DEVELOPMENT OF BLOCK UNIFICATION HYBRID LINEAR MULTISTEP METHODS FOR FLUID FLOW EQUATIONS

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (PhD) IN MATHEMATICS

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ABSTRACT

Many physical problems are modelled as differential equations which are either ordinary or partial. These equations require solutions that can be obtained analytically or by the use of numerical methods. For differential equations of higher order, it is almost impossible to obtain solutions analytically, thus the necessity for numerical techniques/methods. This difficulty is the motivation for this study. The study focus on formulation and development of block unification linear multi-step method for the numerical solution of fluid flow equations. with application to both initial and boundary value problems. For this purpose, a Chebyshev polynomials valid in interval [-1,1] and with respect to weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ was employed as basis function for the development of continuous hybrid schemes in a collocation and interpolation technique. In order to make the continuous methods self-starting, some block methods of discrete hybrid form were derived. The methods were analysed using appropriate existing theorems to investigate their consistency, zero-stability, convergence and the investigation shows that the developed methods were consistent, zero-stable and hence convergent. These methods were of order three, four and five; with minimal error constants of $C_{p+4} = 0.001541, C_{p+3} = -0.0006944, C_{p+2} = -0.001389$ respectively. The methods were implemented on fifteen (15) test problem from the literature to show the accuracy, efficiency and effectiveness of the methods. It is observed that the proposed methods have maximum error of 6.980 x 10^{-28} for the oscillatory problem from ship dynamics compared with maximum error of 2.846 x 10^{-7} obtained from Predictor-Corrector method found in the literature. Also for the purpose of comparison, it was observed that the results obtained from the developed methods were validated with Runge- Kutta method and some results in the existing methods which shows an excellent agreement.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Higher order ordinary differential equations are used in modelling problems arising from many physical problems. Such problems include electronic magnetic waves, the deflection of a curved beam having a constant or varying cross-section, gravity driven flows, draining coating flows, thin film flow, the motion of rocket, etc. As such, numerical methods which offer faster and accurate solutions are necessary.

Significant attention is being given to the solution of differential equations from some of the above physical problems and as such several methods have been developed over the years to solve differential equations resulting from such problems.

This research is aimed at solving boundary layer flow that results into ordinary differential equations of the form shown in equation (1.1) with appropriate initial and boundary value conditions with $\mu = 2,3,4$ and also fourth order partial differential equations of the form of equation (1.2)

$$y^{(\mu)} = f(x, y, y', \ldots, y^{(\mu-1)}) \tag{1.1}$$

$$y_{xxxx} = f(x, t, y_x, y_t, y_{xx}, y_{tt}) \tag{1.2}$$

Many numerical methods have been developed for solving third and fourth order ODEs. Some of these are the works of Jator *et al*. (2018), Ogunlaran and Oladejo, (2014), Mohammed and Adeniyi (2014), Adesanya *et al.* (2013), Tirmizi *et al.* (2004), Reutskiy *et al.* (2008), Li *et al.* (2012), Reddy (2016) and Adeyeye and Omar (2017).

Most common approaches for solving problems in the forms of equation (1.1) is by reducing them to systems of first order equations and applying any of the available methods to solve them. This requires more computational effort.

The Boundary Value Method (BVM) is an application of linear multistep methods. They are a class of linear multistep methods with step number k whose k additional conditions are not only imposed at the beginning but also at the end of the integration process. This makes them form a discrete analog of the continuous boundary problems. The boundary value technique simultaneously generates approximate solution $\left(x_1, x_2, \cdots, x_N\right)^T$ to the exact solution $\left(y(x_1), y(x_2), \cdots, y(x_N)^T\right)$ of equations (1.1) on the entire interval of integration. This approach has the advantage of producing smaller global errors (at the end of the range of integration) than those produced by the step-by-step methods due to the presence of accumulated errors at each step in the step-by-step method (Jator *et al*., 2018). Boundary Value Methods have been used over the years for the solution of first order and second order ordinary differential equations with either initial or boundary conditions. The higher order equation has to go through the reduction method mentioned above.

1.2 Statement of the Research Problem

Numerical methods reduce the solutions of mathematical problems to computations that can be performed manually or by the use of calculating machines. The quality of a numerical method depends, to an extent, on the accuracy of the method. Numerical methods nowadays do not seek for exact answers as exact answers are almost impossible to obtain in practice. Instead, getting approximate solutions is the major concern and this is done while maintaining reasonable bounds when it comes to errors (Lambert, 1973).

Taylor's series has been found to require the evaluation of higher order derivatives frequently even though it has an advantage of working well if the successive derivatives could be calculated easily and as well as giving a correct solution with significant digits of accuracy when the problem is written in variable separable form. Predictor-Corrector methods developed by Familua and Omole (2017) is an advantage of minimising truncation and round off errors because of the step by step method but were found to require larger storage spaces and were also found to be expensive to implement in terms of number of function evaluation per step.

Linear multistep methods (LMMs) also known as discrete linear multistep methods have been widely used for the numerical integration of first order initial value problems (IVPs). They are also used for solving higher order problems by transforming the IVPs into an equivalent system of first order IVPs. These methods suffered the disadvantage of requiring additional starting values and special procedures for changing step length h. Continuous Linear Multistep Methods (CLMMs) are however reported to have advantages over the discrete methods. Some of these are: an estimation of better global errors, their usage to recover standard schemes, provision of a simplified form of coefficients for further analytical work at different points and the guarantee of easy approximation of solutions at all interior points of the integration interval.

In order to improve the storage space, methods for the direct solution of the higher order IVPs have been proposed in Jato *et al*. (2018) implemented in the predictor-corrector mode, hybrid mode or block mode to take the advantage of generating numerical solutions simultaneously and overcoming the zero stability barrier.

In light of the aforementioned problems, this research seeks to develop continuous modified linear multistep methods applied as Block Unification Methods (BUMs) which will take care of the shortcomings of the mentioned methods while putting together the properties of the existing methods.

1.3 Aim and Objectives of the Study

1.3.1 Aim of the study

This work is aimed at developing block unification hybrid linear multistep methods for fluid flow equations.

1.3.2 Objectives of the study

The objectives of this research are to;

i) develop k-step continuous implicit methods with Chebyshev polynomials as basis functions using the interpolation and collocation approach.

ii) analyse basic properties of the developed methods such as order and error constants, consistency and convergence.

iii) test the accuracy and efficiency of the derived schemes by comparing the absolute error with those of recent existing methods found in the literature.

iv) validate the results of the proposed methods.

1.4 Justification for the Study

Many literature propose methods for solving higher order ordinary differential equations that require their being reduced to systems of first order. This could require more computational effort than the direct methods. The problems associated with this are non-economisation of computer time, computational burden and cost of implementation. In addition to these, it does not allow additional information associated with specific ODEs to be utilised. This research intends to propose a method that addresses some of these issues.

With the increased use of boundary value methods (note: with no off grid point) to solve differential equations and the successes being recorded, the research seeks to derive boundary value methods with off grid points to solve boundary layer flow.

1.5 Scope and Limitation of the Study

1.5.1 Scope of the Study

The research develops and applies block unification linear multistep methods for third and fourth order boundary layer flow and partial differential equations. It makes use of the chebyshev polynomials as the basis functions used in deriving the continuous linear multistep methods that are applied as boundary value methods. The collocation and interpolation approaches are applied in the derivation process. It takes into consideration boundary flow problems of third and fourth order ordinary differential equations and fourth order partial differential equations. The derived schemes include an off-grid point.

1.5.2 Limitation of the Study

The research does not consider the solution of other higher order ordinary differential equations (fifth order ODEs and above). As a result, boundary layer flow problems resulting from such differential equations were not considered. It applies the block unification method with one -off- grid point

1.6 Significance of the Study

Time consumption, large storage spaces, etc. are associated with numerical methods that have been developed over the years. The method of implementation of some these methods have led to some of the problems mentioned. This research will come up with a new numerical method implemented as block unification methods that directly solve higher order ordinary differential equations without the need of reducing them to systems of first order problem which is less time consuming to implement. The derived method can be applied to a partial differential equation of fourth order converted to ordinary differential equation.

1.7. Definition of Terms

Initial Value Problems: An nth order scalar initial value problem in ordinary differential equations consists of two parts; an ordinary differential equation which can be written in the form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

And the initial conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, \dots, y^{(n-1)}(a) = \alpha_{n-1}$$

where f is continuous in some open set $\Omega = (a,b)$ in the (x, y) plane, $a, b, \alpha_i, i = 0,1, \dots, n$ are real constants. The order of an ODE is the highest order derivative in the differential equation.

Boundary Value Problem: A boundary value problem is a problem, usually an ordinary differential equation or a partial differential equation, which has values assigned on the physical boundary of the domain in which the problem is specified. For example,

$$\frac{\partial^{(n)}u}{\partial x_n} - \Delta^n u = f \text{ in } \Omega$$

$$u(0,x) = u_1 \text{ on } \partial\Omega$$

$$\frac{\partial u}{\partial x}(0,x) = u_2 \text{ on } \partial\Omega$$

where Ω is the domain and $\partial\Omega$ is its boundary.

Collocation: This is a projection approach for solving differential equations in which the approximate solution is determined from the condition that the equation be satisfied at some chosen points. The points at which the differential equation is satisfied are called the collocation points.

Interpolation: This is the approach where a trial solution of an equation is evaluated as some selected points. These selected points are called interpolation points.

Step-size: Given the sequence of points $\{x_i\}$ in the interval $I = [a,b]$ defined by $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$ such that $h = x_i - x_{i-1}, i = 0,1,2,\ldots,N$. The parameter h is called the mesh size or step size while N is called the number of subintervals.

k-step Method: The most general k-step method takes the form

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^{\mu} \sum_{i=0}^{k} \beta_i f_{n+i}$$

Where $y_{n+i} \approx y(\mathrm{x}_{n+i}) \approx \mathrm{y}(\mathrm{x}_n + ih),\ f_{n+i} = f(x_{n+i}, y_{n+i}) = \mathrm{f}(\mathrm{x}_n + ih, y(x_n + ih))$ and is of implicit type unless $\beta_k = 0$, where it becomes explicit. The coefficients have been normalised so that the coefficient of y_{n+k} is $\alpha_k = 1; \mu$ is the order of the differential equation.

Chebyshev Polynomial of the First Kind: Chebyshev polynomial defined by $T_n(x)$ of the first kind defined on the interval [-1,1] is given by

$$T_n(x) = \cos(n\cos^{-1}x), n \geq 0$$

and satisfies the recurrence relation

$$T_0(x) = 1; T_1(x) = x; T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x); n \geq 1$$

In particular, $T_n(x)$ is indeed an algebraic polynomial of degree n with leading coefficient 2^{n-1}.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Existing Methods

Linear multistep methods (LMMs) are very popular for solving first order initial value problems and also higher order problems which are usually first reduced to systems of first order equations before the application of the LMMs. Many other numerical methods have been developed for solving ODEs.

Researchers like Lambert, (1973), Lie and Norselt, (1989), and Onumaniyi *et al*. (1994) have discussed the solution of ODEs in which Lie and Norselt, (1989) and Onumaniyi *et al*. (1994) made the traditional LMMs continuous through the idea of multistep collocation. The continuous linear multistep methods (CLMMs) just like the constant coefficients linear multistep methods were usually applied to initial value problems as a single formula. This application required the use of known critical points which were obtained by applying a certain kind of formulae known as predictors. The application of the predictor-corrector method requires more human effort and takes long computer time as it requires special technique to supply starting values.

A method known as hybrid was then developed. Hybrid methods came as a result of the desire to increase the order of a linear multistep method without increasing its step number and also without reducing its stability interval. This method retained the characteristics of linear multistep methods and has the property of utilising data at other points other than the step point $x_{n+j} = x_n + jh$. The method is useful in reducing the step number of a method while it remains zero stable.

The block method was proposed by researchers and the first of them was Milne who said they could only be used as a means of obtaining starting values for predictor-corrector methods. Later, the method proposed by Milne were developed to algorithms that were suitable for general use. This developed block method by Milne had been exploited by researchers both in its explicit and implicit form. The block method generates approximations simultaneously at different grid points within the interval of integration. The number of points is dependent on the structure of the block method (Mehrkanoon *et al.*, 2009). This method, compared to linear multistep methods and Runge-Kutta methods, is less expensive in terms of the number of function evaluations.

Block methods also permit easy change of step length. It has a feature of all discrete schemes being of the same order as the schemes get generated from a single continuous formula.

Researchers have applied the above methods to solving ODEs. Some of such researchers are Ogunlaran and Oladejo, (2014) who gave an approximate solution method for third-order multi-point BVPs, Mohammed and Adeniyi, (2014) who derived a three step implicit hybrid linear multistep method, Adesanya *et al.* (2013) developed a new hybrid block method for the solution of third order IVPs. A second-order method developed for the numerical solution of a non-linear, third-order, boundary value problem was given by Tirmizi *et al.* (2004). Jator (2008) derived a continuous linear multistep method which was used to generate multiple finite difference methods assembled in a single block matrix equation that was used to solve third order boundary value problems. An Accurate Five Off-Step Points Implicit Block Method for Direct Solution of Fourth Order Differential Equations was developed by Duromola (2016). Yap and Ismail (2015) derived a Block Hybrid Collocation Method with Application to Fourth Order Differential Equations. They used collocation and interpolation technique on basic polynomials to derive the main and additional methods that were combined and used as a block collocation method. Mohammed (2010) developed a six-step block method for the solution of fourth order ordinary differential equations using interpolation and collocation methods in the method derivation process.

The difference methods have also been applied to the solutions of ODEs by researchers. It involves the generation of methods by defining finite nodes within the interval in which the solution is required using uniform step length h and defining approximations after denoting approximations at the defined nodes. Some researchers that have exploited these methods are Pandey (2017) and Jator (2008). According to Pandey (2017), other researchers have also exploited methods like the Quantic Splines, Non Polynomial Spline, Quintic Splines, Collocation Quantic Splines, B-Splines and all with the application of Finite Difference Methods. Jator (2007) applied a family of Backward Differentiation Formulas (BDFs) to solve stiff second order initial value problems in ODE which was achieved by using the collocation and interpolation methods to develop Multiple Finite Difference Methods applied as BDFs. Noor and Al-Said (2002) developed a general finite difference method to solve a system of third order boundary value problems. Kasi-Viswanadham and Ballem (2015) also used the finite difference method involving Galerkin method and quartic B-splines as basis functions. Salama and Mansour (2005) presented a finite difference method, using four grid points, to solve third order boundary value problems. Jator (2009) presented Multiple Finite Difference Methods obtained from a linear multistep method of step 4, these were used to solve third order boundary value problems directly.

Legendre polynomials were used by Hossain and Islam (2014) as basis functions to extend a Galerkin method to solve fourth order BVPs. Massoun and Benzine (2018) applied homotopy analysis method to solve fourth order initial value problems by reducing them to an equivalent system of first order ordinary differential equations.

Other works by researchers involving ODEs are the Direct Integrators for the General Third Order Ordinary Differential Equation with an Application to Korteweg de-Vries (Jator *et al.*, 2018), Approximate Solution Method for Third-Order Multi-Point Boundary Value Problems (Ogunlaran and Oladejo, 2014). Mohammed *et al.* (2019c) solved third order boundary value problems in ordinary differential equations by using direct integrators of modified multistep method. Mohammed and Adeniyi (2014) derived a three-step implicit hybrid linear multistep method for the solution of third order ordinary differential equations. Mohammed *et al*. (2018) derived a three step continuous hybrid linear multistep method which was applied to solve third order ODEs. Adesanya *et al.* (2013) developed a new hybrid block method for the solution of general third order initial value problems of ordinary differential equations. A second-order method is developed for the numerical solution of a non-linear, third-order, boundary value problem (see Tirmizi *et al.*, 2004). Reutskiy *et al.* (2008) studied two-dimensional heat conduction problems and they used Chebyshev polynomials and trigonometric basis functions to approximate their equations for each time step. Li *et al.* (2012) presented an algorithm for directly solving third order mixed boundary value problems based on the reproducing kernel. Reddy (2016) used quantic B-splines as basis function and septic B-splines as weight function to derive a method through a Petrov-Galerkin method to solve fourth order ordinary differential equations. Adeyeye and Omar (2017) directly solved non-linear fourth order boundary value problems using a numerical approach that is (m+1)th step block method which was developed through a modified Taylor series approach with m being the order of the considered differential equation.

Fourth order ordinary differential equations also receive attention from researchers. A number of researchers have delved into the derivation of methods for the solution of these kinds of ODEs either directly or by reduction. Hussain *et al*. (2016) directly solved fourth order initial value problems. The constructed methods were a three stage fourth order Runge-Kutta method and a three stage fifth order method which they termed as RKFD using rooted trees. Results were compared with existing Runge Kutta Nystrom Method and Runge Kutta method. Kuboye *et al*. (2020) derived hybrid numerical algorithms that were implemented in block form to solve fourth order initial value problems directly. This they achieved using power series as basis function in the derivation of the method and also the interpolation and collocation methods. Also, Ogunlaran and Kehinde (2022) used the Hermite polynomials as basis function to derive a 4-step block method to solve fourth order IVPs. An implementation of a five-step numerical integrator in continuous block method can be found in Adesanya *et al*. (2012). The method was used to solve fourth order initial value problems. The method has the advantage of evaluation at all selected points within the interval of integration. Singh and Singh (2019) considered initial value problems of the fourth order in their work. Laplace Transform Method (LTM) was applied to the problems to derive the solutions to them. These solutions were compared with those from Adomian Decomposition Method and also the exact solutions in which the solutions from Laplace Transform Method were found to give more accurate and efficient solutions fared better. A direct solution of fourth order ODEs using a one-step hybrid block method can be found in Omar and Abdelrahim (2016). The one-step hybrid block method has three off step points. This direct application of the method reduced computational work.

Ndanusa *et al*. (2020) applied the Numerov Method to solve fourth order initial and boundary value problems by transforming the fourth order problem into a couple of second order equations. This is possible because of the missing first derivative term in the problem. Negligible errors were gotten after a comparison of the approximate solutions and the exact solutions. A use of quantic B-splines and a combination of singular value decomposition technique and least square approximation is found in Mushtt *et al*. (2020). This was applied to solve fourth order boundary value problem. Results were compared with those from exact solution and found out those from extended B-splines fared well.

A modified single-step method is proposed by Mohammed *et al.* (2022a) to integrate nonlinear dynamical systems resulting to ordinary differential equations. The higher order A-stable methods were obtained by imposing some special sets of off-grid points in the formulation process of the algorithms. Mohammed *et al.* (2022b) derived a two-step method for solving non-linear dynamical problems employing Bhaskara points as off grid points. Modebei *et al*. (2020b) derived a linear multistep hybrid block method with four off grid points to directly approximate the solution of fourth order linear and non-linear boundary value problems. The method was shown to be flexible and also can be used on fourth order differential equations with either Dirichlet or Neumann Boundary conditions. Filobello-Nino *et al.* (2014) directly solved equation from squeezing flow between two infinite parallel plates slowly approaching each other using a method derived by combining the standard homotopy perturbation method and laplace transform. Hemeda *et al*. (2017) solved a fractional form of unsteady squeeze flow through porous medium analytically by applying the Adomian Decomposition Method and The Picard Method. The Adomian Decomposition method was directly used to solve the fourth order boundary value problem from Beam-Column Theory by Kelesoglu (2014). Results from this that converge rapidly to the analytical solution can be found without changing the nature of the physical phenomena. The deflection of beam determined by the Euler

Bernoullli's equation was considered in Adak and Mandal (2021). The fourth order differential equation with specific boundary conditions (Neumann condition) was solved using the Finite Difference Method. Results showed that a reduction in mesh size helps the Finite Difference Method have better accuracy. Abolarin *et al*. (2020) developed a numerical method that solved second, third and fourth order ordinary differential equations. The schemes for the method were derived using the power series and the method was a two-step method with two off grid points. The methods were only applied to initial value problems. Results show that the method is effective in solving the second, third and fourth order ordinary differential equations.

Mustapha and Salau (2021) compared the Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM) against Finite Difference Method (FDM). This can be found in their paper titled Comparative Solution of Heat Transfer Analysis for Squeezing Flow Between Parallel Disks. The ADM was found to be more efficient and reliable than HPM though it requires more tedious computational effort. Atabo and Adee (2021) used interpolation and collocation to develop a 15-step block method with uniform higher order 16 to directly solve fourth order ordinary differential equations. Adegboye *et al* (2020) modified the fifth Runge-Kutta method for first order ODEs to directly integrate fourth order ODEs. The method was found to require less work as against the classical Runge-Kutta method.

2.2 Partial Differential Equations

Partial differential equations have been solved numerically over the years by the use of several methods that have been available. Higher order PDEs with Dirichlet or Neumann boundary conditions that arise in engineering and sciences have been solved using a wide range of numerical methods. For example, an approximate equation for a long and slender beam called the Euler-Bernoulli equation has its solution as a traverse displacement of the beam from an initially horizontal position. The fourth order parabolic PDE from this has been solved using the sextic spline method by Rashidinia and Mohammadi (2010). Mittal and Jain (2011) described a typical fourth order parabolic PDE as shown

$$y_{tt} + y_{xxxx} = G(x,t) \tag{2.1}$$

Problems that described the nonlinear wave phenomena were solved using the method of line in Saucez *et al*. (2014). The 'good' Boussinesq equation, described in Shokri and Dehghan (2010), is an example of such problem. The described equation is given as

$$y_{tt} = y_{xx} + q y_{xxxx} + y_{xx}^2 \tag{2.2}$$

Subject to appropriate boundary conditions where $q = 1$ or -1. Quintic B-splines for its numerical solution and B-splines methods with redefined basis functions can be found in Siddiqi and Arshed (2014) and Mittal and Jain (2011).

2.3 Collocation Method

Over the years, various researchers considered collocation methods as ways of generating numerical solutions to ordinary differential equations.

A collocation method is a method for the numerical solution of ordinary differential equations, partial differential equations and integral equations. The idea is to choose a finite-dimensional space of candidate solutions usually polynomials up to a certain degree, called trial functions or basis functions, and a number of points in the domain (called *collocation points*), and to determine that solution which satisfies the given equation at the collocation points.

In 1965, Lanczos introduced the standard collocation method with some selected points for the solution of ODEs. In recent times, researchers have employed the methods. Some of them are: Jator and Li, (2009) and Biala and Jator, (2017). Other examples could be seen in the works of Omar and Abdelrahim, (2016), Mohammed *et. al.* (2019a), Mohammed *et. al.* (2019b), Mohammed *et. al.* (2021), Ogunlaran and Oukouomi (2016) and Yap and Ismail (2015).

2.4 Boundary Value Methods

A boundary value method (BVM) is a technique of applying linear multistep methods. In their paper, Brugano and Trigante (1998) described the BVM as a third-way between Linear Multistep and Runge Kutta and also suggested a block version of it. Jator and Li, (2009) applied the BVM technique in which a main method is obtained (i.e. a linear multistep method) and other supporting methods. The main method is then used together with the initial method at n = 0 and also the final method. This they did by applying a combination of compact difference schemes and boundary value methods (Compact Boundary Value Method) to solve the two-dimensional Schrödinger Problem. Biala *et al.* (2017) and Jator *et al.* (2018) applied the technique to the direct solution of second order and third order boundary value problems respectively. More can be found in the works of Biala and Jator, (2015) and Biala and Jator, (2017).

2.5 Boundary Layer Flow

Boundary layer in fluid mechanics is the thin layer of a flowing gas or liquid in contact with a surface such as that of an airplane wing or of the inside of a pipe. The fluid in the boundary layer is subjected to shearing forces. A range of velocities exists across the boundary layer from maximum to zero, provided the fluid is in contact with the surface.

The flow in such boundary layers is generally laminar at the leading or upstream portion and turbulent in the trailing or downstream portion.

Researchers have developed methods for solving the Blasius equation numerically. Bougoffa and Wazwaz (2015) considered the nonlinear Blasius equation with initial boundary conditions. They used a direct method to obtain the exact solution. An approximate analytical solution which contained an auxiliary parameter was also obtained. The parameter makes it convenient for finding analytical solutions. An association of the solution of the Blasius equation with different boundary conditions with practical problems for a flat plate continuously shrinking with a constant velocity into a slot in a stationary fluid with mass transfer at the plate was given by Fang *et al*. (2008). For certain mass suction values, numerical results showed more than two solutions.

Researchers have written on numerical and analytical methods to solve problems related to fluid flow. Ran *et al*. (2009) reduced the governing equations of an axisymmetric Newtonian fluid squeezed between two parallel plates to a nonlinear differential equation and applied an explicit series solution through the homotopy analysis method to solve the equation analytically giving rise to a solution in series form. Inc and Akgul (2014) compared results obtained from solving the reduced fourth order nonlinear boundary value problem from a steady axisymmetric MHD flow of two-dimensional incompressible fluid by applying the reproducing kernel Hilbert space method (RKHSM) with results from Runge-Kutta method (RK-4) and optimal homotopy. The heat transfer analysis for Casson fluid (non-Newtonian) between parallel circular plates was considered by Khan *et al*. (2016). Conservation laws along with viable similarities transformations were used in obtaining the non-linear ordinary differential equations governing the flow. Analytical solution was obtained through the Homotopy Perturbation Method. RK-4 coupled with shooting method were used in obtaining the numerical solution. Numerical values for skin friction coefficient and Nusselt number were tabulated.

The Blasius equation is an equation that is derived from the description of stationary and incompressible flow in two dimensions, forming the boundary layer on a semi-infinite plate parallel to the flow. This is a non-linear third order differential equation.

Considering the Navier Stokes differential equations

$$\begin{cases} \dfrac{\partial U_x}{\partial x}+\dfrac{\partial U_y}{\partial y}=0 \\ U_x\dfrac{\partial U_x}{\partial x}+U_y\dfrac{\partial U_x}{\partial y}=-\dfrac{1}{p}\dfrac{\partial P}{\partial x}+v\left(\dfrac{\partial^2 U_x}{\partial x^2}+\dfrac{\partial^2 U_x}{\partial y^2}\right) \\ U_x\dfrac{\partial U_y}{\partial x}+U_y\dfrac{\partial U_y}{\partial y}=-\dfrac{1}{p}\dfrac{\partial P}{\partial y}+v\left(\dfrac{\partial^2 U_y}{\partial x^2}+\dfrac{\partial^2 U_y}{\partial y^2}\right) \end{cases} \tag{2.3}$$

Along with the boundary conditions

- No-slip condition at the surface: $U_x = U_y$ at $y = 0$
- Uniform flow at infinity $U_x \to U_y$ when $y \to \infty$

The equation above along with the boundary condition expresses a laminar boundary layer over a flat plate from which the Prandtl's equation can be deduced as

$$\begin{cases} \dfrac{\partial U_x}{\partial x}+\dfrac{\partial U_y}{\partial y}=0 \\ U_x\dfrac{\partial U_x}{\partial x}+U_y\dfrac{\partial U_x}{\partial y}=v\dfrac{\partial^2 U_x}{\partial y^2} \\ \dfrac{\partial P}{\partial y}=0 \end{cases} \tag{2.4}$$

Introducing and substituting the stream function Ψ defined by

$$\begin{cases} U_x = \dfrac{\partial \Psi}{\partial y} \\ U_y = -\dfrac{\partial \Psi}{\partial x} \end{cases} \tag{2.3}$$

Implies

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial^2 y} = v\frac{\partial^3 \Psi}{\partial y^3} \tag{2.5}$$

With the boundary conditions which are now

- For any x at $y = 0$, we have $\Psi = 0$ and $\dfrac{\partial \Psi}{\partial y} = 0$
- When $y \to \infty, \dfrac{\partial \Psi}{\partial y} \to U_x$

A dimensionless variable and its function are introduced:

$$\eta = \sqrt{\frac{U_\infty}{vx}} \cdot y \text{ and } \Psi = \sqrt{vU_\infty x}. f(\eta) \tag{2.6}$$

The transformations above lead to the Blasius equation

$$\frac{1}{2} f \cdot f'' + f''' = 0 \tag{2.7}$$

With the boundary conditions as

$$\begin{cases} f = f' = 0 \; at \; \eta = 0 \\ f' \to 1 \; as \; \eta \to \infty \end{cases} \tag{2.8}$$

According to Das and Mohammed (2016) an unsteady two dimensional squeezing flow of an incompressible viscous electrically conducting fluid between the infinite parallel plate were investigated. The coordinate system is chosen such that x - axis is along the plate and y - axis normal to the plate. The two plates are placed at y= ±h(t) where $h(t)=H(1-\alpha t)^{1/2}$ and α is a characteristic parameter having dimensions of time inverse. The two plates are squeezed with a velocity v(t)= dh/dt until they touch. A uniform magnetic field of strength (t) = $B_0(1-\alpha t)^{-1/2}$ is applied perpendicular to the plate, and the electric field is taken as zero. The magnetic Reynolds number is assumed to be small so that the induced magnetic field can be neglected. The fluid considered is a gray, absorbing emitting radiation but non-scattering medium and the Rosseland approximation is used to describe the radiative heat flux in the energy equation. The fluid structure is everywhere in local thermodynamic equilibrium. The plate is maintained at a constant temperature. Also, it is assumed that there exists a homogeneous first order chemical reaction with time dependent reaction rate $K_1(t)=k_1(1-\alpha t)^{-1}$ between the diffusing species and the fluid. Here the symmetric nature of the flow is adopted.

Under the stated assumptions, the governing conservation equations of mass, momentum, energy and mass transfer at unsteady state can be expressed as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B^2(t)}{\rho}u \tag{2.10}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{2.11}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho C_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\nu}{C_p}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]-\frac{1}{\rho C_p}\frac{\partial q_r}{\partial y} \tag{2.12}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-K_1(t)C \tag{2.13}$$

where u and v are velocity components along x and y-axis respectively, ρ is the density of the fluid, υ is the kinematic viscosity, C_p is the specific heat at constant pressure p, κ is the thermal conductivity of the medium, T is the temperature of the fluid, C is the concentration of the solute, D is the molecular diffusivity. Following the Rosseland approximation with the radiative heat flux q_r is modeled as,

$$q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} \tag{2.14}$$

where σ^* is the Stefan-Boltzmann constant and k^* is the mean absorption coefficient. Assuming that the differences in temperature within the flow are such that T^4 can be expressed as a linear combination of the temperature, we expand T^4 in Taylor's series about T_∞ and neglecting higher order terms, we get

$$T^4=4T_\infty{}^3T-3T_\infty{}^4 \tag{2.15}$$

Thus, we have

$$\frac{\partial q_r}{\partial y}=-\frac{16T_\infty{}^3\sigma^*}{3k^*}\frac{\partial^2 T}{\partial y^2} \tag{2.16}$$

Therefore, Eq. (2.12) reduces to

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho C_p}\left[\frac{\partial^2 T}{\partial x^2}+\left(1+\frac{16T_\infty{}^3\sigma^*}{3k^*}\right)\frac{\partial^2 T}{\partial y^2}\right]+\frac{\nu}{C_p}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right] \tag{2.17}$$

The appropriate initial and boundary conditions for the problem are

$$\left.\begin{aligned} &u=0, v=v_w=\frac{dh}{dt}, T=T_H, C=C_H \text{ at } y=h(t), \\ &v=\frac{\partial u}{\partial y}=\frac{\partial T}{\partial y}=\frac{\partial C}{\partial y}=0 \text{ at } y=0 \end{aligned}\right\} \tag{2.18}$$

Using the following dimensionless quantities

$$\left.\begin{aligned} &\eta=\frac{y}{H\sqrt{1-\alpha t}}, \ \mathrm{u}=\frac{\alpha x}{2(1-\alpha t)} f'(\eta), \quad \mathrm{v}= -\frac{\alpha H}{2\sqrt{1-\alpha t}} f(\eta), \\ &\theta=\frac{T}{T_H}, \ \phi=\frac{C}{C_H}, \end{aligned}\right\} \tag{2.19}$$

into Eqs. (2.10) and (2.11) and then eliminating the pressure gradient from the resulting equations, one may finally obtain

$$f'''' - S\left(\eta f''' + 3f'' + ff' - f' f'''\right) - M^2 f'' = 0 \tag{2.20}$$

Now Eqs. (2.17) and (2.13) take the following forms:

$$(1+N_r)\theta'' + \Pr S\left(f\theta' - \eta\theta'\right) + \Pr Ec\left(\mathrm{f}''^2 + 4\delta^2 f'^2\right) = 0, \tag{2.21}$$

$$\phi'' + ScS(\mathrm{f}\,\phi' - \eta\phi') - Sc\gamma\phi = 0, \tag{2.22}$$

with the associated boundary conditions

$$\left.\begin{aligned} &f=0, \ f''=0, \ \theta'=0, \ \phi'=0 \text{ for } \eta=0 \\ &f=1, \ f'=0, \ \theta=1, \ \phi=1 \text{ for } \eta=1 \end{aligned}\right\} \tag{2.23}$$

2.6 Method of Lines

To apply Block Unification Methods or Boundary Value Methods to Partial Differential Equations, these PDEs need to be converted to system of Ordinary Differential Equations using method of Lines. These PDEs have to be well posed as an initial value problem (Cauchy) in at least one dimension ruling out purely elliptic equations like Laplace's equation but leaving a large class of evolution equations that can be solved quite efficiently. The method of lines converts the PDEs into ODEs by replacing the derivatives with their finite, difference approximations (Vigo-Aguiar and Ramos, 2007). The process of converting (1.2) into a system of ODEs will be demonstrated following Biala *et al.* (2017). For real numbers $L_1, L_2, L_3, L_4,$ solution $y(x,t)$ of (1.2) where (x,t) is in the rectangle $[L_1, L_2] \times [L_3, L_4]$, the t variable is discretised with mesh spacing $\Delta t = \dfrac{L_4 - L_3}{M}, \mathrm{t}_m = L_3 + m\Delta t, m = 0,1,\ldots, M, \Delta x = \dfrac{L_2 - L_1}{N}, x_n = L_1 + n\Delta x, n = 0,1,\ldots,\mathrm{N}$ with the vector $\bar{y} = \left[y_{1,1}, \mathrm{y}_{1,2}, y_{2,1}, \ldots, \mathrm{y}_{n-1,m-1} \right]^T$ and $\bar{G} = \left[G_{1,1}, G_{1,2}, G_{2,1}, \ldots, G_{n-1,m-1} \right]^T$ where

$y_m \approx y(x, t_m)$ and $G_m \approx G(x, t_m)$.

2.6.1 Central difference approximations

The best two-point formulae involving abscissae that are chosen symmetrically on both sides of x on the real line is applied when a function $u(x)$ can be evaluated at values to the left and right of x.

Theorem 2.1 Centred Formula of Order $O(h^2)$. *Assume that* $u \in C^3[a,b]$ *and that* $x-h, x, x+h \in [a,b]$, *then*

$$u'(x) \approx \frac{u_{n+1} - u_{n-1}}{2h} \qquad (2.24)$$

Furthermore, there exists a number $\xi = \xi(x) \in [a,b]$ *such that*

$$u'(x) = \frac{u_{n+1} - u_{n-1}}{2h} + T(u,h)$$

Where

$$T(u,h) = -\frac{h^2 u'''(\xi)}{6} = O(h^2)$$

The term $T(u,h)$ *is called the truncation error,* $u_{n+1} = u(x_n + h), u_{n-1} = u(x_n - h)$

By similar derivation of (2.24), $u''(x)$ has the formula

$$u''(x) \approx \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}$$

For a PDE where $u = u_{n,m} = un(x_n, t_m)$, the central difference method by discretising the space variable t becomes

$$\frac{\partial u}{\partial t} = u_t(x, t_m) \approx \frac{u(x, t_{m+1}) - u(x, t_{m-1})}{2h} \tag{2.25}$$

$$\frac{\partial u^2}{\partial t^2} = u_t(x, t_m) \approx \frac{u(x, t_{m+1}) - 2u(x, t_m) + u(x, t_{m-1})}{h^2} \tag{2.26}$$

The formulae (2.25) and (2.26) are called semi-discretisation of the dependent variable u.

Using the central difference approximation (2.26)

$$y_{tt}(x, t_m) \approx \frac{y(x, t_{m+1}) - 2y(x, t_m) + y(x, t_{m-1})}{(\Delta^2 t)}$$

Then (1.2) has the following semi-discretised form

$$\frac{dy_m^4}{dx^2} = -\left(\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2}\right) + G_m \tag{2.27}$$

which can be written in the form

$$y^{(iv)} = f(x, y) \tag{2.28}$$

Subject to the appropriate initial and boundary conditions, where $f(x, y) = Ay + g$ and A is an $(M-1)\times(M-1)$ matrix arising from the semi discretised system (2.28).

CHAPTER THREE

3.0 MATERIALS AND METHOD

3.1 Materials

3.1.1. System Requirements

MAPLE15 software or above on a system with a RAM of a minimum of 512mb space, 1GHZ Processor, a hard disk space of 512MB with basic input and output devices (eg, Mouse, keyboard, etc.) and uninterrupted power supply.

3.1.2 Chebyshev orthogonal polynomial basis function

Two functions are said to be orthogonal to one another if their inner product is zero ,hence a family of functions forms an orthogonal system on an interval (a,b) with weight function $\omega(x)$ if for any two distinct members of the family the following holds

$$< \varphi_1 \varphi_2 >= \int_a^b \varphi_1(x)\varphi_2(x)\omega(x)dx = 0 \tag{3.1}$$

The special orthogonal polynomial known as Chebyshev polynomial $\varphi_1(x)$ over the interval (1,1) was adopted with respect to the weight function $\omega(x) = \dfrac{1}{\sqrt{1-x^2}}$ as

$$\varphi_1(x) = \sum_{i=0}^{n} C_i^n x^i \tag{3.2}$$

The first seven Chebyshev polynomials of the first kind are as follows

$$\left.\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2-1\\
T_3(x) &= 4x^3-3x\\
T_4(x) &= 8x^4-8x^2+1\\
T_5(x) &= 16x^5-20x^3+5x\\
T_6(x) &= 32x^6-48x^4+18x^2-1\\
T_7(x) &= 64x^7-112x^5+56x^3-7x
\end{aligned}\right\} \tag{3.3}$$

The recursive formula is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{3.4}$$

3.2 Methods

In this section, the construction of the continuous linear multistep methods via the interpolation and collocation approach is discussed, which will be used to produce several discrete schemes for solving equations (1.1) and (1.2).

Algorithm 1:

Step 1: Construct the continuous linear multistep method (CLMM) with continuous coefficients as:

$$U(x) = \sum_{i=0}^{v} \alpha_i(x)\,\mathrm{y}_{n+i} + h^{\mu}\left(\sum_{j=0}^{k} \beta_j(x)\, f_{n+j} + \beta_w(x)\, f_{n+w}\right) \tag{3.5}$$

Where $v = \begin{cases} \dfrac{k}{2} & for\ \ even\, k \\ \dfrac{k+1}{2} & for\ \ odd\ \ k \end{cases}$

$\alpha_i(x), \beta_j(\mathrm{x}), \beta_w(\mathrm{x})$ are continuous coefficients and v is chosen to be half the step number so that each formula is derived from equation (3.5) satisfies the root condition.

Step 2: Obtain the main and additional methods by evaluating equation (3.5) in step 1 at x_{n+j} where $j=1(1)2v, j\neq v-1(1)v$ for equation (3.5) to obtain the formulas of the following forms:

$$y_{n+j}+\sum_{i=0}^{v-1}\alpha_i y_{n+i}=h^{\mu}\sum_{i=o}^{k}\beta_i f_{n+i}+h^{\mu}\beta_w f_{n+w} \tag{3.6}$$

Step 3: Obtain the first derivative for $\mu=2, k=3$, first and second derivative formulas for $\mu=3, k=3$ and the first, second and third derivatives for $\mu=4, k=4$ from equation (3.5), which are used to generate additional methods by evaluating $U'(x)$, $U'(x)$ and $U''(x)$, and $U'(x)$, $U''(x)$ and $U'''(x)$ of (3.5) at x_{n+j}, $j=0(1)k$ as given with *m* being the derivative

$$U^{(m)}(x)=\frac{1}{h^{(m)}}\left(\sum_{i=0}^{v}\alpha_i^{(m)}(\mathrm{x})\,\mathrm{y}_{n+i}+h^{\mu}\sum_{i=0}^{k}\beta_i^{(m)}(x)\,f_{n+i}+h^{\mu}\beta_w^{(m)}(x)\,f_{n+w},\right) \tag{3.7}$$

by imposing that

$U^{(m)}(a)=y_0^{(m)}, U^{(m)}(b)=y_N^{(m)}$ for equation (3.5)

Step 4: Combine the schemes obtained in steps 2 and 3 above for equation (3.5) to form a system of equations with form equivalent to $Ax=B$ where

$x=(M_0, M_1, M_2, \ldots, M_{N-1})^T$ and

$M_0=(y_0, y_1, y_2, y_3)^T$, $M_1=(y_0', y_1', y_2', y_3')^T$, $M_2=(y_0'', y_1'', y_2'', y_3''),\ldots$

Step 5: Adopt matrix inversion algorithm to the system of equations in step 4 to obtain the values of the unknowns in the expected block method.

Theorem 3.1 *Let* $T_j(x), j = 0(1)(k+\rho)$ *be the Chebyshev Polynomial used as basis function where* ρ is the number of required step *and W a vector given by* $W = (y_n, y_{n+v-1}, y_{n+v}, f_n, f_{n+1}, ..., f_k)^T$ *. Consider the matrix V defined as*

$$V = \begin{pmatrix} T_0(x_n) & T_1(x_n) & \dots & T_{k+\rho}(x_n) \\ T_0(x_{n+v-1}) & T_1(x_{n+v-1}) & \dots & T_{k+\rho}(x_{n+v-1}) \\ \vdots & \vdots & \dots & \vdots \\ T_0^{(m)}(x_n) & T_1^{(m)}(x_n) & \dots & T_{k+\rho}^{(m)}(x_n) \\ T_0^{(m)}(x_{n+1}) & T_1^{(m)}(x_{n+1}) & \dots & T_{k+\rho}^{(m)}(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots \\ T_0^{(m)}(x_{n+k}) & T_1^{(m)}(x_{n+k}) & \dots & T_{k+\rho}^{(m)}(x_{n+\rho}) \end{pmatrix} \tag{3.8}$$

and obtained by replacing the jth column of V by the vector W and let equation (3.5) satisfy

$U(x_{n+j}) = y_{n+j}$ z and $j = 0, v-2, v-1, v$

$U^{(m)}(x_{n+j}) = f_{n+j} \qquad j = 0(1)k$

$$U^{(m)}(x_{n+j}) = f_{n+j} \qquad j = 0(1)k \tag{3.9}$$

then the continuous representation, equation (3.5) is equivalent to

$$U(x) = \sum_{j=0}^{k+\rho} \frac{\det(V_j)}{\det(V)} T_j(x) \tag{3.10}$$

respectively

Proof: The basis function for equation (3.5) is taken as

$$\begin{cases} \alpha_j(x) = \sum_{i=0}^{k+\rho} \alpha_{i+1,j} T_i(x), & j = 0, v-1(1)v \\ h^{\mu}\beta_j(x) = \sum_{i=0}^{k+\rho} h^{\mu}\beta_{i+1.j} T_i(x), & j = 0(1)k \end{cases} \tag{3.11}$$

where $\alpha_{i+1,j}, h^{\mu}\beta_{i+1,j}$ are coefficients to be determined.

Inserting equation (3.11) into equation (3.5) gives

$$U(x) = \sum_{i=0}^{k+\rho} \alpha_{i+1,v} T_i(x) y_{n+v} + \sum_{i=0}^{k+\rho} \alpha_{i+1,v-1} T_i(x) y_{n+v-1} + \ldots + \sum_{i=0}^{k+\rho} \alpha_{i+1,0} T_i(x) y_n +$$
$$h^{\mu} \sum_{j=0}^{k} \sum_{i=0}^{k+\rho} \beta_{i+1,j} T_i(x) f_{n+j} + h^{\mu} \sum_{i=1}^{k+\rho} \beta_{i+1,w} T_i(x) f_{n+w},$$

Simplified to

$$U(x) = \sum_{i=0}^{k+\rho} \left\{ \alpha_{i+1,v} y_{n+v} + \alpha_{i+1,v-1} y_{n+v-1} + \ldots + \alpha_{i+1,0} y_n + \sum_{j=0}^{k} h^{\mu}\beta_{i+1,j} f_{n+j} + h^{\mu}\beta_{i+1,w} f_{n+w} \right\} T_i(x),$$

expressed in the form

$$U(x) = \sum_{i=0}^{k+\rho} \eta_i T_i(x) \tag{3.12}$$

Imposing conditions in equation (3.9) on equation (3.12) a system of (k+υ) [where $\upsilon = 4, 5, 6$ respectively for $\mu = 2, 3, 4$] equations respectively are obtained which could be expressed in the form $VH = W$ where

$H = (\eta_0, \eta_1, \eta_2, \ldots, \eta_{k+\rho})^T$ are vectors of (k+υ) undetermined coefficients.

The elements of H are found using the Cramer's rule

$$\eta_i = \frac{\det(V_j)}{\det(V)}, \; j = 0(1)(k+\rho) \tag{3.13}$$

where V_j is obtained by replacing the jth column of V by W. Using the newly found elements of H, equation (3.12) is re-written as

$$U(x)=\sum_{i=0}^{k+\rho}\frac{\det(V_j)}{\det(V)}T_i(x) \tag{3.14}$$

3.3 Specification of Methods

3.3.1 Development of block unification method for $k=3, \mu=3$

To derive an implicit three step method with one off-grid point, the following specifications were considered r = 3, s= 5, k = 3, v = $\frac{7}{3}$, to give the continuous form as:

$$y(x)=\alpha_0 y_n+\alpha_1 y_{n+1}+\alpha_2 y_{n+2}+h^3[\beta_0 f_n+\beta_1 f_{n+1}+\beta_2 f_{n+2}+\beta_{\frac{7}{3}} f_{n+\frac{7}{3}}+\beta_3 f_{n+3}] \tag{3.15}$$

Where the α_i's and β_i's are continuous coefficients expressed as functions of x and h as follows

$$\left.\begin{aligned}
\alpha_0 &= \frac{1}{4}\frac{4h^2+1}{h^2}-\frac{3}{2}\frac{t}{h}+\frac{1}{4}\frac{2t^2-1}{h^2}\\
\alpha_1 &= -\frac{1}{2h^2}+\frac{2t}{h}-\frac{1}{2}\frac{2t^2-1}{h^2}\\
\alpha_2 &= \frac{1}{4h^2}-\frac{1}{2}\frac{t}{h}+\frac{1}{4}\frac{2t^2-1}{h^2}\\
\beta_0 &= \frac{1}{564480}\frac{1}{h^4}\left(192t^7-2800t^6h+16800t^5h^2-53200t^4h^3+94080t^3h^4-87136t^2h^5+32064th^6+70560a+10500b+105c\right)\\
\beta_1 &= -\frac{1}{107520}\frac{192t^7-2464t^6h+11872t^5h^2-23520t^4h^3+50848t^2h^5-36928th^6+7420b+105c}{h^4}\\
\beta_2 &= \frac{1}{26880}\frac{192t^7-2128t^6h+8288t^5h^2-11760t^4h^3+11872t^2h^5-6464th^6+5180b+105c}{h^4}\\
\beta_{\frac{7}{3}} &= -\frac{27}{250880}\frac{64t^7-672t^6h+2464t^5h^2-3360t^4h^3+3360t^2h^5-1856th^6+1540b+35c}{h^4}\\
\beta_3 &= \frac{1}{161280}\frac{192x^7-1792t^6h+6048t^5h^2-7480t^4h^3+7616t^2h^5-4224th^6+3780b+105c}{h^4}
\end{aligned}\right\} \tag{3.16}$$

where $t=x-x_n$, $a=th^4-h^4x+h^4x_n$, $b=th^2-h^2x+h^2x_n$ and $c=t-x+x_n$

Evaluating equation (3.15) at points $x = x_{n+3}, x = x_{n+\frac{7}{3}}$ gives

$$y_{n+3} = y_n - 3y_{n+1} + 3y_{n+2} + \frac{1}{140}h^3 f_n + \frac{37}{80}h^3 f_{n+1} + \frac{13}{20}h^3 f_{n+2} - \frac{81}{560}h^3 f_{n+\frac{7}{3}} + \frac{1}{40}h^3 f_{n+3} \tag{3.17}$$

$$y_{n+\frac{7}{3}} = \frac{2}{9}y_n - \frac{7}{9}y_{n+1} + \frac{14}{9}y_{n+2} + \frac{137}{76545}h^3 f_n + \frac{2911}{29160}h^3 f_{n+1} + \frac{139}{1215}h^3 f_{n+2} - \frac{1067}{22680}h^3 f_{n+\frac{7}{3}} + \frac{169}{43740}h^3 f_{n+3} \tag{3.18}$$

For $n = 0(3)(N-3)$

The first derivative formulae are:

$$hy'_n = -\frac{3}{2}y_n + 2y_{n+1} - \frac{1}{2}y_{n+2} + \frac{167}{2940}h^3 f_n + \frac{577}{1680}h^3 f_{n+1} - \frac{101}{420}h^3 f_{n+2} + \frac{783}{3920}h^3 f_{n+\frac{7}{3}} - \frac{11}{420}h^3 f_{n+3} \tag{3.19}$$

$$hy'_{n+1} = -\frac{1}{2}y_n + \frac{1}{2}y_{n+2} - \frac{11}{1470}h^3 f_n - \frac{173}{1120}h^3 f_{n+1} + \frac{1}{105}h^3 f_{n+2} - \frac{27}{1568}h^3 f_{n+\frac{7}{3}} + \frac{1}{336}h^3 f_{n+3} \tag{3.20}$$

$$hy'_{n+2} = \frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} + \frac{13}{2940}h^3 f_n + \frac{367}{1680}h^3 f_{n+1} + \frac{27}{140}h^3 f_{n+2} - \frac{351}{3920}h^3 f_{n+\frac{7}{3}} + \frac{1}{140}h^3 f_{n+3} \tag{3.21}$$

$$hy'_{n+\frac{7}{3}} = \frac{5}{6}y_n - \frac{8}{3}y_{n+1} + \frac{11}{6}y_{n+2} + \frac{680}{107163}h^3 f_n + \frac{310459}{816480}h^3 f_{n+1} + \frac{25679}{51030}h^3 f_{n+2} - \frac{38813}{211680}h^3 f_{n+\frac{7}{3}} + \frac{3865}{244944}h^3 f_{n+3} \tag{3.22}$$

$$hy'_{n+3} = \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} + \frac{9}{980}h^3 f_n + \frac{2393}{3360}h^3 f_{n+1} + \frac{89}{84}h^3 f_{n+2} - \frac{27}{1568}h^3 f_{n+\frac{7}{3}} + \frac{39}{560}h^3 f_{n+3} \tag{3.23}$$

for $n = 0(3)(N-3)$

And the second derivative formulae are:

$$h^2 y''_n = y_n - 2y_{n+1} + y_{n+2} - \frac{389}{1260}h^3 f_n - \frac{227}{240}h^3 f_{n+1} + \frac{53}{60}h^3 f_{n+2} - \frac{81}{112}h^3 f_{n+\frac{7}{3}} + \frac{17}{180}h^3 f_{n+3} \tag{3.24}$$

$$h^2 y''_{n+1} = y_n - 2y_{n+1} + y_{n+2} + \frac{53}{2520}h^3 f_n + \frac{1}{12}h^3 f_{n+1} - \frac{11}{40}h^3 f_{n+2} + \frac{27}{140}h^3 f_{n+\frac{7}{3}} - \frac{1}{45}h^3 f_{n+3} \tag{3.25}$$

$$h^2 y''_{n+2} = y_n - 2y_{n+1} + y_{n+2} + \frac{1}{180}h^3 f_n + \frac{39}{80}h^3 f_{n+1} + \frac{49}{60}h^3 f_{n+2} - \frac{27}{80}h^3 f_{n+\frac{7}{3}} + \frac{1}{36}h^3 f_{n+3} \tag{3.26}$$

$$h^2 y''_{n+\frac{7}{3}} = y_n - 2y_{n+1} + y_{n+2} + \frac{407}{68040}h^3 f_n + \frac{29}{60}h^3 f_{n+1} + \frac{641}{648}h^3 f_{n+2} - \frac{71}{420}h^3 f_{n+\frac{7}{3}} + \frac{29}{1215}h^3 f_{n+3} \tag{3.27}$$

$$h^2 y''_{n+3} = y_n - 2y_{n+1} + y_{n+2} + \frac{1}{540}h^3 f_n + \frac{31}{60}h^3 f_{n+1} + \frac{79}{120}h^3 f_{n+2} + \frac{81}{140}h^3 f_{n+\frac{7}{3}} + \frac{11}{45}h^3 f_{n+3} \tag{3.28}$$

for $n = 0(3)(N-3)$

Remark 3.3.1. *The equations (3.17) to (3.28) together form the Block Unification Method (BUM) used to solve equation (1.1).*

3.3.2 Development of block unification method for $k = 3, \mu = 2$

To derive the accompanying implicit three step method for second order ordinary differential equations with one off-grid point, the following specifications were considered, r = 2, s= 5, k = 3, v = $\frac{7}{3}$, to give the continuous form as:

$$s(x) = \alpha_1 s_{n+1} + \alpha_2 s_{n+2} + h^2[\beta_0 m_n + \beta_1 m_{n+1} + \beta_2 m_{n+2} + \beta_{\frac{7}{3}} m_{n+\frac{7}{3}} + \beta_3 m_{n+3}] \tag{3.29}$$

Where the α_i 's and β_i 's are continuous coefficients expressed as functions of x and h as follows

$$
\left.
\begin{aligned}
\alpha_1 &= -\frac{t}{h}+2 \\
\alpha_2 &= \frac{t}{h}-1 \\
\beta_0 &= \frac{1}{20160}\frac{48t^6-600t^5h+3000t^4h^2-7600t^3h^3+10080t^2h^4-6464th^5+1536h^6-5700a-375b}{h^4} \\
\beta_1 &= -\frac{1}{1920}\frac{24t^6-264t^5h+1060t^4h^2-1680t^3h^3+2532th^5-1672h^6-1260a-165b}{h^4} \\
\beta_2 &= \frac{1}{960}\frac{t^6-456t^5h+1480t^4h^2-1680t^3h^3+672th^5-64h^6-1260a-285b}{h^4} \\
\beta_{\frac{7}{3}} &= -\frac{27}{4480}\frac{8t^6-72t^5h+220t^4h^2-240t^3h^3+108th^5-24h^6-180a-45b}{h^4} \\
\beta_3 &= \frac{1}{720}\frac{6t^6-48t^5h+135t^4h^2-140t^3h^3+65th^5-18h^6-105a-30b}{h^4}
\end{aligned}
\right\} \tag{3.30}
$$

where $t = x - x_n$, $a = th^3 - h^3x + h^3x_n$ and $b = th - hx + hx_n$

Evaluating equation (3.29) at points $x = x_{n+3}, x = x_{n+\frac{7}{3}}, x = x_n$ gives

$$s_{n+\frac{7}{3}} = -\frac{1}{3}s_{n+1} + \frac{4}{3}s_{n+2} - \frac{313}{153090}h^2m_n + \frac{1093}{29160}h^2m_{n+1} + \frac{1817}{7290}h^2m_{n+2} - \frac{527}{7560}h^2m_{n+\frac{7}{3}} + \frac{317}{43740}h^2m_{n+3} \tag{3.31}$$

$$s_{n+3} = -s_{n+1} + 2s_{n+2} - \frac{1}{140}h^2m_n + \frac{29}{240}h^2m_{n+1} + \frac{41}{60}h^2m_{n+2} + \frac{81}{560}h^2m_{n+\frac{7}{3}} + \frac{7}{120}h^2m_{n+3} \tag{3.32}$$

$$s_n = 2s_{n+1} - s_{n+2} + \frac{8}{105}h^2m_n + \frac{209}{240}h^2m_{n+1} - \frac{1}{15}h^2m_{n+2} + \frac{81}{560}h^2m_{n+\frac{7}{3}} - \frac{1}{40}h^2m_{n+3} \tag{3.33}$$

For $n = 0(3)(N-3)$

The first derivative formulae are:

$$hs'_n = -s_{n+1} + s_{n+2} - \frac{101}{315}h^2m_n - \frac{211}{160}h^2m_{n+1} + \frac{7}{10}h^2m_{n+2} - \frac{729}{1120}h^2m_{n+\frac{7}{3}} + \frac{13}{144}h^2m_{n+3} \tag{3.34}$$

$$hs'_{n+1} = -s_{n+1} + s_{n+2} + \frac{23}{2520}h^2m_n - \frac{139}{480}h^2m_{n+1} - \frac{11}{24}h^2m_{n+2} + \frac{297}{1120}h^2m_{n+\frac{7}{3}} - \frac{19}{720}h^2m_{n+3} \tag{3.35}$$

$$hs'_{n+2}=-s_{n+1}+s_{n+2}-\frac{2}{315}h^2m_n+\frac{11}{96}h^2m_{n+1}+\frac{19}{30}h^2m_{n+2}-\frac{297}{1120}h^2m_{n+\frac{7}{3}}+\frac{17}{720}h^2m_{n+3} \tag{3.36}$$

$$hs'_{n+\frac{7}{3}}=-s_{n+1}+s_{n+2}-\frac{403}{68040}h^2m_n+\frac{53}{480}h^2m_{n+1}+\frac{2611}{3240}h^2m_{n+2}-\frac{65}{672}h^2m_{n+\frac{7}{3}}+\frac{383}{19440}h^2m_{n+3} \tag{3.37}$$

$$hs'_{n+3}=-s_{n+1}+s_{n+2}-\frac{5}{504}h^2m_n+\frac{23}{10}h^2m_{n+1}+\frac{19}{40}h^2m_{n+2}+\frac{729}{1120}h^2m_{n+\frac{7}{3}}+\frac{173}{720}h^2m_{n+3} \tag{3.38}$$

for $n=0(3)(N-3)$

Remark 3.3.2. *The equations (3.31) to (3.38) together form the Block Unification Method (BUM) used to solve* equation *(1.1).*

3.3.3 Development of block unification method $k=4,\mu=4$

To derive an implicit four step method with one off grid point, the following specifications were considered $r=4,s=6,k=4,v=\frac{10}{3}$, to give the continuous form as:

$$\mathrm{u}(x)=\alpha_0u_n+\alpha_1u_{n+1}+\alpha_2u_{n+2}+\alpha_3u_{n+3}+h^4\left(\beta_0b_n+\beta_1b_{n+1}+\beta_2b_{n+2}+\beta_3b_{n+3}+\beta_{\frac{10}{3}}b_{\frac{10}{3}}+\beta_4b_{n+4}\right) \tag{3.39}$$

Evaluating equation (3.39) at the points $x=x_{n+4},x=x_{n+\frac{10}{3}}$ gives

$$u_{n+4}=-u_n+4u_{n+1}-6u_{n+2}+4u_{n+3}+\frac{982}{150885}h^4b_n+\frac{81623}{603540}h^4b_{n+1}+\frac{593479}{804720}h^4b_{n+2}-\frac{31}{86220}h^4b_{n+3}+\frac{7533}{53648}h^4b_{n+\frac{10}{3}}-\frac{23299}{1207080}h^4b_{n+4} \tag{3.40}$$

$$u_{n+\frac{10}{3}}=-\frac{14}{81}u_n+\frac{20}{27}u_{n+1}-\frac{35}{27}u_{n+2}+\frac{140}{81}u_{n+3}+\frac{2909857}{14253733840}h^4b_n+\frac{29269579}{1018240956}h^4b_{n+1}+\frac{254898661}{2375895564}h^4b_{n+2}-\frac{14801}{1018240956}h^4b_{n+3}+\frac{2814473}{293320440}h^4b_{n+\frac{10}{3}}-\frac{45580303}{28510746768}h^4b_{n+4} \tag{3.41}$$

For $n=0(4)(\mathrm{N}-4)$

The first derivative formulae are:

$$hu'_n = -\frac{11}{6}u_n + 3u_{n+1} - \frac{3}{2}u_{n+2} + \frac{1}{3}u_{n+3} - \frac{237971}{24141600}h^4b_n - \frac{431323}{2414160}h^4b_{n+1} - \frac{202709}{3218880}h^4b_{n+2} + \frac{11}{344880}h^4b_{n+3} + \frac{1539}{766400}h^4b_{n+\frac{10}{3}} - \frac{1313}{2414160}h^4b_{n+4} \tag{3.42}$$

$$hu'_{n+1} = -\frac{1}{3}u_n - \frac{1}{2}u_{n+1} + u_{n+2} - \frac{1}{6}u_{n+3} - \frac{2269889}{2027894400}h^4b_n + \frac{45643}{905310}h^4b_{n+1} + \frac{966517}{27038592}h^4b_{n+2} + \frac{181}{3621240}h^4b_{n+3} - \frac{185733}{75107200}h^4b_{n+\frac{10}{3}} + \frac{57817}{81115776}h^4b_{n+4} \tag{3.43}$$

$$hu'_{n+2} = \frac{1}{6}u_n - u_{n+1} + \frac{1}{2}u_{n+2} + \frac{1}{3}u_{n+3} + \frac{1548223}{1013947200}h^4b_n - \frac{1772147}{50697360}h^4b_{n+1} - \frac{346681}{6759648}h^4b_{n+2} - \frac{377}{7242480}h^4b_{n+3} + \frac{9747}{4694200}h^4b_{n+\frac{10}{3}} - \frac{130261}{202789440}h^4b_{n+4} \tag{3.44}$$

$$hu'_{n+3} = -\frac{1}{3}u_n + \frac{3}{2}u_{n+1} - 3u_{n+2} + \frac{11}{6}u_{n+3} - \frac{759043}{675964800}h^4b_n + \frac{516401}{8449560}h^4b_{n+1} + \frac{8232487}{45064320}h^4b_{n+2} + \frac{13}{603540}h^4b_{n+3} + \frac{648567}{75107200}h^4b_{n+\frac{10}{3}} - \frac{180161}{135192960}h^4b_{n+4}$$

$$hu'_{n+\frac{10}{3}} = -\frac{13}{18}u_n + 3u_{n+1} - \frac{29}{6}u_{n+2} + \frac{23}{9}u_{n+3} + \frac{473404049}{246389169600}h^4b_n + \frac{195652973}{1759922640}h^4b_{n+1} + \frac{3973831609}{8212972320}h^4b_{n+2} - \frac{245647}{1759922640}h^4b_{n+3} + \frac{2749694}{47528775}h^4b_{n+\frac{10}{3}} - \frac{472713611}{49277833920}h^4b_{n+4} \tag{3.45}$$

$$hu'_{n+4} = -\frac{11}{6}u_n + 7u_{n+1} - \frac{19}{2}u_{n+2} + \frac{13}{3}u_{n+3} + \frac{10072369}{506973600}h^4b_n + \frac{10605461}{50697360}h^4b_{n+1} + \frac{101084371}{67596480}h^4b_{n+2} - \frac{7237}{7242480}h^4b_{n+3} + \frac{15198111}{37553600}h^4b_{n+\frac{10}{3}} - \frac{2273267}{50697360}h^4b_{n+4} \tag{3.46}$$

for $n = 0(4)(N-4)$

The second derivative formulae are:

$$h^2u''_n = 2u_n - 5u_{n+1} + 4u_{n+2} - u_{n+3} + \frac{16103449}{202789440}h^4b_n + \frac{34275233}{50697360}h^4b_{n+1} + \frac{2696563}{16899120}h^4b_{n+2} - \frac{3733}{7242480}h^4b_{n+3} + \frac{351}{107296}h^4b_{n+\frac{10}{3}} - \frac{231683}{202789440}h^4b_{n+4} \tag{3.47}$$

$$h^2u''_{n+1} = u_n - 2u_{n+1} + u_{n+2} - \frac{1158673}{405578880}h^4b_n - \frac{1966163}{25348680}h^4b_{n+1} - \frac{398119}{135192960}h^4b_{n+2} + \frac{11}{452655}h^4b_{n+3} - \frac{297}{15021440}h^4b_{n+\frac{10}{3}} + \frac{11561}{405578880}h^4b_{n+4} \tag{3.48}$$

$$h^2u''_{n+2} = u_{n+1} - 2u_{n+2} + u_{n+3} + \frac{67}{3168585}h^4b_n - \frac{139549}{50697360}h^4b_{n+1} - \frac{5292653}{67596480}h^4b_{n+2} + \frac{53}{7242480}h^4b_{n+3} - \frac{4293}{1502144}h^4b_{n+\frac{10}{3}} + \frac{55409}{101394720}h^4b_{n+4} \tag{3.49}$$

$$h^2u''_{n+3} = -u_n + 4u_{n+1} - 5u_{n+2} + 2u_{n+3} + \frac{192239}{57939840}h^4b_n + \frac{1908581}{12674340}h^4b_{n+1} + \frac{13531829}{19313280}h^4b_{n+2} - \frac{113}{517320}h^4b_{n+3} + \frac{1139157}{15021440}h^4b_{n+\frac{10}{3}} - \frac{111769}{8277120}h^4b_{n+4} \tag{3.50}$$

$$h^2u''_{n+\frac{10}{3}} = -\frac{4}{3}u_n + 5u_{n+1} - 6u_{n+2} + \frac{7}{3}u_{n+3} + \frac{379421111}{24638916960}h^4b_n + \frac{1819245629}{12319458480}h^4b_{n+1} + \frac{18241613111}{16425944640}h^4b_{n+2} - \frac{1353949}{1759922640}h^4b_{n+3} + \frac{9599899}{40557888}h^4b_{n+\frac{10}{3}} - \frac{114919289}{3079864620}h^4b_{n+4} \tag{3.51}$$

$$h^2u''_{n+4} = -2u_n + 7u_{n+1} - 8u_{n+2} + 3u_{n+3} + \frac{7343143}{202789440}h^4b_n + \frac{1130099}{7242480}h^4b_{n+1} + \frac{64469357}{33798240}h^4b_{n+2} - \frac{12289}{7242480}h^4b_{n+3} + \frac{323217}{375536}h^4b_{n+\frac{10}{3}} - \frac{1705189}{40557888}h^4b_{n+4} \tag{3.52}$$

for $n = 0(4)(N-4)$

The third derivative formulae are:

$$h^3u'''_n = -u_n + 3u_{n+1} - 3u_{n+2} + u_{n+3} - \frac{47872339}{135192960}h^4b_n - \frac{18007093}{16899120}h^4b_{n+1} - \frac{519415}{9012864}h^4b_{n+2} + \frac{4919}{2414160}h^4b_{n+3} - \frac{533331}{15021440}h^4b_{n+\frac{10}{3}} + \frac{1455383}{135192960}h^4b_{n+4} \tag{3.53}$$

$$h^3u'''_{n+1} = -u_n + 3u_{n+1} - 3u_{n+2} + u_{n+3} + \frac{3341887}{135192960}h^4b_n - \frac{4745537}{16899120}h^4b_{n+1} - \frac{3926283}{15021440}h^4b_{n+2} - \frac{1493}{2414160}h^4b_{n+3} + \frac{381807}{15021440}h^4b_{n+\frac{10}{3}} - \frac{990179}{135192960}h^4b_{n+4} \tag{3.54}$$

$$h^3u'''_{n+2} = -u_n + 3u_{n+1} - 3u_{n+2} + u_{n+3} - \frac{239451}{15021440}h^4b_n + \frac{1435033}{5633040}h^4b_{n+1} + \frac{12884237}{45064320}h^4b_{n+2} + \frac{89}{160944}h^4b_{n+3} - \frac{515187}{15021440}h^4b_{n+\frac{10}{3}} + \frac{135551}{15021440}h^4b_{n+4} \tag{3.55}$$

$$h^3u'''_{n+3} = -u_n + 3u_{n+1} - 3u_{n+2} + u_{n+3} + \frac{4181663}{135192960}h^4b_n + \frac{53539}{3379824}h^4b_{n+1} + \frac{53128063}{45064320}h^4b_{n+2} - \frac{3397}{2414160}h^4b_{n+3} + \frac{5008527}{15021440}h^4b_{n+\frac{10}{3}} - \frac{7804483}{135192960}h^4b_{n+4} \tag{3.56}$$

$$h^3u'''_{n+\frac{10}{3}} = -u_n + 3u_{n+1} - 3u_{n+2} + u_{n+3} + \frac{1280470423}{32851889280}h^4b_n - \frac{87243599}{4106486160}h^4b_{n+1} + \frac{2747786155}{2190125952}h^4b_{n+2} - \frac{1044683}{586640880}h^4b_{n+3} + \frac{86821621}{13192960}h^4b_{n+\frac{10}{3}} - \frac{2610001931}{32851889280}h^4b_{n+4} \tag{3.57}$$

$$h^3 u'''_{n+4} = -u_n + 3u_{n+1} - 3u_{n+2} + u_{n+3} + \frac{3004333}{135192960} h^4 b_n + \frac{866251}{16899120} h^4 b_{n+1} + \frac{16965903}{15021440} h^4 b_{n+2} - \frac{2249}{2414160} h^4 b_{n+3} + \frac{16884909}{15021440} h^4 b_{n+\frac{10}{3}} + \frac{4703339}{27038592} h^4 b_{n+4} \tag{3.58}$$

for $n = 0(4)(N-4)$

Remark 3.3.1. *The equations (3.40) to (3.58) together form the Block Unification Method (BUM) used to solve equations (1.1) and (1.2).*

3.4 Analysis of the Methods

The properties to be analysed for the derived methods are the order and error constant, the local truncation errors and the consistency of the methods.

3.4.1 Order and error constants

The linear difference operator L associated with the continuous implicit k step method developed, with ρ as the step number, is defined as:

$$\sum_{j=k-\rho}^{k} \alpha_j(x) y(x_{n+j}) = h^{\mu} \sum_{j=0}^{k} \beta_j(x) f(x_{n+j}) + h^{\mu} \beta_v f(x_{n+v}) \quad \text{and} \tag{3.59}$$

With

$$L[y(x);h] = \sum_{j=k-\rho}^{k} \alpha_j(x) y(x+jh) - h^{\mu} \left(\sum_{j=0}^{k} \beta_j(x) y^{(m)}(x+jh) + h^{\mu} \beta_v y^{(m)}(x+vh) \right) \tag{3.60}$$

Where y(x) is an arbitrary test function that is continuously differentiable in the interval [a,b].

The Taylors series expansion about the point x gives

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_p h^p y^p(x) \tag{3.61}$$

The difference operator L and the associated continuous implicit hybrid k step methods are of order p $C_0 = C_1 = C_2 = ... = C_p = ... = C_{p+\mu-1} = 0$ and $C_{p+\mu} \neq 0$. The term $C_{p+k} \neq 0$ is called the error constant.

Definition 3.4.1. *The equations from (3.17) to (3.28), (3.31) to (3.38) and (3.40) to (3.58) form a block of order p if* $C_0 = C_1 = C_2 = ... = C_p = ... = C_{p+\mu-1} = 0$ *and* $C_{p+\mu} \neq 0$ *in which*

$$L[y(x);h] = C_p h^p y^p(x) + O(h^{p+1})$$

In this case, C_p *is a vector of error constants*

Definition 3.4.2. *The equation (3.59) is said to be consistent if the order p > 1.*

Theorem 3.4.1. *(Henrichi (1962)) The implicit method of* equation *(3.59) is said to be consistent if it satisfies the following conditions:*

1. $\sum_{j=o}^{k} \alpha_j = 0$
2. $\rho(1) = \rho'(1) = 0$ *and*
3. $\rho^{(\mu)} = \mu! \sigma(1)$

where μ *is the order of the differential equation.*

Table 3.1: Order and Error Constants for $k=3$ and $\mu=3$

Formulae	C_{p+3}	Formulae	C_{p+3}
τ_{n+3}	$-\frac{1}{1440}$	-	-
$\tau_{n+\frac{7}{3}}$	$-\frac{89}{787320}$	-	-
τ'_n	$\frac{139}{60480}$	τ''_n	$-\frac{169}{20160}$
τ'_{n+1}	$-\frac{1}{3024}$	τ''_{n+1}	$\frac{19}{12096}$
τ'_{n+2}	$-\frac{11}{60480}$	τ''_{n+2}	$-\frac{59}{60480}$
$\tau'_{n+\frac{7}{3}}$	$-\frac{337}{688905}$	$\tau''_{n+\frac{7}{3}}$	$-\frac{2551}{2939328}$
τ'_{n+3}	$-\frac{43}{30240}$	τ''_{n+3}	$-\frac{43}{20160}$

Where the formulae $\tau^{(m)}_{n+j}$ represent each LMM $y^{(m)}_{n+j}, m=0,1,2; j=0,...,3,\frac{7}{3}$. From the definition of error constant given, it follows that all formulae are of order $p=5$.

Table 3.2: Order and Error Constants for $k = 3$ and $\mu = 2$

Formulae	C_{p+2}	Formulae	C_{p+2}
τ_{n+3}	$-\frac{1}{720}$	τ'_{n+1}	$\frac{43}{30240}$
$\tau_{n+\frac{7}{3}}$	$-\frac{281}{787320}$	τ'_{n+2}	$-\frac{17}{15120}$
τ_n	$\frac{1}{360}$	$\tau'_{n+\frac{7}{3}}$	$-\frac{7471}{7348320}$
τ'_n	$-\frac{43}{5040}$	τ'_{n+3}	$-\frac{23}{10080}$

Where the formulae $\tau_{n+j}^{(m)}$ represent each LMM $y_{n+j}^{(m)}, m = 0,1; j = 0,...,3,\frac{7}{3}$. From the definition of error constant given, it follows that all formulae are of order $p = 4$.

Table 3.3: Order and Error Constants for $k=4$ and $\mu=4$

Formulae	C_{p+4}	Formulae	C_{p+4}	Formulae	C_{p+4}
τ_{n+4}	$\frac{31}{20118}$	-	-	-	-
$\tau_{n+\frac{10}{3}}$	$\frac{74005}{1187947782}$	-	-	-	-
τ'_n	$-\frac{11}{80472}$	τ''_n	$\frac{3733}{1689912}$	τ'''_n	$-\frac{4919}{563304}$
τ'_{n+1}	$-\frac{181}{844956}$	τ''_{n+1}	$-\frac{22}{211239}$	τ'''_{n+1}	$\frac{1493}{563304}$
τ'_{n+2}	$\frac{377}{1689912}$	τ''_{n+2}	$-\frac{53}{1689912}$	τ'''_{n+2}	$-\frac{445}{187768}$
τ'_{n+3}	$-\frac{13}{140826}$	τ''_{n+3}	$\frac{113}{120708}$	τ'''_{n+3}	$\frac{3397}{563304}$
$\tau'_{n+\frac{10}{3}}$	$\frac{245647}{410648616}$	$\tau''_{n+\frac{10}{3}}$	$\frac{1353949}{410648616}$	$\tau'''_{n+\frac{10}{3}}$	$\frac{1044683}{136882872}$
τ'_{n+4}	$\frac{7237}{1689912}$	τ''_{n+4}	$\frac{12289}{1689912}$	τ'''_{n+4}	$\frac{2249}{563304}$

Where the formulae $\tau^{(m)}_{n+j}$ represent each LMM $y^{(m)}_{n+j}, m=0,\dots,3; j=0,\dots,4,\frac{10}{3}$. From the definition of error constant given, it follows that all formulae are of order $p=3$.

3.4.2 Consistency of the main method

The first and second characteristic polynomials of equation (3.17) are

$$\rho(z)=z^3-3z^2+3z-1$$

And

$$\sigma=\frac{1}{560}\left(14z^3-81z^{\frac{7}{3}}+364z^2+259z+4\right)$$

By Definition 3.4.2 and Theorem 3.4.1, the equation (3.17) is consistent since it satisfies the following:

1. The order is p = 5 > 1;

2. $\sum_{j=o}^{k} \alpha_j = 0, (\alpha_0 = -1, \alpha_1 = 3, \alpha_2 = -3, \alpha_3 = 1)$

3. $\rho(1) = 0$

   $\rho'(1) = 3z^2 - 6z + 3$

   $\rho'(1) = 0$

   Therefore, $\rho(1) = \rho'(1) = 0$

4. $\rho'''(1) = 6; \sigma(1) = 1$

   Therefore, $\rho'''(1) = 3!\sigma(1)$

The first and second characteristic polynomials of equation (3.31) are

$\rho(z) = z^3 - 2z^2 + z$

And

$$\sigma = \frac{1}{1680}\left(98z^3 + 243z^{\frac{7}{3}} + 1148z^2 + 203z - 12\right)$$

By Definition 3.4.2 and Theorem 3.4.1, the formula of equation (3.31) is consistent since it satisfies the following:

1. The order is p = 4 > 1;

2. $\sum_{j=o}^{k} \alpha_j = 0, (\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = -2, \alpha_3 = 1)$

3. $\rho(1) = 0$

   $\rho'(1) = 3z^2 - 4z + 1$

   $\rho'(1) = 0$

   Therefore, $\rho(1) = \rho'(1) = 0$

4. $\rho'''(1) = 6; \sigma(1) = 1$

   Therefore, $\rho'''(1) = 3!\sigma(1)$

The first and second characteristic polynomials of equation (3.40) are

$$\rho(z) = z^4 - 4z^3 + 6z^2 - 4z + 1$$

And

$$\sigma = \frac{1}{2414160}\left(-46598z^4 + 338985z^{\frac{10}{3}} - 868z^3 + 1780437z^2 + 326492z + 15712\right)$$

By Definition 3.4.2 and Theorem 3.4.1, the formula of equation (3.40) is consistent since it satisfies the following:

1. The order is p = 3 > 1;
2. $\sum_{j=o}^{k} \alpha_j = 0, (\alpha_0 = 1, \alpha_1 = -4, \alpha_2 = 6, \alpha_3 = -4, \alpha_4 = 1)$
3. $\rho(1) = 0$

   $\rho'(1) = 4z^3 - 12z^2 + 12z - 4$

   $\rho'(1) = 0$

   Therefore, $\rho(1) = \rho'(1) = 0$
4. $\rho^{(iv)}(1) = 24; \sigma(1) = 1$

   Therefore, $\rho^{(iv)}(1) = 4!\sigma(1)$

3.5 Convergence of the Methods

Theorem 3.5.1. *Let* (y_i, y_i', y_i'') *be an approximation to the solution vector* $(y(x_i), y'(x_i), \ldots, y^{(s-1)}(x_i))$ *for the system of equation (1.1). If* $e_i = |y(x_i) - y_i|$, $e_i' = |y'(x_i) - y_i'|, \cdots,$ $e_i = |y^{(s-1)}(x_i) - y_i^{(s-1)}|$, *where the exact solution given by the vector* $(y(x), y'(x), \ldots, y^{(s-1)}(x))$ *is several times differentiable and if* $\|E\| = \|Y - \overline{Y}\|$, *then the BUMs in equations (3.17) to (3.28), (3.31) to (3.38) and (3.40) to (3.58) are said to be convergent of orders 5, 4 and 3 respectively which implies that*

$\|E\| = O(h^5)$, $\|E\| = O(h^4)$ *and* $\|E\| = O(h^3)$ *respectively where k is the step number.*

3.5.1 Proof of convergence for $k = 3$

In order to show the block unification method equations (3.17) to (3.28) converges, they are compactly written as

$$EY - h^3 GF(Y) + C + L(h) = 0$$

where E is a $3N \times 3N$ defined by

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}$$ where E_{ij} are $N \times N$ matrices given as

$$E_{11}=\begin{pmatrix} -2 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{7}{9} & -\frac{14}{9} & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & -3 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{3}{2} & -2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\frac{2}{9} & \frac{7}{9} & -\frac{14}{9} & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 3 & -3 & 0 & 1 \end{pmatrix} \tag{3.62}$$

$$E_{21}=\begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\ 2 & -\frac{3}{2} & 0 & 0 & \cdots & 0 \\ \frac{8}{3} & -\frac{11}{6} & 0 & 0 & \cdots & 0 \\ 4 & -\frac{5}{2} & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & & \cdots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \cdots & 0 & -\frac{5}{6} & \frac{8}{3} & -\frac{11}{6} \\ 0 & \cdots & 0 & -\frac{3}{2} & 4 & -\frac{5}{2} \end{pmatrix} \tag{3.63}$$

$$
E_{31} = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
2 & -1 & 0 & 0 & \cdots & 0 \\
2 & -1 & 0 & 0 & \cdots & 0 \\
2 & -1 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \ddots & & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 2 & -1
\end{pmatrix} \tag{3.64}
$$

E_{12}, E_{13}, E_{23}, E_{32}, are $N \times N$ null matrices and E_{22}, E_{33} are $N \times N$ identity matrices. Similarly, another matrix G which is a $3N \times 3N$ matrix defined as

$$
G = \begin{pmatrix}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{pmatrix}
$$

Where G_{ij} are $N \times N$ matrices given as

$$
G_{11} = \begin{pmatrix}
\frac{577}{1680} & -\frac{101}{420} & \frac{783}{3920} & -\frac{11}{420} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{227}{240} & \frac{53}{60} & -\frac{81}{112} & \frac{17}{180} & 0 & 0 & \cdots & 0 & 0 \\
\frac{2911}{29160} & \frac{139}{1215} & -\frac{1067}{22680} & \frac{169}{43740} & 0 & 0 & \cdots & 0 & 0 \\
\frac{37}{80} & \frac{13}{20} & -\frac{81}{560} & \frac{1}{40} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & \vdots & \vdots \\
 & & \cdots & 0 & \frac{167}{2940} & \frac{577}{1680} & -\frac{101}{420} & \frac{783}{3920} & -\frac{11}{420} \\
 & & \cdots & 0 & -\frac{389}{1260} & -\frac{227}{240} & \frac{53}{60} & -\frac{81}{112} & \frac{17}{180} \\
 & & \cdots & 0 & \frac{137}{76545} & \frac{2911}{29160} & \frac{139}{1215} & -\frac{1067}{22680} & \frac{169}{43740} \\
 & & \cdots & 0 & \frac{1}{140} & \frac{37}{80} & \frac{13}{20} & -\frac{81}{560} & \frac{1}{40}
\end{pmatrix}
$$

(3.65)

$$G_{21}=\begin{pmatrix} -\frac{173}{1120} & \frac{1}{105} & -\frac{27}{1568} & \frac{1}{336} & 0 & 0 & \cdots & 0 & 0 \\ \frac{367}{1680} & \frac{27}{140} & -\frac{351}{3920} & \frac{1}{140} & 0 & 0 & \cdots & 0 & 0 \\ \frac{310459}{816480} & \frac{25679}{51030} & -\frac{38813}{211680} & \frac{3865}{244944} & 0 & 0 & \cdots & 0 & 0 \\ \frac{2393}{3360} & \frac{89}{84} & -\frac{27}{1568} & \frac{39}{560} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{11}{1470} & -\frac{173}{1120} & \frac{1}{105} & -\frac{27}{1568} & \frac{1}{336} \\ 0 & 0 & \cdots & 0 & \frac{13}{2940} & \frac{367}{1680} & \frac{27}{140} & -\frac{351}{3920} & \frac{1}{140} \\ 0 & 0 & \cdots & 0 & \frac{680}{107163} & \frac{310459}{816480} & \frac{25679}{51030} & -\frac{38813}{211680} & \frac{3865}{244944} \\ 0 & 0 & \cdots & 0 & \frac{9}{980} & \frac{2393}{3360} & \frac{89}{84} & -\frac{27}{1568} & \frac{39}{560} \end{pmatrix} \quad (3.66)$$

$$G_{31}=\begin{pmatrix} \frac{1}{12} & -\frac{11}{40} & \frac{27}{140} & -\frac{1}{45} & 0 & 0 & \cdots & 0 & 0 \\ \frac{39}{80} & \frac{49}{60} & -\frac{27}{140} & \frac{1}{36} & 0 & 0 & \cdots & 0 & 0 \\ \frac{29}{60} & \frac{641}{648} & -\frac{71}{420} & \frac{29}{1215} & 0 & 0 & \cdots & 0 & 0 \\ \frac{31}{60} & \frac{79}{120} & -\frac{81}{140} & \frac{11}{45} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{53}{2520} & \frac{1}{12} & -\frac{11}{40} & \frac{27}{140} & -\frac{1}{45} \\ 0 & 0 & \cdots & 0 & \frac{1}{180} & \frac{39}{80} & \frac{49}{60} & -\frac{27}{140} & \frac{1}{36} \\ 0 & 0 & \cdots & 0 & \frac{407}{68040} & \frac{29}{60} & \frac{641}{648} & -\frac{71}{420} & \frac{29}{1215} \\ 0 & 0 & \cdots & 0 & \frac{1}{540} & \frac{31}{60} & \frac{79}{120} & \frac{81}{140} & \frac{11}{45} \end{pmatrix} \quad (3.67)$$

G_{12}, G_{13}, G_{22}, G_{23}, G_{32}, G_{33} are $N \times N$ null matrices

Similarly, we establish the convergence of equations (3.31) to (3.38) which are compactly written as

$$EY - h^2 GF(Y) + C + L(h) = 0$$

where E is a $2N \times 2N$ defined by

$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$ where E_{ij} are $N \times N$ matrices given as

$$E_{11} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{4}{3} & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{3} & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & -2 & 0 & 1 \end{pmatrix} \tag{3.68}$$

$$E_{21} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ \cdots & \ddots & \vdots \\ \cdots & 1 & -1 \\ \cdots & 1 & -1 \\ \cdots & 1 & -1 \\ \cdots & 1 & -1 \end{pmatrix} \tag{3.69}$$

E_{12} is an $N \times N$ null matrix and E_{22} is an $N \times N$ identity matrix. Similarly, another matrix G which is a $2N \times 2N$ matrix defined as

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

Where G_{ij} are $N \times N$ matrices given as

$$G_{11}=\begin{pmatrix} -\frac{211}{160} & \frac{7}{10} & -\frac{729}{1120} & \frac{13}{144} & 0 & 0 & \cdots & 0 & 0 \\ \frac{209}{240} & -\frac{1}{15} & \frac{81}{560} & -\frac{1}{40} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1093}{29160} & \frac{1817}{7290} & -\frac{527}{7560} & \frac{317}{43740} & 0 & 0 & \cdots & 0 & 0 \\ \frac{29}{240} & \frac{41}{60} & \frac{81}{560} & \frac{7}{120} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & \vdots & \vdots \\ & & \cdots & 0 & -\frac{101}{315} & -\frac{211}{160} & \frac{7}{10} & -\frac{729}{1120} & \frac{13}{144} \\ & & \cdots & 0 & \frac{8}{105} & \frac{209}{240} & -\frac{1}{15} & \frac{81}{560} & \frac{7}{120} \\ & & \cdots & 0 & -\frac{313}{153090} & \frac{1093}{29160} & \frac{1817}{7290} & -\frac{527}{7560} & \frac{317}{43740} \\ & & \cdots & 0 & -\frac{1}{140} & \frac{29}{240} & \frac{41}{60} & \frac{81}{560} & \frac{1}{120} \end{pmatrix} \quad (3.70)$$

$$G_{21}=\begin{pmatrix} -\frac{139}{480} & -\frac{11}{24} & -\frac{297}{1120} & -\frac{19}{720} & 0 & 0 & \cdots & 0 & 0 \\ \frac{11}{96} & \frac{19}{30} & -\frac{297}{1120} & \frac{17}{720} & 0 & 0 & \cdots & 0 & 0 \\ \frac{53}{480} & \frac{2611}{3240} & -\frac{65}{672} & \frac{383}{19440} & 0 & 0 & \cdots & 0 & 0 \\ \frac{23}{10} & \frac{19}{40} & \frac{729}{1120} & \frac{173}{720} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{23}{2520} & -\frac{139}{480} & -\frac{11}{24} & \frac{297}{1120} & -\frac{19}{720} \\ 0 & 0 & \cdots & 0 & -\frac{2}{315} & \frac{11}{96} & \frac{19}{30} & -\frac{297}{1120} & \frac{17}{720} \\ 0 & 0 & \cdots & 0 & -\frac{403}{68040} & \frac{53}{480} & \frac{2611}{3240} & -\frac{65}{672} & \frac{383}{19440} \\ 0 & 0 & \cdots & 0 & -\frac{5}{504} & \frac{23}{10} & \frac{19}{40} & \frac{729}{1120} & \frac{173}{720} \end{pmatrix} \quad (3.71)$$

G_{12} and G_{22} are $N \times N$ null matrices

3.5.2 Proof of convergence for $k = 4$

In order to show the block unification method equations (3.40) to (3.58) converges, they are compactly written as

$$EY - h^4 GF(Y) + C + L(h) = 0$$

where E is a $4N \times 4N$ defined by

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{pmatrix}$$ where E_{ij} are $N \times N$ matrices given as

$$E_{11} = \begin{pmatrix}
-3 & \frac{3}{2} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
5 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{20}{27} & \frac{35}{27} & -\frac{140}{81} & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-4 & 6 & -4 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{11}{6} & -3 & \frac{3}{2} & -\frac{1}{3} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & -2 & 5 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & -3 & 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \frac{14}{81} & -\frac{20}{27} & \frac{3}{27} & -\frac{140}{81} & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & -4 & 6 & -4 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \ddots & & \ddots & \ddots & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{11}{6} & -3 & \frac{3}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 5 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{14}{81} & -\frac{20}{27} & -\frac{140}{80} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & -4 & 0 & 1
\end{pmatrix} \tag{3.72}$$

$$E_{21} = \begin{pmatrix}
\frac{1}{2} & -1 & \frac{1}{6} & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & -\frac{1}{2} & -\frac{1}{3} & 0 & 0 & 0 & 0 & \cdots & 0 \\
-\frac{3}{2} & 3 & -\frac{11}{6} & 0 & 0 & 0 & 0 & \cdots & 0 \\
-3 & \frac{29}{6} & -\frac{23}{9} & 0 & 0 & 0 & 0 & \cdots & 0 \\
-7 & \frac{19}{2} & -\frac{13}{3} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{6} & 0 & \cdots & 0 \\
0 & 0 & -\frac{1}{6} & 1 & -\frac{1}{2} & -\frac{1}{3} & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{11}{6} & 0 & \cdots & 0 \\
0 & 0 & \frac{13}{18} & -3 & \frac{29}{6} & -\frac{23}{9} & 0 & \cdots & 0 \\
0 & 0 & \frac{11}{6} & -7 & \frac{19}{2} & -\frac{13}{3} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{6} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{6} & 1 & -\frac{1}{2} & -\frac{1}{3} \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{11}{6} \\
0 & 0 & 0 & \cdots & 0 & \frac{13}{18} & -3 & \frac{29}{6} & -\frac{23}{9} \\
0 & 0 & 0 & \cdots & 0 & \frac{11}{6} & -7 & \frac{19}{2} & -\frac{13}{3}
\end{pmatrix} \tag{3.73}$$

$$
E_{31} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-4 & 5 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-5 & 6 & -\frac{7}{3} & 0 & 0 & 0 & 0 & \cdots & 0 \\
-7 & 8 & -3 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -4 & 5 & -2 & 0 & \cdots & 0 \\
0 & 0 & \frac{4}{3} & -5 & 6 & -\frac{7}{3} & 0 & \cdots & 0 \\
0 & 0 & 2 & -7 & 8 & -3 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1 & -4 & 5 & -2 \\
0 & 0 & 0 & \cdots & 0 & \frac{4}{3} & -5 & 6 & -\frac{7}{3} \\
0 & 0 & 0 & \cdots & 0 & 2 & -7 & 8 & -3
\end{pmatrix} \tag{3.74}
$$

$$
E_{41} = \begin{pmatrix}
-3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1
\end{pmatrix} \tag{3.75}
$$

E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{32}, E_{34}, E_{42}, E_{43} are $N \times N$ null matrices and E_{22}, E_{33}, E_{44} are $N \times N$ identity matrices. Similarly, another matrix G which is a $4N \times 4N$ matrix defined as

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix}$$ where G_{ij} are $N \times N$ matrices given as

$$
G_{11} = \begin{pmatrix}
-\frac{431323}{2414160} & -\frac{202709}{3218880} & \frac{11}{344880} & \frac{1539}{766400} & -\frac{13130}{2414160} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{34275233}{50697360} & \frac{2696563}{16899120} & -\frac{3733}{7242480} & \frac{351}{107296} & -\frac{231683}{202789440} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-\frac{18007093}{16899120} & -\frac{519415}{9012864} & \frac{4919}{2414160} & -\frac{533331}{15021440} & \frac{1455383}{135192960} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{29269579}{1018240956} & \frac{254898661}{2375895564} & -\frac{14801}{1018240956} & \frac{2814473}{293320440} & -\frac{45580303}{28510746768} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{81623}{603540} & \frac{593479}{804720} & -\frac{31}{86220} & \frac{7533}{53648} & -\frac{23299}{1207080} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -\frac{237971}{24141600} & -\frac{431323}{2414160} & -\frac{202709}{3218880} & \frac{11}{344880} & \frac{1539}{766400} & -\frac{1313}{2414160} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{16103449}{202789440} & \frac{3475233}{50697360} & \frac{2696563}{16899120} & -\frac{3733}{7242480} & \frac{351}{107296} & -\frac{231683}{202789440} & 0 & \cdots & 0 \\
0 & 0 & 0 & -\frac{47872339}{135192960} & -\frac{18007093}{16899120} & -\frac{519415}{9012864} & \frac{4919}{2414160} & -\frac{533331}{15021440} & \frac{1455383}{135192960} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{2909857}{14253733840} & \frac{29269579}{1018240956} & \frac{254898661}{2375895564} & -\frac{14801}{1018240956} & \frac{2814473}{293320440} & -\frac{45580303}{28510746768} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{982}{150885} & \frac{81623}{603540} & \frac{593479}{804720} & -\frac{31}{86220} & \frac{7533}{53648} & -\frac{23299}{1207080} & 0 & \cdots & 0 \\
\vdots & & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{237971}{24141600} & -\frac{431323}{2414160} & -\frac{202709}{3218880} & \frac{11}{344880} & \frac{1539}{766400} & -\frac{1313}{2414160} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{16103449}{202789440} & \frac{34275233}{50697360} & \frac{26965633}{16899120} & -\frac{3733}{7242480} & \frac{351}{107296} & -\frac{231683}{202789440} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{47872339}{135192960} & -\frac{18007093}{16899120} & -\frac{519415}{9012864} & \frac{4919}{2414160} & -\frac{533331}{15021440} & \frac{1455383}{135192960} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{2909857}{14253733840} & \frac{29269579}{1018240956} & \frac{254898661}{2375895564} & -\frac{14801}{1018240956} & \frac{2814473}{293320440} & -\frac{45580303}{28510746768} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{982}{150885} & \frac{81623}{603540} & \frac{593479}{804720} & -\frac{31}{86220} & \frac{7533}{53648} & -\frac{23299}{1207080}
\end{pmatrix}
\tag{3.76}
$$

$$
G_{12} = \begin{pmatrix}
\frac{45643}{905310} & \frac{966517}{27038592} & \frac{181}{3621240} & -\frac{185733}{75107200} & \frac{57817}{81115776} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-\frac{1772147}{50697360} & -\frac{346681}{6759648} & -\frac{377}{7242480} & \frac{9747}{4694200} & -\frac{130261}{202789440} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{516401}{8449560} & \frac{8232487}{45064320} & \frac{13}{603540} & \frac{648567}{75107200} & -\frac{180161}{135192960} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{195652973}{1759922640} & \frac{3973831609}{8212972320} & -\frac{245647}{1759922640} & \frac{2749694}{47528775} & -\frac{472713611}{49277833920} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{10605461}{50697360} & \frac{101084371}{67596480} & -\frac{7237}{7242480} & \frac{15198111}{37553600} & -\frac{2273267}{50697360} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -\frac{2269889}{2027894400} & \frac{45643}{905310} & \frac{966517}{27038592} & \frac{181}{3621240} & -\frac{185733}{75107200} & \frac{57817}{81115776} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1548223}{1013947200} & -\frac{1772147}{50697360} & -\frac{346681}{6759648} & -\frac{377}{7242480} & \frac{9747}{4694200} & -\frac{130261}{202789440} & 0 & \cdots & 0 \\
0 & 0 & 0 & -\frac{759043}{675964800} & \frac{516401}{8449560} & \frac{8232487}{45064320} & \frac{13}{603540} & \frac{648567}{75107200} & -\frac{180161}{135192960} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{473404049}{246389169600} & \frac{195652973}{1759922640} & \frac{3973831609}{8212972320} & -\frac{245647}{1759922640} & \frac{2749694}{47528775} & -\frac{472713611}{49277833920} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{10072369}{506973600} & \frac{10605461}{50697360} & \frac{101084371}{67596480} & -\frac{7237}{7242480} & \frac{15198111}{37553600} & -\frac{2273267}{50697360} & 0 & \cdots & 0 \\
\vdots & & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{2269889}{2027894400} & \frac{45643}{905310} & \frac{966517}{27038592} & \frac{181}{3621240} & -\frac{185733}{75107200} & \frac{57817}{81115776} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1548223}{1013947200} & -\frac{1772147}{50697360} & -\frac{346681}{6759648} & -\frac{377}{7242480} & \frac{9747}{4694200} & -\frac{130261}{202789440} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{759043}{675964800} & \frac{516401}{8449560} & \frac{8232487}{45064320} & \frac{13}{603540} & \frac{648567}{75107200} & -\frac{180161}{135192960} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{473404049}{246389169600} & \frac{195652973}{1759922640} & \frac{3973831609}{8212972320} & -\frac{245647}{1759922640} & \frac{2749694}{47528775} & -\frac{472713611}{49277833920} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{10072369}{506973600} & \frac{10605461}{50697360} & \frac{101084371}{67596480} & -\frac{7237}{7242480} & \frac{15198111}{37553600} & -\frac{2273267}{50697360}
\end{pmatrix}
\tag{3.77}
$$

$$
G_{31}=\begin{pmatrix}
-\frac{1966163}{25348680} & -\frac{398119}{135192960} & \frac{11}{452655} & -\frac{297}{15021440} & \frac{11561}{405578880} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
-\frac{139549}{50697360} & -\frac{5292653}{67596480} & \frac{53}{7242480} & -\frac{4293}{1502144} & \frac{55409}{101394720} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{1908581}{12674340} & \frac{13531829}{19313280} & -\frac{113}{517320} & \frac{1139157}{15021440} & -\frac{111769}{8277120} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{1819245629}{12319458480} & \frac{18241613111}{16425944640} & -\frac{1353949}{1759922640} & \frac{9599899}{40557888} & -\frac{114919289}{3079864620} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{1130099}{7242480} & \frac{64469357}{33798240} & -\frac{12289}{7242480} & \frac{323217}{375536} & -\frac{1705189}{40557888} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & -\frac{1158673}{405578880} & -\frac{1966163}{25348680} & -\frac{398119}{135192960} & \frac{11}{452655} & -\frac{297}{15021440} & \frac{11561}{405578880} & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{67}{3168585} & -\frac{139549}{50697360} & -\frac{5292653}{67596480} & \frac{53}{7242480} & -\frac{4293}{1502144} & \frac{55409}{101394720} & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{192239}{57939840} & \frac{1908581}{12674340} & \frac{13531829}{19313280} & -\frac{113}{517320} & \frac{1139157}{15021440} & -\frac{111769}{8277120} & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{379421111}{24638916960} & \frac{1819245629}{12319458480} & \frac{18241613111}{16425944640} & -\frac{1353949}{1759922640} & \frac{9599899}{40557888} & -\frac{114919289}{3079864620} & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{7343143}{202789440} & \frac{1130099}{7242480} & \frac{64469357}{33798240} & -\frac{12289}{7242480} & \frac{323217}{375536} & -\frac{1705189}{40557888} & 0 & \cdots & 0\\
\vdots & & & \ddots & & \ddots & & & & \ddots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1158673}{405578880} & -\frac{1966163}{25348680} & -\frac{398119}{135192960} & \frac{11}{452655} & -\frac{297}{15021440} & \frac{11561}{405578880}\\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{67}{3168585} & -\frac{139549}{50697360} & -\frac{5292653}{67596480} & \frac{53}{7242480} & -\frac{4293}{1502144} & \frac{55409}{101394720}\\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{192239}{57939840} & \frac{1908581}{12674340} & \frac{13531829}{19313280} & -\frac{113}{517320} & \frac{1139157}{15021440} & -\frac{111769}{8277120}\\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{379421111}{24638916960} & \frac{1819245629}{12319458480} & \frac{18241613111}{16425944640} & -\frac{1353949}{1759922640} & \frac{9599899}{40557888} & -\frac{114919289}{3079864620}\\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{7343143}{202789440} & \frac{1130099}{7242480} & \frac{64469357}{33798240} & -\frac{12289}{7242480} & \frac{323217}{375536} & -\frac{1705189}{40557888}
\end{pmatrix}
\tag{3.78}
$$

$$
G_{41}=\begin{pmatrix}
-\frac{4745537}{16899120} & -\frac{3926283}{15021440} & -\frac{1493}{2414160} & \frac{381807}{15021440} & -\frac{990179}{135192960} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1435033}{5633040} & \frac{12884237}{45064320} & \frac{89}{160944} & -\frac{515187}{15021440} & \frac{135551}{15021440} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{53539}{3379824} & \frac{53128063}{45064320} & -\frac{3397}{2414160} & \frac{5008527}{15021440} & -\frac{7804483}{135192960} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-\frac{87243599}{4106486160} & \frac{2747786155}{2190125952} & -\frac{1044683}{586640880} & \frac{86821621}{13192960} & -\frac{2610001931}{32851889280} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{866251}{16899120} & \frac{16965903}{15021440} & -\frac{2249}{2414160} & \frac{16884909}{15021440} & \frac{4703339}{27038592} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{3341887}{135192960} & -\frac{4745537}{16899120} & -\frac{3926283}{15021440} & -\frac{1493}{2414160} & \frac{381807}{15021440} & -\frac{990179}{135192960} & 0 & \cdots & 0 \\
0 & 0 & 0 & -\frac{239451}{15021440} & \frac{1435033}{5633040} & \frac{12884237}{45064320} & \frac{89}{160944} & -\frac{515187}{15021440} & \frac{135551}{15021440} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{4181663}{135192960} & \frac{53539}{3379824} & \frac{53128063}{45064320} & -\frac{3397}{2414160} & \frac{5008527}{15021440} & -\frac{7804483}{135192960} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1280470423}{32851889280} & -\frac{87243599}{4106486160} & \frac{2747786155}{2190125952} & -\frac{1044683}{586640880} & \frac{86821621}{13192960} & -\frac{2610001931}{32851889280} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{3004333}{135192960} & \frac{866251}{16899120} & \frac{16965903}{15021440} & -\frac{2249}{2414160} & \frac{16884909}{15021440} & \frac{4703339}{27038592} & 0 & \cdots & 0 \\
\vdots & & & \ddots & & \ddots & & & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{3341887}{135192960} & -\frac{4745537}{16899120} & -\frac{3926283}{15021440} & -\frac{1493}{2414160} & \frac{381807}{15021440} & -\frac{990179}{135192960} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{239451}{15021440} & \frac{1435033}{5633040} & \frac{12884237}{45064320} & \frac{89}{160944} & -\frac{515187}{15021440} & \frac{135551}{15021440} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{4181663}{135192960} & \frac{53539}{3379824} & \frac{53128063}{45064320} & -\frac{3397}{2414160} & \frac{5008527}{15021440} & -\frac{7804483}{135192960} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1280470423}{32851889280} & -\frac{87243599}{4106486160} & \frac{2747786155}{2190125952} & -\frac{1044683}{586640880} & \frac{86821621}{13192960} & -\frac{2610001931}{32851889280} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{3004333}{135192960} & \frac{866251}{16899120} & \frac{16965903}{15021440} & -\frac{2249}{2414160} & \frac{16884909}{15021440} & \frac{4703339}{27038592}
\end{pmatrix}
\tag{3.79}
$$

Where all other matrices are null matrices and then the following vectors are defined with s being the order of the differential equation the method is applied to

$$\bar{Y} = (y_{n+1}, \ldots, y_{n+k}, hy'_{n+1}, \ldots, hy'_{n+k}, \ldots, h^{(s)} y^{(s)}_{n+1}, \ldots, h^{(s)} y^{(s)}_{n+k})^T$$

$$Y = (y(x_{n+1}), \ldots, y(x_{n+k}), hy'(x_{n+1}), \ldots, hy'(x_{n+k}), \ldots, h^{(s-1)} y''(x_{n+1}), \ldots, h^{(s-1)} y''(x_{n+k}))^T$$

$$F = (f_{n+1}, \ldots, f_{n+2v}, hf'_{n+1}, \ldots, hf'_{n+k}, \ldots, h^{(s-1)} f''_{n+1}, \ldots, h^{(s-1)} f''_{n+k})^T$$

$$L(h) = (l_1, \ldots, l_N, l'_1, \ldots, l'_N, \ldots, l_1^{(s-1)}, \ldots, l_N^{(s-1)})^T$$

$$C = (\beta'^{(0)}_0 h^3 f_0 - hy'_0, \beta''^{(0)}_0 h^3 f_0 - hy''_0, \beta^{(0)}_0 h^3 f_0 - y_0, \beta^{(1)}_0 h^3 f_0, \ldots, \beta^{(v-2)}_0 h^3 f_0, \beta^{(v+1)}_0 h^3 f_0, \ldots,$$
$$\beta^{(k)}_0 h^3 f_0, 0, \ldots, 0, \beta'^{(1)}_0 h^3 f_0 - \alpha'^{(1)}_0 y_0, \beta'^{(k)}_0 h^3 f_0 - \alpha'^{(k)}_0 y_0, 0, \ldots 0, \beta''^{(1)}_0 h^3 f_0 - \alpha''^{(1)}_0 y_0,$$
$$\beta''^{(k)}_0 h^3 f_0 - \alpha''^{(k)}_0 y_0, 0, \ldots 0)^T$$

With $L(h)$ representing the local truncation error vector at the point x_n of the methods in equations (3.17) to (3.28), (3.31) to (3.38) and (3.40) to (3.58).

3.5.3 General proof of theorem 3.5.1

Consider the exact form of the systems formed from equations (3.17) to (3.28), (3.31) to (3.38) and (3.40) to equation (3.58) given by the general form below with s being the order of the differential equation to be solved

$$EY - h^s GF(Y) + C + L(h) = 0 \tag{3.80}$$

where $L(h)$ is the truncation error vector obtained from the systems formed. The approximate form of the system is given by

$$E\bar{Y} - h^3 GF(\bar{Y}) + C = 0 \tag{3.81}$$

where $\bar{Y}$ is the approximate solution of vector Y.

Subtracting equation (3.81) from equation (3.80) and letting $e = |\bar{y} - y| = (e_1, \dots e_N, e'_1, \dots e'_N, e''_1, \dots e''_N, \dots)^T$ and using the mean value theorem, we have the error system

$$(\mathrm{E} - h^s GB)\,\mathrm{e} = L(h) \tag{3.82}$$

where B is the Jacobian matrix and its entries $B_{ab}, \mathrm{a}, \mathrm{b} = 1, 2, 3,$ are defined as

$$B_{ab} = \begin{pmatrix} \frac{\partial f_1^{(\mathrm{a}-1)}}{\partial y_1^{(\mathrm{b}-1)}} & \cdots & \frac{\partial f_1^{(a-1)}}{\partial f_N^{(-1)}} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N^{(\mathrm{a}-1)}}{\partial f_1^{(\mathrm{b}-1)}} & \cdots & \frac{\partial f_N^{(\mathrm{a}-1)}}{\partial f_N^{(\mathrm{b}-1)}} \end{pmatrix} \tag{3.83}$$

Using equation (3.82), we let

$$\mathrm{S} = (\mathrm{E} - h^s GB)^{-1} = E(I - h^s E^{-1} GB)^{-1}$$

$$\mathrm{SE}^{-1} = (I - h^s E^{-1} GB)^{-1} = I + (I - h^s E^{-1} GB) + (I - h^s E^{-1} GB)^2 + \dots$$

The above is an infinite series which converges at $\|h^s E^{-1} GB\| < 1$. We claim that E is invertible if and only if E_{11} is invertible. Since E_{11} is a block diagonal matrix with elements in its main diagonal as non-zero (note: E_{ii}, with i = 2(1)s are all identity matrices), then it is invertible. Therefore, S is monotone for any sufficiently small h and thus singular if

$$\|E^{-1}\| < \frac{1}{\|h^s GB\|}$$

From (3.81) and $L(h)$

$$e = (\mathrm{E} - h^s GB)^{-1} L(h)$$

$$e = SL(h)$$

$$\|e\| = \|SL(h)\|$$

$$= O(h^{-s})O(h^{k+s+1})$$

$$= O(h^{k+1})$$

Which show that the methods are convergent and the global errors are of order $O(h^{k+1})$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Method Implementation

The blocks for $\mu = 3$ are implemented as follows while noting that a block of step k makes use of each of the methods in the block in steps of k i.e. n = 0, k, …, N-k. This approach has an advantage of generating approximate solutions simultaneously to the exact solution on the entire interval of integration.

The steps are:

Step 1: Combine the derived blocks to obtain Y_1 for n = 0 on the interval $[y_n, y_{n+k}]$ and also S_1 for n = 0 on the same interval, Y_2 and S_2 for n = 1 on the interval $[y_{n+k}, y_{n+2k}]$, Y_3 and S_3 for n = 2 on the interval $[y_{n+2k}, y_{n+3k}]$ and so on

Step 2: A unified block of system of kN equations in kN unknowns which can be solved easily is formed from the union of the equations derived in step 1.

Step 3: The values of the solution(s), the first and second derivatives, are generated by the sequence $\{y_n\},\{s_n\},\{y'_n\},\{s'_n\}$ and $\{y''_n\}$ $n = 0,...,N$ respectively which are the solutions from Step 2.

In order to apply the BUM to the PDEs, the equivalent form of the derived BUM from (3.40) to (3.58) for the direct numerical integration of (1.2) is given as

$$u_{i,n+4} = -u_{i,n} + 4u_{i,n+1} - 6u_{i,n+2} + 4u_{i,n+3} + \frac{982}{150885}h^4 b_{i,n} + \frac{81623}{603540}h^4 b_{i,n+1} + \frac{593479}{804720}h^4 b_{i,n+2} - \frac{31}{86220}h^4 b_{i,n+3} + \frac{7533}{53648}h^4 b_{i,n+\frac{10}{3}} - \frac{23299}{1207080}h^4 b_{i,n+4}$$

$$u_{i,n+\frac{10}{3}} = -\frac{14}{81}u_{i,n} + \frac{20}{27}u_{i,n+1} - \frac{35}{27}u_{i,n+2} + \frac{140}{81}u_{i,n+3} + \frac{2909857}{14253733840}h^4 b_{i,n} + \frac{29269579}{1018240956}h^4 b_{i,n+1} + \frac{254898661}{2375895564}h^4 b_{i,n+2} - \frac{14801}{1018240956}h^4 b_{i,n+3} + \frac{2814473}{293320440}h^4 b_{i,n+\frac{10}{3}} - \frac{45580303}{28510746768}h^4 b_{i,n+4}$$

$$hu'_{i,n} = -\frac{11}{6}u_{i,n} + 3u_{i,n+1} - \frac{3}{2}u_{i,n+2} + \frac{1}{3}u_{i,n+3} - \frac{237971}{24141600}h^4 b_{i,n} - \frac{431323}{2414160}h^4 b_{i,n+1} - \frac{202709}{3218880}h^4 b_{i,n+2} + \frac{11}{344880}h^4 b_{i,n+3} + \frac{1539}{766400}h^4 b_{i,n+\frac{10}{3}} - \frac{1313}{2414160}h^4 b_{i,n+4}$$

$$hu'_{i,n+1} = -\frac{1}{3}u_{i,n} - \frac{1}{2}u_{i,n+1} + u_{i,n+2} - \frac{1}{6}u_{i,n+3} - \frac{2269889}{2027894400}h^4 b_{i,n} + \frac{45643}{905310}h^4 b_{i,n+1} + \frac{966517}{27038592}h^4 b_{i,n+2} + \frac{181}{3621240}h^4 b_{i,n+3} - \frac{185733}{75107200}h^4 b_{i,n+\frac{10}{3}} + \frac{57817}{81115776}h^4 b_{i,n+4}$$

$$hu'_{i,n+2} = \frac{1}{6}u_{i,n} - u_{i,n+1} + \frac{1}{2}u_{i,n+2} + \frac{1}{3}u_{i,n+3} + \frac{1548223}{1013947200}h^4 b_{i,n} - \frac{1772147}{50697360}h^4 b_{i,n+1} - \frac{346681}{6759648}h^4 b_{i,n+2} - \frac{377}{7242480}h^4 b_{i,n+3} + \frac{9747}{4694200}h^4 b_{i,n+\frac{10}{3}} - \frac{130261}{202789440}h^4 b_{i,n+4}$$

$$hu'_{i,n+3} = -\frac{1}{3}u_{i,n} + \frac{3}{2}u_{i,n+1} - 3u_{i,n+2} + \frac{11}{6}u_{i,n+3} - \frac{759043}{675964800}h^4 b_{i,n} + \frac{516401}{8449560}h^4 b_{i,n+1} + \frac{8232487}{45064320}h^4 b_{i,n+2} + \frac{13}{603540}h^4 b_{i,n+3} + \frac{648567}{75107200}h^4 b_{i,n+\frac{10}{3}} - \frac{180161}{135192960}h^4 b_{i,n+4}$$

$$hu'_{i,n+\frac{10}{3}} = -\frac{13}{18}u_{i,n} + 3u_{i,n+1} - \frac{29}{6}u_{i,n+2} + \frac{23}{9}u_{\mathrm{i},n+3} + \frac{473404049}{246389169600}h^4 b_{\mathrm{i},n} + \frac{195652973}{1759922640}h^4 b_{i,n+1} + \frac{3973831609}{8212972320}h^4 b_{i,n+2} - \frac{245647}{1759922640}h^4 b_{i,n+3} + \frac{2749694}{47528775}h^4 b_{i,n+\frac{10}{3}} - \frac{472713611}{49277833920}h^4 b_{n+4,m}$$

$$hu'_{i,n+4} = -\frac{11}{6}u_{i,n} + 7u_{i,n+1} - \frac{19}{2}u_{i,n+2} + \frac{13}{3}u_{i,n+3} + \frac{10072369}{506973600}h^4 b_{i,n} + \frac{10605461}{50697360}h^4 b_{i,n+1} + \frac{101084371}{67596480}h^4 b_{i,n+2} - \frac{7237}{7242480}h^4 b_{i,n+3} + \frac{15198111}{37553600}h^4 b_{i,n+\frac{10}{3}} - \frac{2273267}{50697360}h^4 b_{i,n+4}$$

$$h^2u''_{i,n}=2u_{i,n}-5u_{i,n+1}+4u_{i,n+2}-u_{i,n+3}+\frac{16103449}{202789440}h^4b_{i,n}+\frac{34275233}{50697360}h^4b_{i,n+1}+\frac{2696563}{16899120}h^4b_{i,n+2}-\frac{3733}{7242480}h^4b_{i,n+3}+\frac{351}{107296}h^4b_{i,n+\frac{10}{3}}-\frac{231683}{202789440}h^4b_{i,n+4}$$

$$h^2u''_{i,n+1}=u_{i,n}-2u_{i,n+1}+u_{i,n+2}-\frac{1158673}{405578880}h^4b_{i,n}-\frac{1966163}{25348680}h^4b_{i,n+1}-\frac{398119}{135192960}h^4b_{i,n+2}+\frac{11}{452655}h^4b_{i,n+3}-\frac{297}{15021440}h^4b_{i,n+\frac{10}{3}}+\frac{11561}{405578880}h^4b_{i,n+4}$$

$$h^2u''_{i,n+2}=u_{i,n+1}-2u_{i,n+2}+u_{i,n+3}+\frac{67}{3168585}h^4b_{i,n}-\frac{139549}{50697360}h^4b_{i,n+1}-\frac{5292653}{67596480}h^4b_{i,n+2}+\frac{53}{7242480}h^4b_{i,n+3}-\frac{4293}{1502144}h^4b_{i,n+\frac{10}{3}}+\frac{55409}{101394720}h^4b_{i,n+4}$$

$$h^2u''_{i,n+3}=-u_{i,n}+4u_{i,n+1}-5u_{i,n+2}+2u_{i,n+3}+\frac{192239}{57939840}h^4b_{i,n}+\frac{1908581}{12674340}h^4b_{i,n+1}+\frac{13531829}{19313280}h^4b_{i,n+2}-\frac{113}{517320}h^4b_{i,n+3}+\frac{1139157}{15021440}h^4b_{i,n+\frac{10}{3}}-\frac{111769}{8277120}h^4b_{i,n+4}$$

$$h^2u''_{i,n+\frac{10}{3}}=-\frac{4}{3}u_{i,n}+5u_{i,n+1}-6u_{i,n+2}+\frac{7}{3}u_{i,n+3}+\frac{379421111}{24638916960}h^4b_{i,n}+\frac{1819245629}{12319458480}h^4b_{i,n+1}+\frac{18241613111}{16425944640}h^4b_{i,n+2}-\frac{1353949}{1759922640}h^4b_{i,n+3}+\frac{9599899}{40557888}h^4b_{i,n+\frac{10}{3}}-\frac{114919289}{3079864620}h^4b_{i,n+4}$$

$$h^2u''_{i,n+4}=-2u_{i,n}+7u_{i,n+1}-8u_{i,n+2}+3u_{i,n+3}+\frac{7343143}{202789440}h^4b_{i,n}+\frac{1130099}{7242480}h^4b_{i,n+1}+\frac{64469357}{33798240}h^4b_{i,n+2}-\frac{12289}{7242480}h^4b_{i,n+3}+\frac{323217}{375536}h^4b_{i,n+\frac{10}{3}}-\frac{1705189}{40557888}h^4b_{i,n+4}$$

$$h^3u'''_{i,n}=-u_{i,n}+3u_{i,n+1}-3u_{i,n+2}+u_{i,n+3}-\frac{47872339}{135192960}h^4b_{i,n}-\frac{18007093}{16899120}h^4b_{i,n+1}-\frac{519415}{9012864}h^4b_{i,n+2}+\frac{4919}{2414160}h^4b_{i,n+3}-\frac{533331}{15021440}h^4b_{i,n+\frac{10}{3}}+\frac{1455383}{135192960}h^4b_{i,n+4}$$

$$h^3u'''_{i,n+1} = -u_{i,n} + 3u_{i,n+1} - 3u_{i,n+2} + u_{i,n+3} + \frac{3341887}{135192960}h^4b_{i,n} - \frac{4745537}{16899120}h^4b_{i,n+1} - \frac{3926283}{15021440}h^4b_{i,n+2} - \frac{1493}{2414160}h^4b_{i,n+3} + \frac{381807}{15021440}h^4b_{i,n+\frac{10}{3}} - \frac{990179}{135192960}h^4b_{i,n+4}$$

$$h^3u'''_{i,n+2} = -u_{i,n} + 3u_{i,n+1} - 3u_{i,n+2} + u_{i,n+3} - \frac{239451}{15021440}h^4b_{i,n} + \frac{1435033}{5633040}h^4b_{i,n+1} + \frac{12884237}{45064320}h^4b_{i,n+2} + \frac{89}{160944}h^4b_{i,n+3} - \frac{515187}{15021440}h^4b_{i,n+\frac{10}{3}} + \frac{135551}{15021440}h^4b_{i,n+4}$$

$$h^3u'''_{i,n+3} = -u_{i,n} + 3u_{i,n+1} - 3u_{i,n+2} + u_{i,n+3} + \frac{4181663}{135192960}h^4b_{i,n} + \frac{53539}{3379824}h^4b_{i,n+1} + \frac{53128063}{45064320}h^4b_{i,n+2} - \frac{3397}{2414160}h^4b_{i,n+3} + \frac{5008527}{15021440}h^4b_{i,n+\frac{10}{3}} - \frac{7804483}{135192960}h^4b_{i,n+4}$$

$$h^3u'''_{i,n+\frac{10}{3}} = -u_{i,n} + 3u_{i,n+1} - 3u_{i,n+2} + u_{i,n+3} + \frac{1280470423}{32851889280}h^4b_{i,n} - \frac{87243599}{4106486160}h^4b_{i,n+1} + \frac{2747786155}{2190125952}h^4b_{i,n+2} - \frac{1044683}{586640880}h^4b_{i,n+3} + \frac{86821621}{13192960}h^4b_{i,n+\frac{10}{3}} - \frac{2610001931}{32851889280}h^4b_{i,n+4}$$

$$h^3u'''_{i,n+4} = -u_{i,n} + 3u_{i,n+1} - 3u_{i,n+2} + u_{i,n+3} + \frac{3004333}{135192960}h^4b_{i,n} + \frac{866251}{16899120}h^4b_{i,n+1} + \frac{16965903}{15021440}h^4b_{i,n+2} - \frac{2249}{2414160}h^4b_{i,n+3} + \frac{16884909}{15021440}h^4b_{i,n+\frac{10}{3}} + \frac{4703339}{27038592}h^4b_{i,n+4}$$

4.2 Numerical Examples

In this section, seven numerical examples are considered. The examples were solved using the third order and second order CLMMs of step 3 derived in this research and also the fourth order CLMMs of step 4. Some of these examples were solved using the Runge Kutta Method. Comparisons are made between the proposed methods and the Runge Kutta by obtaining the results and by obtaining errors $E = Maximum\left|y(x_n) - y_n\right|$ in the interval of integration when comparing with the work in Jator *et al.*, (2018). It should be noted that the number of function evaluations (NFEs) involved in implementing the

CLMMs is $N \times k$ in the entire range of integration. Other examples considered include Blasius equation, Sakiadis equation, Falkner-Skan equation and Squeezing flow equation.

Remark 4.2.1. *The BUM from equations (3.17) to (3.28) will be referred to as BUM3, the BUM from* equations *(3.31) to (3.38) will be referred to as BUM2 and the BUM from equations (3.40) to (3.58) will be referred to as BUM4 in this section.*

Problem 1: Consider the third order boundary value problem (Jator *et al.*, 2018)

$$y'''(x) - xy(x) = (x^3 - 2x^2 - 5x - 3)e^x, 0 < x < 1$$

$$y(0) = y(1) = 0, y'(0) = 1$$

Exact: $y(x) = (x - x^2)e^x$

Problem 2: Blasius Equation

$$2y''' + yy'' = 0$$

$$y(0) = 0, y'(0) = 0, y'(\infty) = 1$$

Problem 3: Sakiadis flow

$$2y''' + yy'' = 0$$

$$y(0) = 0, y'(0) = 1, y'(\infty) = 0$$

Problem 4: Falkner-Skan Equation

$$y'''(\eta) + \beta_0 y(\eta) y''(\eta) + \beta\left(1 - y'(\eta)^2\right) = 0$$

$$y(0) = 0, y'(0) = 0, \lim_{\eta \to \infty} y'(\eta) = 1$$

Problem 5: Stagnation Point Flow under the influence of MHD

$$y''' - y'^2 + 1 + yy' - M(y' - 1) = 0$$

$$s'' + ys' - y's - Ms = 0$$

$$y(0) = k, y'(0) = \alpha, y'(\infty) = 1$$

$$s(0) = 1, s(\infty) = 0$$

Problem 6: MHD Stagnation Point Flow and Heat Transfer due to Nanofluid

$$y''' + yy'' - y'^2 + M(A - y') + A^2 = 0$$

$$s'' + \Pr ys' + \Pr Nbh's' + \Pr Nts'^2 = 0$$

$$h'' + Lefh' + \frac{Nt}{Nb} s'' = 0$$

With boundary conditions

$$y(0) = 0, y'(0) = 1, s(0) = 1, h(0) = 1$$

$$y'(\infty) = A, s(\infty) = 0, h(\infty) = 0$$

Problem 7: MHD Stagnation Point Flow through a Porous Stretching/Shrinking Sheet

$$y''' - [y' + M + K]y' + 1 + M + K + yy'' = 0,$$

$$s'' - [2y's - EcM(y' - 1)^2 - Ecy''^2 - ys']\Pr = 0$$

$$y(0) = \beta, y'(0) = 0, y'(\infty) = 1$$

$$s(0) = 1, s'(\infty) = 0$$

Problem 8: Squeezing Flow

$$u^{iv} - s(\eta u''' + 3u'' + u''u' - uu''') - M^2 u'' = 0$$

$$u(0)=0, u''(0)=0, u(1)=1, u'(1)=0$$

Problem 9: Consider the oscillatory problem arising from ship dynamics

$$u^{iv} + 3u'' + u(2 + \varepsilon \cos \lambda t) = 0;$$

$$\mathrm{u}(0) = 1, \mathrm{u}'(0) = 0, \mathrm{u}''(0) = 0, \mathrm{u}'''(0) = 0$$

Where $\lambda = 0$ for existence of the exact solution

$$\mathrm{u}(t) = 2\cos t - \cos\left(\sqrt{2t}\right)$$

Problem 10: Consider the non-linear problem

$$u^{iv} = (u')^2 - uu''' + 4t + e^t(1 - 4t + t^2);$$

$$\mathrm{u}(0) = 1, \mathrm{u}'(0) = 1, \mathrm{u}''(0) = 3, \mathrm{u}'''(0) = 1$$

Exact solution $u(t) = \mathrm{t}^2 + e^t$

Problem 11: Consider the non-linear two-point BVP

$$u^{iv}(x) + \left(\mathrm{u}''(x)\right)^2 = \sin(x) + \sin^2(x), x \in [0,1]$$

$$\mathrm{u}(0) = 0, \mathrm{u}'(0) = 1, \mathrm{u}(1) = \sin(1), \mathrm{u}'(1) = \cos(1)$$

with exact solution $u(x) = \sin x$

Problem 12: Consider the linear BVP

$$u^{iv}(x) - \mathrm{u}''(\mathrm{x}) - \mathrm{u}(\mathrm{x}) = \mathrm{e}^x(x - 3), x \in [0,1]$$

$$u(0) = u'(0) = 0, u(1) = 3, u'(1) = -e$$

with exact solution $u(x) = (1 - x)e^x$

Problem 13: Consider the “good” Boussinesq equation

$$u_{tt} = u_{xx} + u_{xx}^2 - u_{xxxx}, 0 \le x \le 1, t \ge 0$$

with appropriate boundary conditions

$$\left.\begin{array}{l} \mathrm{u}(0,t) = 0, \mathrm{u}(1,t) = 0 \\ u_{xx}(0,t) = 0, \mathrm{u}_{xx}(1,t) = 0, t > 0 \end{array}\right\}$$

The exact solution for this problem is

$$\mathrm{u}(x,t) = -A\sec h^2\left(\sqrt{\frac{A}{6}}(x - ct + v_0)\right) - \left(b + \frac{1}{2}\right)$$

Here c is the velocity, A is amplitude of the pulse, b is an arbitrary parameter and v_0 is the initial position. Using the same theoretical parameters as in (Mohanty and Kaur, 2016) A = 0.369, b = −1/2 and c = 0.868

Upon semi discretisation of the time variable, we obtain

$$\frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta t)^2} + \frac{d^4u_m}{dx^4} - \frac{d^2u_m}{dx^2} - \frac{d^2u_m^2}{dx^2} = g_m, 0 \le x \le 1, m = 1, \ldots, M - 1$$

where $\Delta t, t_m, m = 0,1,\ldots, M, u, u_m(x) \approx u(x,t_m)$, $g = \left[g_1(x), \ldots, g_m(x)\right]^T$ and $g_m \approx g(x,t_m) = 0$, which is expressed in the form

$$u^{(iv)} = f(x,u,u',u'',u''') = Au + g$$

A is an $M-1 \times M-1$ matrix and $g_m = 0$.

Problem 14: Consider the following homogenous fourth-order parabolic equation

$$u_{tt} = u_{xxxx} = 0, 0 \le x \le 1, t \ge 0$$

subject to the initial condition

$$u(x,0)=\sin \pi x, u_t(x,0)=-\pi^2 \sin \pi x$$

and with the appropriate boundary conditions

$$\left.\begin{aligned} &u(0,t)=0, u(1,t)=0, \\ &u_{xx}(0,t)=0, u_{xx}(1,t)=0 \end{aligned}\right\}$$

The exact solution for this problem is

$$u(x,t)=e^{-\pi^2 t}\sin \pi x$$

Upon semi discretisation of the time variable, we obtain

$$\frac{u_{m+1}-2u_m+u_{m-1}}{(\Delta t)^2}-\frac{d^4 u_m}{dx^4}=g_m, 0\le x\le 1, m=1,\ldots,M-1$$

Where

$$\Delta t=(L_4-L_3)/\mathrm{M}, t_m=L_3+m\Delta t, m=0,1,\ldots,M, u=\left[u_1(x),\ldots,u_M(x)\right]^T, u_m(x)\approx u(x,t_m)$$

g and g_m are as expressed in problem 13, which is expressed in the form

$$u^{(iv)}=f(x,u,u',u'',u''')=Au+g$$

A is as expressed in problem 14 and $g_m=0$

Problem 15: Consider the fourth order parabolic equation with constant coefficient

$$u_{tt}+u_{xxxx}=(\pi^4-1)\sin \pi x \cos t, 0\le x\le 1, t\ge 0$$

subject to the initial conditions

$$u(x,0)=\sin \pi x, u_t(\mathrm{x},0)=0$$

and with the boundary conditions

$$\mathrm{u}(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0$$

with the exact solution

$$\mathrm{u}(x,t) = \sin \pi x \cos t$$

Upon semi discretisation of the time variable, we obtain

$$\frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta t)^2} + \frac{d^4 u_m}{dx^4} = g_m, 0 \le x \le 1, m = 1, \ldots, M - 1$$

Where

$$\Delta t = (L_4 - L_3)/\mathrm{M}, t_m = L_3 + m\Delta t, m = 0,1,\ldots,M, u = [u_1(x),\ldots,u_M(x)]^T, u_m(x) \approx u(x,t_m)$$

g as expressed in problem 13 and $g_m(x) \approx g(x,t_m) = (\pi^4 - 1)\sin \pi x \cos t_m$, which is expressed in the form

$$u^{(iv)} = f(x,u,u',u'',u''') = Au + g$$

A is as expressed in problem 14 and g is a vector of constants.

Table 4.1: Comparison of the Errors from BUM3, Jator *et al*. (2018) and ETRs (in Jator *et al*. (2018)) for Problem 1.

N	BUM3	CPU	Jator *et al*. (2018)	CPU	ETRs
6	8.33E-07	0.240	1.525E-05	0.288	1.089E-02
12	2.06E-08	0.253	9.257E-07	0.257	7.666E-05
24	7.0E-10	0.261	5.873E-08	0.263	5.108E-06
48	6.52E-11	0.328	3.683E-09	0.344	3.300E-07
96	4.31E-12	0.354	2.305E-10	0.359	2.098E-08
192	2.13E-13	0.416	1.428E-11	0.438	1.323E-09

The Maximum of the absolute errors were obtained in the entire interval of integration. Tables 4.1 shows the comparison between the ETRs, Jator et. al (2018) and BUM3. It is observed that the BUM3 performs better than the ETRs and Jator et. al (2018) respectively in terms of accuracy as well as CPU time. Hence, the BUM3 are quite accurate and efficient.

Table 4.2: Comparison of the Solutions from BUM3 and Runge-Kutta Method for Problem 2

	BUM3				Runge-Kutta			
x	N	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	N
1.0	9	1.021157329	0.5063049940	0.9381906626	1.021157016	0.506305291	0.93810698	27
2.0	17	0.5442717691	1.051664551	0.3810337080	0.5442717609	1.051664633	0.381033607	51
3.0	25	0.4045496973	1.679698960	0.1689551177	0.4045497078	1.6796990467	0.168955073	75
4.0	33	0.3527462516	2.432249676	0.06202511200	0.3527462779	2.432249926	0.0620251103	99
5.0	41	0.33256595103	3.3170985421	0.0155692563	0.3325659529	3.3170985488	0.0155692560	123

Table 4.2 shows the validity of the propose method (BUM3) and classical R-K method for problem 2. It is observed that with small values of N, the propose methods shows an excellent agreement with R-K method.

Table 4.3: Comparison of the Solutions from BUM3 and Runge-Kutta Method for Problem 3.

	BUM3				Runge-Kutta			
x	**N**	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	**N**
1.0	9	-1.062106604	0.4858145149	-0.9021137979	-1.0621056881	0.4858148417	-0.9021137490	27
2.0	17	-0.6214631716	0.8954882570	-0.3357451645	-0.6214629182	0.895488335	-0.3357452060	51
3.0	25	-0.5078781704	1.190534705	-0.1428727781	-0.5078780256	1.190534757	-0.1428727865	75
4.0	33	-0.4687973723	1.377935656	-0.06161582430	-0.4687972558	1.3779357168	-0.0616581740	99
5.0	41	-0.4539702818	1.487355776	-0.02661787579	-0.4539701772	1.487355831	-0.0266178690	123

Table 4.3 shows the validity of the propose method (BUM3) and classical R-K method for problem 3. It is observed that with small values of N, the propose methods shows an excellent agreement with R-K method

Table 4.4: Comparison of the Solutions from BUM3 and Runge-Kutta Method for Problem 4.

	BUM3			Runge-Kutta Method		
x	**N**	$y''(x_\infty)$	$y(x_\infty)$	**N**	$y''(x_\infty)$	$y(x_\infty)$
0.1	9	0.5223955323	0.6065298823	27	0.522394253	0.606530550
0.2	17	0.03825982349	1.510386946	51	0.0382595394	1.510388234
0.3	25	0.0014085063	2.502848721	75	0.0014082032	2.502849911
0.4	33	0.0000245898	3.502571462	99	0.0000245779	3.502571249

Table 4.4 shows the validity of the proposed method (BUM3) and classical R-K method for problem 4. It is observed that with small values of N (number of function evaluations), the proposed method shows an excellent agreement with R-K method

Table 4.5: Numerical Comparison for the Stretching Case $(\alpha > 0)$ with the Existing Results for Problem 5

$\alpha \succ 0$	$y''(0)$ **Bhatti *et al.* (2018)**	$y''(0)$ **BUM2**	$-s'(0)$ **Bhatti *et al.* (2018)**	$-s'(0)$ **BUM2**
0	1.23258765	1.232583905	0.81130132	0.8113170417
0.1	1.14656100	1.146557577	0.863451660	0.8634652926
0.2	1.051129994	1.051127244	0.91330283	0.9133157941
0.3	0.94681611	0.9468142651	0.96111587	0.30112933847
0.5	0.71329495	0.7132950814	1.05145843	1.051476251
1	0	0	1.25331413	1.253359472
2	-1.88730667	-1.887402684	1.58956678	1.589740624
3	-10.26474931	-10.26844767	2.33809899	2.3399380450

Table 4.5 show the numerical comparison for Hartmann number (M) and suction/injection parameter (k) for different values of stretching parameter (α > 0). From the table, it is observed that when M = 0 and k = 0, the propose method (BUM3) and (BUM2) are in good agreement with Successive linearization method applied by Bhatti *et al.* (2018).

Table 4.6: Numerical comparison for the Shrinking Case $(\alpha < 0)$ with the Existing Results for Problem 5

$\alpha \prec 0$	$y''(0)$ **Bhatti *et al.* (2018)**	$y''(0)$ **BUM2**	$-s'(0)$ **Bhatti *et al.* (2018)**	$-s'(0)$ **BUM2**
-0.25	1.40224081	1.402238699	0.66857275	0.6686022783
-0.5	1.49566976	1.495675888	0.50144758	0.5015139670
-0.75	1.48929824	1.489330566	0.29376251	0.2939313809
-1.0	1.32881688	1.328961913	0	0
-1.15	1.08223117	1.082786939	-0.29799548	-0.2961961037
-1.2465	0.58428167	0.6173065669	-0.94776590	-0.8869752488
-1.2474	*	0.5741833003	*	-0.9561670930

Note: * implies no given solution for the given value of α .

Table 4.6 show the numerical comparison for Hartmann number (M) and suction/injection parameter (k) for different values of stretching parameter (α<0). From the table, it is observed that when M = 0 and k = 0, the propose method (BUM3) and (BUM2) are in good agreement with Successive linearization method applied by Bhatti *et al.* (2018).

Table 4.7: Comparison of Values of $-y''(0)$ with Ibrahim *et al*. (2013) when M=0 for Problem 6

A	**BUM3**	**Ibrahim *et al.* (2013)**
0.01	1.001814580	0.9980
0.1	0.9707740182	0.9694
0.2	0.9185865930	0.9181
0.5	0.6672900043	0.6673
2.0	-2.017484069	-2.0175
3.0	-4.728934014	-4.7292

In order to assess the accuracy of the present method (BUM3), a comparison with previously reported data available in the literature has been made. It is clear from Table 4.7 that the numerical values of the skin friction coefficient $-y''(0)$ in this paper for different values of A, when M = 0 are in excellent agreement with the result published in Ibrahim et. al (2013). A comparison of the results with literature values shown in Table 4.7 shows excellent agreement and therefore it is confident that the proposed method is highly accurate

Table 4.8: Comparison of Results for Local Nusselt Number with Ibrahim *et al*. (2013) for Problem 6

Pr	A	BUM2	Ibrahim *et al.* (2013)
1	0.1	0.6121655816	0.6022
	0.2	0.6306184745	0.6245
	0.5	0.6936280163	0.6924
1.5	0.1	0.7785162969	0.7768
	0.2	0.7981803167	0.7971
	0.5	0.8650307873	0.8648

To further validate the proposed method (BUM2), comparison of local Nusselt number $-s'(0)$ for different values of velocity ratio parameter A and Prandtl number Pr by ignoring the effects of M, Nb and Nt parameters has been shown in Table 4.8, which is also in excellent agreement with Ibrahim et. al (2013). A comparison of the results with literature values shown in Table 4.7 shows excellent agreement and therefore it is confident that the proposed method is highly accurate

Table 4.9: Numerical comparison with Ali Abbas *et al*. (2019) with Different values of $y''(0)$ for Shrinking Case ($\alpha < 0$) for Problem 7

	BUM2	Ali Abbas *et al*. (2019)
α	$M = 0, K = 0$	$M = 0, K = 0$
-0.25	1.4022	1.4023
-0.50	1.4957	1.4957
-1.0	1.3289	1.3289
-1.10	1.1870	1.1868
-1.15	1.0828	1.0823
-1.18	1.0013	1.0004
-1.20	0.9337	0.9324

Numerical comparability has been brought through Table 4.9 with the existing literature of Ali Abbas *et al.* (2019) by taking M =0,K =0 for shrinking case ($\alpha < 0$). It is found that the current methos (BUMs) are in excellent agreement with the existing literature which assures the validity of the present flow problem.

Table 4.10: Effects of S on Skin Friction for Problem 8

S	BUM4 $-f''(1)$	Mustapfa *et al.* (2012) $-f''(1)$	Das and Mohammed (2016) $-f''(1)$
-1	2.170092	2.170090	2.170091
-0.5	2.614038	2.614038	2.614038
0.01	3.007134	3.007134	3.007134
0.5	3.336449	3.336449	3.336449
2	4.167389	4.167389	4.167389

In order to ascertain the accuracy of the numerical results of (BUM4) of problem 8 with the absence of a magnetic field, the comparison was made with the data of Mustafa *et al*. (2012) and Das and Mohammed (2016). The $-y''(1)$ values were calculated for various S values. Excellent agreement was found between the results, as shown in Table 4.10. Thus, the use of the present numerical code for the current model was justified.

Table 4.11: Comparison of Errors from for Problem 9 (h = 0.003125)

t	Error in BUM4	Error in Familua and Omole (2017) [Block Mode]	Error in Familua and Omole (2017) [P-C Mode]	Error in Ukpebor *et al*. (2020)
0.003125	6.980×10^{-28}	6.686×10^{-13}	5.686×10^{-10}	1.900×10^{-19}
0.006250	1.119×10^{-26}	1.458×10^{-11}	1.768×10^{-10}	2.300×10^{-19}
0.009375	4.342×10^{-26}	1.083×10^{-10}	5.910×10^{-09}	8.600×10^{-19}
0.001250	1.159×10^{-25}	3.918×10^{-10}	5.768×10^{-09}	1.380×10^{-18}
0.015625	2.481×10^{-22}	1.025×10^{-09}	1.100×10^{-08}	3.530×10^{-18}
0.018750	3.818×10^{-21}	2.217×10^{-09}	6.899×10^{-08}	5.310×10^{-18}
0.021875	1.516×10^{-20}	4.226×10^{-09}	4.636×10^{-08}	8.880×10^{-18}
0.025000	3.950×10^{-20}	7.358×10^{-09}	5.788×10^{-07}	3.922×10^{-17}
0.028125	8.459×10^{-20}	1.197×10^{-08}	2.246×10^{-07}	5.846×10^{-17}
0.031250	1.632×10^{-20}	1.846×10^{-08}	2.846×10^{-07}	8.477×10^{-17}

Table 4.11 shows the comparison of absolute errors between between BUM4, Familua and Omole (2017) and Ukpebor *et al.* (2020). It is shown that the newly block method gives a better approximation to the application problem in the dynamics of ship

Table 4.12: Comparison of Errors for Problem 10

t	BUM4	Error in Familua and Omole (2017)	Error in Kuboye *et al.* (2020)
0.003125	1.402×10^{-14}	$1.149 \text{ x } 10^{-12}$	1.788×10^{-10}
0.006250	2.140×10^{-13}	$1.885 \text{ x } 10^{-11}$	1.134×10^{-08}
0.009375	8.424×10^{-13}	$9.780 \text{ x } 10^{-11}$	1.196×10^{-07}
0.012500	2.179×10^{-12}	$3.166 \text{ x } 10^{-10}$	6.401×10^{-07}
0.015625	4.622×10^{-12}	$7.909 \text{ x } 10^{-10}$	2.349×10^{-06}
0.018750	8.700×10^{-12}	$1.676 \text{ x } 10^{-09}$	6.573×10^{-06}
0.021875	1.500×10^{-11}	$3.169 \text{ x } 10^{-09}$	1.610×10^{-05}
0.025000	2.414×10^{-11}	$5.512 \text{ x } 10^{-09}$	3.501×10^{-05}
0.028125	3.685×10^{-11}	$8.995 \text{ x } 10^{-09}$	6.985×10^{-05}
0.031250	5.401×10^{-11}	$1.396 \text{ x } 10^{-08}$	1.245×10^{-04}

From the Table 4.12, the BUM4s reveal that the method is superior in terms of accuracy when compared with other existing methods in the literature.

Table 4.13: Error of methods for Problem 11

x	Error in BUM4	Error in Noor and Mohyud-Din (2007)
0.0	0.0	0.0
0.1	9.25×10^{-10}	7.78×10^{-8}
0.2	3.01×10^{-9}	2.72×10^{-7}
0.3	5.25×10^{-9}	5.24×10^{-7}
0.4	6.82×10^{-9}	7.77×10^{-7}
0.5	7.31×10^{-9}	9.71×10^{-7}
0.6	6.68×10^{-9}	1.05×10^{-6}
0.7	5.14×10^{-9}	9.63×10^{-7}
0.8	2.99×10^{-9}	6.84×10^{-7}
0.9	9.48×10^{-10}	2.71×10^{-7}
1.0	0.0	0.0

Table 4.13 shows the comparison of the exact solution and the numerical solutions obtained using BUM for Problem 12 with N=10, at the points x = 0(0.1)1.0. it is observed that the BUM4 methods is perform better than the method found in Noor and Mohyud-Din (2007).

Table 4.14: Error of methods for Problem 12

x	Error in BUM4	Error in Noor and Mohyud-Din (2007)	Error in Costabile and Napoli (2015) m = 5	Error in Costabile and Napoli (2015) m = 12
0.1	9.33×10^{-14}	2.00×10^{-10}	6.33×10^{-9}	1.97×10^{-13}
0.2	3.09×10^{-14}	7.00×10^{-10}	7.50×10^{-9}	1.56×10^{-13}
0.3	5.53×10^{-14}	1.35×10^{-9}	3.12×10^{-10}	1.83×10^{-13}
0.4	7.42×10^{-14}	2.00×10^{-9}	3.91×10^{-9}	2.06×10^{-13}
0.5	8.21×10^{-14}	2.51×10^{-9}	9.89×10^{-10}	2.02×10^{-13}
0.6	7.76×10^{-14}	2.72×10^{-9}	1.75×10^{-9}	2.25×10^{-13}
0.7	6.22×10^{-14}	2.21×10^{-9}	2.91×10^{-9}	2.04×10^{-13}
0.8	3.82×10^{-14}	1.80×10^{-9}	1.01×10^{-8}	1.98×10^{-13}
0.9	1.27×10^{-14}	7.25×10^{-10}	7.80×10^{-9}	2.18×10^{-13}
1.0	0.0	0.0	0.0	0.0

Table 4.14 shows the comparison of the exact solution and the numerical solutions obtained using BUM at the points x = 0(0.1)1.0 for Problem 13. Table 4.8 show the errors obtained in Problem 4 with Er = |yapp − yex|, for h = 0.1, as compared with methods in Costabile and Napoli (2015) and Noor & MohyudDin (2007). This shows the superiority of the BVM developed over the methods in the citedis a comparison of errors from BUM4, Noor and Mohyud-Din (2007) and Constabile and Napoli (2015).

Table 4.15: Maximum Error for Problem 13

Time	Parameter	BUM4	CPU (s.)	Mohanty and Kaur (2016)	Modebei *et al*. (2020a)
t = 0.5	$h = \frac{1}{40}$	1.6741(-12)	0.59	7.8998(-07)	1.7274(-10)
t = 1.0	$h = \frac{1}{60}$	1.6842(-14)	1.88	7.5071(-09)	1.7531(-12)
t = 1.5	$h = \frac{1}{80}$	3.2191(-16)	2.11	5.7588(-11)	2.2691(-14)
t = 2.0	$h = \frac{1}{100}$	2.7605(-18)	3.48	2.9068(-13)	1.6471(-16)

Table 4.15 for different values of t and h, shows the maximum absolute errors obtained, in comparison with the Modebei *et al.* (2020a) and the method in Mohanty & Kaur (2016). This shows the superiority of the BUM4 over both methods.

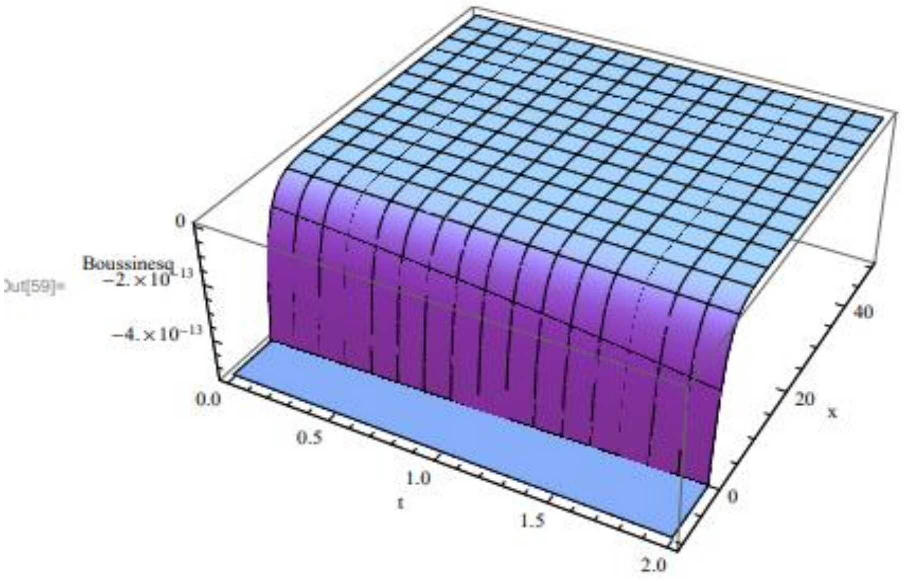

Figure 4.1: Surface plot for the Numerical Solution for Problem 13

Figure 4.1 shows the graphical representation with surface plot for the numerical solution (shaded region) for Problem 13.

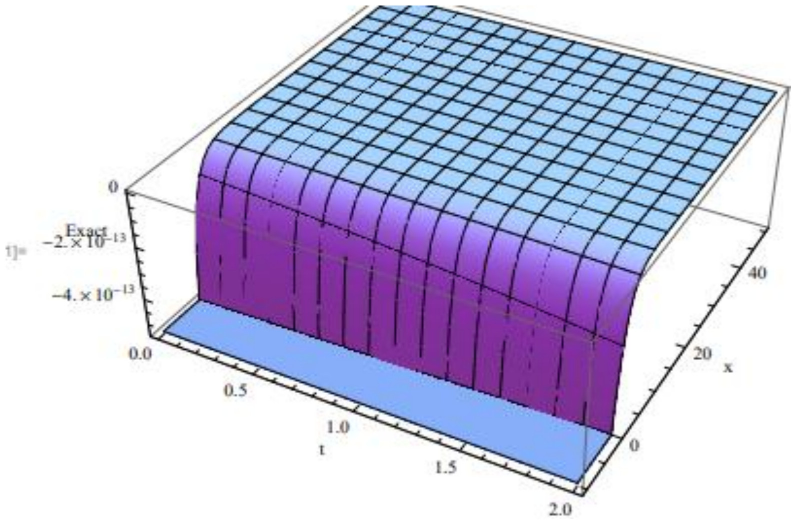

Figure 4.2: Surface plot for the Exact Solution for Problem 13

Figure 4.2 shows the graphical representation with surface plot for the exact solution (shaded region) for Problem 13.

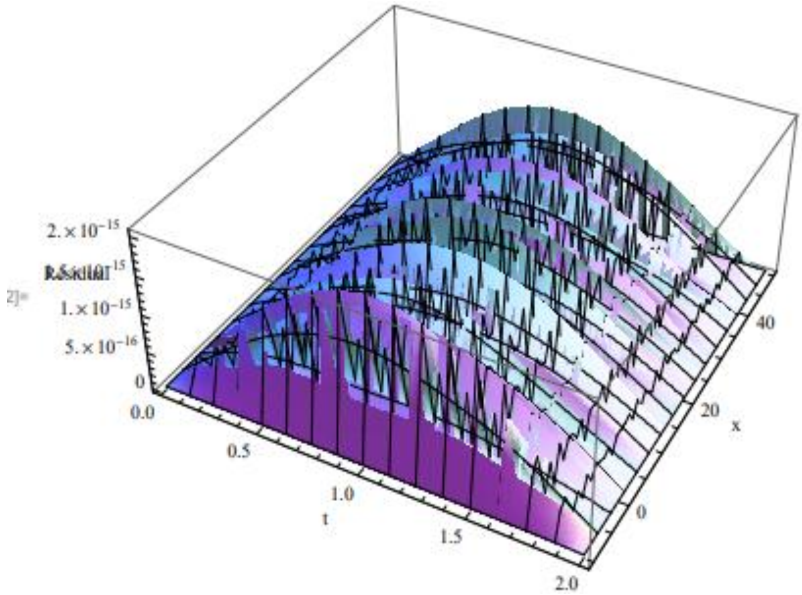

Figure 4.3: Surface plot for the Residual for Problem 13

Figure 4.3 shows the graphical representation with surface plot for the residual or error (shaded region) for Problem 13.

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CPUTime = TimeUsed
18.235
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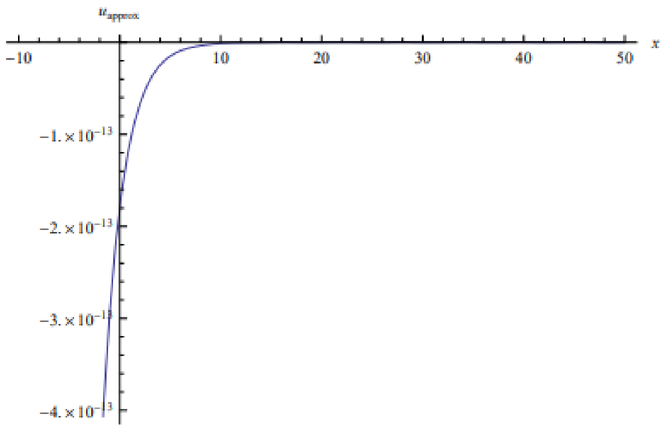

Figure 4.4: Global Error Plot for Problem 13

Figure 4.4 shows the cumulative error caused by many iterations for problem at discrete time steps which is equally spaced for problem 13.

Table 4.16: Numerical Results with t = 0 for Problem 14

x	Exact	Approximate	Error in BUM4
0.1	0.3090169944	0.3090169936	1.49941×10^{-7}
0.2	0.5877852524	0.5877849672	2.85204×10^{-7}
0.3	0.8090169944	0.5877848598	3.9255×10^{-7}
0.4	0.9510565165	0.9510560550	4.6147×10^{-7}
0.5	1.0000000000	0.9999995148	4.85219×10^{-7}
0.6	0.9510565163	0.9510560548	4.6147×10^{-7}
0.7	0.8090169941	0.8090166016	3.9255×10^{-7}
0.8	0.5877852522	0.5877849670	2.85204×10^{-7}
0.9	0.3090169936	0.3090168437	1.49941×10^{-7}
1.0	1.8892×10^{-16}	1.8892×10^{-16}	0

Table 4.16 shows the comparison of the exact solution, the numerical solutions obtained using BUM4 and the absolute error at the points x = 0((0.1)1 for Problem 14

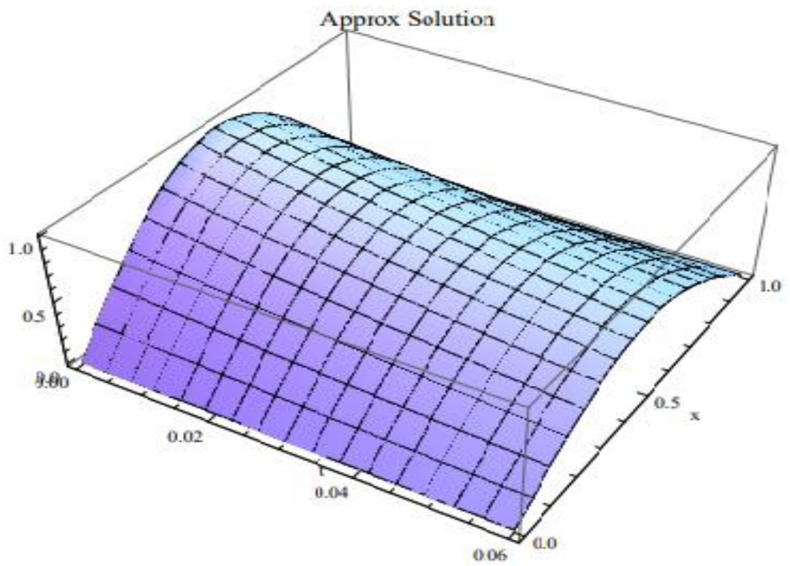

Figure 4.5: Surface Plot for the Approximate Solution of Problem 14

Figure 4.5 shows the graphical representation with surface plot for the approximate solution (shaded region) for Problem 14.

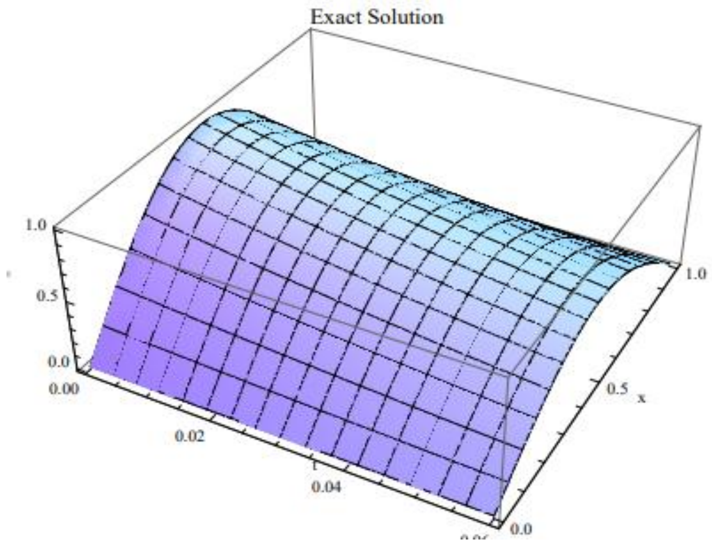

Figure 4.6: Surface Plot for the Exact Solution of Problem 14

Figure 4.6 shows the graphical representation with surface plot for the exact solution (shaded region) for Problem 14.

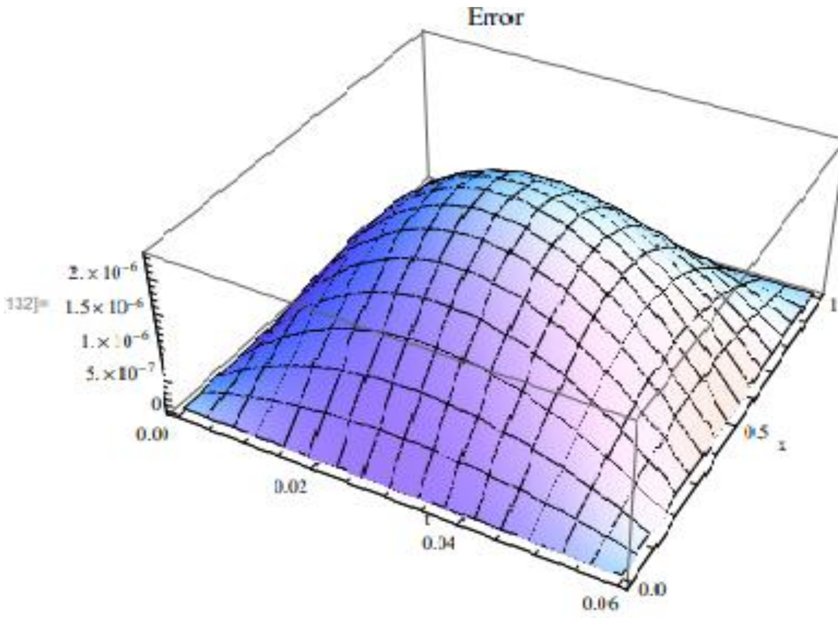

Figure 4.7: Surface Plot for the Error of Problem 14

Figure 4.7 shows the graphical representation with surface plot for error (shaded region) for Problem 14.

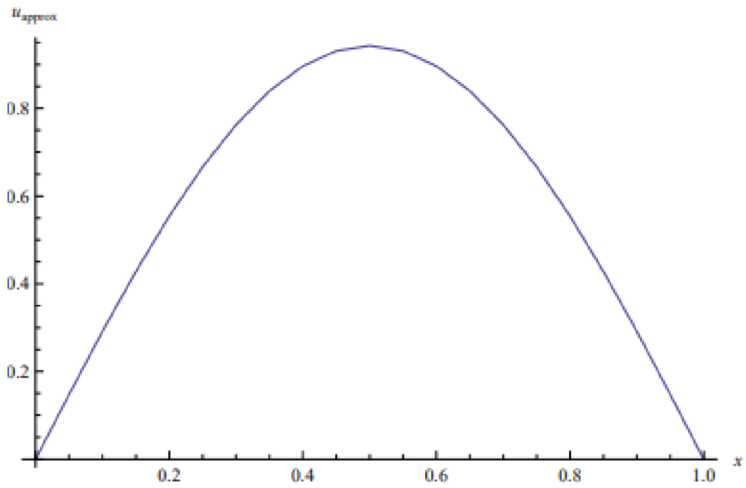

Figure 4.8: Global Error Plot for Problem 14

Figure 4.8 shows the cumulative error caused by many iterations for problem at discrete time steps which is equally spaced for problem 14.

Table 4.17 Numerical Results with t = 0 for Problem 15

x	Exact	Approximate	Error in BUM4
0.1	0.3090169944	0.3090168478	1.49941×10^{-14}
0.2	0.587785252292473	0.587785252292347	1.26232×10^{-13}
0.3	0.809016994374947	0.809016994374773	1.74416×10^{-13}
0.4	0.951056516295154	0.951056516294949	2.04947×10^{-13}
0.5	1.000000000000000	0.999999999999785	2.15272×10^{-13}
0.6	0.951056516295155	0.951056516294949	2.06168×10^{-13}
0.7	0.809016994374951	0.809016994374775	1.75637×10^{-13}
0.8	0.587785252292477	0.587785252292350	1.27121×10^{-13}
0.9	0.309016994374951	0.309016994374883	6.83897×10^{-14}
1.0	1.22465×10^{-16}	1.22465×10^{-16}	0

Table 4.17 shows the comparison of the exact solution, the numerical solutions obtained using BUM4 and the absolute error at the points x = 0((0.1)1 for Problem 15.

Table 4.18: Maximum Error for Problem 15

Method	k	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
BUM4	10	6.92E-14	1.23E-13	1.77E-13	2.04E-13	2.16E-13
	16	9.10E-14	1.62E-13	2.36E-13	2.68E-13	2.85E-13
Rashidinia and	10	2.91E-6	1.73E-6	1.60E-6	2.23E-6	2.60E-7
Mohammadi (2010)	16	4.47E-7	2.66E-7	1.37E-7	1.55E-7	1.57E-7
Modebei *et al*.	10	4.88E-10	8.56E-10	1.28E-9	1.39E-9	1.58E-9
(2020a)	16	8.28E-10	1.50E-9	2.17E-9	2.43E-9	2.48E-9

Table 4.18 at different values of k and points of x, shows the maximum absolute errors obtained, in comparison with the BUM4 and the method in Rashidinia and Mohammadi, (2010) and Modebei *et al*. (2020a). This shows the superiority of the BUM4.

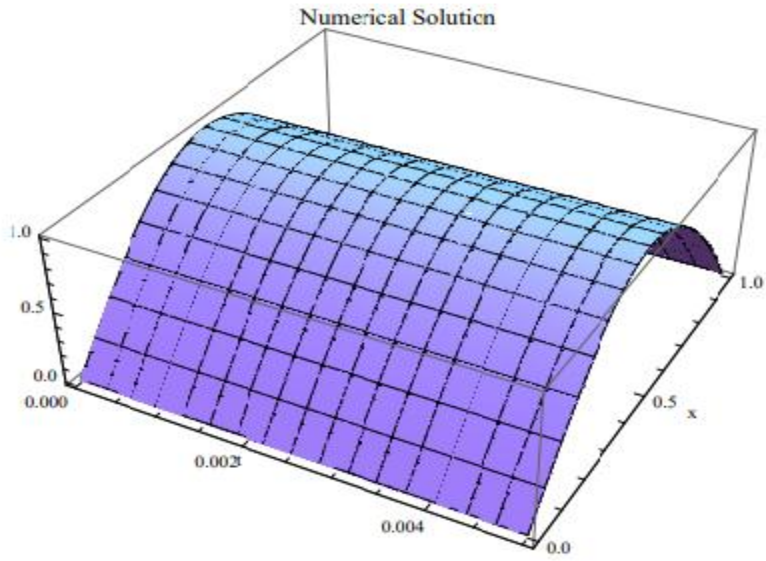

Figure 4.9: Surface Plots for the Numerical Solution of Problem 15

Figure 4.9 shows the graphical representation with surface plot for the numerical solution (shaded region) for Problem 15.

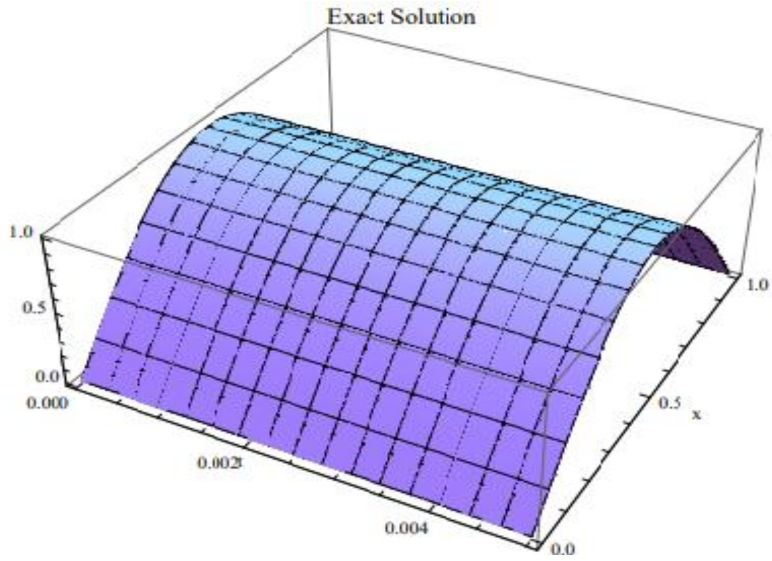

Figure 4.10: Surface Plot for the Exact Solution of Problem 15

Figure 4.10 shows the graphical representation with surface plot for the exact solution (shaded region) for Problem 15.

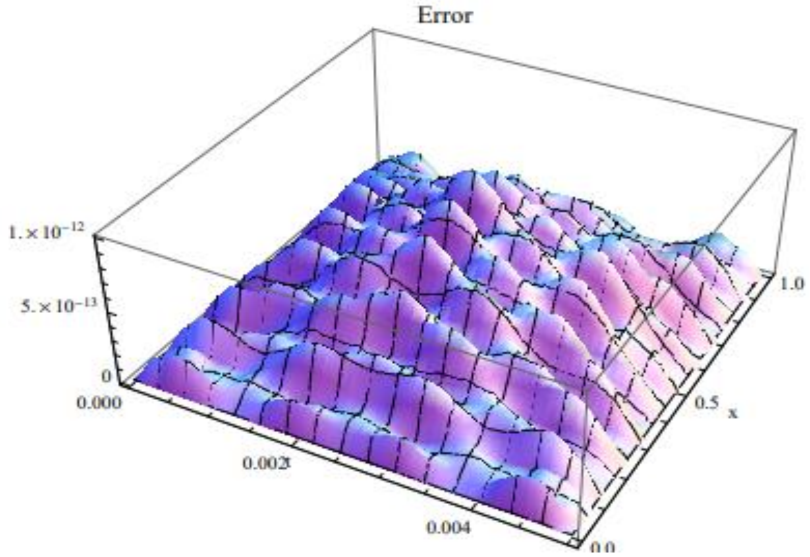

Figure 4.11: Surface Plot for the Error of Problem 15

Figure 4.11 shows the graphical representation with surface plot for the error (shaded region) for Problem 15.

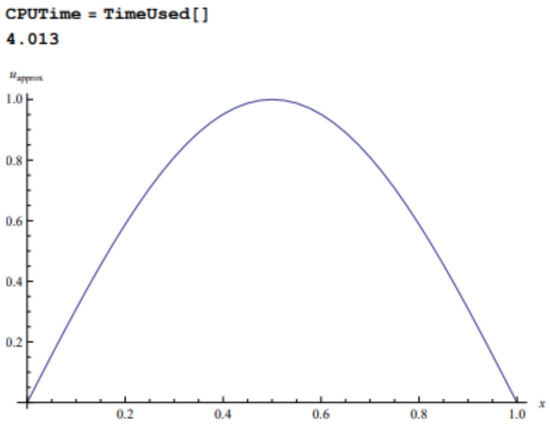

Figure 4.12: Global Error Plot for Problem 15

Figure 4.12 shows the cumulative error caused by many iterations for problem at discrete time steps which is equally spaced for problem 15.

4.3 Discussion of Results

Table 4.1 shows a comparison of the errors from Proposed Methods, Jator *et al*. (2018) and ETRs (in Jator *et al*. (2018)) for Problem 1. The behaviour of the proposed methods was compared using the results obtained from the existing method in the literature. These results were compared to results in Jator *et al*. (2018) and the Extended Trapezoidal Methods (ETRs). The results of BUM3 perform better than the methods of Jator *et al*. (2018) and the ETRs.

Table 4.2 shows the validity of the propose method (BUM3) and classical R-K method for problem 2. It is observed that with small values of N, the propose methods shows an excellent agreement with R-K Method.

Table 4.3 shows the validity of the propose method (BUM3) and classical R-K method for problem 3. It is observed that with small values of N, the propose methods shows an excellent agreement with R-K

Table 4.4 shows the validity of the propose method (BUM3) and classical R-K method for problem 4. It is observed that with small values of N, the propose methods shows an excellent agreement with R-K method

Tables 4.5 and 4.6 show the numerical comparison for the stretching case $(\alpha \succ 0)$ and the shrinking case $(\alpha \prec 0)$ with the existing results for M = k = 0 respectively for Problem 5. Both tables show the numerical comparison for Hartmann number, M, and suction/injection parameter, k, for different values of stretching and shrinking parameter, it can be observed that when $M = k = 0$ for both cases of α, the results from the proposed method are in good agreement with those in existing literature.

Tables 4.7 and 4.8 show a comparison of values of $-y''(0)$ when M = 0 and local Nusselt number $-s'(0)$ at $Nt = 0, Nb \to 0,$, for different values of Pr with those found in Ibrahim *et al*. (2013) for Problem 6. It is clear from Table 4.7 that the numerical values of the skin friction coefficient $-y''(0)$ in this research for different values of A, when M = 0 are in agreement with the result published in Ibrahim *et al*. (2013). A further validation of the method used is found in table 4.8 which is a comparison of local Nusselt number $-s'(0)$ for different values of velocity ratio parameter A and Prandtl number Pr by ignoring the effects of M, Nb and Nt parameters. Results are in agreement with Ibrahim *et al*. (2013).

Table 4.9 is for Problem 7 which is a numerical comparison with Ali Abbas *et al*. (2019) for different values of $y''(0)$ for the shrinking case ($\alpha \prec 0$). The table clearly shows that results from the proposed method are in agreement with those from Ali Abbas *et al*. (2019).

Table 4.10 shows a comparison of the effects of various values of squeeze number, S, on the skin friction, local Nusselt number and local Sherwood number. Excellent agreement was found in the comparison. This now validates the use of BUM4.

Table 4.11 shows a comparison of errors from BUM4, Familua and Omole (2017) and Ukpebor *et al.* (2020) with h = 0.003125. Values in the table show that the BUM4 gives a better approximation of the problem in Ship Dynamics.

Table 4.12 compares the errors from BUM4, Familua and Omole (2017) and Kuboye *et al*. (2020) for Problem 10. The superiority of BUM4 in terms of accuracy is seen in the table.

Table 4.13 shows errors obtained from BUM4 and methods found in Noor and Mohyud-Din (2007) and Constabile and Napoli (2015). The table shows that BUM4 gives a better approximation compared to other methods.

Table 4.14 shows a comparison of BUM4, Noor and Mohyud-Din (2007) and Costabile and Napoli (2015). Even though the methods compared with seemed to have produced better results at several points of evaluation, it should be noticed that the method in Noor and Mohyud-Din (2007) and Costabile and Napoli (2015) with m = 12 used polynomials of degree 15 and for m = 5 a polynomial of degree 8 was used. The BUM compares favourably with other methods.

The table 4.15 shows the maximum errors for h and t for the BUM and the method in Modebei *et al.* (2020). The BUM4 is seen to show a good performance for such a problem as Problem 13. Figures 4.1, 4.2 and 4.3 show the graphical representations of surface plots for numerical, exact and residual (all shaded regions) for Problem 13 and Figure 4.4 shows the graph for computer time.

Table 4.16 shows Problem 14's comparison of results for exact solution, numerical solution obtained using BUM4 and error at points x = 0, 0.1, ... ,1.0. Figures 4.5 4.6 and 4.7 show the graphical representation of surface plots for analytical, numerical and error (all shaded regions) for Problem 14. The CPU time graph is shown in figure 4.8.

Table 4.17 shows Problem 16's comparison of results for exact solution, numerical solution obtained using BUM4 and error at points x = 0, 0.1, ... ,1.0. The table 4.18 that follows shows maximum absolute errors obtained at different values of k and point x. The superiority of BUM4 is seen in the table. Figures 4.9, 4.10 and 4.11 show the graphical representation of surface plots for exact, numerical and error (all shaded regions) for Problem 15 and figure 4.2 shows the time graph for the solution of Problem 15.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

This thesis aimed to solve boundary layer flow equations through the use of numerical method. These equations and higher order ordinary differential equations are a common occurrence in science and engineering as they model some problems arising in these fields. Analytical solutions to higher order differential equations are almost impossible to obtain and this necessitates the need for numerical methods to solve them.

Block unification methods consisting of linear multistep methods were developed and applied to solve third (coupled with second) and fourth order ordinary equations. They were also applied to solve fourth order partial differential equations. The linear multistep methods that make up the block unification method are continuous linear multistep methods. The continuous linear multistep methods were used because they have advantages of better global error estimation, ease of use in recovering standard schemes and also have a guarantee of easy approximation of solutions at all interior points of the integration interval. Chebyshev polynomials with respect to weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ were used as basis functions in deriving the schemes. The Collocation and Interpolation Techniques were employed and an off grid point (hybrid point) was included in the derivation process of the continuous linear multistep methods in order to incorporate a function evaluation at the off grid point which helps in avoiding the barrier of not attaining higher order convergence and also taking the advantage of generating numerical solutions simultaneously and overcoming the zero stability barrier.

Analysis of basic properties such as consistency, zero-stability and convergence of the numerical methods was carried out with the use of existing theorems and findings showed that the methods are of higher order and convergent. The derived methods are self-starting which gave the advantage of minimising truncation and round off errors and were implemented as boundary value method to simultaneously produce approximations solution of $\{y_n, y_{n+k}\}$ at block points $\{x_n, x_{n+k}\}$, on non-overlapping interval.

The effectiveness of the derived methods is demonstrated by considering fifteen (15) test problems that include Blasius equation, Sakiadis equation, Falkner-Skan equation, Squeezing Flow equation, the Oscillatory problem arising from Ship Dynamics and the "good" Boussinesq equation. These equations come as partial differential equations but have been reduced to ordinary differential equations through similarity transformation and method of lines.

The desirable property of a numerical solution is for it to behave like that of the exact solution of the problem which can be seen in the tables and figures presented. From the results obtained, it was observed that, in the Blasius equation, the number of function evaluation per step of the method was 41 while that of Runge-Kutta method was 123. The method gave a maximum error of 1.6741×10^{-12} for the "good" Boussinesq equation as against a maximum error of 1.7274×10^{-10} from the method in Modebei *et al.* (2020a). Therefore, these results and others presented show good performance of the developed methods in terms of efficiency and accuracy when compared with the results obtained using existing methods having the overall least error. Also for the purpose of comparison, it was observed that the results obtained from the developed methods were validated with some results in the existing methods which shown an excellent agreement.

5.2 Recommendations

1. The proposed methods have shown better performance in terms of accuracy and efficiency after comparison with those in existing literature. Hence, it is suggested to apply them to problems from physical phenomena that lead into third and fourth order boundary value problems and also fourth order partial differential equations.
2. Other orthogonal polynomials could be used as basis functions for the trial solution, this could lead to an improvement in solutions obtained.
3. Methods with more than one off-grid point could be derived and applied to the aforementioned problems. This may lead to improved results.
4. Higher order differential equations could be considered and the suggested method of solution in the research could be applied.

5.3 Contribution to knowledge

The following are contributions made to the body of knowledge:

1. new classes of continuous implicit three-step methods applied as Block Unification Method for the direct solution of third order BVPs of ODE were developed.
2. the methods minimize storage space and improves computer time by 30%.
3. a new class of continuous implicit four step Block Unification Method for the solution of fourth order BVPs of ODE and IBVPs of PDE was developed.
4. the new class methods have a wide scope of application viz-a-viz linear, nonlinear, singular, nonsingular, homogeneous, nonhomogeneous, constant coefficients, variable coefficients, oscillatory problems

REFERENCES

Abolarin, O. E., Kuboye, J. O., Adeyefa, E. O., & Ogunware, B. G. (2020). New Efficient Numerical Model for Solving Second, Third and Fourth Order Ordinary Differential Equations Directly *Gazi University Journal of Science*, *33*(4), 821-833.

Adak, M., & Mandal, A. (2021). Numerical Solution of Fourth-Order Boundary Value Problems for Euler–Bernoulli Beam Equation using FDM. *Journal of Physics: Conference Series*, 2070 (1), 1-6.

Adegboye, Z. A., Yahaya, Y. A., & Ahmed, U. I. (2020). Direct Integration of General Fourth Order Ordinary Differential Equations Using Fifth Order Ruge-Kutta Method. *Journal of the Nigerian Mathematical Society, 39*(1), 69-78.

Adesanya, A. O., Momoh, A. A., Alkali, Adamu, M., & Tahir, A. (2012). Five Steps Block Method for the Solution of Fourth Order Ordinary Differential Equations. *International Journal of Engineering Research and Applications (IJERA),2*(5), .991-998.

Adesanya, A. O., Udoh, M. D., & Ajileye, A. M. (2013). A New Hybrid Block Method for the Solution of General Third Order Initial Value Problems of Ordinary Differential Equations. *International Journal of Pure and Applied Mathematics, 86*(2), 365-375.

Adeyeye, O., & Omar, Z. (2017). Solving Nonlinear Fourth-Order Boundary Value Problems Using a Numerical Approach:(m + 1)th-Step Block Method. *International Journal of Differential Equations*, 2017(*1*),1-9.

Ali Abbas, M., Bhatti, M. M., & Rashidi, M. M. (2019). Heat Transfer on MHD Stagnation Point Flow through a Porous Shrinking/Stretching Sheet: A Numerical Study. *Thermal Science*, *24*(2), 1335-1344.

Atabo, V. O., & Adee, S. O. (2021). A New Special 15-Block Method for Solving General Fourth Order Ordinary Equations. *Journal of the Nigerian Society of Physical Sciences*, *3*(1), 308-333.

Bhatti, M. M., Ali Abbas, M., & Rashidi, M. M. (2018). A robust numerical method for solving stagnation point flow over a permeable shrinking sheet under the influence of MHD. *Applied Computation and Mathematics,* 316(1): 381-389.

Biala, T. A., & Jator, S. N. (2015). A boundary Value Approach for Solving Three-Dimensional Elliptic and Hyperbolic Partial Differential Equations. *Springer Plus*,4(1), 1-16.

Biala, T. A., & Jator, S. N. (2017). A Family of Boundary Value Methods for Systems of Second-Order Boundary Value Problems. *International Journal of Differential Equations*, 2017(1), 1-12.

Biala, T. A., Jator, S. N. & Adeniyi, R. B. (2017). Boundary Value Methods for Second-Order PDEs via the Lanczos-Chebyshev Reduction Technique. *Mathematical Problems in Engineering*. 2017(1), 1-12.

Bougoffa, L., & Wazwaz, A. (2015). New Approximate Solutions of the Blasius Equation. *International Journal for Numerical Heat and Fluid Flow, 25*(7), 1590-1599.

Brugano, L., & Trigante, D. (1998). *Solving Differential Problems by Multistep Initial and Boundary Value Methods*. Amsterdam: Gordon and Breach science Publisher.

Costabile, F., & Napoli, A. (2015). Collocation for High Order Differential Equations with Two-Points Hermite Boundary Conditions. *Journal of Applied Numerical Mathematics*, 87(1), 157-167.

Das, K., & Mohammed, U. (2016). Radiation Effects on an Unsteady Squeezing Flow in the Presence of a Magnetic Field. *Moldavian Journal of the Physical Sciences, 15*(2), 101-110.

Duromola, M. K. (2016). An Accurate Five Off-Step Points Implicit Block Method for Direct Solution of Fourth Order Differential Equations. *Open Access Library Journal*, *3*(6), 1-14.

Familua, A. B., & Omole, E. O. (2017). Five Points Mono Hybrid Linear Multistep Method for Solving nth Order Ordinary Differential Equations Using Power Series Function. *Asian Research Journal of Mathematics,* 3(1), 1-17.

Fang, T., Liang, W., & Chia-fon, F. L. (2008). A new solution branch for the Blasius equation—a shrinking sheet problem. *Computers & Mathematics with Applications*, *56*(12), 3088-3095.

Filobello-Nino, U., Vazquez-Leal, H., Cervantes-Perez, J., Benhammouda, B., Perez-Sesma, A., Luis Hernandez-Martinez, L., Jimenez-Fernandez, V. M., Herrera-May, A. L., Pereyra-Diaz, D., Antonio Marin-Hernandez & Chua, J. H. (2014). A Handy Approximate Solution for a Squeezing Flow Between Two Infinite Plates by Using of Laplace Transform-Homotopy Perturbation Method. *SpringerPlus*, 3(1), 1-10.

Hemeda, A. A., Eladdad, E. E. & Lairje, I. A. (2017). Solution of the Fractional Form of Unsteady Squeezing Flow through Porous Medium. *Mathematical Problems in Engineering*, 2017(1), 1-16.

Hossain, B., & Islam, S. (2014). Numerical Solutions of General Fourth Order Two Point Boundary Value Problems by Galerkin Method with Legendre Polynomials. *Dhaka University Journal of Science*, 62(2), 103-108.

Hussain, K., Ismail, F., & Senu, N. (2016). Solving directly special fourth-order ordinary differential equations using Runge–Kutta type method. *Journal of Computational and Applied Mathematics*, *306*(1), 179-199.

Ibrahim, W., Shankar, B., & Nandeppanavar, M. M. (2013). MHD stagnation point flow and heat transfer due to nanofluid towards a stretching sheet. *International Journal of Heat and Mass Transfer*, 56(1), 1-9.

Inc, M., & Akgul, A. (2014). Approximate Solutions for MHD squeezing fluid flow by a Novel Method. *A Springer Open Journal, Boundary Value Problems,* 2014(1), 1-17.

Jator, S. N. (2007). Solving Stiff Second Order Initial Value Problems Directly by Backward Differentiation Formulas. *Proceedings of the International Conference on Computational and Mathematical Methods in Science and Engineering, , Illinois, Chicago, USA*, 223-232.

Jator, S. N. (2008). On the Numerical Integration of Third Order Boundary Value Problems by a Linear Multistep Method. *International Journal of Pure and Applied Mathematics, 46*(3), 375-388.

Jator, S. N. (2009). Novel Finite Difference Schemes for Third Order Boundary Value Problems. *International Journal of Pure and Applied Mathematics, 53*(1), 37-54.

Jator, S. N., & Li, J. (2009). Boundary Value Methods via a Multistep Method with Variable Coefficients for Second Order Initial and Boundary Value Problems. *International Journal of Pure and Applied Mathematics, 50*(3), 403-420

Jator, S., Okunlola, T., Biala, T., & Adeniyi, R. (2018). Direct Integrators for the General Third-Order Ordinary Differential Equations with an Application to the Korteweg–de Vries Equation. *International Journal of Applied Computational Mathematics,* 4(1), 1-26.

Kasi-Viswanadham, K. N. S., & Ballem, S. (2015). Numerical Solution of Fifth Order Boundary Value Problems by Galerkin Method with Quartic B-splines. *International Journal of Computer Applications, 77*(1), 1-15.

Kelesoglu, O. (2014). The Solution of Fourth Order Boundary Value Problem Arising out of the Beam-Column Theory Using Adomian Decomposition Method. *Mathematical Problems in Engineering*, 2014(1), 1-6.

Khan, U., Khan, S. I., Ahmed, N., Bano, S., & Mohyud-Din, S. T. (2016). Heat Transfer Analysis for Squeezing Flow of a Casson Fuid Between Parallel Plates. *Ain Shams Engineering Journal 7*(1), 497-504.

Kuboye, J. O., Elusakin, O. R., & Quadri, O. F. (2020) Numerical Algorithms for Direct Solution of Fourth Order Ordinary Differential Equations. *Journal of the Nigerian Society of Physical Sciences*, 2 (2020) 218–227.

Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations*. New York: John Wiley.

Li, Z., Wang, Y., & Tan, F. (2012). The Solution of a Class of Third Order Boundary Value Problems by the Reproducing Kernel Method. *Abstract and Applied Analysis,* 2011(1), 1-11.

Lie, I., & Norselt, S. P. (1989). Super Convergence for Multistep Collocation. *Mathematical Computations*, *52*(185), 65-79.

Massoun, Y., & Benzine, R. (2018). The Homotopy Analysis Method for Fourth-Order Initial Value problems. *Journal of Physical Mathematics, 9*(1), 1-4.

Mehrkanoon, S., Seleiman, M., & Majid, Z. A. (2009). Block Method for Numerical Solution of Fuzzy Differential Equations. *International Mathematical Forum, 4*(46), 2269-2280.

Mittal, R. C., & Jain, R. K. (2011). B-splines Methods with Redefined Basis Functions for Solving Fourth Order Parabolic Partial Differential Equations. *Applied Mathematics and Computation, 217*(23), 9741-9755.

Modebei, M. I., Adeniyi, R. B., & Jator, S. N. (2020a). Numerical Approximations of Fourth-Order PDEs Using Block Unification Method. *Journal of the Nigerian Mathematical Society,* 39(1), 47-68.

Modebei, M. I., Jator, S. N. & Ramos, H. (2020b). Block Hybrid Method for the Numerical solution of Fourth Order Boundary Value Problems. *Journal of Computational and Applied Mathematics*, 377(1), 1-15.

Mohammed, U. (2010). A Six Step Block Method for Solution of Fourth Order Ordinary Differential Equations. *The Pacific Journal of Science and Technology*, 11(1), 259-265.

Mohammed, U., & Adeniyi, R. B. (2014). A Three Step Implicit Hybrid Linear Multistep Method for the Solution of Third Order Ordinary Differential Equations. *Gen. Math. Notes,* 25(1), 62-74.

Mohammed, U., Adeniyi, R. B., Semenov, M., Jiya, M., & Ma'ali, I. (2018). A Family of Hybrid Linear Multistep Method Type for Special Third Order Ordinary Differential Equations. *Journal of the Nigerian Mathematical Society,* 37(1),1-22.

Mohammed, U., Oyelami, O & Semenov, M. (2019a). An Orthogonal Based Self-starting Numerical Integrator for Second Order Initial and Boundary Value Problems ODEs. *IOP Conf. Series: Journal of Physics: Conference Series,* 1145(1) ,1-8.

Mohammed, U., Oyelami, O., Semenov, M.E. & Ma'ali , A.I.(2019b). Development and Implementation of Block Unification Multi-Step Methods for the Solution of Second Order Ordinary Differential Equations. *IOP Conf. Series Journal of Physics,* 1366(1), 1-10.

Mohammed, U., Semenov, M., Ma'ali, A. I., Abubakar, A., & Abdullah, H. (2019c). Direct Integrators of Modified Multistep Method for the Solution of Third Order Boundary Value Problem in Ordinary Differential Equations. *IOP Conference Series Materials Science and Engineering,* 597(1), 1-6.

Mohammed, U., Garba, J & Semenov, M.E. (2021). One-Step Second Derivative Block Intra-Step Method for Stiff System of Ordinary Differential Equations. *Journal of Nigerian Mathematical Society (JNMS)***,** 40(1), 47-57.

Mohammed, U., Garba, J., Abdullahi, A. A., Ma'ali, A. I., & Semenov, M. (2022a). Second Derivatives Single Step Block Hybrid Method for Non-Linear Dynamical System. *Nigerian Journal of Mathematics and Applications*, 32(2), 42-55.

Mohammed, U., Oyelami, O., Garba, J., Abdullahi, A.A., & Semenov, M. (2022b). Optimization of Two-Step Method with Bhaskara Points for Solving Non-linear Dynamical Problems. *Nigerian Journal of Mathematics and Applications, 32*(2), 151-168.

Mohanty, R. K., & Kaur, D. (2016). High Accuracy Implicit Variable Mesh Methods for Numerical Study of Special Types of Fourth Order Nonlinear Parabolic Equations. *Applied Mathematics and Computer,* 273(1), 678-696.

Mushtt, I. Z., Mahmood, S. S., & Hamed, D. M. (2020). Solving Fourth Order Boundary Value Problem by Using Extended Quantic B-spline Interpolation. *Applied Mathematics,* 10(2): 28- 33.

Mustafa, M., Hayat, T., & Obaidat, S. (2012). On Heat and Mass Transfer in the Unsteady Squeezing Flow Between Parallel Plates. *Meccanica*, 47(1), 1581-1589.

Mustapha, R. A. & Salau, A. M. (2021). Comparative Solution of Heat Transfer Analysis for Squeezing Flow Between Parallel Disks. *Iconic Research and Engineering (IRE) Journals*, 4 (10), 136-142.

Ndanusa, A., Adeboye, K. R., Mustapha, A. U. & Abdullahi, R. (2020). On Numerov Method for Solving Fourth Order Ordinary Differential Equations. *FUDMA Journal of Science, 4*(4), 355 – 362.

Noor, M. A., & Al-Said, E. A. (2002). Finite-Difference Method for a System of Third-Order Boundary-Value Problems. *Journal of Optimization Theory and Applications*, 112(3), 627-637.

Noor, M. A., & Mohyud-Din, S. T. (2007). An Efficient Method for Fourth Order Boundary Value Problems. *Computer and Mathematics Application,* 54(1), 1101-1111.

Ogunlaran, O. M., & Kehinde, M. A. (2022). A- 4 Step Hermite Based Multiderivative Block Integrator for Solving Fourth Order Ordinary Differential Equations. *Abacus (Mathematics Science Series)*, 49(2), 1-19.

Ogunlaran, O. M., & Oladejo, N. K. (2014). Approximate Solution Method for Third-Order Multi-Point Boundary Value Problems. *International Journal of Mathematical Sciences*, 34(2), 1571-1580.

Ogunlaran, O. M., & Oukouomi, N. S. C. (2016). Chebyshev Collocation Method for the Solution of a System of Second-Order Boundary Value Problems. *Journal of Algebra and Applied Mathematics, 14*(2), 121-134

Omar, Z. & Abdelrahim, R. (2016). Direct Solution of Fourth Order Ordinary Differential Equations Using a One Step Hybrid Block Method of Order Five. *International Journal of Pure and Applied Mathematics*, *109*(4), 763-777.

Onumaniyi, P., Awoyemi, D. O., Jator, S. N., & Sirisena, U. W. (1994). New Linear Multistep Methods with Continuous Coefficients for First Order Initial Value Problems. *Journal of Nigerian Mathematical Society*, 13(1), 37-51.

Pandey, P.K. (2017). A Numerical Method for the Solution of General Third Order Boundary Value Problem in Ordinary Differential Equations. *Bulletin of the International Mathematical Virtual Institute, 7*(1), 129-138.

Ran, X. J., Zhu, Q. Y., & Li, Y. (2009). An Explicit Series Solution of the Squeezing Flow between Two Infinite Plates by Means of the Homotopy Analysis Method. *Communications in Nonlinear Science and Numerical Simulation*, 14(2), 119-132.

Rashidinia, J., & Mohammadi, R. (2010). Sextic Spline Solution of Variable Coefficient Fourth Order Parabolic Equations. *International Journal of Computational Mathematics*, 87(15), 3443-3454.

Reddy, S. M. (2016). Collocation Method for Fourth Order Boundary Value Problems using Quartic B-splines. *IOSR Journal of Mathematics (IOSR-JM*), 12(5), 38-45

Reutskiy, S., Chen, C. S., & Tian, H. Y. (2008). A boundary meshless method using Chebyshev interpolation and trigonometric basis function for solving heat conduction problems. *International Journal for Numerical Methods in Engineering*, 74(1), 1621-1644.

Salama, A. A., & Mansour, A. A. (2005). Fourth-Order Finite-Difference Method forThird-Order Boundary-Value Problems. *Numerical Heat Transfer*, 47, 383–401.

Saucez, P., Vande-Wouwer, A., Schiesser, W. E., & Zegeling, P. (2014). Method of Lines Study of Non-linear Dispersive Waves. *Journal of Computational and Applied Mathematics,* 168, 413-423.

Shokri, A., & Dehghan, M. (2010). A Not-a-Knot Meshless Method Using Radial Basis Functions and Predictor-Corrector Scheme to te Numerical Solution of Improved Boussinesq Equation. *Computer Physics Communications,* 181, 1990-2000.

Siddiqi, S. S., & Arshed, S. (2014). Quintic B-spline for the Numerical Solutions of Fourth Order Boundary Value Problems. *International Journal of Numerical Analysis and Modeling*, *5*(1), 101-111.

Singh, I. & Singh, G. (2019). Numerical Study for Solving Fourth Order Ordinary Differential Equations. *International Journal for Research in Engineering Application & Management*, *5*(1), 638-642.

Tirmizi, I. A., Twizell E. H., & Siraj-ul-Islam (2004). A Numerical Method for Third-Order Non-Linear Boundary-Value Problems in Engineering. *International Journal of Computer Mathematics.* 82(1), 103-109.

Ukpebor, L. A., Omole, E. O., & L. O. Adoghe, L. O. (2020). An Order Four Numerical Scheme for Fourth-Order Initial Value Problems Using Lucas Polynomial with Application in Ship Dynamics. *International Journal of Mathematical Research,* 9(1): 28-41.

Vigo-Aguiar, J., & Ramos, H. (2007). A Family of A-stable collocation Methods of Higher Order for Initial Value Prolems. *IMA Journal of Numerical Analysis,* 27, 798-817.

Yap, L. K., & Ismail, F. (2015). Block Hybrid Collocation Method with Application to Fourth Order Differential Equations. *Mathematical Problems in Engineering,* 2015(1), 1-6.