Distance

It is observed that concentration of the fluid does not change much with an increase in Thermal Grashof number $\left(G_{r\theta}\right)$ at steady time.

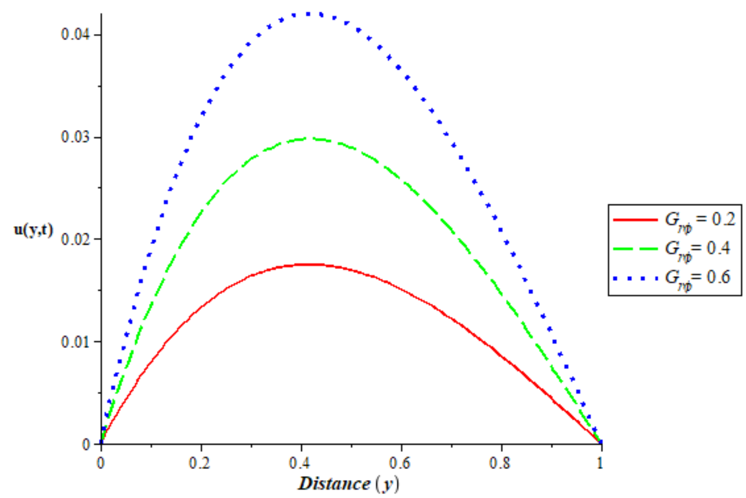

Figure 4.17: Effect of Solutal Grashof Number $\left(G_{r\phi}\right)$ on Velocity of the Fluid $u\left(y,t\right)$ Along Distance

It is observed that velocity of the fluid increases with an increase in Solutal Grashof number $\left(G_{r\phi}\right)$ at steady time.

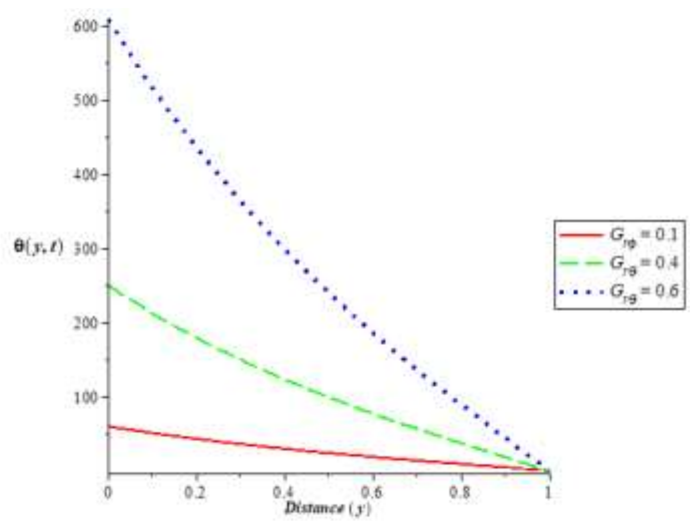

Figure 4.18: Effect of Solutal Grashof Number $\left(G_{r\phi}\right)$ on Temperature Profile $\theta\left(y,t\right)$ Along Distance

It is observed that the temperature of the fluid increases with an increase in Solutal Grashof number $\left(G_{r\phi}\right)$ at steady time.

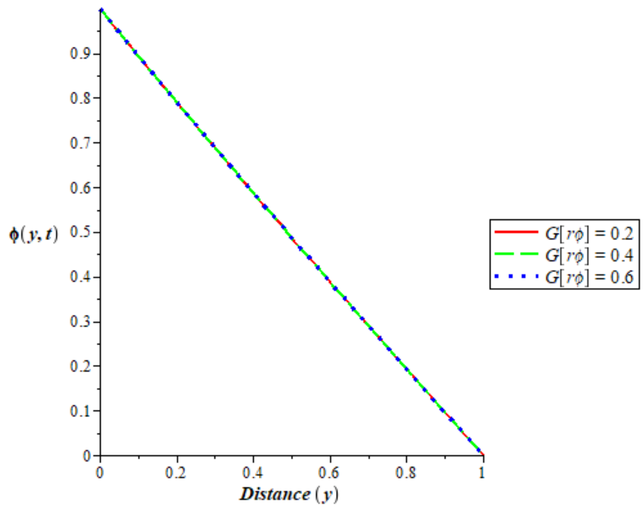

Figure 4.19: Effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on Concentration Profile $\phi(y,t)$ Along Distance

It is observed that concentration of the fluid does not change much with an increase in Solutal Grashof number $\left(G_{r\phi}\right)$ at steady time. time (t)

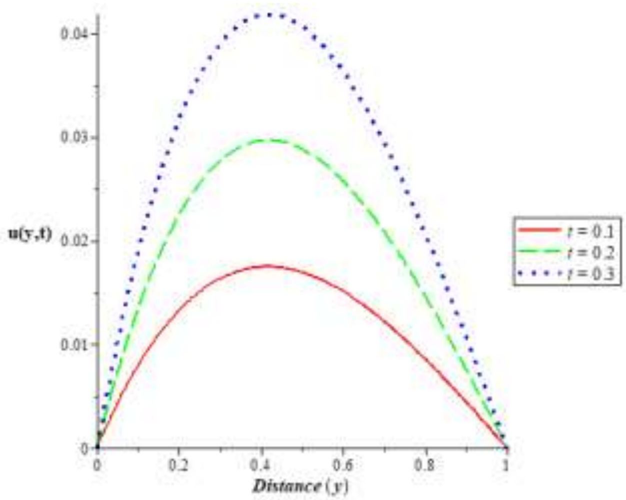

Figure 4.20: Effect of Time (t) on Velocity of the Fluid $u(y,t)$

It is observed that velocity of the fluid increases with an increase in time (t).

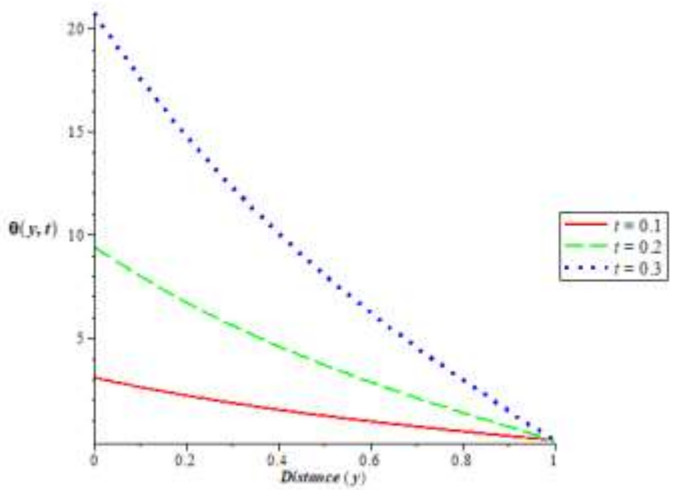

Figure 4.21: Effect of Time (t) on Temperature Profile $\theta(y,t)$

It is observed that the temperature of the fluid increases with an increase in time (t).

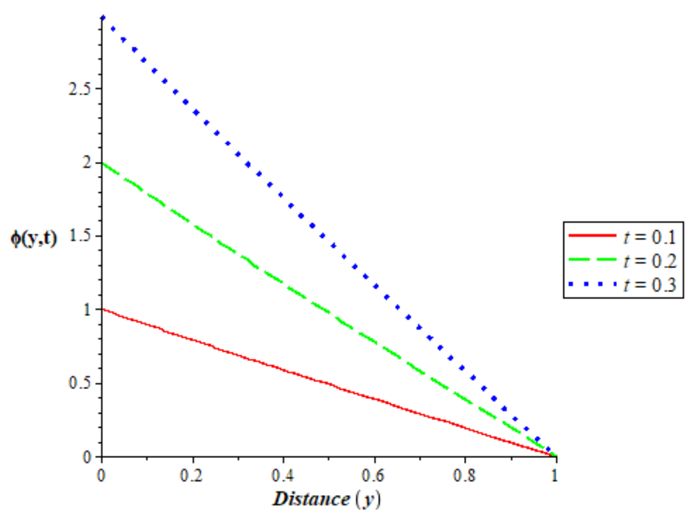

Figure 4.22: Effect of time (t) on concentration profile $\phi(y,t)$

It is observed that concentration of the fluid increases with an increase in time (t).

Table 4.1. Numerical values of skin-friction coefficient at the plate $y = 0$ for various values of physical parameters

P_e	H_a	S	E_c	P_{em}	R_e	$G_{r\theta}$	$G_{r\phi}$	t	CF_0
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	0.09376453347
2	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	0.09111573447
3	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	0.08908191868
1	0.6	0.3	1	0.2	0.1	0.1	0.2	0.1	0.09259884228
1	0.8	0.3	1	0.2	0.1	0.1	0.2	0.1	0.09103592697
1	0.4	0.6	1	0.2	0.1	0.1	0.2	0.1	0.09376453347
1	0.4	0.9	1	0.2	0.1	0.1	0.2	0.1	0.09376453347
1	0.4	0.3	2	0.2	0.1	0.1	0.2	0.1	0.09378194994
1	0.4	0.3	3	0.2	0.1	0.1	0.2	0.1	0.09379943198
1	0.4	0.3	1	0.4	0.1	0.1	0.2	0.1	0.09294406100
1	0.4	0.3	1	0.6	0.1	0.1	0.2	0.1	0.09215296717
1	0.4	0.3	1	0.2	0.2	0.1	0.2	0.1	0.06103630928
1	0.4	0.3	1	0.2	0.3	0.1	0.2	0.1	0.05001863507
1	0.4	0.3	1	0.2	0.1	0.2	0.2	0.1	0.1228671986
1	0.4	0.3	1	0.2	0.1	0.4	0.2	0.1	0.1810725292
1	0.4	0.3	1	0.2	0.1	0.1	0.4	0.1	0.1584787222
1	0.4	0.3	1	0.2	0.1	0.1	0.6	0.1	0.2232279061
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.2	0.1582187806
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.3	0.2223278641

Table 4.2: Numerical values of Nusselt number at the plate $y = 0$ for various values of physical parameters.

P_e	H_a	S	E_c	P_{em}	R_e	$G_{r\theta}$	$G_{r\phi}$	t	Nu
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	18300.84406
2	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	3569.959011
3	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	3044.911915
1	0.6	0.3	1	0.2	0.1	0.1	0.2	0.1	-134.4319549
1	0.8	0.3	1	0.2	0.1	0.1	0.2	0.1	-17.43137658
1	0.4	0.6	1	0.2	0.1	0.1	0.2	0.1	18300.84491
1	0.4	0.9	1	0.2	0.1	0.1	0.2	0.1	18300.84491
1	0.4	0.3	2	0.2	0.1	0.1	0.2	0.1	65052.27514
1	0.4	0.3	3	0.2	0.1	0.1	0.2	0.1	138487.9284
1	0.4	0.3	1	0.4	0.1	0.1	0.2	0.1	478.3549511
1	0.4	0.3	1	0.6	0.1	0.1	0.2	0.1	27.24311949
1	0.4	0.3	1	0.2	0.2	0.1	0.2	0.1	-26.56457031
1	0.4	0.3	1	0.2	0.3	0.1	0.2	0.1	0.2112692571
1	0.4	0.3	1	0.2	0.1	0.2	0.2	0.1	36600.10279
1	0.4	0.3	1	0.2	0.1	0.4	0.2	0.1	73198.61598
1	0.4	0.3	1	0.2	0.1	0.1	0.4	0.1	119026.2452
1	0.4	0.3	1	0.2	0.1	0.1	0.6	0.1	372733.8326
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.2	118479.3884
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.3	368119.9574

Table 4.3: Numerical values of Sherwood number at the plate $y = 0$ for various values of physical parameters.

P_e	H_a	S	E_c	P_{em}	R_e	$G_{r\theta}$	$G_{r\phi}$	t	S_h
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	1.065368116
2	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	1.065368116
3	0.4	0.3	1	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.6	0.3	1	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.8	0.3	1	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.4	0.6	1	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.4	0.9	1	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.4	0.3	2	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.4	0.3	3	0.2	0.1	0.1	0.2	0.1	1.065368116
1	0.4	0.3	1	0.4	0.1	0.1	0.2	0.1	1.129456117
1	0.4	0.3	1	0.6	0.1	0.1	0.2	0.1	1.191954372
1	0.4	0.3	1	0.2	0.2	0.1	0.2	0.1	0.5326840581
1	0.4	0.3	1	0.2	0.3	0.1	0.2	0.1	0.3551227054
1	0.4	0.3	1	0.2	0.1	0.2	0.2	0.1	1.065368116
1	0.4	0.3	1	0.2	0.1	0.4	0.2	0.1	1.065368116
1	0.4	0.3	1	0.2	0.1	0.1	0.4	0.1	1.065368116
1	0.4	0.3	1	0.2	0.1	0.1	0.6	0.1	1.065368116
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.2	2.128180031
1	0.4	0.3	1	0.2	0.1	0.1	0.2	0.3	3.185886358

Table 4.4: Comparison between present and Rajput and Sahu (2011) results.

y	$\theta(y,t)$ **Harmonic Results**	$\theta(y,t)$ **Laplace Results**	$\lvert\theta_{Lapl} - \theta_{harm}\rvert$
0	1.0000000000	1.0000000000	0
0.1	0.8332534696	0.8502840152	1.703 x10^{-2}
0.2	0.6900599584	0.7056395321	1.558 x10^{-2}
0.3	0.5663719114	0.5705631300	4.191 x10^{-3}
0.4	0.4586931209	0.4484879662	1.021 x10^{-2}
0.5	0.3639799028	0.3414480166	2.253 x10^{-2}
0.6	0.2795550617	0.2499276713	2.963 x10^{-2}
0.7	0.2030322162	0.1728926221	3.014 x10^{-2}
0.8	0.1322483475	0.1079690244	2.428 x10^{-2}
0.9	0.0652026544	0.0517237204	1.348 x10^{-2}
1.0	0	0	0

4.2 Discussion of Results

Figure 4.1 shows the effect of Peclet number (P_e) on the velocity of the fluid $u(y,t)$. It is observed that velocity of the fluid reduces with an increase in the Peclet number (P_e) at steady time.

Figure 4.2 displays the effect of Peclet number (P_e) on the temperature of the fluid $\theta(y,t)$. It is observed that the temperature of the fluid $\theta(y,t)$ reduces with increase in the Peclet number (P_e) at steady time.

Figure 4.3 shows the effect of Peclet number (P_e) on the concentration of the fluid $\phi(y,t)$. It is observed that the concentration of the fluid $\phi(y,t)$ does not change much with an increase in Peclet number (P_e) at steady time.

Figure 4.4 shows the effect of Hartman number (H_a) on the velocity of the fluid $u(y,t)$. It is observed that velocity of the fluid reduces with an increase in the Hartman number (H_a) at steady time.

Figure 4.5 shows the effect of Hartman number (H_a) on the concentration of the fluid $\phi(y,t)$. It is observed that the concentration of the fluid $\phi(y,t)$ does not change much with an increase in the Hartman number (H_a) at steady time.

Figure 4.6 shows the effect of Eckert number $\left(E_c\right)$ on the velocity of the fluid $u\left(y,t\right)$. It is observed that velocity of the fluid $u\left(y,t\right)$ does not change much with an increase in the Eckert number $\left(E_c\right)$ at steady time.

Figure 4.7 shows the effect of Eckert number $\left(E_c\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It shows that an increase in Eckert number from 0 (no viscous heating) through 0.5 to 1 (high viscous heating) clearly boost temperature in the porous regime. Eckert number signifies the quantity of mechanical energy converted via internal friction to thermal energy.

Figure 4.8 shows the effect of Eckert number $\left(E_c\right)$ on the concentration of the fluid $\phi\left(y,t\right)$. It is observed that the concentration of the fluid $\phi\left(y,t\right)$ does not change much with an increase in the Eckert number $\left(E_c\right)$ at steady time.

Figure 4.9 shows the effect of Peclet mass number $\left(P_{em}\right)$ on the velocity of the $u\left(y,t\right)$. It is observed that the velocity of the $u\left(y,t\right)$ reduces with an increase in the Peclet mass number $\left(P_{em}\right)$ at steady time.

Figure 4.10 shows the effect of Peclet mass number $\left(P_{em}\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It is observed that temperature of the fluid $\theta\left(y,t\right)$ does not change much with an increase in the Peclet mass number $\left(P_{em}\right)$ at steady time.

Figure 4.11 shows the effect of Peclet mass number (P_{em}) on the concentration of the fluid $\phi(y,t)$. It is observed that concentration of the fluid $\phi(y,t)$ reduces with an increase in the Peclet mass number (P_{em}) at steady time.

Figure 4.12 shows the effect of Reynold number (R_e) on the velocity of the fluid $u(y,t)$. It is observed that velocity of the fluid reduces with an increase in Reynold number (R_e) at steady time.

Figure 4.13 shows the effect of Reynold number (R_e) on the concentration of the fluid $\phi(y,t)$. It is observed that the concentration of the fluid reduces with an increase in Reynold number (R_e) at steady time.

Figure 4.14 shows the effect of Thermal Grashof number $(G_{r\theta})$ on the velocity of the fluid $u(y,t)$. It is observed that velocity of the fluid increases with an increase in Thermal Grashof number $(G_{r\theta})$ at steady time.

Figure 4.15 shows the effect of Thermal Grashof number $(G_{r\theta})$ on the temperature of the fluid $\theta(y,t)$. It is observed that the temperature of the fluid increases with an increase in Thermal Grashof number $(G_{r\theta})$ at steady time.

Figure 4.16 shows the effect of Thermal Grashof number $(G_{r\theta})$ on the concentration of the fluid $\phi(y,t)$. It is observed that concentration of the fluid does not change much with an increase in Thermal Grashof number $(G_{r\theta})$ at steady time.

Figure 4.17 shows the effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on the velocity of the fluid $u\left(y,t\right)$. It is observed that velocity of the fluid increases with an increase in Solutal Grashof number $\left(G_{r\phi}\right)$ at steady time.

Figure 4.18 shows the effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It is observed that the temperature of the fluid increases with an increase in Solutal Grashof number $\left(G_{r\phi}\right)$ at steady time.

Figure 4.19 shows the effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on the concentration of the fluid $\phi\left(y,t\right)$. It is observed that concentration of the fluid does not change much with an increase in Solutal Grashof number $\left(G_{r\phi}\right)$ at steady time. time $\left(t\right)$

Figure 4.20 shows the effect of time $\left(t\right)$ on the velocity of the fluid $u\left(y,t\right)$. It is observed that velocity of the fluid increases with an increase in time $\left(t\right)$.

Figure 4.21 shows the effect of time $\left(t\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It is observed that the temperature of the fluid increases with an increase in time $\left(t\right)$.

Figure 4.22 shows the effect of time $\left(t\right)$ on the concentration of the fluid $\phi\left(y,t\right)$. It is observed that concentration of the fluid increases with an increase in time $\left(t\right)$.

Table 4.1 shows that at the plate $\left(y=0\right)$ when the Eckert number $\left(E_c\right)$, Solutal Grashof number $\left(G_{r\phi}\right)$ and Grashof thermal number $\left(G_{r\theta}\right)$ increase, the skin friction $\left(CF_0\right)$ is increasing. The rate of skin friction (CF_0) decreases for increasing values of Reynold number (R_e), Hatmann number (H_a), Peclet number (P_e) and Peclet mass number (P_{em})

Table 4.2 shows that the rate of heat transfer at the plate $(y=0)$ increases for increasing values of Eckert number (E_c) Reynold number (R_e), Hatmann number (H_a), Grashof thermal $(G_{r\theta})$ and Solutal Grashof number $(G_{r\phi})$, but a reverse trend is observed for increasing values of Peclet number (P_e) and Peclet mass number (P_{em}).

Table 4.3 shows that the rate of mass transfer at the plate $(y=0)$ increases for increasing values of Peclet mass number (P_{em}), but a reverse trend is observed for increasing values of the Reynold number (R_e).

Table 4.4 on page 84 demonstrates agreement between the results obtained using harmonic solution technique and the results obtained by Rajput and Sahu (2011) using Laplace transform method. Generally, the difference is of order $10^{-2}$and 10^{-3}

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATION

5.1 Conclusion

A mathematical analysis has been carried out to model transient magnetohydrodynamic free convection flow between two long vertical parallel plates with viscous energy dissipation. The dimensionless governing coupled non-linear partial differential equations for this investigation were solved analytically using harmonic solution technique. The effects of the dimensionless parameters as shown on the graph were analyzed. It is concluded that:

(i) Peclet energy number, Hartman number., Peclet mass number, Reynold number reduce the velocity of the fluid.

(ii) Thermal Grashof number and Solutal Grashof number enhance the velocity of the fluid.

(iii) Peclet number reduces the temperature of the fluid.

(iv) Eckert number, Thermal Grashof number and Solutal Grashof number enhance the temperature of the fluid.

(v) Peclet mass number and Reynold number reduce the concentration of the fluid.

(vi) Eckert number $\left(E_c\right)$, Solutal Grashof number $\left(G_{r\phi}\right)$ and Grashof thermal number $\left(G_{r\theta}\right)$ increase the skin friction $\left(CF_0\right)$ at the plate $\left(y=0\right)$.

(vii) Reynold number $\left(R_e\right)$, Hatmann number $\left(H_a\right)$, Peclet energy number $\left(P_e\right)$ and Peclet mass number $\left(P_{em}\right)$ reduces the rate of skin friction $\left(CF_0\right)$ at the plate $\left(y=0\right)$.

(viii) Eckert number $\left(E_c\right)$ Reynold number $\left(R_e\right)$, Hatmann number $\left(H_a\right)$, Grashof thermal $\left(G_{r\theta}\right)$ and Solutal Grashof number $\left(G_{r\phi}\right)$ increases the rate of heat transfer at the plate $\left(y=0\right)$.

(ix) Peclet energy number $\left(P_e\right)$ and Peclet mass number $\left(P_{em}\right)$ reduces the rate of heat transfer at the plate $\left(y=0\right)$.

5.2 Recommendation

For further study, it is recommended that other analytical methods (separation of variables, polynomial approximation, method of lines and so on) can be used to analyse transient magnetohydrodynamic free convection flow between two long vertical parallel plates with viscous energy dissipation to ascertain how best the result can be obtained.

Magnetohydrodynamics (MHD) finds its application in meteorology, solar physics, geophysics and motion of the earth core. MHD free convection flow have also significant applications in the field of stellar and planetary magnetospheres, aeronautical plasma flows, chemical engineering and electronics. The need for study of transient magnetohydrodynamics free convectional flow with viscous energy dissipation also has it application in the efficiency of the devices used in industries and engineering.

5.3 Contribution to Knowledge

In this study, the following contribution was made to knowledge.

1. This present research work extends the work of Rajput and Sahu (2011) by incorporating viscous energy dissipation term in the heat process.
2. Transient magnetohydrodynamic free convention flow between two long vertical parallel plates with viscous energy dissipation was solved using harmonic solution technique.

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