DEVELOPMENT OF BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULAE FOR SOLVING CLASS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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AUGUST, 2023

ABSTRACT

In this research, the Block Hybrid Backward Differentiation Formulae (BHBDF) for the step number K= 4,5 and 6 were developed for the solution of general second order ordinary differential equations ODE. The Order of the Block methods are 5,6 and 7 respectively. The Continuous formulations of this methods were done through interpolation and collocation approaches. The power series polynomial was used as basis function at some selected grids and off-grids points. The continuous schemes were further evaluated at those points to produce discrete schemes which are combined to form block method. Analysis of the basic properties of the discrete schemes investigated showed consistency, zero stability and convergence of the proposed block methods. Numerical examples were solved to examine the efficiency and accuracy of the proposed method. The results showed that the proposed methods with relatively small errors performed favorably in comparison with the existing methods.

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CHAPER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Some real-life physical problems that arise in various fields of study, be they engineering, medicine, the sciences, or others, when modeled mathematically, lead to differential equations (Abada *et al.,* 2017). Many of these equations do not have solution in closed forms. There is need to provide good numerical methods to approximate their solutions. Development of linear multi-step methods (LMMs) for solving ordinary differential equations (ODEs) can be generated using some methods such as Taylor series, numerical interpolation, numerical integration and collocation techniques. Block methods for solving ordinary differential equations (ODEs) have been proposed by Milne (1953) The Milnes idea of proceeding in blocks was developed by Rosser (1967)

The general second-order ordinary differential equation of the form

$$y'' = f\left(x, y, y'\right), \; y\left(a\right) = y_o, \; y'\left(a\right) = \delta_o \qquad (1.1)$$

Where f is assumed to satisfy a Lipchitz condition as given in Henrici (1962).

A differential equation can be defined as an equation that contains a derivative or involve a dependent variable (y) an independent variable (x) and one or more differential co-efficient of y with respect to x. An example of a differential equation is $y'' = 2xy$. Differential equations are categorized to two forms which are: Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). An ordinary differential equation (ODEs) is one in which the unknown function is dependent variable with a single independent variable. An Ordinary Differential Equations (ODEs) is classified according to the order of the highest derivative with respect to the dependent variable appearing in the equations. The most important cases for application are the first and second orders. Partial Differential Equations (PDEs) are differential equation in which the unknown function is a function of multiple independent variables and the equation involves its partial derivative.

Several numerical methods have been designed and proposed in literature for solving second order ordinary differential equations. For example, Areo and Adeniyi (2013) developed a self-starting linear multistep method and applied it to solve second order IVPs of ODEs directly. Two intra step grid points were considered by means of collocation and interpolation approach. Omar and Abeldrahim (2016) proposed a single-step hybrid block method of order five to solve second order ODEs. In the work, three off step points were approximated by collocation approach.

In the work by Olabode and Momoh (2016), continuous hybrid multistep method with Legendre polynomial as the approximate solutions was investigated to obtain the approximation of the solution of second order ODEs. Also, two intra step grid points were considered by means of collocation and interpolation approach. Moreso, Sunday *et al.* (2014) developed numerical solution of stiff and oscillatory first order differential equations, using the combination of power series and exponential function to produce a new numerical integrator for the solution of stiff first order ODEs. Most of the methods proposed for the solution of stiff problems are numerically unstable unless the step size is taken to be extremely small and the adoption of implicit A-stable schemes is better for the solution of stiffness problems.

Especially methods for the numerical solutions of the second order Ordinary Differential Equations (ODEs). The integer k is called the step number of the method for $k \geq 1$ is called a multi step or k –step method. Linear multi step method of step number k or a linear k-step method, can be written in the general form as follow:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j \ f(x_{n+j}) \quad (1.2)$$

Where α_j and $\beta_j$are continuous coefficients to be determined h is the step size

k is the step number. M is the order of the differential equation.

1.2 Statement of the Research Problem

Numerical analyst are usually faced with the challenge of obtaining starting or initial values for Linear Multistep Methods (LMMs) when step number $k \geq 2$ in solving differential equations numerically before now, one step methods like Taylor series, Euler method Runge-Kutta or Trapezoidal methods are used to obtained the starting values for such methods (Omar and Abeldrahim, 2016). The hybrid method is not exempted from this problem as it shares the same standard methods (Ibrahim *et al*., 2020). The need for special predictors to predict the off-grid values in hybrid forms of the Linear Multistep Methods (LMMs) (Sunday *et al*., 2014). The discrete schemes obtained from the continuous formulation of k-step block hybrid backward differentiation formulae can be used in block form to obtain the block solution.

1.3 Aim and Objectives of the Study

The aim of this research is to developed block hybrid backward differentiation formulae for solving class of second order ordinary differential equations (ODEs). Hence the following objectives are to:

1. Developed block hybrid backward differentiation formulae for step numbers k=4, 5 and 6 with two off-grid points at interpolation.
2. Perform analysis of the basic properties of the proposed method in terms of order, error constant, zero stability, consistency and convergence.
3. Apply the developed block hybrid backward differentiation formulae to solve some second order ordinary differential equations (ODEs).

4. Compare the results of the proposed methods with some existing methods and exact solutions.

1.4 Significance of the Study

The development of Block Hybrid Backward Differentiation Formulae (BHBDF) for solving a class of second-order ordinary differential equations (ODEs) holds considerable significance within the realm of numerical methods and mathematical modelling (Kayode and Obarhua, 2017;). This research addresses a crucial need for efficient and accurate techniques to tackle the complexities inherent in solving general second-order ODEs.

Advancement of Numerical Methods: The study contributes to the advancement of numerical methods for solving ODEs. The creation of Block Hybrid Backward Differentiation Formulae introduces a novel approach that combines the strengths of backward differentiation and block methods. This innovation expands the toolkit available to researchers and practitioners, enabling them to tackle a wider array of differential equations more effectively.

Enhanced Solution Accuracy: The development of Block Hybrid Backward Differentiation Formulae has the potential to offer enhanced accuracy in solving second-order ODEs. By incorporating power series polynomials as basis functions and utilizing interpolation and collocation approaches, the study seeks to provide solutions that are not only accurate but also adaptive to the characteristics of the problem at hand.

Broad Applicability: The research's focus on general second-order ODEs underscores its broad applicability. Many scientific disciplines, including physics, engineering, biology, and economics, rely on ODEs to model real-world phenomena. The methods developed in this study could find practical utility across these domains, enabling researchers to derive more accurate and insightful results from their models.

In conclusion, the development of Block Hybrid Backward Differentiation Formulae for solving second-order ordinary differential equations signifies a substantial contribution to the field of numerical methods and mathematical modeling. Its potential to enhance accuracy, broaden applicability, and strike a balance between efficiency and precision reinforces its importance in addressing complex ODEs prevalent across various scientific disciplines. The rigorous analysis and validation offered by the study further solidify the significance of the proposed methods within the realm of practical implementation and theoretical advancement.

1.5 Scope and Limitation of the Study

This research is restricted to solving second order ordinary differential equations (ODEs). The major focus of this study is on developing k-step of block hybrid backward differentiation formulae (BHBDF) for solving some class of second order ordinary differential equations (ODEs) with two off-grids points. The performance of these schemes in the solution of differential equations shall be checked and it is restricted to numerical solutions of $k = 4, 5\ and\ 6$ with two off-grids points at interpolation point which is only used to obtain numerical schemes.

1.6 Justification for the Study

The justification for this study rests upon the need to address existing challenges in numerical methods for solving second-order ODEs and the potential benefits that the development of Block Hybrid Backward Differentiation Formulae offers. The diverse applications, complexities of real-world problems, and the advancement of computational techniques all underscore the relevance and significance of this research. The empirical validation and potential for future research further affirm the importance of exploring this innovative approach to solving ODEs (Jator, 2001). This research would contribute to numerical analysis through the formulation of new classes of efficient consistent block hybrid backward differentiation formulae for the direct solution of ordinary differential equations.

1.7 Definition of Terms

Linear Multistep Methods (LMMs): Given a sequence of to y_n be an approximation to $y(x_n)$ and let $f_n = (x_n, y_n)$ .If a computational method for determining the sequence $\{y_n\}$ takes the form of the linear relationship between

$$y_{n+j,}f_{n+j\ i.e} \sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j\ f(x_{n+j})$$

(1.2) Then (1.2) is a linear multistep method (LMM) of step number k.

Order of Linear Multistep Methods: A LMMs is said to be of order ρ if

$$L[y(x):h] = \sum_{j=0}^{k} [\alpha_j y(x_n + jh) - h\beta_j y'(x_n + jh)]$$

$c_0 = c_1 = c_2 \ldots = c_p = 0\ , c_{p+2} \neq 0$ is the error constant

Error Constant: The term C_{p+1} is called error constant and it implies that the local truncation error is given by $E_{n+k} = C_{p+1} h^{p+1} y^{p+1}(x_n) + 0(h^{p+2})$

Consistency of LMM: A Linear Multistep Method is said to be consistent if it has the order $p \geq 1$and satisfies the following axioms or conditions

i. $\sum_{i=o}^{k} \alpha_i = 0$

ii. $\rho(r) = \rho'(r) = 0$

iii. $\rho''(r) = 2!\,\sigma(r)$

Where ρ(r) and $\sigma(r)$ are the first and second characteristics polynomial of our method respectively

Zero stability of Hybrid Block Methods: The Hybrid Block Method is said to be zero stable if the roots of R of the characteristic polynomial $\bar{p}(R) = \det\left[RA^{o} - A'\right]$ satisfies $|R| \leq 1$ and every root with $|R_o| \leq 1$ has multiplicity not exceeding two in the limit as $h \to 0$

Convergence of linear Multistep Methods: a LMMs is said to be convergent if and only if it satisfies both consistency and zero stability.

Absolute Stability: The linear multi-step method is said to be absolutely stable if its region of absolute stability contains the whole of the left hand half $Re(h\boldsymbol{\lambda}) < 0$ $or$for a given h and for all h the root r_s of $\pi(r,h) = p(r) - h\delta(r)$ satisfy $|r_s| < 1$ $s = 1, 2, 3, \ldots, k$ and it is absolutely unstable for that h otherwise.

Absolute Error: Let X^* be any estimate to the number X, the absolute error in X^* is referred to $\Delta = |\ X - X^*|$, in which Δ is distant between the numbers X and X^*.

Collocation Point: a point at which the derivative of the function is evaluated.

Interpolation Point: a point at which the solution function is evaluated.

Degree of Differential Equation: the highest power which the differential equation is raised.

For example$\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right) + y = 0$

The degree of the differential equation above is two

Ordinary Differential equation (ODE): a differential equation that consists of functions of an independent variable and its derivatives.

For example $\frac{dy}{dx} - xy = 0.$

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Previous Related works

2.1.1 Linear multistep methods

A linear multistep method (LMMs) is known for solving ordinary differential equation (ODEs) and also higher order differential equations. The introduction of continuous collocation schemes is of more importance as better global error can be estimated and approximations can be equally obtained, the gap between the discrete collocations' methods and the conventional multistep method is bridged (Yahaya and Tijjani, 2015). In recent times, discrete methods have been extended to continuous forms based on multistep collocation and by this extension, there is increase in their ability to solve the ordinary differential equations (ODEs): the discrete ones are self-starting they overcome the problems of overlap solution models usually related with multistep finite difference methods and on the same fixed meshes. The higher order methods can be applied successively by selected different points of the step number. Differential equations first came in to existence with the invention of calculus by Newton and Leibnitz. Isaac (1671) listed three kinds of differential equations:

$$\frac{dy}{dx} = f(x) \tag{2.1}$$

$$\frac{dy}{dx} = f(x, y) \tag{2.2}$$

$$x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y \tag{2.3}$$

In all the three classes, y is an unknown of x (or of x_1 and x_2), and f is a given function. He solved these equations using infinite series and discussed the non-uniqueness of solutions.

The first two classes contained only ordinary derivatives of one or more dependent variables, and are called ordinary differential equations (ODE). The third class involved the partial derivatives of one dependent variable which is known as system of partial differential equations (PDEs). Jacob Bernoulli proposed the Bernoulli differential equation in 1695. This is an ordinary differentiation equation of the form $y^1 + p(x)y = Q(x)y^n$. Differential equations are among the most important mathematical tools used in producing models in physical sciences, Biological sciences, and Engineering. Over the years, several researchers developed methods in finding analytical solutions of initial value problem (IVP) in ordinary differential equations (ODEs) of the form.

$$y'' = f\left(x, y, y'\right), y\left(x_0\right) = y_0, y'\left(x_0\right) = y'_0 \quad (2.4)$$

The improvement of numerical methods for the solution of initial value problem (IVP) in ordinary differential equations (ODEs) of the form (2.4) gave mount to two major discrete variable methods namely: single step (one step) methods and multistep methods, most especially the linear multistep method. The single step methods are very low order of accuracy and they are suitable for first order IVPs of ODEs. Such as Euler's methods, Runge-kutta methods etc.

The numerical solution of higher order single step methods such as Runge-kutta methods, in terms of the number of function evaluation per step, is sacrificed since more function evaluations are required. Hence, solving (2.4) using any single step methods means reducing it to an equivalent system of first order IVPs in ODEs which increase the scale of the problem, thus increasing its size, reducing to first order is ineffective due to computational burden and also uneconomical arising from computer time wastage and gives results of low accuracy.

However, Linear Multistep Methods include methods such as Numerov method, Adams-Bashforth method, Adam-Multon method. These methods give more accuracy and are appropriate for the direct solution of (2.4) without necessarily reducing it to an equivalent system of first order IVPs of ODES.

Ordinary Differential Equations of the form (2.4) are examined by some authors including Jator (2001), Mohammed (2010); Areo and Adeniyi (2013), Adamu *et al*. (2019) and Ra'ft *et al.* (2020) among others, by first reducing them to an equivalent system of first order ordinary differential equations and then using any appropriate numerical method to solve the resultant system. The disadvantage of this is that it consumes more time, human efforts and computer program to check the accuracy of these methods are usually complicated (Adamu *et al*., 2019).

Moreso, in consideration of these setbacks, we considered a method that can solve LMM without reduction. Some prominent scholars have made efforts to solve higher order initial value problems of second order ordinary differential equations by a number of different methods, these includes the work Momoh *et al.* (2014), Abdelrahim *et al.* (2016), Adamu *et al.* (2019) and Ibrahim *et al.* (2020) among others. The direct methods are self-starting methods which are formulated in terms of LMMs called block methods. The block method offers the traditional advantage single step methods for instance, Rung-Kutta methods of been self starting and allow easy change of step length. Another important attribute of the block method is that all the discrete schemes are of uniform order and are obtained from a single continuous formula unlike the non-starting predictor corrector technique.

Ibrahim *et al*. (2020) construct two-step second derivative hybrid block backward Differentiation formula. The newly proposed scheme was derived based on interpolation and collocation approach. The discrete schemes were obtained from the continuous schemes. The derived method is applied to solve non-linear systems of stiff ordinary differential equations. Numerical experiments show that the method is suitable for stiff differential equations. In this research, we shall adopt the block method approach to formulate a second order numerical scheme using power series approximation as basis function.

Other numerical methods that are useful while solving ODEs are the collation methods and hybrid methods. In mathematics, collocation method for ordinary differential equation is a method for the numerical solution of ordinary differential equations, partial differential equations and integral equations. Collocation methods were used over the past decades in search of solution to a wide class of ordinary differential equations, partial differential equations, Integra-differential equations and functional equations. The attractiveness of such methods is owing to their abstract simplicity and also large applicability. According to Popov *et al.* (2017), the method was first proposed by Frazer, Jones. The work of Frazer *et al.* (1938) was dedicated to the solution of PDEs. Collocation at the family of orthogonal polynomials is often called orthogonal collocation. Orthogonal collocation is the method for the numerical solution of partial differential equations. It uses collocation at the zeros of some orthogonal polynomials to transform the partial differential equation (PDE) to a set of ordinary differential equations (ODEs). The ODE can then be solved by any method (Ramos, 2017).

Chebyshev orthogonal collocation methods are described by Fox and Parker (1968). Special collocation methods are very much related to this form of collocation, Henrici (1962). There is a quick improvement as reported in the literature on the use of collocation methods on the use of numerical solutions of first order ODEs. The multistep collocation techniques involve obtaining solution of a set of function of a linear combination of a function known as the trial function. The analytical solution of an IVP is assumed to be approximated by the basis function. The linear combination of this basis is required to satisfy the approximation at some certain grid points called the collocation points.

The hybrid method has been anticipated in the literature. The methods share the property of utilizing data at other points other than the points the step points $\left\{x_{n+j;}x_{n+j=}x_{n+jh}\right\}$ while retraining uniqueness of the continuous linear multistep methods.

The Method involves the determination of an approximate solution in a suitable set of functions, sometimes called basis function. Hybrid method is not a method in its own accurate since particular predictors were needed to estimate the solution of the off-step point \and the derivative function as well. In view of the disadvantage mentioned above, many researchers focused on efforts in improving the numerical solution of IVPs of ODEs. One of the outcomes is the development of a class of methods called Block method. The contribution of Bolaji (2017) also proposed for a family of Hybrid Backward Differentiation Formulae and a three step Hybrid Linear Multistep method for a direct solution of second and third order ODEs and the solution of second order IVPs.

2.2 Collocation Method

A collocation method can simply be described as a method, which involves the determination of an approximate solution in a suitable set of functions, sometimes called basis function. The approximate solution is required to satisfy the initial or boundary conditions along with the differential equations (2.4) at certain points called the collocation points.

Continuous collocation schemes is of more importance as better global error can be estimated and approximations can be equally obtained, the gap between the discrete collocations methods and the conventional multistep method is bridged.

In recent times discrete methods have been extended to continuous forms based on multistep collocation and by this extension, there is this an increase in their ability to solve the ordinary differential equations (ODEs) the discrete one are self-starting they overcome the problems of overlap solution models usually related with multistep finite difference methods and on the same fixed meshes the higher order methods can be applied successively by selected different points of the step number. Obviously over the past years, collocation methods evolved as valuable methods for the solution of abroad class of problems covering ordinary and partial differential equations, functional equations and Butcher (2008) first proposed the collocation method, specifically intended for the solution of partial differential equations in two variables, with collocation being applied in two variables, with collocation being applied in one variable for each fixed value of the second. This actually is a method of lines procedure. The work of Kayode and Obarhua (2017) was dedicated to the solution of ODEs. While the applicability of collocation method to the solution of partial differential equations was mentioned in (Kayode and Obarhua, 2017), not only discussed collocation for both ordinary and partial differential equations, but also provided some numerical examples. These methods have in common the option of polynomial for the basis function.

2.3 Block Methods

The narrative property of the method that can be briefly discussed in this chapter is that of simultaneously producing approximations to the solution of initial value problem at k points $x_{n+1,} x_{n+2,} x_{n+N}$. Although these methods will be formulated in terms of linear multistep methods, it can be observed that they are equivalent to certain Runge- Kutta method and preserve the traditional Runge – Kutta advantage of being self –starting and permitting easy change of step length. Their advantage over conventional Runge – Kutta method lies in the fact that they are less expensive in terms of function evaluations forgiven order blocks method appear to have been proposed by Mohammed and Adeniyi (2014) use to obtaining starting values of corrector of block method consists of a set of all new functions values which are evaluated during each application of the relative formula to produce k new set of values of solution in each computational step (Akinfenwa, 2011). Although these methods is formulated in terms of linear multi-step methods, it can be observed that they are equivalent to certain Runge- Kutta methods advantage of being self- starting and permitting easy change of step length their advantage over conventional Runge – Kutta method lies in the fact that they are less expensive in terms of function evaluations forgiven order method, predictor – corrector algorithms (Areo and Adeniyi, 2013).

2.4 Hybrid Methods

According to Kayode and Obarhua (2017) numerical analysis has over the years been determined on solution at the grid points ignoring what happens at other points than the grid points. Searching for higher order numerical methods has led to researchers throwing in additional off-step points in the process of formulation. Methods formulated using this approach are called hybrid methods, they preserve the self-starting property of Runge-Kutta methods as well as being able to provide more solutions at a single application. They are also said to be capable of overcoming Dalquist barrier theorem which states that a linear multistep method cannot have order greater than k+1 for k odd and k + 2 for k even. There have been successful methods developed in this area too. Like the methods in (Areo and Adeniyi, 2013; Badmus *et al*., 2014; Kuboye and Omar, 2015 and Kayode and Obarhua, 2017).

2.5 The Backward Differentiation Formula (BDF)

Backward differentiation formula (BDF) is a linear multistep method suitable for solving differential equations and stiff initial value problems. The Backward Differentiation Formulae is an example implicit multistep method with a strange uniqueness of function evaluation at a single point. There are other modifications of this method such as the blended backward differentiation formula and the extended backward differentiation formula.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Derivation of the Numerical Schemes

We present the derivation of some Hybrid Backward Differentiation Formula (HBDF) for solving some class of second-order ordinary differential equations of the form

$$\frac{d^2y(x)}{dx^2} = f\left(x, y, \frac{dy(x)}{dx}\right) \tag{3.1}$$

coupled with appropriate initial conditions

$$y(x_0) = \varphi_1, \frac{dy(x_0)}{dx} = \varphi_2 \tag{3.2}$$

where f is a continuous function such that $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$, x_0 is the initial point, $y \in \mathbb{R}$ is an $n-$dimensional vector, x is a scalar variable, φ_1 and φ_2 are the initial values.

In this research, we seek to develop numerical schemes in the form of HBDF as:

$$Y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_v(x) y_{n+v} + h^2 \beta_k(x) f_{n+k} \tag{3.3}$$

where h is the chosen step size and $\alpha_j(x): j = 0,1,2, \dots, k,\ \alpha_v(x), \beta_k(x)$ are unknown continuous coefficients to be determined. For Backward Differentiation Formula, we note that $\alpha_k = 1$ and $\beta_k \neq 0$. In this study, we will derive HBDF for the step numbers $k = 4, 5\ and\ 6$ step numbers of the proposed method using power series function as the basis function.

3.2 Specifications of the Method

3.2.1 4-Step block hybrid backward differentiation formulae (4SBHBDF)

We seek an approximation of the form;

$$Y(x) = \sum_{j=0}^{t+c-1} \alpha_j x^j \qquad (3.4)$$

where t is the interpolation points, c is the collocation points and α_j are unknown coefficients to be determined. Then, we take

$$Y(x) = y_{n+j}, j = 0,1,2, \dots, k-1 \qquad (3.5)$$

$$Y''(x_{n+k}) = f_{n+k} \qquad (3.6)$$

To derive 4SBHBDF, we take $t = 6, c = 1$ and $x \in [x_n, x_{n+4}]$. Therefore, (3.4) becomes;

$$Y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6$$

$$Y'(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + 5\alpha_5 x^4 + 6\alpha_6 x^5$$

$$Y''(x) = 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + 20\alpha_5 x^3 + 30\alpha_6 x^4 \qquad (3.7)$$

Interpolating (3.5) at $x_{n+i}; i = 0, \frac{1}{2}, 1, \frac{3}{2} 2, 3$ and collocate (3.6) at $x_{n+i}; i = 4$. This results in a system of equations;

$$Y''(x_{n+4}) = 2\alpha_2 + 6\alpha_3 x_{n+4} + 12\alpha_4 x_{n+4}^2 + 20\alpha_5 x_{n+4}^3 + 30\alpha_6 x_{n+4}^4$$

$$Y(x_{n+i}) = \alpha_0 + \alpha_1 x_{n+i} + \alpha_2 x_{n+i}^2 + \alpha_3 x_{n+i}^3 + \alpha_4 x_{n+i}^4 + \alpha_5 x_{n+i}^5 + \alpha_6 x_{n+i}^6$$

$$Y(x_n) = \alpha_0 + \alpha_1 x_n + \alpha_2 x_n^2 + \alpha_3 x_n^3 + \alpha_4 x_n^4 + \alpha_5 x_n^5 + \alpha_6 x_n^6$$

$$Y\left(x_{n+\frac{1}{2}}\right) = \alpha_0 + \alpha_1 x_{n+\frac{1}{2}} + \alpha_2 x^2_{n+\frac{1}{2}} + \alpha_3 x^3_{n+\frac{1}{2}} + \alpha_4 x^4_{n+\frac{1}{2}} + \alpha_5 x^5_{n+\frac{1}{2}} + \alpha_6 x^6_{n+\frac{1}{2}}$$

$$Y(x_{n+1}) = \alpha_0 + \alpha_1 x_{n+1} + \alpha_2 x^2_{n+1} + \alpha_3 x^3_{n+1} + \alpha_4 x^4_{n+1} + \alpha_5 x^5_{n+1} + \alpha_6 x^6_{n+1}$$

$$Y\left(x_{n+\frac{3}{2}}\right) = \alpha_0 + \alpha_1 x_{n+\frac{3}{2}} + \alpha_2 x^2_{n+\frac{3}{2}} + \alpha_3 x^3_{n+\frac{3}{2}} + \alpha_4 x^4_{n+\frac{3}{2}} + \alpha_5 x^5_{n+\frac{3}{2}} + \alpha_6 x^6_{n+\frac{3}{2}}$$

$$Y(x_{n+2}) = \alpha_0 + \alpha_1 x_{n+2} + \alpha_2 x^2_{n+2} + \alpha_3 x^3_{n+2} + \alpha_4 x^4_{n+2} + \alpha_5 x^5_{n+2} + \alpha_6 x^6_{n+2}$$

$$Y(x_{n+3}) = \alpha_0 + \alpha_1 x_{n+3} + \alpha_2 x^2_{n+3} + \alpha_3 x^3_{n+3} + \alpha_4 x^4_{n+3} + \alpha_5 x^5_{n+3} + \alpha_6 x^6_{n+3}$$

(3.8)

$$D\psi = Y$$

where

$\psi = \left(\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \alpha_2, \alpha_3, \beta_4\right)^T, Y = \left(y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+3}, f_{n+4}\right)^T$ and the matrix D of the proposed method is expressed as

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & \left(x_n+\frac{1}{2}h\right) & \left(x_n+\frac{1}{2}h\right)^2 & \left(x_n+\frac{1}{2}h\right)^3 & \left(x_n+\frac{1}{2}h\right)^4 & \left(x_n+\frac{1}{2}h\right)^5 & \left(x_n+\frac{1}{2}h\right)^6 \\ 1 & (x_n+1h) & (x_n+1h)^2 & (x_n+1h)^3 & (x_n+1h)^4 & (x_n+1h)^5 & (x_n+1h)^6 \\ 1 & \left(x_n+\frac{3}{2}h\right) & \left(x_n+\frac{3}{2}h\right)^2 & \left(x_n+\frac{3}{2}h\right)^3 & \left(x_n+\frac{3}{2}h\right)^4 & \left(x_n+\frac{3}{2}h\right)^5 & \left(x_n+\frac{3}{2}h\right)^6 \\ 1 & (x_n+2h) & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 & (x_n+2h)^6 \\ 1 & (x_n+3h) & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 & (x_n+3h)^6 \\ 0 & 0 & 2 & 6(x_n+4h) & 12(x_n+4h)^2 & 20(x_n+4h)^3 & 30(x_n+4h)^4 \end{bmatrix} \quad (3.9)$$

Solving (3.8) using matrix inversion method with the aid of Maple 2017 software to obtain the following continuous coefficients;

$$\alpha_0 = \frac{1}{45378}\frac{1}{h^6}\left(45378\,h^6 + 213165\,h^5\,x_n + 368068\,h^4\,x_n^2 + 304235\,h^3\,x_n^3 + 127982\,h^2\,x_n^4 + 26020\,h\,x_n^5 + 1992\,x_n^6\right)$$
$$- \frac{1}{45378}\frac{1}{h^6}\left(\left(213165\,h^5 + 736136\,h^4\,x_n + 912705\,h^3\,x_n^2 + 511928\,h^2\,x_n^3 + 130100\,h\,x_n^4 + 11952\,x_n^5\right)x\right)$$
$$+ \frac{1}{45378}\frac{\left(368068\,h^4 + 912705\,h^3\,x_n + 767892\,h^2\,x_n^2 + 260200\,h\,x_n^3 + 29880\,x_n^4\right)x^2}{h^6}$$
$$- \frac{1}{45378}\frac{\left(304235\,h^3 + 511928\,h^2\,x_n + 260200\,h\,x_n^2 + 39840\,x_n^3\right)x^3}{h^6}$$
$$+ \frac{1}{22689}\frac{\left(63991\,h^2 + 65050\,h\,x_n + 14940\,x_n^2\right)x^4}{h^6} - \frac{2}{22689}\frac{\left(6505\,h + 2988\,x_n\right)x^5}{h^6}$$
$$+ \frac{332}{7563}\frac{x^6}{h^6} \tag{3.10}$$

$$\alpha_{\frac{1}{2}} = -\frac{16}{37815}\frac{x_n\left(25164\,h^5 + 67860\,h^4\,x_n + 68295\,h^3\,x_n^2 + 31970\,h^2\,x_n^3 + 6921\,h\,x_n^4 + 550\,x_n^5\right)}{h^6}$$
$$+ \frac{16}{37815}\frac{1}{h^6}\left(\left(25164\,h^5 + 135720\,h^4\,x_n + 204885\,h^3\,x_n^2 + 127880\,h^2\,x_n^3 + 34605\,h\,x_n^4 + 3300\,x_n^5\right)x\right)$$
$$- \frac{16}{2521}\frac{\left(4524\,h^4 + 13659\,h^3\,x_n + 12788\,h^2\,x_n^2 + 4614\,h\,x_n^3 + 550\,x_n^4\right)x^2}{h^6}$$
$$+ \frac{16}{7563}\frac{\left(13659\,h^3 + 25576\,h^2\,x_n + 13842\,h\,x_n^2 + 2200\,x_n^3\right)x^3}{h^6}$$
$$- \frac{16}{7563}\frac{\left(6394\,h^2 + 6921\,h\,x_n + 1650\,x_n^2\right)x^4}{h^6} + \frac{16}{12605}\frac{\left(2307\,h + 1100\,x_n\right)x^5}{h^6}$$
$$- \frac{1760}{7563}\frac{x^6}{h^6} \tag{3.11}$$

$$\alpha_1 = \frac{1}{5042}\frac{1}{h^6}\Big(x_n\big(56412\,h^5 + 208476\,h^4 x_n + 248597\,h^3 x_n^2 + 128823\,h^2 x_n^3 + 29700\,h\, x_n^4 + 2452\,x_n^5\big)\Big) - \frac{3}{5042}\frac{1}{h^6}\Big(\big(18804\,h^5 + 138984\,h^4 x_n + 248597\,h^3 x_n^2 + 171764\,h^2 x_n^3 + 49500\,h\, x_n^4 + 4904\,x_n^5\big)\,x\Big) + \frac{3}{5042}\frac{\big(69492\,h^4 + 248597\,h^3 x_n + 257646\,h^2 x_n^2 + 99000\,h\,x_n^3 + 12260\,x_n^4\big)\,x^2}{h^6} - \frac{1}{5042}\frac{\big(248597\,h^3 + 515292\,h^2 x_n + 297000\,h\,x_n^2 + 49040\,x_n^3\big)\,x^3}{h^6} + \frac{3}{5042}\frac{\big(42941\,h^2 + 49500\,h\,x_n + 12260\,x_n^2\big)\,x^4}{h^6} - \frac{6}{2521}\frac{\big(2475\,h + 1226\,x_n\big)\,x^5}{h^6} + \frac{1226}{2521}\frac{x^6}{h^6} \tag{3.12}$$

$$\alpha_{\frac{3}{2}} = -\frac{16}{22689}\frac{x_n\big(10668\,h^5 + 42964\,h^4 x_n + 57719\,h^3 x_n^2 + 32774\,h^2 x_n^3 + 8041\,h\,x_n^4 + 690\,x_n^5\big)}{h^6} + \frac{16}{22689}\frac{1}{h^6}\Big(\big(10668\,h^5 + 85928\,h^4 x_n + 173157\,h^3 x_n^2 + 131096\,h^2 x_n^3 + 40205\,h\,x_n^4 + 4140\,x_n^5\big)\,x\Big) - \frac{16}{22689}\frac{\big(42964\,h^4 + 173157\,h^3 x_n + 196644\,h^2 x_n^2 + 80410\,h\,x_n^3 + 10350\,x_n^4\big)\,x^2}{h^6} + \frac{16}{22689}\frac{\big(57719\,h^3 + 131096\,h^2 x_n + 80410\,h\,x_n^2 + 13800\,x_n^3\big)\,x^3}{h^6} - \frac{16}{22689}\frac{\big(32774\,h^2 + 40205\,h\,x_n + 10350\,x_n^2\big)\,x^4}{h^6} + \frac{16}{22689}\frac{\big(8041\,h + 4140\,x_n\big)\,x^5}{h^6} - \frac{3680}{7563}\frac{x^6}{h^6} \tag{3.13}$$

$$
\begin{aligned}
\alpha_2 \\
&= \frac{1}{15126}\,\frac{1}{h^6}\Big(x_n\big(36801\,h^5 + 154260\,h^4\,x_n + 220407\,h^3\,x_n^2 + 134984\,h^2\,x_n^3 \\
&+ 35172\,h\,x_n^4 + 3136\,x_n^5\big)\Big) \\
&- \frac{1}{15126}\,\frac{1}{h^6}\Big(\big(36801\,h^5 + 308520\,h^4\,x_n + 661221\,h^3\,x_n^2 + 539936\,h^2\,x_n^3 \\
&+ 175860\,h\,x_n^4 + 18816\,x_n^5\big)\,x\Big) \\
&+ \frac{1}{5042}\,\frac{\big(51420\,h^4 + 220407\,h^3\,x_n + 269968\,h^2\,x_n^2 + 117240\,h\,x_n^3 + 15680\,x_n^4\big)\,x^2}{h^6} \\
&- \frac{1}{15126}\,\frac{\big(220407\,h^3 + 539936\,h^2\,x_n + 351720\,h\,x_n^2 + 62720\,x_n^3\big)\,x^3}{h^6} \\
&+ \frac{2}{7563}\,\frac{\big(33746\,h^2 + 43965\,h\,x_n + 11760\,x_n^2\big)\,x^4}{h^6} - \frac{2}{2521}\,\frac{\big(2931\,h + 1568\,x_n\big)\,x^5}{h^6} \\
&+ \frac{1568}{7563}\,\frac{x^6}{h^6}
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
\alpha_3 = \\
&-\frac{1}{226890}\,\frac{1}{h^6}\Big(x_n\big(33756\,h^5 + 146860\,h^4\,x_n + 222785\,h^3\,x_n^2 + 148745\,h^2\,x_n^3 \\
&+ 43204\,h\,x_n^4 + 4140\,x_n^5\big)\Big) \\
&+ \frac{1}{226890}\,\frac{1}{h^6}\Big(\big(33756\,h^5 + 293720\,h^4\,x_n + 668355\,h^3\,x_n^2 + 594980\,h^2\,x_n^3 \\
&+ 216020\,h\,x_n^4 + 24840\,x_n^5\big)\,x\Big) \\
&- \frac{1}{45378}\,\frac{\big(29372\,h^4 + 133671\,h^3\,x_n + 178494\,h^2\,x_n^2 + 86408\,h\,x_n^3 + 12420\,x_n^4\big)\,x^2}{h^6} \\
&+ \frac{1}{45378}\,\frac{\big(44557\,h^3 + 118996\,h^2\,x_n + 86408\,h\,x_n^2 + 16560\,x_n^3\big)\,x^3}{h^6} \\
&- \frac{1}{45378}\,\frac{\big(29749\,h^2 + 43204\,h\,x_n + 12420\,x_n^2\big)\,x^4}{h^6} + \frac{2}{113445}\,\frac{\big(10801\,h + 6210\,x_n\big)\,x^5}{h^6} \\
&- \frac{46}{2521}\,\frac{x^6}{h^6}
\end{aligned}
\tag{3.15}
$$

$$\beta_4 = \frac{1}{5042}\frac{x_n\left(18h^5 + 81h^4x_n + 130h^3x_n^2 + 95h^2x_n^3 + 32hx_n^4 + 4x_n^5\right)}{h^4}$$

$$- \frac{1}{2521}\frac{\left(9h^5 + 81h^4x_n + 195h^3x_n^2 + 190h^2x_n^3 + 80hx_n^4 + 12x_n^5\right)x}{h^4}$$

$$+ \frac{1}{5042}\frac{\left(81h^4 + 390h^3x_n + 570h^2x_n^2 + 320hx_n^3 + 60x_n^4\right)x^2}{h^4}$$

$$- \frac{5}{2521}\frac{\left(13h^3 + 38h^2x_n + 32hx_n^2 + 8x_n^3\right)x^3}{h^4} + \frac{5}{5042}\frac{\left(19h^2 + 32hx_n + 12x_n^2\right)x^4}{h^4}$$

$$- \frac{4}{2521}\frac{\left(4h + 3x_n\right)x^5}{h^4} + \frac{2}{2521}\frac{x^6}{h^4}$$

(3.16)

The values of the continuous coefficients are then substituted in to the proposed method in (3.3) to obtain

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_1(x)y_{n+y_{n+1}} + \alpha_{\frac{3}{2}}(x)y_{n+\frac{3}{2}} + \alpha_2(x)y_{n+2} +$$
$$\alpha_3(x)y_{n+3} + h^2\beta_4(x)f_{n+4} \quad (3.17)$$

$$y(x) = f_{n+4}\left(\frac{1}{5042}\,\frac{x_n\left(18\,h^5 + 81\,h^4\,x_n + 130\,h^3\,x_n^2 + 95\,h^2\,x_n^3 + 32\,h\,x_n^4 + 4\,x_n^5\right)}{h^4}\right.$$

$$-\frac{1}{2521}\,\frac{\left(9\,h^5 + 81\,h^4\,x_n + 195\,h^3\,x_n^2 + 190\,h^2\,x_n^3 + 80\,h\,x_n^4 + 12\,x_n^5\right)x}{h^4}$$

$$+\frac{1}{5042}\,\frac{\left(81\,h^4 + 390\,h^3\,x_n + 570\,h^2\,x_n^2 + 320\,h\,x_n^3 + 60\,x_n^4\right)x^2}{h^4}$$

$$-\frac{5}{2521}\,\frac{\left(13\,h^3 + 38\,h^2\,x_n + 32\,h\,x_n^2 + 8\,x_n^3\right)x^3}{h^4} + \frac{5}{5042}\,\frac{\left(19\,h^2 + 32\,h\,x_n + 12\,x_n^2\right)x^4}{h^4}$$

$$\left.-\frac{4}{2521}\,\frac{\left(4\,h + 3\,x_n\right)x^5}{h^4} + \frac{2}{2521}\,\frac{x^6}{h^4}\right) + y_n\left(\frac{1}{45378}\,\frac{1}{h^6}\left(45378\,h^6\right.\right.$$

$$\left.+ 213165\,h^5\,x_n + 368068\,h^4\,x_n^2 + 304235\,h^3\,x_n^3 + 127982\,h^2\,x_n^4 + 26020\,h\,x_n^5 + 1992\,x_n^6\right)$$

$$-\frac{1}{45378}\,\frac{1}{h^6}\left(\left(213165\,h^5 + 736136\,h^4\,x_n + 912705\,h^3\,x_n^2 + 511928\,h^2\,x_n^3\right.\right.$$

$$\left.\left.+ 130100\,h\,x_n^4 + 11952\,x_n^5\right)x\right)$$

$$+\frac{1}{45378}\,\frac{\left(368068\,h^4 + 912705\,h^3\,x_n + 767892\,h^2\,x_n^2 + 260200\,h\,x_n^3 + 29880\,x_n^4\right)x^2}{h^6}$$

$$-\frac{1}{45378}\,\frac{\left(304235\,h^3 + 511928\,h^2\,x_n + 260200\,h\,x_n^2 + 39840\,x_n^3\right)x^3}{h^6}$$

$$+\frac{1}{22689}\,\frac{\left(63991\,h^2 + 65050\,h\,x_n + 14940\,x_n^2\right)x^4}{h^6} - \frac{2}{22689}\,\frac{\left(6505\,h + 2988\,x_n\right)x^5}{h^6}$$

$$\left.+\frac{332}{7563}\,\frac{x^6}{h^6}\right)$$

$$+ y_{n+1}\left(\frac{1}{5042}\,\frac{1}{h^6}\left(x_n\left(56412\,h^5 + 208476\,h^4\,x_n + 248597\,h^3\,x_n^2 + 128823\,h^2\,x_n^3\right.\right.\right.$$

$$\left.\left.+ 29700\,h\,x_n^4 + 2452\,x_n^5\right)\right)$$

$$-\frac{3}{5042}\,\frac{1}{h^6}\left(\left(18804\,h^5 + 138984\,h^4\,x_n + 248597\,h^3\,x_n^2 + 171764\,h^2\,x_n^3 + 49500\,h\right.\right.$$

$$\left.\left.x_n^4 + 4904\,x_n^5\right)x\right)$$

$$+\frac{3}{5042}\,\frac{\left(69492\,h^4 + 248597\,h^3\,x_n + 257646\,h^2\,x_n^2 + 99000\,h\,x_n^3 + 12260\,x_n^4\right)x^2}{h^6}$$

$$-\frac{1}{5042}\,\frac{\left(248597\,h^3 + 515292\,h^2\,x_n + 297000\,h\,x_n^2 + 49040\,x_n^3\right)x^3}{h^6}$$

$$+\frac{3}{5042}\,\frac{\left(42941\,h^2 + 49500\,h\,x_n + 12260\,x_n^2\right)x^4}{h^6} - \frac{6}{2521}\,\frac{\left(2475\,h + 1226\,x_n\right)x^5}{h^6}$$

$$\left.+\frac{1226}{2521}\,\frac{x^6}{h^6}\right)$$

$$+y_{n+2}\Bigg(\frac{1}{15126}\frac{1}{h^6}\big(x_n\big(36801\,h^5+154260\,h^4\,x_n+220407\,h^3\,x_n^2+134984\,h^2\,x_n^3$$

$$+35172\,h\,x_n^4+3136\,x_n^5\big)\big)$$

$$-\frac{1}{15126}\frac{1}{h^6}\big(\big(36801\,h^5+308520\,h^4\,x_n+661221\,h^3\,x_n^2+539936\,h^2\,x_n^3$$

$$+175860\,h\,x_n^4+18816\,x_n^5\big)\,x\big)$$

$$+\frac{1}{5042}\frac{\big(51420\,h^4+220407\,h^3\,x_n+269968\,h^2\,x_n^2+117240\,h\,x_n^3+15680\,x_n^4\big)\,x^2}{h^6}$$

$$-\frac{1}{15126}\frac{\big(220407\,h^3+539936\,h^2\,x_n+351720\,h\,x_n^2+62720\,x_n^3\big)\,x^3}{h^6}$$

$$+\frac{2}{7563}\frac{\big(33746\,h^2+43965\,h\,x_n+11760\,x_n^2\big)\,x^4}{h^6}-\frac{2}{2521}\frac{\big(2931\,h+1568\,x_n\big)\,x^5}{h^6}$$

$$+\frac{1568}{7563}\frac{x^6}{h^6}\Bigg)+y_{n+3}\Bigg($$

$$-\frac{1}{226890}\frac{1}{h^6}\big(x_n\big(33756\,h^5+146860\,h^4\,x_n+222785\,h^3\,x_n^2+148745\,h^2\,x_n^3$$

$$+43204\,h\,x_n^4+4140\,x_n^5\big)\big)$$

$$+\frac{1}{226890}\frac{1}{h^6}\big(\big(33756\,h^5+293720\,h^4\,x_n+668355\,h^3\,x_n^2+594980\,h^2\,x_n^3$$

$$+216020\,h\,x_n^4+24840\,x_n^5\big)\,x\big)$$

$$-\frac{1}{45378}\frac{\big(29372\,h^4+133671\,h^3\,x_n+178494\,h^2\,x_n^2+86408\,h\,x_n^3+12420\,x_n^4\big)\,x^2}{h^6}$$

$$+\frac{1}{45378}\frac{\big(44557\,h^3+118996\,h^2\,x_n+86408\,h\,x_n^2+16560\,x_n^3\big)\,x^3}{h^6}$$

$$-\frac{1}{45378}\frac{\big(29749\,h^2+43204\,h\,x_n+12420\,x_n^2\big)\,x^4}{h^6}+\frac{2}{113445}\frac{\big(10801\,h+6210\,x_n\big)\,x^5}{h^6}$$

$$-\frac{46}{2521}\frac{x^6}{h^6}\Bigg)+y_{n+\frac{1}{2}}\Bigg($$

$$-\frac{16}{37815}\frac{x_n\big(25164\,h^5+67860\,h^4\,x_n+68295\,h^3\,x_n^2+31970\,h^2\,x_n^3+6921\,h\,x_n^4+550\,x_n^5\big)}{h^6}$$

$$+\frac{16}{37815}\frac{1}{h^6}\big(\big(25164\,h^5+135720\,h^4\,x_n+204885\,h^3\,x_n^2+127880\,h^2\,x_n^3+34605\,h$$

$$x_n^4+3300\,x_n^5\big)\,x\big)$$

$$-\frac{16}{2521}\frac{\big(4524\,h^4+13659\,h^3\,x_n+12788\,h^2\,x_n^2+4614\,h\,x_n^3+550\,x_n^4\big)\,x^2}{h^6}$$

$$+\frac{16}{7563}\frac{\big(13659\,h^3+25576\,h^2\,x_n+13842\,h\,x_n^2+2200\,x_n^3\big)\,x^3}{h^6}$$

$$-\frac{16}{7563}\frac{\big(6394\,h^2+6921\,h\,x_n+1650\,x_n^2\big)\,x^4}{h^6}+\frac{16}{12605}\frac{\big(2307\,h+1100\,x_n\big)\,x^5}{h^6}$$

$$-\frac{1760}{7563}\frac{x^6}{h^6}\Bigg)$$

$$
\begin{aligned}
&+y_{n+\frac{3}{2}}\Bigg(\\
&-\frac{16}{22689}\frac{x_n\left(10668h^5+42964h^4x_n+57719h^3x_n^2+32774h^2x_n^3+8041hx_n^4+690x_n^5\right)}{h^6} \\
&+\frac{16}{22689}\frac{1}{h^6}\Big(\big(10668h^5+85928h^4x_n+173157h^3x_n^2+131096h^2x_n^3+40205hx_n^4 \\
&+4140x_n^5\big)x\Big) \\
&-\frac{16}{22689}\frac{\left(42964h^4+173157h^3x_n+196644h^2x_n^2+80410hx_n^3+10350x_n^4\right)x^2}{h^6} \\
&+\frac{16}{22689}\frac{\left(57719h^3+131096h^2x_n+80410hx_n^2+13800x_n^3\right)x^3}{h^6} \\
&-\frac{16}{22689}\frac{\left(32774h^2+40205hx_n+10350x_n^2\right)x^4}{h^6}+\frac{16}{22689}\frac{\left(8041h+4140x_n\right)x^5}{h^6} \\
&-\frac{3680}{7563}\frac{x^6}{h^6}\Bigg)
\end{aligned}
$$

(3.18)

Evaluate (3.18) at $x = x_{n+4}$, gives the discrete scheme as

$$y_{n+4} = -\frac{18515}{7563}y_n + \frac{38144}{2521}y_{n+\frac{1}{2}} - \frac{95480}{2521}y_{n+1} - \frac{66710}{2521}y_{n+2} + \frac{41608}{7563}y_{n+3} + \frac{356608}{7563}y_{n+\frac{3}{2}} + \frac{420}{2521}h^2f_{n+2} \quad (3.19)$$

To obtain the sufficient schemes required, we obtain the first derivative of (3.18) and evaluate the continuous function at $x = x_n, x = x_{n+\frac{1}{2}}, x = x_{n+1}, x = x_{n+\frac{3}{2}}, x = x_{n+2}, x = x_{n+3}$ and $x = x_{n+4}$

$$hz_{n+4} = -\frac{175261}{45378}y_n + \frac{892864}{37815}y_{n+\frac{1}{2}} - \frac{146742}{2521}y_{n+1} - \frac{578929}{15126}y_{n+2} + \frac{669398}{113445}y_{n+3} + \frac{1607104}{22689}y_{n+\frac{3}{2}} + \frac{1163}{2521}h^2f_{n+4}$$

$$hz_{n+3} = -\frac{5235}{7563}y_n + \frac{55008}{12605}y_{n+\frac{1}{2}} - \frac{58275}{5042}y_{n+1} - \frac{26055}{2521}y_{n+2} + \frac{62759}{25210}y_{n+3} + \frac{118880}{7563}y_{n+\frac{3}{2}} + \frac{45}{2521}h^2f_{n+4}$$

$$hz_{n+2} = \frac{1525}{15126}y_n - \frac{27136}{37815}y_{n+\frac{1}{2}} + \frac{5724}{2521}y_{n+1} + \frac{43195}{15126}y_{n+2} + \frac{3556}{37815}y_{n+3} - \frac{34816}{7563}y_{n+\frac{3}{2}} - \frac{3}{2521}h^2 f_{n+4}$$

$$hz_{n+1} = \frac{2023}{22689}y_n - \frac{35936}{37815}y_{n+\frac{1}{2}} - \frac{3747}{5054}y_{n+1} - \frac{3305}{7563}y_{n+2} + \frac{4591}{226890}y_{n+3} + \frac{45856}{22689}y_{n+\frac{3}{2}} - \frac{1}{2521}h^2 f_{n+4}$$

$$hz_n = -\frac{23685}{5054}y_n - \frac{134208}{12605}y_{n+\frac{1}{2}} - \frac{28206}{2521}y_{n+1} - \frac{12267}{5054}y_{n+2} + \frac{5626}{37815}y_{n+3} + \frac{56896}{7563}y_{n+\frac{3}{2}} - \frac{9}{2521}h^2 f_{n+4}$$

$$hz_{n+\frac{3}{2}} = -\frac{887}{15126}y_n + \frac{5913}{12605}y_{n+\frac{1}{2}} - \frac{39861}{20168}y_{n+1} + \frac{8739}{10084}y_{n+2} - \frac{8147}{302520}y_{n+3} + \frac{1831}{2521}y_{n+\frac{3}{2}} + \frac{9}{20168}h^2 f_{n+4}$$

$$hz_{n+\frac{1}{2}} = -\frac{2840}{7563}y_n - \frac{86401}{37815}y_{n+\frac{1}{2}} + \frac{84825}{20168}y_{n+1} + \frac{18485}{30252}y_{n+2} - \frac{10217}{302520}y_{n+3} + \frac{56896}{7563}y_{n+\frac{3}{2}} + \frac{15}{20168}h^2 f_{n+4} \quad (3.20)$$

where z is the first derivative of y.

Likewise, we further obtain the second derivatives of (3.18), thereafter, evaluating at $x = x_{n+\frac{1}{2}}, x = x_{n+\frac{3}{2}}, x = x_{n+1}, x = x_{n+2}$ and $x = x_{n+3}$ to obtain;

$$y_{n+3} = \frac{64139}{92191}y_n - \frac{400032}{92191}y_{n+\frac{1}{2}} - \frac{1010475}{92191}y_{n+1} + \frac{679401}{92191}y_{n+2} - \frac{1261792}{92191}y_{n+\frac{3}{2}} - \frac{81}{3179}h^2 f_{n+4} + \frac{22689}{92191}h^2 f_{n+3}$$

$$y_{n+\frac{1}{2}} = \frac{10433}{14364}y_n - \frac{1145}{2128}y_{n+1} - \frac{473}{1064}y_{n+2} + \frac{1637}{57456}y_{n+3} + \frac{4409}{3591}y_{n+\frac{3}{2}} - \frac{31}{44688}h^2 f_{n+4} - \frac{2521}{11172}h^2 f_{n+\frac{1}{2}}$$

$$y_{n+1} = -\frac{785}{24597}y_n + \frac{4384}{8199}y_{n+\frac{1}{2}} - \frac{193}{8199}y_{n+2} - \frac{23}{24597}y_{n+3} + \frac{12832}{24597}y_{n+\frac{3}{2}} + \frac{1}{24597}h^2 f_{n+4} - \frac{2521}{24597}h^2 f_{n+1}$$

$$y_{n+2} = -\frac{5353}{15471}y_n + \frac{1312}{573}y_{n+\frac{1}{2}} - \frac{8269}{15471}y_{n+3} + \frac{88736}{15471}y_{n+\frac{3}{2}} + \frac{19}{3438}h^2 f_{n+4} + \frac{2521}{3438}h^2 f_{n+2}$$

(3.21)

3.2.2 5-Step block hybrid backward differentiation formulae (5SBHBDF)

To derive 5SBHBDF, we take $t = 7, c = 1$and $x \in [x_n, x_{n+5}]$. Therefore, (3.4) becomes;

$$Y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6 + \alpha_7 x^7 \qquad (3.22)$$

Interpolating (3.5) at $x_{n+i}; i = 0, \frac{2}{5} 1\ \frac{6}{5}, 2, 3,4$ and collocate (3.6) at $x_{n+i}; i = 5$. This results in a system of equations;

$$Y''(x_{n+5}) = 2\alpha_2 + 6\alpha_3 x_{n+5} + 12\alpha_4 x_{n+5}^2 + 20\alpha_5 x_{n+5}^3 + 30\alpha_6 x_{n+5}^4 + 42\alpha_7 x_{n+5}^5$$

$$Y(x_{n+i}) = \alpha_0 + \alpha_1 x_{n+i} + \alpha_2 x_{n+i}^2 + \alpha_3 x_{n+i}^3 + \alpha_4 x_{n+i}^4 + \alpha_5 x_{n+i}^5 + \alpha_6 x_{n+i}^6 + \alpha_7 x_{n+i}^7$$

$$Y(x_n) = \alpha_0 + \alpha_1 x_n + \alpha_2 x_n^2 + \alpha_3 x_n^3 + \alpha_4 x_n^4 + \alpha_5 x_n^5 + \alpha_6 x_n^6 + \alpha_7 x_n^7$$

$$Y\left(x_{n+\frac{2}{5}}\right) = \alpha_0 + \alpha_1 x_{n+\frac{2}{5}} + \alpha_2 x_{n+\frac{2}{5}}^2 + \alpha_3 x_{n+\frac{2}{5}}^3 + \alpha_4 x_{n+\frac{2}{5}}^4 + \alpha_5 x_{n+\frac{2}{5}}^5 + \alpha_6 x_{n+\frac{2}{5}}^6 + \alpha_7 x_{n+\frac{2}{5}}^7$$

$$Y(x_{n+1}) = \alpha_0 + \alpha_1 x_{n+1} + \alpha_2 x_{n+1}^2 + \alpha_3 x_{n+1}^3 + \alpha_4 x_{n+1}^4 + \alpha_5 x_{n+1}^5 + \alpha_6 x_{n+1}^6 + \alpha_7 x_{n+1}^7$$

$$Y\left(x_{n+\frac{6}{5}}\right) = \alpha_0 + \alpha_1 x_{n+\frac{6}{5}} + \alpha_2 x_{n+\frac{6}{5}}^2 + \alpha_3 x_{n+\frac{6}{5}}^3 + \alpha_4 x_{n+\frac{6}{5}}^4 + \alpha_5 x_{n+\frac{6}{5}}^5 + \alpha_6 x_{n+\frac{6}{5}}^6 + \alpha_7 x_{n+\frac{6}{5}}^7$$

$$Y(x_{n+2}) = \alpha_0 + \alpha_1 x_{n+2} + \alpha_2 x_{n+2}^2 + \alpha_3 x_{n+2}^3 + \alpha_4 x_{n+2}^4 + \alpha_5 x_{n+2}^5 + \alpha_6 x_{n+2}^6 + \alpha_7 x_{n+2}^7$$

$$Y(x_{n+3}) = \alpha_0 + \alpha_1 x_{n+3} + \alpha_2 x_{n+3}^2 + \alpha_3 x_{n+3}^3 + \alpha_4 x_{n+3}^4 + \alpha_5 x_{n+3}^5 + \alpha_6 x_{n+3}^6 + \alpha_7 x_{n+3}^7$$

$$Y(x_{n+4}) = \alpha_0 + \alpha_1 x_{n+4} + \alpha_2 x_{n+4}^2 + \alpha_3 x_{n+4}^3 + \alpha_4 x_{n+4}^4 + \alpha_5 x_{n+4}^5 + \alpha_6 x_{n+4}^6 + \alpha_7 x_{n+4}^7$$

(3.23)

$$D\psi = Y$$

where

$$\psi = \left(\alpha_0, \alpha_{\frac{2}{5}}, \alpha_1, \alpha_{\frac{6}{5}}, \alpha_2, \alpha_3, \alpha_4, \beta_5\right)^T, Y = \left(y_n, y_{n+\frac{2}{5}}, y_{n+1}, y_{n+\frac{6}{5}}, y_{n+2}, y_{n+3}, y_{n+4}, f_{n+5}\right)^T$$

and the matrix D of the proposed method is expressed as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+\frac{2}{5}} & x_{n+\frac{2}{5}}^2 & x_{n+\frac{2}{5}}^3 & x_{n+\frac{2}{5}}^4 & x_{n+\frac{2}{5}}^5 & x_{n+\frac{2}{5}}^6 & x_{n+\frac{2}{5}}^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+\frac{6}{5}} & x_{n+\frac{6}{5}}^2 & x_{n+\frac{6}{5}}^3 & x_{n+\frac{6}{5}}^4 & x_{n+\frac{6}{5}}^5 & x_{n+\frac{6}{5}}^6 & x_{n+\frac{6}{5}}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5} & 20x_{n+5} & 30x_{n+5} & 42x_{n+5} \end{bmatrix} \quad (3.24)$$

$$\alpha_0 := \frac{1}{9150624}\frac{1}{h^7}\Big(9150624\,h^7 + 51086232\,h^6\,x_n + 104189700\,h^5\,x_n^2 + 103953790\,h^4\,x_n^3 + 55514681\,h^3\,x_n^4 + 16008243\,h^2\,x_n^5 + 2325235\,h\,x_n^6 + 131975\,x_n^7\Big)$$
$$- \frac{1}{9150624}\frac{1}{h^7}\Big(\big(51086232\,h^6 + 208379400\,h^5\,x_n + 311861370\,h^4\,x_n^2 + 222058724\,h^3\,x_n^3 + 80041215\,h^2\,x_n^4 + 13951410\,h\,x_n^5 + 923825\,x_n^6\big)\,x\Big)$$
$$+ \frac{1}{3050208}\frac{1}{h^7}\Big(\big(34729900\,h^5 + 103953790\,h^4\,x_n + 111029362\,h^3\,x_n^2 + 53360810\,h^2\,x_n^3 + 11626175\,h\,x_n^4 + 923825\,x_n^5\big)\,x^2\Big)$$
$$- \frac{1}{9150624}\frac{1}{h^7}\Big(\big(103953790\,h^4 + 222058724\,h^3\,x_n + 160082430\,h^2\,x_n^2 + 46504700\,h\,x_n^3 + 4619125\,x_n^4\big)\,x^3\Big)$$
$$+ \frac{1}{9150624}\frac{\big(55514681\,h^3 + 80041215\,h^2\,x_n + 34878525\,h\,x_n^2 + 4619125\,x_n^3\big)\,x^4}{h^7}$$
$$- \frac{1}{3050208}\frac{\big(5336081\,h^2 + 4650470\,h\,x_n + 923825\,x_n^2\big)\,x^5}{h^7}$$
$$+ \frac{5}{9150624}\frac{\big(465047\,h + 184765\,x_n\big)\,x^6}{h^7} - \frac{131975}{9150624}\frac{x^7}{h^7}$$

$$\alpha_1 := \frac{1}{2859570}\frac{1}{h^7}\Big(x_n\big(54818496\,h^6 + 251180400\,h^5\,x_n + 372756560\,h^4\,x_n^2 + 249551320\,h^3\,x_n^3 + 82610124\,h^2\,x_n^4 + 13099955\,h\,x_n^5 + 786925\,x_n^6\big)\Big)$$
$$- \frac{1}{2859570}\frac{1}{h^7}\Big(\big(54818496\,h^6 + 502360800\,h^5\,x_n + 1118269680\,h^4\,x_n^2 + 998205280\,h^3\,x_n^3 + 413050620\,h^2\,x_n^4 + 78599730\,h\,x_n^5 + 5508475\,x_n^6\big)\,x\Big)$$
$$+ \frac{1}{190638}\frac{1}{h^7}\Big(\big(16745360\,h^5 + 74551312\,h^4\,x_n + 99820528\,h^3\,x_n^2 + 55073416\,h^2\,x_n^3 + 13099955\,h\,x_n^4 + 1101695\,x_n^5\big)\,x^2\Big)$$
$$- \frac{1}{571914}\frac{1}{h^7}\Big(\big(74551312\,h^4 + 199641056\,h^3\,x_n + 165220248\,h^2\,x_n^2 + 52399820\,h\,x_n^3 + 5508475\,x_n^4\big)\,x^3\Big)$$
$$+ \frac{1}{571914}\frac{\big(49910264\,h^3 + 82610124\,h^2\,x_n + 39299865\,h\,x_n^2 + 5508475\,x_n^3\big)\,x^4}{h^7}$$
$$- \frac{1}{953190}\frac{\big(27536708\,h^2 + 26199910\,h\,x_n + 5508475\,x_n^2\big)\,x^5}{h^7}$$
$$+ \frac{1}{571914}\frac{\big(2619991\,h + 1101695\,x_n\big)\,x^6}{h^7} - \frac{157385}{571914}\frac{x^7}{h^7}$$

$$
\begin{aligned}
\alpha_2 := {} & \frac{1}{6778240}\,\frac{1}{h^7}\Big(x_n\,\big(11349648\,h^6 + 57664500\,h^5\,x_n + 100267600\,h^4\,x_n^2 + 78553555\,h^3\,x_n^3 \\
& + 29351397\,h^2\,x_n^4 + 5073865\,h\,x_n^5 + 323275\,x_n^6\big)\Big) - \frac{1}{6778240}\,\frac{1}{h^7}\Big(\big(11349648\,h^6 \\
& + 115329000\,h^5\,x_n + 300802800\,h^4\,x_n^2 + 314214220\,h^3\,x_n^3 + 146756985\,h^2\,x_n^4 \\
& + 30443190\,h\,x_n^5 + 2262925\,x_n^6\big)\,x\Big) + \frac{3}{1355648}\,\frac{1}{h^7}\Big(\big(3844300\,h^5 + 20053520\,h^4\,x_n \\
& + 31421422\,h^3\,x_n^2 + 19567598\,h^2\,x_n^3 + 5073865\,h\,x_n^4 + 452585\,x_n^5\big)\,x^2\Big) \\
& - \frac{1}{1355648}\,\frac{1}{h^7}\Big(\big(20053520\,h^4 + 62842844\,h^3\,x_n + 58702794\,h^2\,x_n^2 + 20295460\,h \\
& x_n^3 + 2262925\,x_n^4\big)\,x^3\Big) \\
& + \frac{1}{1355648}\,\frac{\big(15710711\,h^3 + 29351397\,h^2\,x_n + 15221595\,h\,x_n^2 + 2262925\,x_n^3\big)\,x^4}{h^7} \\
& - \frac{3}{6778240}\,\frac{\big(9783799\,h^2 + 10147730\,h\,x_n + 2262925\,x_n^2\big)\,x^5}{h^7} \\
& + \frac{1}{1355648}\,\frac{\big(1014773\,h + 452585\,x_n\big)\,x^6}{h^7} - \frac{64655}{1355648}\,\frac{x^7}{h^7}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_3 := {} & -\frac{1}{111523230}\,\frac{1}{h^7}\Big(x_n\,\big(29579904\,h^6 + 155140800\,h^5\,x_n + 284336120\,h^4\,x_n^2 \\
& + 239807140\,h^3\,x_n^3 + 98188086\,h^2\,x_n^4 + 18399995\,h\,x_n^5 + 1243825\,x_n^6\big)\Big) \\
& + \frac{1}{111523230}\,\frac{1}{h^7}\Big(\big(29579904\,h^6 + 310281600\,h^5\,x_n + 853008360\,h^4\,x_n^2 \\
& + 959228560\,h^3\,x_n^3 + 490940430\,h^2\,x_n^4 + 110399970\,h\,x_n^5 + 8706775\,x_n^6\big)\,x\Big) \\
& - \frac{1}{7434882}\,\frac{1}{h^7}\Big(\big(10342720\,h^5 + 56867224\,h^4\,x_n + 95922856\,h^3\,x_n^2 + 65458724\,h^2\,x_n^3 \\
& + 18399995\,h\,x_n^4 + 1741355\,x_n^5\big)\,x^2\Big) \\
& + \frac{1}{22304646}\,\frac{1}{h^7}\Big(\big(56867224\,h^4 + 191845712\,h^3\,x_n + 196376172\,h^2\,x_n^2 \\
& + 73599980\,h\,x_n^3 + 8706775\,x_n^4\big)\,x^3\Big) \\
& - \frac{1}{22304646}\,\frac{\big(47961428\,h^3 + 98188086\,h^2\,x_n + 55199985\,h\,x_n^2 + 8706775\,x_n^3\big)\,x^4}{h^7} \\
& + \frac{1}{37174410}\,\frac{\big(32729362\,h^2 + 36799990\,h\,x_n + 8706775\,x_n^2\big)\,x^5}{h^7} \\
& - \frac{1}{22304646}\,\frac{\big(3679999\,h + 1741355\,x_n\big)\,x^6}{h^7} + \frac{248765}{22304646}\,\frac{x^7}{h^7}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_4 := {} & \frac{1}{137259360}\,\frac{1}{h^7}\Big(x_n\Big(4364568\,h^6 + 23232900\,h^5\,x_n + 43656710\,h^4\,x_n^2 + 38222635\,h^3\,x_n^3 \\
& + 16514097\,h^2\,x_n^4 + 3316865\,h\,x_n^5 + 237025\,x_n^6\Big)\Big) - \frac{1}{137259360}\,\frac{1}{h^7}\Big(\Big(4364568\,h^6 \\
& + 46465800\,h^5\,x_n + 130970130\,h^4\,x_n^2 + 152890540\,h^3\,x_n^3 + 82570485\,h^2\,x_n^4 \\
& + 19901190\,h\,x_n^5 + 1659175\,x_n^6\Big)\,x\Big) + \frac{1}{9150624}\,\frac{1}{h^7}\Big(\Big(1548860\,h^5 + 8731342\,h^4\,x_n \\
& + 15289054\,h^3\,x_n^2 + 11009398\,h^2\,x_n^3 + 3316865\,h\,x_n^4 + 331835\,x_n^5\Big)\,x^2\Big) \\
& - \frac{1}{27451872}\,\frac{1}{h^7}\Big(\Big(8731342\,h^4 + 30578108\,h^3\,x_n + 33028194\,h^2\,x_n^2 + 13267460\,h \\
& x_n^3 + 1659175\,x_n^4\Big)\,x^3\Big) \\
& + \frac{1}{27451872}\,\frac{\Big(7644527\,h^3 + 16514097\,h^2\,x_n + 9950595\,h\,x_n^2 + 1659175\,x_n^3\Big)\,x^4}{h^7} \\
& - \frac{1}{45753120}\,\frac{\Big(5504699\,h^2 + 6633730\,h\,x_n + 1659175\,x_n^2\Big)\,x^5}{h^7} \\
& + \frac{1}{27451872}\,\frac{\Big(663373\,h + 331835\,x_n\Big)\,x^6}{h^7} - \frac{47405}{27451872}\,\frac{x^7}{h^7}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{\frac{2}{5}} := {} & -\frac{3125}{1427497344}\,\frac{1}{h^7}\Big(x_n\Big(4900464\,h^6 + 15105900\,h^5\,x_n + 18025160\,h^4\,x_n^2 \\
& + 10593985\,h^3\,x_n^3 + 3232311\,h^2\,x_n^4 + 486275\,h\,x_n^5 + 28225\,x_n^6\Big)\Big) \\
& + \frac{3125}{1427497344}\,\frac{1}{h^7}\Big(\Big(4900464\,h^6 + 30211800\,h^5\,x_n + 54075480\,h^4\,x_n^2 + 42375940\,h^3\,x_n^3 \\
& + 16161555\,h^2\,x_n^4 + 2917650\,h\,x_n^5 + 197575\,x_n^6\Big)\,x\Big) - \frac{15625}{475832448}\,\frac{1}{h^7}\Big(\Big(1007060\,h^5 \\
& + 3605032\,h^4\,x_n + 4237594\,h^3\,x_n^2 + 2154874\,h^2\,x_n^3 + 486275\,h\,x_n^4 + 39515\,x_n^5\Big)\,x^2\Big) \\
& + \frac{15625}{1427497344}\,\frac{1}{h^7}\Big(\Big(3605032\,h^4 + 8475188\,h^3\,x_n + 6464622\,h^2\,x_n^2 + 1945100\,h\,x_n^3 \\
& + 197575\,x_n^4\Big)\,x^3\Big) \\
& - \frac{15625}{1427497344}\,\frac{\Big(2118797\,h^3 + 3232311\,h^2\,x_n + 1458825\,h\,x_n^2 + 197575\,x_n^3\Big)\,x^4}{h^7} \\
& + \frac{3125}{475832448}\,\frac{\Big(1077437\,h^2 + 972550\,h\,x_n + 197575\,x_n^2\Big)\,x^5}{h^7} \\
& - \frac{78125}{1427497344}\,\frac{\Big(19451\,h + 7903\,x_n\Big)\,x^6}{h^7} + \frac{88203125}{1427497344}\,\frac{x^7}{h^7}
\end{aligned}
$$

$$\alpha_{\frac{6}{5}} := -\frac{3125}{3229632}\frac{1}{h^7}\Big(x_n\big(15984h^6 + 75900h^5x_n + 118660h^4x_n^2 + 82535h^3x_n^3 + 28041h^2x_n^4 + 4525hx_n^5 + 275x_n^6\big)\Big) + \frac{3125}{3229632}\frac{1}{h^7}\Big(\big(15984h^6 + 151800h^5x_n + 355980h^4x_n^2 + 330140h^3x_n^3 + 140205h^2x_n^4 + 27150hx_n^5 + 1925x_n^6\big)x\Big) - \frac{15625}{1076544}\frac{1}{h^7}\Big(\big(5060h^5 + 23732h^4x_n + 33014h^3x_n^2 + 18694h^2x_n^3 + 4525hx_n^4 + 385x_n^5\big)x^2\Big) + \frac{15625}{3229632}\frac{\big(23732h^4 + 66028h^3x_n + 56082h^2x_n^2 + 18100hx_n^3 + 1925x_n^4\big)x^3}{h^7} - \frac{15625}{3229632}\frac{\big(16507h^3 + 28041h^2x_n + 13575hx_n^2 + 1925x_n^3\big)x^4}{h^7} + \frac{3125}{1076544}\frac{\big(9347h^2 + 9050hx_n + 1925x_n^2\big)x^5}{h^7} - \frac{78125}{3229632}\frac{\big(181h + 77x_n\big)x^6}{h^7} + \frac{859375}{3229632}\frac{x^7}{h^7}$$

$$\beta_5 := -\frac{1}{317730}\frac{1}{h^5}\Big(x_n\big(288h^6 + 1560h^5x_n + 3020h^4x_n^2 + 2770h^3x_n^3 + 1287h^2x_n^4 + 290hx_n^5 + 25x_n^6\big)\Big) + \frac{1}{317730}\frac{1}{h^5}\Big(\big(288h^6 + 3120h^5x_n + 9060h^4x_n^2 + 11080h^3x_n^3 + 6435h^2x_n^4 + 1740hx_n^5 + 175x_n^6\big)x\Big) - \frac{1}{21182}\frac{\big(104h^5 + 604h^4x_n + 1108h^3x_n^2 + 858h^2x_n^3 + 290hx_n^4 + 35x_n^5\big)x^2}{h^5} + \frac{1}{63546}\frac{\big(604h^4 + 2216h^3x_n + 2574h^2x_n^2 + 1160hx_n^3 + 175x_n^4\big)x^3}{h^5} - \frac{1}{63546}\frac{\big(554h^3 + 1287h^2x_n + 870hx_n^2 + 175x_n^3\big)x^4}{h^5} + \frac{1}{105910}\frac{\big(429h^2 + 580hx_n + 175x_n^2\big)x^5}{h^5} - \frac{1}{63546}\frac{\big(35x_n + 58h\big)x^6}{h^5} + \frac{5}{63546}\frac{x^7}{h^5}$$

(3.25)

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{2}{5}}(x)y_{n+\frac{2}{5}} + \alpha_1(x)y_{n+y_{n+1}} + \alpha_{\frac{6}{5}}(x)y_{n+\frac{6}{5}} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} + h^2\beta_5(x)f_{n+5} \quad (3.26)$$

$$y(x) = \Bigg(\frac{1}{9150624} \frac{1}{h^7} \big(9150624 h^7 + 51086232 h^6 x_n + 104189700 h^5 x_n^2 + 103953790 h^4 x_n^3 + 55514681 h^3 x_n^4 + 16008243 h^2 x_n^5 + 2325235 h x_n^6 + 131975 x_n^7\big) - \frac{1}{9150624} \frac{1}{h^7} \big(\big(51086232 h^6 + 208379400 h^5 x_n + 311861370 h^4 x_n^2 + 222058724 h^3 x_n^3 + 80041215 h^2 x_n^4 + 13951410 h x_n^5 + 923825 x_n^6\big) x\big) + \frac{1}{3050208} \frac{1}{h^7} \big(\big(34729900 h^5 + 103953790 h^4 x_n + 111029362 h^3 x_n^2 + 53360810 h^2 x_n^3 + 11626175 h x_n^4 + 923825 x_n^5\big) x^2\big) - \frac{1}{9150624} \frac{1}{h^7} \big(\big(103953790 h^4 + 222058724 h^3 x_n + 160082430 h^2 x_n^2 + 46504700 h x_n^3 + 4619125 x_n^4\big) x^3\big) + \frac{1}{9150624} \frac{\big(55514681 h^3 + 80041215 h^2 x_n + 34878525 h x_n^2 + 4619125 x_n^3\big) x^4}{h^7} - \frac{1}{3050208} \frac{\big(5336081 h^2 + 4650470 h x_n + 923825 x_n^2\big) x^5}{h^7} + \frac{5}{9150624} \frac{\big(465047 h + 184765 x_n\big) x^6}{h^7} - \frac{131975}{9150624} \frac{x^7}{h^7} \Bigg) y_n$$

$$+ \Bigg(\frac{1}{2859570} \frac{1}{h^7} \big(x_n \big(54818496 h^6 + 251180400 h^5 x_n + 372756560 h^4 x_n^2 + 249551320 h^3 x_n^3 + 82610124 h^2 x_n^4 + 13099955 h x_n^5 + 786925 x_n^6\big)\big) - \frac{1}{2859570} \frac{1}{h^7} \big(\big(54818496 h^6 + 502360800 h^5 x_n + 1118269680 h^4 x_n^2 + 998205280 h^3 x_n^3 + 413050620 h^2 x_n^4 + 78599730 h x_n^5 + 5508475 x_n^6\big) x\big) + \frac{1}{190638} \frac{1}{h^7} \big(\big(16745360 h^5 + 74551312 h^4 x_n + 99820528 h^3 x_n^2 + 55073416 h^2 x_n^3 + 13099955 h x_n^4 + 1101695 x_n^5\big) x^2\big) - \frac{1}{571914} \frac{1}{h^7} \big(\big(74551312 h^4 + 199641056 h^3 x_n + 165220248 h^2 x_n^2 + 52399820 h x_n^3 + 5508475 x_n^4\big) x^3\big) + \frac{1}{571914} \frac{\big(49910264 h^3 + 82610124 h^2 x_n + 39299865 h x_n^2 + 5508475 x_n^3\big) x^4}{h^7} - \frac{1}{953190} \frac{\big(27536708 h^2 + 26199910 h x_n + 5508475 x_n^2\big) x^5}{h^7} + \frac{1}{571914} \frac{\big(2619991 h + 1101695 x_n\big) x^6}{h^7} - \frac{157385}{571914} \frac{x^7}{h^7} \Bigg)$$

$$
\begin{aligned}
& y_{n+1} + \Bigg(\frac{1}{6778240} \frac{1}{h^7} \Big(x_n \big(11349648\, h^6 + 57664500\, h^5 x_n + 100267600\, h^4 x_n^2 \\
& + 78553555\, h^3 x_n^3 + 29351397\, h^2 x_n^4 + 5073865\, h\, x_n^5 + 323275\, x_n^6 \big) \Big) \\
& - \frac{1}{6778240} \frac{1}{h^7} \Big(\big(11349648\, h^6 + 115329000\, h^5 x_n + 300802800\, h^4 x_n^2 + 314214220\, h^3 \\
& x_n^3 + 146756985\, h^2 x_n^4 + 30443190\, h\, x_n^5 + 2262925\, x_n^6 \big)\, x \Big) \\
& + \frac{3}{1355648} \frac{1}{h^7} \Big(\big(3844300\, h^5 + 20053520\, h^4 x_n + 31421422\, h^3 x_n^2 + 19567598\, h^2 x_n^3 \\
& + 5073865\, h\, x_n^4 + 452585\, x_n^5 \big)\, x^2 \Big) \\
& - \frac{1}{1355648} \frac{1}{h^7} \Big(\big(20053520\, h^4 + 62842844\, h^3 x_n + 58702794\, h^2 x_n^2 + 20295460\, h \\
& x_n^3 + 2262925\, x_n^4 \big)\, x^3 \Big) \\
& + \frac{1}{1355648} \frac{\big(15710711\, h^3 + 29351397\, h^2 x_n + 15221595\, h\, x_n^2 + 2262925\, x_n^3 \big)\, x^4}{h^7} \\
& - \frac{3}{6778240} \frac{\big(9783799\, h^2 + 10147730\, h\, x_n + 2262925\, x_n^2 \big)\, x^5}{h^7} \\
& + \frac{1}{1355648} \frac{\big(1014773\, h + 452585\, x_n \big)\, x^6}{h^7} - \frac{64655}{1355648} \frac{x^7}{h^7} \Bigg)\, y_{n+2} + \Bigg(\\
& - \frac{1}{111523230} \frac{1}{h^7} \Big(x_n \big(29579904\, h^6 + 155140800\, h^5 x_n + 284336120\, h^4 x_n^2 \\
& + 239807140\, h^3 x_n^3 + 98188086\, h^2 x_n^4 + 18399995\, h\, x_n^5 + 1243825\, x_n^6 \big) \Big) \\
& + \frac{1}{111523230} \frac{1}{h^7} \Big(\big(29579904\, h^6 + 310281600\, h^5 x_n + 853008360\, h^4 x_n^2 \\
& + 959228560\, h^3 x_n^3 + 490940430\, h^2 x_n^4 + 110399970\, h\, x_n^5 + 8706775\, x_n^6 \big)\, x \Big) \\
& - \frac{1}{7434882} \frac{1}{h^7} \Big(\big(10342720\, h^5 + 56867224\, h^4 x_n + 95922856\, h^3 x_n^2 + 65458724\, h^2 x_n^3 \\
& + 18399995\, h\, x_n^4 + 1741355\, x_n^5 \big)\, x^2 \Big) \\
& + \frac{1}{22304646} \frac{1}{h^7} \Big(\big(56867224\, h^4 + 191845712\, h^3 x_n + 196376172\, h^2 x_n^2 \\
& + 73599980\, h\, x_n^3 + 8706775\, x_n^4 \big)\, x^3 \Big) \\
& - \frac{1}{22304646} \frac{\big(47961428\, h^3 + 98188086\, h^2 x_n + 55199985\, h\, x_n^2 + 8706775\, x_n^3 \big)\, x^4}{h^7} \\
& + \frac{1}{37174410} \frac{\big(32729362\, h^2 + 36799990\, h\, x_n + 8706775\, x_n^2 \big)\, x^5}{h^7} \\
& - \frac{1}{22304646} \frac{\big(3679999\, h + 1741355\, x_n \big)\, x^6}{h^7} + \frac{248765}{22304646} \frac{x^7}{h^7} \Bigg)
\end{aligned}
$$

$$y_{n+3} + \left(\frac{1}{137259360} \frac{1}{h^7} \left(x_n \left(4364568\, h^6 + 23232900\, h^5 x_n + 43656710\, h^4 x_n^2 \right.\right.\right.$$

$$\left.\left. + 38222635\, h^3 x_n^3 + 16514097\, h^2 x_n^4 + 3316865\, h\, x_n^5 + 237025\, x_n^6 \right)\right)$$

$$- \frac{1}{137259360} \frac{1}{h^7} \left(\left(4364568\, h^6 + 46465800\, h^5 x_n + 130970130\, h^4 x_n^2 + 152890540\, h^3 \right.\right.$$

$$\left.\left. x_n^3 + 82570485\, h^2 x_n^4 + 19901190\, h\, x_n^5 + 1659175\, x_n^6 \right) x \right)$$

$$+ \frac{1}{9150624} \frac{1}{h^7} \left(\left(1548860\, h^5 + 8731342\, h^4 x_n + 15289054\, h^3 x_n^2 + 11009398\, h^2 x_n^3 \right.\right.$$

$$\left.\left. + 3316865\, h\, x_n^4 + 331835\, x_n^5 \right) x^2 \right)$$

$$- \frac{1}{27451872} \frac{1}{h^7} \left(\left(8731342\, h^4 + 30578108\, h^3 x_n + 33028194\, h^2 x_n^2 + 13267460\, h \right.\right.$$

$$\left.\left. x_n^3 + 1659175\, x_n^4 \right) x^3 \right)$$

$$+ \frac{1}{27451872} \frac{\left(7644527\, h^3 + 16514097\, h^2 x_n + 9950595\, h\, x_n^2 + 1659175\, x_n^3 \right) x^4}{h^7}$$

$$- \frac{1}{45753120} \frac{\left(5504699\, h^2 + 6633730\, h\, x_n + 1659175\, x_n^2 \right) x^5}{h^7}$$

$$\left. + \frac{1}{27451872} \frac{\left(663373\, h + 331835\, x_n \right) x^6}{h^7} - \frac{47405}{27451872} \frac{x^7}{h^7} \right) y_{n+4} + \Bigg($$

$$- \frac{3125}{1427497344} \frac{1}{h^7} \left(x_n \left(4900464\, h^6 + 15105900\, h^5 x_n + 18025160\, h^4 x_n^2 \right.\right.$$

$$\left.\left. + 10593985\, h^3 x_n^3 + 3232311\, h^2 x_n^4 + 486275\, h\, x_n^5 + 28225\, x_n^6 \right)\right)$$

$$+ \frac{3125}{1427497344} \frac{1}{h^7} \left(\left(4900464\, h^6 + 30211800\, h^5 x_n + 54075480\, h^4 x_n^2 + 42375940\, h^3 x_n^3 \right.\right.$$

$$\left.\left. + 16161555\, h^2 x_n^4 + 2917650\, h\, x_n^5 + 197575\, x_n^6 \right) x \right) - \frac{15625}{475832448} \frac{1}{h^7} \left(\left(1007060\, h^5 \right.\right.$$

$$\left.\left. + 3605032\, h^4 x_n + 4237594\, h^3 x_n^2 + 2154874\, h^2 x_n^3 + 486275\, h\, x_n^4 + 39515\, x_n^5 \right) x^2 \right)$$

$$+ \frac{15625}{1427497344} \frac{1}{h^7} \left(\left(3605032\, h^4 + 8475188\, h^3 x_n + 6464622\, h^2 x_n^2 + 1945100\, h\, x_n^3 \right.\right.$$

$$\left.\left. + 197575\, x_n^4 \right) x^3 \right)$$

$$- \frac{15625}{1427497344} \frac{\left(2118797\, h^3 + 3232311\, h^2 x_n + 1458825\, h\, x_n^2 + 197575\, x_n^3 \right) x^4}{h^7}$$

$$+ \frac{3125}{475832448} \frac{\left(1077437\, h^2 + 972550\, h\, x_n + 197575\, x_n^2 \right) x^5}{h^7}$$

$$\left. - \frac{78125}{1427497344} \frac{\left(19451\, h + 7903\, x_n \right) x^6}{h^7} + \frac{88203125}{1427497344} \frac{x^7}{h^7} \right) y_{n+\frac{2}{5}} + \Bigg($$

$$- \frac{3125}{3229632} \frac{1}{h^7} \left(x_n \left(15984\, h^6 + 75900\, h^5 x_n + 118660\, h^4 x_n^2 + 82535\, h^3 x_n^3 + 28041\, h^2 x_n^4 \right.\right.$$

$$\left.\left. + 4525\, h\, x_n^5 + 275\, x_n^6 \right)\right) + \frac{3125}{3229632} \frac{1}{h^7} \left(\left(15984\, h^6 + 151800\, h^5 x_n + 355980\, h^4 x_n^2 \right.\right.$$

$$y_{n+\frac{6}{5}} + \Bigg(-\frac{1}{317730}\frac{1}{h^5}\left(x_n\left(288h^6 + 1560h^5x_n + 3020h^4x_n^2 + 2770h^3x_n^3 + 1287h^2x_n^4 + 290hx_n^5 + 25x_n^6\right)\right) + \frac{1}{317730}\frac{1}{h^5}\left(\left(288h^6 + 3120h^5x_n + 9060h^4x_n^2 + 11080h^3x_n^3 + 6435h^2x_n^4 + 1740hx_n^5 + 175x_n^6\right)x\right) - \frac{1}{21182}\frac{\left(104h^5 + 604h^4x_n + 1108h^3x_n^2 + 858h^2x_n^3 + 290hx_n^4 + 35x_n^5\right)x^2}{h^5} + \frac{1}{63546}\frac{\left(604h^4 + 2216h^3x_n + 2574h^2x_n^2 + 1160hx_n^3 + 175x_n^4\right)x^3}{h^5} - \frac{1}{63546}\frac{\left(554h^3 + 1287h^2x_n + 870hx_n^2 + 175x_n^3\right)x^4}{h^5} + \frac{1}{105910}\frac{\left(429h^2 + 580hx_n + 175x_n^2\right)x^5}{h^5} - \frac{1}{63546}\frac{\left(35x_n + 58h\right)x^6}{h^5} + \frac{5}{63546}\frac{x^7}{h^5}\Bigg)f_{n+5} \qquad (3.27)$$

Evaluate (3.27) at $x = x_{n+5}$ gives the discrete scheme as

$$y_{n+5} = \frac{1175093}{1900638}y_n - \frac{861828125}{29739528}y_{n+\frac{2}{5}} + \frac{14402209}{95319}y_{n+1} + \frac{3094397}{84728}y_{n+2} + \frac{51879766}{3717441}y_{n+3} + \frac{2887259}{571914}y_{n+4} - \frac{10421875}{67284}y_{n+\frac{6}{5}} + \frac{1748}{10591}h^2f_{n+5} \qquad (3.28)$$

To obtain the sufficient schemes required, we obtain the first derivative of (3.27) and evaluate the continuous function at $x = x_n, x = x_{n+\frac{2}{5}}, x = x_{n+1}, x = x_{n+\frac{6}{5}}, x = x_{n+2}, x = x_{n+3}, x = x_{n+4}$ *and* $x = x_{n+5}$ to obtain;

$$hz_{n+5} = \frac{22312067}{2287656}y_n - \frac{16280496875}{356874336}y_{n+\frac{2}{5}} + \frac{673500379}{2859570}y_{n+1} + \frac{93913213}{1694560}y_{n+2} - \frac{2218967221}{111523230}y_{n+3} + \frac{181271233}{34314840}y_{n+4} - \frac{194153125}{807408}y_{n+\frac{6}{5}} + \frac{144}{7565}h^2f_{n+5}$$

$$hz_{n+\frac{6}{5}} = -\frac{42397}{945625}y_n + \frac{136285}{472056}y_{n+\frac{2}{5}} - \frac{2411136}{4728125}y_{n+1} + \frac{8545473}{37825000}y_{n+2} - \frac{13249088}{553190625}y_{n+3} + \frac{246209}{99290625}y_{n+4} + \frac{26435}{5607}y_{n+\frac{6}{5}} - \frac{288}{4728125}h^2f_{n+5}$$

$$hz_{n+\frac{2}{5}} = -\frac{3855982}{6619375}y_n - \frac{18653885}{9913176}y_{n+\frac{2}{5}} + \frac{246481664}{33096875}y_{n+1} + \frac{129246273}{264775000}y_{n+2} - \frac{92231296}{1290778125}y_{n+3} + \frac{2462486}{297871875}y_{n+4} - \frac{20215}{3738}y_{n+\frac{6}{5}} - \frac{7488}{33096875}h^2f_{n+5}$$

$$hz_{n+4} = \frac{10657}{6052}y_n - \frac{11928125}{1416168}y_{n+\frac{2}{5}} + \frac{343648}{7565}y_{n+1} + \frac{731511}{60520}y_{n+2} - \frac{1745792}{295035}y_{n+3} + \frac{4385621}{1906380}y_{n+4} - \frac{353125}{7476}y_{n+\frac{6}{5}} + \frac{144}{7565}h^2f_{n+5}$$

$$hz_{n+3} = -\frac{4316}{10591}y_n + \frac{6678125}{3304392}y_{n+\frac{2}{5}} - \frac{1246843}{105910}y_{n+1} - \frac{1756053}{423640}y_{n+2} + \frac{18095591}{12391470}y_{n+3} + \frac{26143}{158865}y_{n+4} + \frac{40625}{3204}y_{n+\frac{6}{5}} - \frac{117}{52955}h^2f_{n+5}$$

$$hz_{n+2} = \frac{42430}{28957}y_n - \frac{71065625}{89218584}y_{n+\frac{2}{5}} + \frac{8152832}{1429785}y_{n+1} + \frac{690877}{423640}y_{n+2} + \frac{13351552}{55761615}y_{n+3} - \frac{11902}{612765}y_{n+4} - \frac{696875}{100926}y_{n+\frac{6}{5}} + \frac{64}{158865}h^2f_{n+5}$$

$$hz_{n+1} = \frac{13247}{254184}y_n - \frac{4928125}{13217568}y_{n+\frac{2}{5}} - \frac{1386829}{317730}y_{n+1} - \frac{296487}{1694560}y_{n+2} + \frac{258371}{12391470}y_{n+3} - \frac{947}{423640}y_{n+4} + \frac{434375}{89712}y_{n+\frac{6}{5}} + \frac{3}{52955}h^2f_{n+5}$$

$$hz_n = -\frac{709531}{127092}y_n + \frac{106346875}{9913176}y_{n+\frac{2}{5}} - \frac{3045472}{158865}y_{n+1} - \frac{709353}{423640}y_{n+2} + \frac{182592}{688415}y_{n+3} - \frac{60619}{1906380}y_{n+4} + \frac{115625}{7476}y_{n+\frac{6}{5}} + \frac{48}{52955}h^2f_{n+5} \qquad (3.29)$$

where z is the first derivative of y.

Likewise, we further obtain the second derivatives of (3.27), thereafter, evaluating at $x = x_{n+\frac{2}{5}}, x = x_{n+1,}\ x = x_{n+\frac{6}{5}}, x = x_{n+2}\ , x = x_{n+3}$ and $x = x_{n+4}$ to obtain;

$$y_{n+\frac{2}{5}} = -\frac{1650327432}{2487990625}y_n - \frac{5163285504}{123439953125}y_{n+1} - \frac{1979742141}{123439953125}y_{n+2} + \frac{344635904}{123439953125}y_{n+3} - \frac{42573544}{123439953125}y_{n+4} + \frac{705874}{796157}y_{n+\frac{6}{5}} + \frac{28664064}{286118921875}h^2f_{n+5} - \frac{59479056}{457790275}h^2f_{n+\frac{2}{5}}$$

$$y_{n+\frac{6}{5}} = -\frac{5236008}{106409375}y_n + \frac{221984256}{532046875}y_{n+1} + \frac{405735723}{1064093750}y_{n+2} - \frac{247228928}{6916609375}y_{n+3} + \frac{1897816}{532046875}y_{n+4} + \frac{250423}{885326}y_{n+\frac{2}{5}} - \frac{857088}{101108890625}h^2f_{n+5} - \frac{134568}{951425}h^2f_{n+\frac{6}{5}}$$

$$y_{n+1} = -\frac{213247}{4242320}y_n + \frac{1794717}{16969280}y_{n+2} - \frac{149567}{10340655}y_{n+3} + \frac{20551}{12726960}y_{n+4} + \frac{2815625}{5090784}y_{n+\frac{6}{5}} + \frac{53509375}{132360384}y_{n+\frac{2}{5}} - \frac{9}{212116}h^2f_{n+5} - \frac{95319}{1060580}h^2f_{n+1}$$

$$y_{n+2} = -\frac{56632}{314037}y_n + \frac{404480}{104679}y_{n+1} + \frac{8344064}{12247443}y_{n+3} - \frac{4712}{104679}y_{n+4} - \frac{2656250}{942111}y_{n+\frac{6}{5}} + \frac{390625}{453609}y_{n+\frac{2}{5}} + \frac{256}{314037}h^2f_{n+5} - \frac{169456}{314037}h^2f_{n+2}$$

$$y_{n+3} = -\frac{26878449}{23292272}y_n - \frac{41370927}{1455767}y_{n+1} - \frac{444206997}{93169088}y_{n+2} + \frac{24262771}{23292272}y_{n+4} + \frac{1343265625}{46584544}y_{n+\frac{6}{5}} + \frac{509046875}{93169088}y_{n+\frac{2}{5}} - \frac{29133}{2911534}h^2f_{n+5} - \frac{3717441}{2911534}h^2f_{n+3}$$

$$y_{n+4} = -\frac{7947519}{3991321}y_n - \frac{197072832}{3991321}y_{n+1} - \frac{192559275}{15965284}y_{n+2} + \frac{222129536}{51887173}y_{n+3} + \frac{405078125}{7982642}y_{n+\frac{6}{5}} + \frac{1951328125}{207548692}y_{n+\frac{2}{5}} - \frac{118368}{3991321}h^2f_{n+5} + \frac{1143828}{3991321}h^2f_{n+4}$$

(3.30)

The equation (3.30) is the proposed 5SBHBDFfor solving second order ordinary differential equations.

3.2.3 6-Step block hybrid backward differentiation formulae (6SBHBDF)

To derive 6SBHBDF, we take $t = 8, c = 1$and $x \in [x_n, x_{n+6}]$. Therefore, (3.4) becomes:

$$Y(x) = \alpha\text{-}_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6 + \alpha_7 x^7 + \alpha_8 x^8 \qquad (3.31)$$

Interpolating (3.5) at $x_{n+i}; i = 0, 1, \frac{3}{2}, 2, \frac{5}{2}, 3,4,5,6$ and collocate (3.6) at $x_{n+i}; i = 6$. This results in a system of equations;

$$Y''(x_{n+6}) = 2\alpha_2 + 6\alpha_3 x_{n+6} + 12\alpha_4 x_{n+6}^2 + 20\alpha_5 x_{n+6}^3 + 30\alpha_6 x_{n+6}^4 + 42\alpha_7 x_{n+6}^5 + 56\alpha_8 x_{n+6}^8$$

$$Y(x_{n+i}) = \alpha_0 + \alpha_1 x_{n+i} + \alpha_2 x_{n+i}^2 + \alpha_3 x_{n+i}^3 + \alpha_4 x_{n+i}^4 + \alpha_5 x_{n+i}^5 + \alpha_6 x_{n+i}^6 + \alpha_7 x_{n+i}^7 + \alpha_8 x_{n+i}^8$$

$$Y(x_n) = \alpha_0 + \alpha_1 x_n + \alpha_2 x_n^2 + \alpha_3 x_n^3 + \alpha_4 x_n^4 + \alpha_5 x_n^5 + \alpha_6 x_n^6 + \alpha_7 x_n^7 + \alpha_8 x_n^8$$

$$Y\left(x_{n+\frac{3}{2}}\right) = \alpha_0 + \alpha_1 x_{n+\frac{3}{2}} + \alpha_2 x_{n+\frac{3}{2}}^2 + \alpha_3 x_{n+\frac{3}{2}}^3 + \alpha_4 x_{n+\frac{3}{2}}^4 + \alpha_5 x_{n+\frac{3}{2}}^5 + \alpha_6 x_{n+\frac{3}{2}}^6 + \alpha_7 x_{n+\frac{3}{2}}^7 + \alpha_8 x_{n+\frac{3}{2}}^8$$

$$Y(x_{n+1}) = \alpha_0 + \alpha_1 x_{n+1} + \alpha_2 x_{n+1}^2 + \alpha_3 x_{n+1}^3 + \alpha_4 x_{n+1}^4 + \alpha_5 x_{n+1}^5 + \alpha_6 x_{n+1}^6 + \alpha_7 x_{n+1}^7 + \alpha_8 x_{n+1}^8$$

$$Y(x_{n+2}) = \alpha_0 + \alpha_1 x_{n+2} + \alpha_2 x_{n+2}^2 + \alpha_3 x_{n+2}^3 + \alpha_4 x_{n+2}^4 + \alpha_5 x_{n+2}^5 + \alpha_6 x_{n+2}^6 + \alpha_7 x_{n+2}^7$$

$$Y\left(x_{n+\frac{5}{2}}\right) = \alpha_0 + \alpha_1 x_{n+\frac{5}{2}} + \alpha_2 x_{n+\frac{5}{2}}^2 + \alpha_3 x_{n+\frac{5}{2}}^3 + \alpha_4 x_{n+\frac{5}{2}}^4 + \alpha_5 x_{n+\frac{5}{2}}^5 + \alpha_6 x_{n+\frac{5}{2}}^6 + \alpha_7 x_{n+\frac{5}{2}}^7 + \alpha_8 x_{n+\frac{5}{2}}^8$$

$$Y(x_{n+3}) = \alpha_0 + \alpha_1 x_{n+3} + \alpha_2 x_{n+3}^2 + \alpha_3 x_{n+3}^3 + \alpha_4 x_{n+3}^4 + \alpha_5 x_{n+3}^5 + \alpha_6 x_{n+3}^6 + \alpha_7 x_{n+3}^7 + \alpha_8 x_{n+3}^8$$

$$Y(x_{n+4}) = \alpha_0 + \alpha_1 x_{n+4} + \alpha_2 x_{n+4}^2 + \alpha_3 x_{n+4}^3 + \alpha_4 x_{n+4}^4 + \alpha_5 x_{n+4}^5 + \alpha_6 x_{n+4}^6 + \alpha_7 x_{n+4}^7 + \alpha_8 x_{n+4}^8$$

$$Y(x_{n+5}) = \alpha_0 + \alpha_1 x_{n+5} + \alpha_2 x_{n+5}^2 + \alpha_3 x_{n+5}^3 + \alpha_4 x_{n+5}^4 + \alpha_5 x_{n+5}^5 + \alpha_6 x_{n+5}^6 + \alpha_7 x_{n+5}^7 + \alpha_8 x_{n+5}^8$$

(3.32)

$$D\psi = Y$$

where

$$\psi = \left(\alpha_0, \alpha_1, \alpha_{\frac{3}{2}}, \alpha_2, \alpha_{\frac{5}{2}}, \alpha_3, \alpha_4, \alpha_{5,} \beta_6\right)^T, \; Y = \left(y_n, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}, y_{n+4}, y_{n+5,} f_{n+6}\right)^T$$

and the matrix D of the proposed method is expressed as

$$D=\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\
1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 & x_{n+\frac{3}{2}}^8 \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\
1 & x_{n+\frac{5}{2}} & x_{n+\frac{5}{2}}^2 & x_{n+\frac{5}{2}}^3 & x_{n+\frac{5}{2}}^4 & x_{n+\frac{5}{2}}^5 & x_{n+\frac{5}{2}}^6 & x_{n+\frac{5}{2}}^7 & x_{n+\frac{5}{2}}^8 \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 \\
0 & 0 & 2 & 6x_{n+6} & 12x_{n+6} & 20x_{n+6} & 30x_{n+6} & 42x_{n+6} & 56x_{n+6}
\end{bmatrix}$$

$$\alpha_0 := \frac{1}{290952000}\,\frac{1}{h^8}\Big(290952000\,h^8 + 1016850600\,h^7 x_n + 1472992650\,h^6 x_n^2 + 1158134537\,h^5 x_n^3 + 541534133\,h^4 x_n^4 + 154381955\,h^3 x_n^5 + 26221325\,h^2 x_n^6 + 2426708\,h x_n^7 + 93692\,x_n^8\Big) - \frac{1}{290952000}\,\frac{1}{h^8}\Big(\big(1016850600\,h^7 + 2945985300\,h^6 x_n + 3474403611\,h^5 x_n^2 + 2166136532\,h^4 x_n^3 + 771909775\,h^3 x_n^4 + 157327950\,h^2 x_n^5 + 16986956\,h x_n^6 + 749536\,x_n^7\big)\,x\Big) + \frac{1}{290952000}\,\frac{1}{h^8}\Big(\big(1472992650\,h^6 + 3474403611\,h^5 x_n + 3249204798\,h^4 x_n^2 + 1543819550\,h^3 x_n^3 + 393319875\,h^2 x_n^4 + 50960868\,h x_n^5 + 2623376\,x_n^6\big)\,x^2\Big) - \frac{1}{290952000}\,\frac{1}{h^8}\Big(\big(1158134537\,h^5 + 2166136532\,h^4 x_n + 1543819550\,h^3 x_n^2 + 524426500\,h^2 x_n^3 + 84934780\,h x_n^4 + 5246752\,x_n^5\big)\,x^3\Big) + \frac{1}{290952000}\,\frac{1}{h^8}\Big(\big(541534133\,h^4 + 771909775\,h^3 x_n + 393319875\,h^2 x_n^2 + 84934780\,h x_n^3 + 6558440\,x_n^4\big)\,x^4\Big) - \frac{1}{290952000}\,\frac{\big(154381955\,h^3 + 157327950\,h^2 x_n + 50960868\,h x_n^2 + 5246752\,x_n^3\big)\,x^5}{h^8} + \frac{1}{290952000}\,\frac{\big(26221325\,h^2 + 16986956\,h x_n + 2623376\,x_n^2\big)\,x^6}{h^8} - \frac{1}{72738000}\,\frac{\big(606677\,h + 187384\,x_n\big)\,x^7}{h^8} + \frac{23423}{72738000}\,\frac{x^8}{h^8}$$

$$\alpha_1 := -\frac{1}{323280}\frac{1}{h^8}\Big(x_n\big(9448200\,h^7 + 23569470\,h^6 x_n + 24253811\,h^5 x_n^2 + 13341369\,h^4 x_n^3 + 4234385\,h^3 x_n^4 + 774885\,h^2 x_n^5 + 75644\,h\,x_n^6 + 3036\,x_n^7\big)\Big)$$
$$+ \frac{1}{323280}\frac{1}{h^8}\Big(\big(9448200\,h^7 + 47138940\,h^6 x_n + 72761433\,h^5 x_n^2 + 53365476\,h^4 x_n^3 + 21171925\,h^3 x_n^4 + 4649310\,h^2 x_n^5 + 529508\,h\,x_n^6 + 24288\,x_n^7\big)\,x\Big)$$
$$- \frac{1}{323280}\frac{1}{h^8}\Big(\big(23569470\,h^6 + 72761433\,h^5 x_n + 80048214\,h^4 x_n^2 + 42343850\,h^3 x_n^3 + 11623275\,h^2 x_n^4 + 1588524\,h\,x_n^5 + 85008\,x_n^6\big)\,x^2\Big) + \frac{1}{323280}\frac{1}{h^8}\Big(\big(24253811\,h^5 + 53365476\,h^4 x_n + 42343850\,h^3 x_n^2 + 15497700\,h^2 x_n^3 + 2647540\,h\,x_n^4 + 170016\,x_n^5\big)\,x^3\Big)$$
$$- \frac{1}{323280}\frac{1}{h^8}\Big(\big(13341369\,h^4 + 21171925\,h^3 x_n + 11623275\,h^2 x_n^2 + 2647540\,h\,x_n^3 + 212520\,x_n^4\big)\,x^4\Big)$$
$$+ \frac{1}{323280}\,\frac{\big(4234385\,h^3 + 4649310\,h^2 x_n + 1588524\,h\,x_n^2 + 170016\,x_n^3\big)\,x^5}{h^8}$$
$$- \frac{1}{323280}\,\frac{\big(774885\,h^2 + 529508\,h\,x_n + 85008\,x_n^2\big)\,x^6}{h^8}$$
$$+ \frac{1}{80820}\,\frac{\big(18911\,h + 6072\,x_n\big)\,x^7}{h^8} - \frac{253}{26940}\,\frac{x^8}{h^8}$$

$$\alpha_{\frac{3}{2}} := \frac{32}{6364575}\frac{1}{h^8}\Big(x_n\big(15474600\,h^7 + 43757910\,h^6 x_n + 49143983\,h^5 x_n^2 + 28797377\,h^4 x_n^3 + 9577295\,h^3 x_n^4 + 1815065\,h^2 x_n^5 + 181922\,h\,x_n^6 + 7448\,x_n^7\big)\Big)$$
$$- \frac{32}{6364575}\frac{1}{h^8}\Big(\big(15474600\,h^7 + 87515820\,h^6 x_n + 147431949\,h^5 x_n^2 + 115189508\,h^4 x_n^3 + 47886475\,h^3 x_n^4 + 10890390\,h^2 x_n^5 + 1273454\,h\,x_n^6 + 59584\,x_n^7\big)\,x\Big)$$
$$+ \frac{32}{909225}\frac{1}{h^8}\Big(\big(6251130\,h^6 + 21061707\,h^5 x_n + 24683466\,h^4 x_n^2 + 13681850\,h^3 x_n^3 + 3889425\,h^2 x_n^4 + 545766\,h\,x_n^5 + 29792\,x_n^6\big)\,x^2\Big) - \frac{32}{909225}\frac{1}{h^8}\Big(\big(7020569\,h^5 + 16455644\,h^4 x_n + 13681850\,h^3 x_n^2 + 5185900\,h^2 x_n^3 + 909610\,h\,x_n^4 + 59584\,x_n^5\big)\,x^3\Big)$$
$$+ \frac{32}{909225}\frac{1}{h^8}\Big(\big(4113911\,h^4 + 6840925\,h^3 x_n + 3889425\,h^2 x_n^2 + 909610\,h\,x_n^3 + 74480\,x_n^4\big)\,x^4\Big)$$
$$- \frac{32}{909225}\,\frac{\big(1368185\,h^3 + 1555770\,h^2 x_n + 545766\,h\,x_n^2 + 59584\,x_n^3\big)\,x^5}{h^8}$$
$$+ \frac{32}{909225}\,\frac{\big(259295\,h^2 + 181922\,h\,x_n + 29792\,x_n^2\big)\,x^6}{h^8}$$
$$- \frac{64}{6364575}\,\frac{\big(90961\,h + 29792\,x_n\big)\,x^7}{h^8} + \frac{34048}{909225}\,\frac{x^8}{h^8}$$

$$\alpha_2 := -\frac{1}{215520}\frac{1}{h^8}\Big(x_n\big(22710600\,h^7+67998510\,h^6x_n+80917763\,h^5x_n^2+49859847\,h^4x_n^3+17281585\,h^3x_n^4+3385295\,h^2x_n^5+348252\,h\,x_n^6+14548\,x_n^7\big)\Big)$$

$$+\frac{1}{215520}\frac{1}{h^8}\Big(\big(22710600\,h^7+135997020\,h^6x_n+242753289\,h^5x_n^2+199439388\,h^4x_n^3+86407925\,h^3x_n^4+20311770\,h^2x_n^5+2437764\,h\,x_n^6+116384\,x_n^7\big)x\Big)$$

$$-\frac{1}{215520}\frac{1}{h^8}\Big(\big(67998510\,h^6+242753289\,h^5x_n+299159082\,h^4x_n^2+172815850\,h^3x_n^3+50779425\,h^2x_n^4+7313292\,h\,x_n^5+407344\,x_n^6\big)x^2\Big)+\frac{1}{215520}\frac{1}{h^8}\Big(\big(80917763\,h^5+199439388\,h^4x_n+172815850\,h^3x_n^2+67705900\,h^2x_n^3+12188820\,h\,x_n^4+814688\,x_n^5\big)x^3\Big)$$

$$-\frac{1}{215520}\frac{1}{h^8}\Big(\big(49859847\,h^4+86407925\,h^3x_n+50779425\,h^2x_n^2+12188820\,h\,x_n^3+1018360\,x_n^4\big)x^4\Big)$$

$$+\frac{1}{215520}\frac{\big(17281585\,h^3+20311770\,h^2x_n+7313292\,h\,x_n^2+814688\,x_n^3\big)x^5}{h^8}$$

$$-\frac{1}{215520}\frac{\big(3385295\,h^2+2437764\,h\,x_n+407344\,x_n^2\big)x^6}{h^8}$$

$$+\frac{1}{53880}\frac{\big(87063\,h+29096\,x_n\big)x^7}{h^8}-\frac{3637}{53880}\frac{x^8}{h^8}$$

$$\alpha_{\frac{5}{2}} := \frac{32}{505125}\frac{1}{h^8}\Big(x_n\big(1261800\,h^7+3903750\,h^6x_n+4821863\,h^5x_n^2+3087417\,h^4x_n^3+1108895\,h^3x_n^4+224025\,h^2x_n^5+23642\,h\,x_n^6+1008\,x_n^7\big)\Big)$$

$$-\frac{32}{505125}\frac{1}{h^8}\Big(\big(1261800\,h^7+7807500\,h^6x_n+14465589\,h^5x_n^2+12349668\,h^4x_n^3+5544475\,h^3x_n^4+1344150\,h^2x_n^5+165494\,h\,x_n^6+8064\,x_n^7\big)x\Big)$$

$$+\frac{32}{505125}\frac{1}{h^8}\Big(\big(3903750\,h^6+14465589\,h^5x_n+18524502\,h^4x_n^2+11088950\,h^3x_n^3+3360375\,h^2x_n^4+496482\,h\,x_n^5+28224\,x_n^6\big)x^2\Big)-\frac{32}{505125}\frac{1}{h^8}\Big(\big(4821863\,h^5+12349668\,h^4x_n+11088950\,h^3x_n^2+4480500\,h^2x_n^3+827470\,h\,x_n^4+56448\,x_n^5\big)x^3\Big)$$

$$+\frac{32}{505125}\frac{1}{h^8}\Big(\big(3087417\,h^4+5544475\,h^3x_n+3360375\,h^2x_n^2+827470\,h\,x_n^3+70560\,x_n^4\big)x^4\Big)$$

$$-\frac{32}{505125}\frac{\big(1108895\,h^3+1344150\,h^2x_n+496482\,h\,x_n^2+56448\,x_n^3\big)x^5}{h^8}$$

$$+\frac{32}{505125}\frac{\big(224025\,h^2+165494\,h\,x_n+28224\,x_n^2\big)x^6}{h^8}$$

$$-\frac{64}{505125}\frac{\big(11821\,h+4032\,x_n\big)x^7}{h^8}+\frac{3584}{56125}\frac{x^8}{h^8}$$

$$
\begin{aligned}
\alpha_3 := & -\frac{1}{1454760}\frac{1}{h^8}\Big(x_n\big(42539400\,h^7+134424990\,h^6x_n+170331587\,h^5x_n^2\\
&+112246433\,h^4x_n^3+41540285\,h^3x_n^4+8633225\,h^2x_n^5+934028\,hx_n^6+40652\,x_n^7\big)\Big)\\
&+\frac{1}{1454760}\frac{1}{h^8}\Big(\big(42539400\,h^7+268849980\,h^6x_n+510994761\,h^5x_n^2\\
&+448985732\,h^4x_n^3+207701425\,h^3x_n^4+51799350\,h^2x_n^5+6538196\,hx_n^6+325216\,x_n^7\big)x\Big)\\
&-\frac{1}{1454760}\frac{1}{h^8}\Big(\big(134424990\,h^6+510994761\,h^5x_n+673478598\,h^4x_n^2\\
&+415402850\,h^3x_n^3+129498375\,h^2x_n^4+19614588\,hx_n^5+1138256\,x_n^6\big)x^2\Big)\\
&+\frac{1}{1454760}\frac{1}{h^8}\Big(\big(170331587\,h^5+448985732\,h^4x_n+415402850\,h^3x_n^2+172664500\,h^2\\
&x_n^3+32690980\,hx_n^4+2276512\,x_n^5\big)x^3\Big)\\
&-\frac{1}{1454760}\frac{1}{h^8}\Big(\big(112246433\,h^4+207701425\,h^3x_n+129498375\,h^2x_n^2\\
&+32690980\,hx_n^3+2845640\,x_n^4\big)x^4\Big)\\
&+\frac{1}{1454760}\frac{\big(41540285\,h^3+51799350\,h^2x_n+19614588\,hx_n^2+2276512\,x_n^3\big)x^5}{h^8}\\
&-\frac{1}{1454760}\frac{\big(8633225\,h^2+6538196\,hx_n+1138256\,x_n^2\big)x^6}{h^8}\\
&+\frac{1}{363690}\frac{\big(233507\,h+81304\,x_n\big)x^7}{h^8}-\frac{10163}{363690}\frac{x^8}{h^8}
\end{aligned}
$$

$$\alpha_4 := \frac{1}{6465600}\frac{1}{h^8}\Big(x_n\big(18671400\,h^7 + 60528690\,h^6 x_n + 79199597\,h^5 x_n^2 + 54254793\,h^4 x_n^3 + 20995055\,h^3 x_n^4 + 4577985\,h^2 x_n^5 + 518948\,h\,x_n^6 + 23532\,x_n^7\big)\Big)$$
$$- \frac{1}{6465600}\frac{1}{h^8}\Big(\big(18671400\,h^7 + 121057380\,h^6 x_n + 237598791\,h^5 x_n^2 + 217019172\,h^4 x_n^3 + 104975275\,h^3 x_n^4 + 27467910\,h^2 x_n^5 + 3632636\,h\,x_n^6 + 188256\,x_n^7\big)\,x\Big)$$
$$+ \frac{1}{6465600}\frac{1}{h^8}\Big(\big(60528690\,h^6 + 237598791\,h^5 x_n + 325528758\,h^4 x_n^2 + 209950550\,h^3 x_n^3 + 68669775\,h^2 x_n^4 + 10897908\,h\,x_n^5 + 658896\,x_n^6\big)\,x^2\Big)$$
$$- \frac{1}{6465600}\frac{1}{h^8}\Big(\big(79199597\,h^5 + 217019172\,h^4 x_n + 209950550\,h^3 x_n^2 + 91559700\,h^2 x_n^3 + 18163180\,h\,x_n^4 + 1317792\,x_n^5\big)\,x^3\Big)$$
$$+ \frac{1}{6465600}\frac{1}{h^8}\Big(\big(54254793\,h^4 + 104975275\,h^3 x_n + 68669775\,h^2 x_n^2 + 18163180\,h\,x_n^3 + 1647240\,x_n^4\big)\,x^4\Big)$$
$$- \frac{1}{6465600}\frac{\big(20995055\,h^3 + 27467910\,h^2 x_n + 10897908\,h\,x_n^2 + 1317792\,x_n^3\big)\,x^5}{h^8}$$
$$+ \frac{1}{6465600}\frac{\big(4577985\,h^2 + 3632636\,h\,x_n + 658896\,x_n^2\big)\,x^6}{h^8}$$
$$- \frac{1}{1616400}\frac{\big(129737\,h + 47064\,x_n\big)\,x^7}{h^8} + \frac{1961}{538800}\frac{x^8}{h^8}$$

$$\alpha_5 := -\frac{1}{18858000}\frac{1}{h^8}\Big(x_n\big(5257800\,h^7 + 17290350\,h^6 x_n + 23047787\,h^5 x_n^2 + 16165233\,h^4 x_n^3 + 6441505\,h^3 x_n^4 + 1455125\,h^2 x_n^5 + 171708\,h\,x_n^6 + 8092\,x_n^7\big)\Big)$$
$$+ \frac{1}{18858000}\frac{1}{h^8}\Big(\big(5257800\,h^7 + 34580700\,h^6 x_n + 69143361\,h^5 x_n^2 + 64660932\,h^4 x_n^3 + 32207525\,h^3 x_n^4 + 8730750\,h^2 x_n^5 + 1201956\,h\,x_n^6 + 64736\,x_n^7\big)\,x\Big)$$
$$- \frac{1}{2694000}\frac{1}{h^8}\Big(\big(2470050\,h^6 + 9877623\,h^5 x_n + 13855914\,h^4 x_n^2 + 9202150\,h^3 x_n^3 + 3118125\,h^2 x_n^4 + 515124\,h\,x_n^5 + 32368\,x_n^6\big)\,x^2\Big) + \frac{1}{2694000}\frac{1}{h^8}\Big(\big(3292541\,h^5 + 9237276\,h^4 x_n + 9202150\,h^3 x_n^2 + 4157500\,h^2 x_n^3 + 858540\,h\,x_n^4 + 64736\,x_n^5\big)\,x^3\Big)$$
$$- \frac{1}{2694000}\frac{1}{h^8}\Big(\big(2309319\,h^4 + 4601075\,h^3 x_n + 3118125\,h^2 x_n^2 + 858540\,h\,x_n^3 + 80920\,x_n^4\big)\,x^4\Big)$$
$$+ \frac{1}{2694000}\frac{\big(920215\,h^3 + 1247250\,h^2 x_n + 515124\,h\,x_n^2 + 64736\,x_n^3\big)\,x^5}{h^8}$$
$$- \frac{1}{2694000}\frac{\big(207875\,h^2 + 171708\,h\,x_n + 32368\,x_n^2\big)\,x^6}{h^8}$$
$$+ \frac{1}{4714500}\frac{\big(42927\,h + 16184\,x_n\big)\,x^7}{h^8} - \frac{289}{673500}\frac{x^8}{h^8}$$

$$\beta_6 := \frac{1}{323280}\frac{1}{h^6}\Big(x_n\big(1800\,h^7 + 6030\,h^6 x_n + 8239\,h^5 x_n^2 + 5971\,h^4 x_n^3 + 2485\,h^3 x_n^4 + 595\,h^2 x_n^5 + 76\,h x_n^6 + 4\,x_n^7\big)\Big) - \frac{1}{323280}\frac{1}{h^6}\Big(\big(1800\,h^7 + 12060\,h^6 x_n + 24717\,h^5 x_n^2 + 23884\,h^4 x_n^3 + 12425\,h^3 x_n^4 + 3570\,h^2 x_n^5 + 532\,h x_n^6 + 32\,x_n^7\big)x\Big) + \frac{1}{323280}\frac{1}{h^6}\Big(\big(6030\,h^6 + 24717\,h^5 x_n + 35826\,h^4 x_n^2 + 24850\,h^3 x_n^3 + 8925\,h^2 x_n^4 + 1596\,h x_n^5 + 112\,x_n^6\big)x^2\Big) - \frac{7}{323280}\frac{\big(1177\,h^5 + 3412\,h^4 x_n + 3550\,h^3 x_n^2 + 1700\,h^2 x_n^3 + 380\,h x_n^4 + 32\,x_n^5\big)x^3}{h^6} + \frac{7}{323280}\frac{\big(853\,h^4 + 1775\,h^3 x_n + 1275\,h^2 x_n^2 + 380\,h x_n^3 + 40\,x_n^4\big)x^4}{h^6} - \frac{7}{323280}\frac{\big(355\,h^3 + 510\,h^2 x_n + 228\,h x_n^2 + 32\,x_n^3\big)x^5}{h^6} + \frac{7}{323280}\frac{\big(85\,h^2 + 76\,h x_n + 16\,x_n^2\big)x^6}{h^6} - \frac{1}{80820}\frac{\big(19\,h + 8\,x_n\big)x^7}{h^6} + \frac{1}{80820}\frac{x^8}{h^6} \tag{3.33}$$

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+y_{n+1}} + \alpha_{\frac{3}{2}}(x)y_{n+\frac{3}{2}} + \alpha_2(x)y_{n+2} + \alpha_{\frac{5}{2}}(x)y_{n+\frac{5}{2}} + \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} + \alpha_5(x)y_{n+5} + h^2\beta_6(x)f_{n+6} \tag{3.34}$$

$$y(x) = \left(\frac{1}{290952000} \frac{1}{h^8} \left(290952000\, h^8 + 1016850600\, h^7 x_n + 1472992650\, h^6 x_n^2 \right.\right.$$

$$+ 1158134537\, h^5 x_n^3 + 541534133\, h^4 x_n^4 + 154381955\, h^3 x_n^5 + 26221325\, h^2 x_n^6$$

$$\left. + 2426708\, h\, x_n^7 + 93692\, x_n^8\right) - \frac{1}{290952000} \frac{1}{h^8} \left(\left(1016850600\, h^7 \right.\right.$$

$$+ 2945985300\, h^6 x_n + 3474403611\, h^5 x_n^2 + 2166136532\, h^4 x_n^3 + 771909775\, h^3 x_n^4$$

$$\left.\left. + 157327950\, h^2 x_n^5 + 16986956\, h\, x_n^6 + 749536\, x_n^7\right) x\right)$$

$$+ \frac{1}{290952000} \frac{1}{h^8} \left(\left(1472992650\, h^6 + 3474403611\, h^5 x_n + 3249204798\, h^4 x_n^2 \right.\right.$$

$$\left.\left. + 1543819550\, h^3 x_n^3 + 393319875\, h^2 x_n^4 + 50960868\, h\, x_n^5 + 2623376\, x_n^6\right) x^2\right)$$

$$- \frac{1}{290952000} \frac{1}{h^8} \left(\left(1158134537\, h^5 + 2166136532\, h^4 x_n + 1543819550\, h^3 x_n^2 \right.\right.$$

$$\left.\left. + 524426500\, h^2 x_n^3 + 84934780\, h\, x_n^4 + 5246752\, x_n^5\right) x^3\right)$$

$$+ \frac{1}{290952000} \frac{1}{h^8} \left(\left(541534133\, h^4 + 771909775\, h^3 x_n + 393319875\, h^2 x_n^2 \right.\right.$$

$$\left.\left. + 84934780\, h\, x_n^3 + 6558440\, x_n^4\right) x^4\right)$$

$$- \frac{1}{290952000} \frac{\left(154381955\, h^3 + 157327950\, h^2 x_n + 50960868\, h\, x_n^2 + 5246752\, x_n^3\right) x^5}{h^8}$$

$$+ \frac{1}{290952000} \frac{\left(26221325\, h^2 + 16986956\, h\, x_n + 2623376\, x_n^2\right) x^6}{h^8}$$

$$\left. - \frac{1}{72738000} \frac{\left(606677\, h + 187384\, x_n\right) x^7}{h^8} + \frac{23423}{72738000} \frac{x^8}{h^8} \right) y_n + \left(\right.$$

$$- \frac{1}{323280} \frac{1}{h^8} \left(x_n \left(9448200\, h^7 + 23569470\, h^6 x_n + 24253811\, h^5 x_n^2 + 13341369\, h^4 x_n^3 \right.\right.$$

$$\left.\left. + 4234385\, h^3 x_n^4 + 774885\, h^2 x_n^5 + 75644\, h\, x_n^6 + 3036\, x_n^7\right)\right)$$

$$+ \frac{1}{323280} \frac{1}{h^8} \left(\left(9448200\, h^7 + 47138940\, h^6 x_n + 72761433\, h^5 x_n^2 + 53365476\, h^4 x_n^3 \right.\right.$$

$$\left.\left. + 21171925\, h^3 x_n^4 + 4649310\, h^2 x_n^5 + 529508\, h\, x_n^6 + 24288\, x_n^7\right) x\right)$$

$$- \frac{1}{323280} \frac{1}{h^8} \left(\left(23569470\, h^6 + 72761433\, h^5 x_n + 80048214\, h^4 x_n^2 + 42343850\, h^3 x_n^3 \right.\right.$$

$$\left.\left. + 11623275\, h^2 x_n^4 + 1588524\, h\, x_n^5 + 85008\, x_n^6\right) x^2\right) + \frac{1}{323280} \frac{1}{h^8} \left(\left(24253811\, h^5 \right.\right.$$

$$\left.\left. + 53365476\, h^4 x_n + 42343850\, h^3 x_n^2 + 15497700\, h^2 x_n^3 + 2647540\, h\, x_n^4 + 170016\, x_n^5\right) x^3\right)$$

$$- \frac{1}{323280} \frac{1}{h^8} \left(\left(13341369\, h^4 + 21171925\, h^3 x_n + 11623275\, h^2 x_n^2 + 2647540\, h\, x_n^3 \right.\right.$$

$$\left.\left. + 212520\, x_n^4\right) x^4\right)$$

$$+\frac{1}{323280}\frac{\left(4234385\,h^3+4649310\,h^2\,x_n+1588524\,h\,x_n^2+170016\,x_n^3\right)x^5}{h^8}$$

$$-\frac{1}{323280}\frac{\left(774885\,h^2+529508\,h\,x_n+85008\,x_n^2\right)x^6}{h^8}$$

$$+\frac{1}{80820}\frac{\left(18911\,h+6072\,x_n\right)x^7}{h^8}-\frac{253}{26940}\frac{x^8}{h^8}\Bigg)y_{n+1}$$

$$+\Bigg(\frac{32}{6364575}\frac{1}{h^8}\Big(x_n\Big(15474600\,h^7+43757910\,h^6\,x_n+49143983\,h^5\,x_n^2+28797377\,h^4$$

$$x_n^3+9577295\,h^3\,x_n^4+1815065\,h^2\,x_n^5+181922\,h\,x_n^6+7448\,x_n^7\Big)\Big)$$

$$-\frac{32}{6364575}\frac{1}{h^8}\Big(\Big(15474600\,h^7+87515820\,h^6\,x_n+147431949\,h^5\,x_n^2+115189508\,h^4\,x_n^3$$

$$+47886475\,h^3\,x_n^4+10890390\,h^2\,x_n^5+1273454\,h\,x_n^6+59584\,x_n^7\Big)x\Big)$$

$$+\frac{32}{909225}\frac{1}{h^8}\Big(\Big(6251130\,h^6+21061707\,h^5\,x_n+24683466\,h^4\,x_n^2+13681850\,h^3\,x_n^3$$

$$+3889425\,h^2\,x_n^4+545766\,h\,x_n^5+29792\,x_n^6\Big)x^2\Big)-\frac{32}{909225}\frac{1}{h^8}\Big(\Big(7020569\,h^5$$

$$+16455644\,h^4\,x_n+13681850\,h^3\,x_n^2+5185900\,h^2\,x_n^3+909610\,h\,x_n^4+59584\,x_n^5\Big)x^3\Big)$$

$$+\frac{32}{909225}\frac{1}{h^8}\Big(\Big(4113911\,h^4+6840925\,h^3\,x_n+3889425\,h^2\,x_n^2+909610\,h\,x_n^3$$

$$+74480\,x_n^4\Big)x^4\Big)$$

$$-\frac{32}{909225}\frac{\left(1368185\,h^3+1555770\,h^2\,x_n+545766\,h\,x_n^2+59584\,x_n^3\right)x^5}{h^8}$$

$$+\frac{32}{909225}\frac{\left(259295\,h^2+181922\,h\,x_n+29792\,x_n^2\right)x^6}{h^8}$$

$$-\frac{64}{6364575}\frac{\left(90961\,h+29792\,x_n\right)x^7}{h^8}+\frac{34048}{909225}\frac{x^8}{h^8}\Bigg)$$

$$
\begin{aligned}
& y_{n+\frac{3}{2}} + \Bigg(-\frac{1}{215520}\,\frac{1}{h^8}\Big(x_n\,\big(22710600\,h^7 + 67998510\,h^6\,x_n + 80917763\,h^5\,x_n^2 \\
&\quad + 49859847\,h^4\,x_n^3 + 17281585\,h^3\,x_n^4 + 3385295\,h^2\,x_n^5 + 348252\,h\,x_n^6 + 14548\,x_n^7\big)\Big) \\
&\quad + \frac{1}{215520}\,\frac{1}{h^8}\Big(\big(22710600\,h^7 + 135997020\,h^6\,x_n + 242753289\,h^5\,x_n^2 \\
&\quad + 199439388\,h^4\,x_n^3 + 86407925\,h^3\,x_n^4 + 20311770\,h^2\,x_n^5 + 2437764\,h\,x_n^6 + 116384\,x_n^7\big)\,x\Big) \\
&\quad - \frac{1}{215520}\,\frac{1}{h^8}\Big(\big(67998510\,h^6 + 242753289\,h^5\,x_n + 299159082\,h^4\,x_n^2 \\
&\quad + 172815850\,h^3\,x_n^3 + 50779425\,h^2\,x_n^4 + 7313292\,h\,x_n^5 + 407344\,x_n^6\big)\,x^2\Big) \\
&\quad + \frac{1}{215520}\,\frac{1}{h^8}\Big(\big(80917763\,h^5 + 199439388\,h^4\,x_n + 172815850\,h^3\,x_n^2 + 67705900\,h^2\,x_n^3 \\
&\quad + 12188820\,h\,x_n^4 + 814688\,x_n^5\big)\,x^3\Big) \\
&\quad - \frac{1}{215520}\,\frac{1}{h^8}\Big(\big(49859847\,h^4 + 86407925\,h^3\,x_n + 50779425\,h^2\,x_n^2 + 12188820\,h\,x_n^3 \\
&\quad + 1018360\,x_n^4\big)\,x^4\Big) \\
&\quad + \frac{1}{215520}\,\frac{\big(17281585\,h^3 + 20311770\,h^2\,x_n + 7313292\,h\,x_n^2 + 814688\,x_n^3\big)\,x^5}{h^8} \\
&\quad - \frac{1}{215520}\,\frac{\big(3385295\,h^2 + 2437764\,h\,x_n + 407344\,x_n^2\big)\,x^6}{h^8} \\
&\quad + \frac{1}{53880}\,\frac{\big(87063\,h + 29096\,x_n\big)\,x^7}{h^8} - \frac{3637}{53880}\,\frac{x^8}{h^8}\Bigg)\,y_{n+2} \\
&\quad + \Bigg(\frac{32}{505125}\,\frac{1}{h^8}\Big(x_n\,\big(1261800\,h^7 + 3903750\,h^6\,x_n + 4821863\,h^5\,x_n^2 + 3087417\,h^4\,x_n^3 \\
&\quad + 1108895\,h^3\,x_n^4 + 224025\,h^2\,x_n^5 + 23642\,h\,x_n^6 + 1008\,x_n^7\big)\Big) \\
&\quad - \frac{32}{505125}\,\frac{1}{h^8}\Big(\big(1261800\,h^7 + 7807500\,h^6\,x_n + 14465589\,h^5\,x_n^2 + 12349668\,h^4\,x_n^3 \\
&\quad + 5544475\,h^3\,x_n^4 + 1344150\,h^2\,x_n^5 + 165494\,h\,x_n^6 + 8064\,x_n^7\big)\,x\Big) \\
&\quad + \frac{32}{505125}\,\frac{1}{h^8}\Big(\big(3903750\,h^6 + 14465589\,h^5\,x_n + 18524502\,h^4\,x_n^2 + 11088950\,h^3\,x_n^3 \\
&\quad + 3360375\,h^2\,x_n^4 + 496482\,h\,x_n^5 + 28224\,x_n^6\big)\,x^2\Big) - \frac{32}{505125}\,\frac{1}{h^8}\Big(\big(4821863\,h^5 \\
&\quad + 12349668\,h^4\,x_n + 11088950\,h^3\,x_n^2 + 4480500\,h^2\,x_n^3 + 827470\,h\,x_n^4 + 56448\,x_n^5\big)\,x^3\Big) \\
&\quad + \frac{32}{505125}\,\frac{1}{h^8}\Big(\big(3087417\,h^4 + 5544475\,h^3\,x_n + 3360375\,h^2\,x_n^2 + 827470\,h\,x_n^3 \\
&\quad + 70560\,x_n^4\big)\,x^4\Big)
\end{aligned}
$$

$$- \frac{32}{505125} \frac{\left(1108895\, h^3 + 1344150\, h^2 x_n + 496482\, h x_n^2 + 56448\, x_n^3\right) x^5}{h^8}$$

$$+ \frac{32}{505125} \frac{\left(224025\, h^2 + 165494\, h x_n + 28224\, x_n^2\right) x^6}{h^8}$$

$$- \frac{64}{505125} \frac{\left(11821\, h + 4032\, x_n\right) x^7}{h^8} + \frac{3584}{56125} \frac{x^8}{h^8} \Bigg) y_{n+\frac{5}{2}} + \Bigg($$

$$- \frac{1}{1454760} \frac{1}{h^8} \Big(x_n \Big(42539400\, h^7 + 134424990\, h^6 x_n + 170331587\, h^5 x_n^2 + 1122464$$

$$x_n^3 + 41540285\, h^3 x_n^4 + 8633225\, h^2 x_n^5 + 934028\, h x_n^6 + 40652\, x_n^7 \Big) \Big)$$

$$+ \frac{1}{1454760} \frac{1}{h^8} \Big(\Big(42539400\, h^7 + 268849980\, h^6 x_n + 510994761\, h^5 x_n^2 + 44898573$$

$$x_n^3 + 207701425\, h^3 x_n^4 + 51799350\, h^2 x_n^5 + 6538196\, h x_n^6 + 325216\, x_n^7 \Big) x \Big)$$

$$- \frac{1}{1454760} \frac{1}{h^8} \Big(\Big(134424990\, h^6 + 510994761\, h^5 x_n + 673478598\, h^4 x_n^2 + 4154028$$

$$x_n^3 + 129498375\, h^2 x_n^4 + 19614588\, h x_n^5 + 1138256\, x_n^6 \Big) x^2 \Big)$$

$$+ \frac{1}{1454760} \frac{1}{h^8} \Big(\Big(170331587\, h^5 + 448985732\, h^4 x_n + 415402850\, h^3 x_n^2 + 1726645$$

$$x_n^3 + 32690980\, h x_n^4 + 2276512\, x_n^5 \Big) x^3 \Big)$$

$$- \frac{1}{1454760} \frac{1}{h^8} \Big(\Big(112246433\, h^4 + 207701425\, h^3 x_n + 129498375\, h^2 x_n^2$$

$$+ 32690980\, h x_n^3 + 2845640\, x_n^4 \Big) x^4 \Big)$$

$$+ \frac{1}{1454760} \frac{\left(41540285\, h^3 + 51799350\, h^2 x_n + 19614588\, h x_n^2 + 2276512\, x_n^3\right) x^5}{h^8}$$

$$- \frac{1}{1454760} \frac{\left(8633225\, h^2 + 6538196\, h x_n + 1138256\, x_n^2\right) x^6}{h^8}$$

$$+ \frac{1}{363690} \frac{\left(233507\, h + 81304\, x_n\right) x^7}{h^8} - \frac{10163}{363690} \frac{x^8}{h^8} \Bigg)$$

$$y_{n+3} + \Bigg(\frac{1}{6465600}\,\frac{1}{h^8}\Big(x_n\,\big(18671400\,h^7 + 60528690\,h^6\,x_n + 79199597\,h^5\,x_n^2$$

$$+ 54254793\,h^4\,x_n^3 + 20995055\,h^3\,x_n^4 + 4577985\,h^2\,x_n^5 + 518948\,h\,x_n^6 + 23532\,x_n^7\big)\Big)$$

$$- \frac{1}{6465600}\,\frac{1}{h^8}\Big(\big(18671400\,h^7 + 121057380\,h^6\,x_n + 237598791\,h^5\,x_n^2$$

$$+ 217019172\,h^4\,x_n^3 + 104975275\,h^3\,x_n^4 + 27467910\,h^2\,x_n^5 + 3632636\,h\,x_n^6 + 188256\,x_n^7\big)\,x\Big)$$

$$+ \frac{1}{6465600}\,\frac{1}{h^8}\Big(\big(60528690\,h^6 + 237598791\,h^5\,x_n + 325528758\,h^4\,x_n^2$$

$$+ 209950550\,h^3\,x_n^3 + 68669775\,h^2\,x_n^4 + 10897908\,h\,x_n^5 + 658896\,x_n^6\big)\,x^2\Big)$$

$$- \frac{1}{6465600}\,\frac{1}{h^8}\Big(\big(79199597\,h^5 + 217019172\,h^4\,x_n + 209950550\,h^3\,x_n^2 + 91559700\,h^2\,x_n^3$$

$$+ 18163180\,h\,x_n^4 + 1317792\,x_n^5\big)\,x^3\Big)$$

$$+ \frac{1}{6465600}\,\frac{1}{h^8}\Big(\big(54254793\,h^4 + 104975275\,h^3\,x_n + 68669775\,h^2\,x_n^2 + 18163180\,h$$

$$x_n^3 + 1647240\,x_n^4\big)\,x^4\Big)$$

$$- \frac{1}{6465600}\,\frac{\big(20995055\,h^3 + 27467910\,h^2\,x_n + 10897908\,h\,x_n^2 + 1317792\,x_n^3\big)\,x^5}{h^8}$$

$$+ \frac{1}{6465600}\,\frac{\big(4577985\,h^2 + 3632636\,h\,x_n + 658896\,x_n^2\big)\,x^6}{h^8}$$

$$- \frac{1}{1616400}\,\frac{\big(129737\,h + 47064\,x_n\big)\,x^7}{h^8} + \frac{1961}{538800}\,\frac{x^8}{h^8}\Bigg)\,y_{n+4} + \Bigg($$

$$-\frac{1}{18858000}\,\frac{1}{h^8}\Big(x_n\,\big(5257800\,h^7 + 17290350\,h^6\,x_n + 23047787\,h^5\,x_n^2 + 16165233\,h^4\,x_n^3$$

$$+ 6441505\,h^3\,x_n^4 + 1455125\,h^2\,x_n^5 + 171708\,h\,x_n^6 + 8092\,x_n^7\big)\Big)$$

$$+ \frac{1}{18858000}\,\frac{1}{h^8}\Big(\big(5257800\,h^7 + 34580700\,h^6\,x_n + 69143361\,h^5\,x_n^2 + 64660932\,h^4\,x_n^3$$

$$+ 32207525\,h^3\,x_n^4 + 8730750\,h^2\,x_n^5 + 1201956\,h\,x_n^6 + 64736\,x_n^7\big)\,x\Big)$$

$$- \frac{1}{2694000}\,\frac{1}{h^8}\Big(\big(2470050\,h^6 + 9877623\,h^5\,x_n + 13855914\,h^4\,x_n^2 + 9202150\,h^3\,x_n^3$$

$$+ 3118125\,h^2\,x_n^4 + 515124\,h\,x_n^5 + 32368\,x_n^6\big)\,x^2\Big) + \frac{1}{2694000}\,\frac{1}{h^8}\Big(\big(3292541\,h^5$$

$$+ 9237276\,h^4\,x_n + 9202150\,h^3\,x_n^2 + 4157500\,h^2\,x_n^3 + 858540\,h\,x_n^4 + 64736\,x_n^5\big)\,x^3\Big)$$

$$- \frac{1}{2694000}\,\frac{1}{h^8}\Big(\big(2309319\,h^4 + 4601075\,h^3\,x_n + 3118125\,h^2\,x_n^2 + 858540\,h\,x_n^3$$

$$+ 80920\,x_n^4\big)\,x^4\Big)$$

$$+\frac{1}{2694000}\frac{\left(920215\,h^3+1247250\,h^2\,x_n+515124\,h\,x_n^2+64736\,x_n^3\right)x^5}{h^8}$$

$$-\frac{1}{2694000}\frac{\left(207875\,h^2+171708\,h\,x_n+32368\,x_n^2\right)x^6}{h^8}$$

$$\left.+\frac{1}{4714500}\frac{\left(42927\,h+16184\,x_n\right)x^7}{h^8}-\frac{289}{673500}\frac{x^8}{h^8}\right)y_{n+5}$$

$$+\left(\frac{1}{323280}\frac{1}{h^6}\left(x_n\left(1800\,h^7+6030\,h^6\,x_n+8239\,h^5\,x_n^2+5971\,h^4\,x_n^3+2485\,h^3\,x_n^4\right.\right.\right.$$

$$\left.\left.+595\,h^2\,x_n^5+76\,h\,x_n^6+4\,x_n^7\right)\right)-\frac{1}{323280}\frac{1}{h^6}\left(\left(1800\,h^7+12060\,h^6\,x_n+24717\,h^5\right.\right.$$

$$\left.\left.x_n^2+23884\,h^4\,x_n^3+12425\,h^3\,x_n^4+3570\,h^2\,x_n^5+532\,h\,x_n^6+32\,x_n^7\right)x\right)$$

$$+\frac{1}{323280}\frac{1}{h^6}\left(\left(6030\,h^6+24717\,h^5\,x_n+35826\,h^4\,x_n^2+24850\,h^3\,x_n^3+8925\,h^2\,x_n^4\right.\right.$$

$$\left.\left.+1596\,h\,x_n^5+112\,x_n^6\right)x^2\right)$$

$$-\frac{7}{323280}\frac{\left(1177\,h^5+3412\,h^4\,x_n+3550\,h^3\,x_n^2+1700\,h^2\,x_n^3+380\,h\,x_n^4+32\,x_n^5\right)x^3}{h^6}$$

$$+\frac{7}{323280}\frac{\left(853\,h^4+1775\,h^3\,x_n+1275\,h^2\,x_n^2+380\,h\,x_n^3+40\,x_n^4\right)x^4}{h^6}$$

$$-\frac{7}{323280}\frac{\left(355\,h^3+510\,h^2\,x_n+228\,h\,x_n^2+32\,x_n^3\right)x^5}{h^6}$$

$$\left.+\frac{7}{323280}\frac{\left(85\,h^2+76\,h\,x_n+16\,x_n^2\right)x^6}{h^6}-\frac{1}{80820}\frac{\left(19\,h+8\,x_n\right)x^7}{h^6}+\frac{1}{80820}\frac{x^8}{h^6}\right)$$

$$f_{n+6}$$

$$y_{n+5}+\left(\frac{1}{323280}\frac{1}{h^6}\left(x_n\left(1800h^7+6030h^6x_n+8239h^5x_n^2+5971h^4x_n^3+2485h^3x_n^4+595h^2x_n^5+76hx_n^6+4x_n^7\right)\right)-\frac{1}{323280}\frac{1}{h^6}\left(\left(1800h^7+12060h^6x_n+24717h^5x_n^2+23884h^4x_n^3+12425h^3x_n^4+3570h^2x_n^5+532hx_n^6+32x_n^7\right)x\right)+\frac{1}{323280}\frac{1}{h^6}\left(\left(6030h^6+24717h^5x_n+35826h^4x_n^2+24850h^3x_n^3+8925h^2x_n^4+1596hx_n^5+112x_n^6\right)x^2\right)-\frac{7}{323280}\frac{\left(1177h^5+3412h^4x_n+3550h^3x_n^2+1700h^2x_n^3+380hx_n^4+32x_n^5\right)x^3}{h^6}+\frac{7}{323280}\frac{\left(853h^4+1775h^3x_n+1275h^2x_n^2+380hx_n^3+40x_n^4\right)x^4}{h^6}-\frac{7}{323280}\frac{\left(355h^3+510h^2x_n+228hx_n^2+32x_n^3\right)x^5}{h^6}+\frac{7}{323280}\frac{\left(85h^2+76hx_n+16x_n^2\right)x^6}{h^6}-\frac{1}{80820}\frac{\left(19h+8x_n\right)x^7}{h^6}+\frac{1}{80820}\frac{x^8}{h^6}\right)f_{n+6}$$

(3.35)

Evaluate (3.35) at $x = x_{n+6}$gives the discrete scheme as

$$y_{n+6} = -\frac{24619}{44900}y_n + \frac{8757}{449}y_{n+1} + \frac{161217}{898}y_{n+2} + \frac{46298}{449}y_{n+3} - \frac{195111}{8980}y_{n+4} - \frac{196096}{2245}y_{n+\frac{3}{2}} - \frac{2216448}{11225}y_{n+\frac{5}{2}} + \frac{63}{449}h^2 f_{n+6} \quad (3.36)$$

To obtain the sufficient schemes required, we obtain the first derivative of (3.35) and evaluate the continuous function at $x = x_n, x = x_{n+\frac{3}{2}}, x = x_{n+1}, x = x_{n+\frac{5}{2}}, x = x_{n+}2x = x_{n+3,}x = x_{n+4}, x = x_{n+5} and\ x = x_{n+6}$ to obtain;

$$hz_n = -\frac{564917}{161640}y_n - \frac{2200832}{28287}y_{n+\frac{3}{2}} + \frac{26245}{898}y_{n+1} + \frac{189255}{1796}y_{n+2} + \frac{118165}{4041}y_{n+3} - \frac{10373}{3592}y_{n+4} + \frac{8763}{31430}y_{n+5} - \frac{179456}{2245}y_{n+\frac{5}{2}} - \frac{5}{161640}h^2 f_{n+6}$$

$$hz_{n+\frac{5}{2}} = \frac{41233}{20689920}y_n - \frac{33515}{344832}y_{n+1} - \frac{484245}{229888}y_{n+2} + \frac{454915}{517248}y_{n+3} - \frac{53569}{1379328}y_{n+4} + \frac{11457}{4023040}y_{n+5} + \frac{16481}{28287}y_{n+\frac{3}{2}} - \frac{5}{114944}h^2 f_{n+6}$$

$$hz_{n+6} = -\frac{7450459}{8082000}y_n - \frac{18002048}{56125}y_{n+\frac{5}{2}} + \frac{291757}{8980}y_{n+1} + \frac{5294343}{17960}y_{n+2} + \frac{6667789}{40410}y_{n+3} - \frac{5885119}{179600}y_{n+4} + \frac{10611327}{1571500}y_{n+5} - \frac{102251648}{707175}y_{n+\frac{3}{2}} + \frac{3727}{8980}h^2 f_{n+6}$$

$$hz_{n+5} = -\frac{12467}{96984}y_n - \frac{353024}{6735}y_{n+\frac{5}{2}} + \frac{25445}{5388}y_{n+1} + \frac{246715}{5388}y_{n+2} + \frac{351470}{12123}y_{n+3} - \frac{84539}{10776}y_{n+4} + \frac{474937}{188580}y_{n+5} - \frac{1835776}{84861}y_{n+\frac{3}{2}} + \frac{35}{2694}h^2 f_{n+6}$$

$$hz_{n+4} = \frac{16987}{808200}y_n + \frac{557824}{141435}y_{n+\frac{3}{2}} - \frac{2213}{2694}y_{n+1} - \frac{16029}{1796}y_{n+2} - \frac{30247}{4041}y_{n+3} + \frac{32731}{17960}y_{n+4} + \frac{19539}{157150}y_{n+5} + \frac{381184}{33675}y_{n+\frac{5}{2}} - \frac{1}{898}h^2 f_{n+6}$$

$$hz_{n+3} = -\frac{30457}{8082000}y_n - \frac{623744}{707175}y_{n+\frac{3}{2}} + \frac{42969}{17960}y_{n+2} + \frac{743}{4490}y_{n+1} + \frac{111067}{40410}y_{n+3} + \frac{23843}{179600}y_{n+4} + \frac{3201}{392875}y_{n+5} - \frac{255104}{56125}y_{n+\frac{5}{2}} + \frac{1}{8980}h^2 f_{n+6}$$

$$hz_{n+2} = -\frac{57397}{24246000}y_n - \frac{2347904}{2121525}y_{n+\frac{3}{2}} - \frac{28871}{53880}y_{n+2} + \frac{3731}{26940}y_{n+1} - \frac{50573}{121230}y_{n+3} + \frac{14863}{538800}y_{n+4} - \frac{10559}{4714500}y_{n+5} + \frac{319616}{168375}y_{n+\frac{5}{2}} + \frac{1}{26940}h^2 f_{n+6}$$

$$hz_{n+1} = -\frac{138217}{4041000}y_n + \frac{5649152}{707175}y_{n+\frac{3}{2}} - \frac{64791}{8980}y_{n+2} - \frac{35193}{8980}y_{n+1} - \frac{30368}{20205}y_{n+3} + \frac{35609}{269400}y_{n+4} - \frac{18873}{1571500}y_{n+5} + \frac{768256}{168375}y_{n+\frac{5}{2}} + \frac{1}{4490}h^2 f_{n+6}$$

$$hz_{n+\frac{3}{2}} = \frac{590359}{103449600}y_n - \frac{264487}{141435}y_{n+\frac{3}{2}} + \frac{830697}{229888}y_{n+2} - \frac{57547}{114944}y_{n+1} + \frac{259721}{517248}y_{n+3} - \frac{91469}{2298880}y_{n+4} + \frac{69423}{20115200}y_{n+5} - \frac{19243}{11225}y_{n+\frac{5}{2}} - \frac{7}{114944}h^2 f_{n+6} \qquad (3.37)$$

Likewise, we further obtain the second derivatives of (3.35), thereafter, evaluating at $x = x_n, x = x_{n+\frac{3}{2}}, x = x_{n+1}, x = x_{n+\frac{5}{2}}, x = x_{n+2}, x = x_{n+3,} x = x_{n+4}\ and\ x = x_{n+5,}$ to obtain;

$$y_{n+4} = -\frac{528391}{1502695} y_n + \frac{3913540}{300539} y_{n+1} + \frac{37786230}{300539} y_{n+2} + \frac{19696840}{300539} y_{n+3} - \frac{5706396}{1502695} y_{n+5} - \frac{17954816}{300539} y_{n+\frac{3}{2}} - \frac{209471488}{1502695} y_{n+\frac{5}{2}} + \frac{1616400}{300539} h^2 f_{n+4} + \frac{7740}{300539} h^2 f_{n+6}$$

$$y_{n+5} = \frac{8388983}{64512072} y_n - \frac{4192775}{896001} y_{n+1} - \frac{52297525}{1194668} y_{n+2} - \frac{206983100}{8064009} y_{n+3} + \frac{37584095}{7168008} y_{n+4} + \frac{170256640}{8064009} y_{n+\frac{3}{2}} + \frac{43578112}{896001} y_{n+\frac{5}{2}} + \frac{67350}{298667} h^2 f_{n+5} + \frac{31175}{1792002} h^2 f_{n+6}$$

$$y_{n+3} = \frac{73411}{1759700} y_n - \frac{60039}{35194} y_{n+1} - \frac{675243}{35194} y_{n+2} - \frac{805401}{351940} y_{n+4} + \frac{107433}{879850} y_{n+5} + \frac{740864}{87985} y_{n+\frac{3}{2}} + \frac{6861312}{439925} y_{n+\frac{5}{2}} + \frac{40410}{17597} h^2 f_{n+3} - \frac{27}{1759700} h^2 f_{n+6}$$

$$y_{n+1} = -\frac{543963}{13678400} y_n - \frac{1361421}{273568} y_{n+2} - \frac{41511}{34196} y_{n+3} + \frac{306113}{2735680} y_{n+4} - \frac{8883}{854900} y_{n+5} + \frac{154848}{42745} y_{n+\frac{3}{2}} + \frac{749344}{213725} y_{n+\frac{5}{2}} + \frac{4041}{34196} h^2 f_{n+1} - \frac{27}{136784} h^2 f_{n+6}$$

$$y_{n+2} = \frac{178627}{376964550} y_n - \frac{334}{8133} y_{n+1} - \frac{282628}{7539291} y_{n+3} - \frac{1393}{8376990} y_{n+4} + \frac{578}{6980825} y_{n+5} + \frac{20417536}{37696455} y_{n+\frac{3}{2}} + \frac{11236352}{20942475} y_{n+\frac{5}{2}} - \frac{26940}{279233} h^2 f_{n+2} - \frac{2}{837699} h^2 f_{n+6}$$

$$y_{n+\frac{3}{2}} = -\frac{2496761}{574791680} y_n + \frac{16704585}{28739584} y_{n+1} + \frac{8124165}{57479168} y_{n+2} - \frac{2600795}{14369792} y_{n+3} + \frac{2011131}{114958336} y_{n+4} - \frac{234279}{143697920} y_{n+5} + \frac{31356}{70165} y_{n+\frac{5}{2}} - \frac{33675}{224528} h^2 f_{n+\frac{3}{2}} + \frac{885}{28739584} h^2 f_{n+6} \quad (3.38)$$

3.3 Analysis of Basic Properties

In this section, we address the order, error constants, consistency, stability and convergence of the developed methods.

3.3.1 Order and error constants of the developed methods

Following the works of (Areo and Adeniyi, 2013) and Ra'ft *et al.* (2020), the Local Truncation Error (LTE) for a block method of the form (3.3) is defined with the linear operator;

$$\mathcal{L}[y(x),h] = \sum_{j=0}^{k}\big[\alpha_j y(x+jh) - \alpha_v(x)y_{n+v}$$

$$+ h^2\beta_k(x)f_{n+k}\big] \qquad (3.39)$$

We assume that $y(x)$ is sufficiently differentiable such that the linear operator defined above can be expanded as a Taylor's series about the point x. Then,

$$\mathcal{L}[y(x),h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_q h^q y^q(x) + \cdots \qquad (3.40)$$

The method above will be consistent if $\mathcal{L}[y(x),h] \to 0$as $h \to 0$. Therefore, we can compare the coefficient to have

$$\left.\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k = \sum_{j=0}^{k}\alpha_j \\ C_1 &= (\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) = \sum_{j=0}^{k}\big(j\alpha_j - \beta_j\big) \\ &\vdots \\ C_q &= \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k) \end{aligned}\right\} \qquad (3.41)$$

The method is consistent if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0$,for $C_{p+2} \neq 0$. The constant C_{p+2} is the error constant. After defining the concept of error constant, we shall obtain the error constants of the proposed discrete hybrid block methods for $k = 4$, $k = 5$, and $k = 6$.

From 4SBHBDF, we developed the proposed method in (3.19) as

$$y_{n+4} = -\frac{18515}{7563}y_n + \frac{38144}{2521}y_{n+\frac{1}{2}} - \frac{95480}{2521}y_{n+1} - \frac{66710}{2521}y_{n+2} + \frac{41608}{7563}y_{n+3}$$

$$+ \frac{356608}{7563}y_{n+\frac{3}{2}} + \frac{420}{2521}h^2 f_{n+2}$$

where;

$$\alpha_0 = \frac{18515}{7563}\alpha_{\frac{1}{2}} = -\frac{38144}{2521}, \alpha_1 = \frac{95480}{2521}, \alpha_{\frac{3}{2}} = -\frac{356608}{7563}, \alpha_2 = \frac{66710}{2521}, \alpha_3 = -\frac{41608}{7563}, \alpha_4 = 1, \beta_4 = \frac{420}{2521}$$

Applying (3.41), we have

$$\left.\begin{array}{l}
C_0 = \sum_{j=0}^{k} \alpha_j = \frac{18515}{7563} - \frac{38144}{2521} + \frac{95480}{2521} - \frac{356608}{7563} + \frac{66710}{2521} - \frac{41608}{7563} + 1 = 0 \\
C_1 = \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^1\left(-\frac{38144}{2521}\right) + 1^1\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{356608}{7563}\right) + (2)^1\left(\frac{66710}{2521}\right) + (3)^1\left(-\frac{41608}{7563}\right) + (4)^1(1) \end{pmatrix} = 0 \\
C_2 = \frac{1}{2!}\begin{pmatrix} 0^2\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^2\left(-\frac{38144}{2521}\right) + 1^2\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{356608}{7563}\right) + (2)^2\left(\frac{66710}{2521}\right) + (3)^2\left(-\frac{41608}{7563}\right) + (4)^2(1) \end{pmatrix} - \left(\frac{420}{2521}\right) = 0 \\
C_3 = \frac{1}{3!}\begin{pmatrix} 0^3\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^3\left(-\frac{38144}{2521}\right) + 1^3\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{356608}{7563}\right) + (2)^3\left(\frac{66710}{2521}\right) + (3)^3\left(-\frac{41608}{7563}\right) + (4)^3(1) \end{pmatrix} - 4^2\left(\frac{420}{2521}\right) = 0 \\
C_4 = \frac{1}{4!}\begin{pmatrix} 0^4\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^4\left(-\frac{38144}{2521}\right) + 1^4\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{356608}{7563}\right) + (2)^4\left(\frac{66710}{2521}\right) + (3)^4\left(-\frac{41608}{7563}\right) + (4)^4(1) \end{pmatrix} - \frac{1}{2!}4^3\left(\frac{420}{2521}\right) = 0 \\
C_5 = \frac{1}{5!}\begin{pmatrix} 0^3\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^5\left(-\frac{38144}{2521}\right) + 1^5\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{356608}{7563}\right) + (2)^5\left(\frac{66710}{2521}\right) + (3)^5\left(-\frac{41608}{7563}\right) + 4^5(1) \end{pmatrix} - \frac{1}{2!}4^4\left(\frac{420}{2521}\right) = 0 \\
C_6 = \frac{1}{6!}\begin{pmatrix} 0^6\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^6\left(-\frac{38144}{2521}\right) + 1^6\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^6\left(-\frac{356608}{7563}\right) + (2)^6\left(\frac{66710}{2521}\right) + (3)^6 - \frac{41608}{7563} + 4^6(1) \end{pmatrix} - \frac{1}{2!}4^5\left(\frac{420}{2521}\right) = 0 \\
C_7 = \frac{1}{7!}\begin{pmatrix} 0^5\left(\frac{18515}{7563}\right) + \left(\frac{1}{2}\right)^7\left(-\frac{38144}{2521}\right) + 1^7\left(\frac{95480}{2521}\right) \\ + \left(\frac{3}{2}\right)^7\left(-\frac{356608}{7563}\right) + (2)^7\left(\frac{66710}{2521}\right) + 3^7\frac{41608}{7563} + (4)^7 \end{pmatrix} - \frac{1}{5!}4^6\left(\frac{420}{2521}\right) = -\frac{1163}{30252}
\end{array}\right\} \quad (3.45)$$

Since $C_{p+2} = C_7$. The $p = 5$ implies the method is of order 5 with error constant $C_7 = -\frac{1163}{30252}$

For the first discrete method in (3.20)

$hz_{n+4} = -\frac{175261}{45378}y_n + \frac{892864}{37815}y_{n+\frac{1}{2}} - \frac{146742}{2521}y_{n+1} - \frac{578929}{15126}y_{n+2} + \frac{669398}{113445}y_{n+3} + \frac{1607104}{22689}y_{n+\frac{3}{2}} + \frac{1163}{2521}h^2 f_{n+4}$

where;

$\alpha_0 = \frac{175261}{45378}\alpha_{\frac{1}{2}} = -\frac{892864}{37815}, \alpha_1 = \frac{146742}{2521}, \alpha_{\frac{3}{2}} = -\frac{1607104}{22689}, \alpha_2 = \frac{578929}{15126}, \alpha_3 = -\frac{669398}{113445}, \gamma_4 = 1, \beta_4 = \frac{1163}{2521}$

Applying (3.41), we have

$$\left.\begin{array}{l}
C_0 = \sum_{j=0}^{k}\alpha_j = \frac{175261}{45378} - \frac{892864}{37815} + \frac{95480}{2521} - \frac{1607104}{22689} + \frac{578929}{15126} - \frac{669398}{113445} + 1 = 0 \\
C_1 = \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^1\left(-\frac{892864}{37815}\right) + 1^1\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^1\left(-\frac{356608}{7563}\right) + (2)^1\left(\frac{578929}{15126}\right) + (3)^1\left(-\frac{669398}{113445}\right)\end{pmatrix} = 0 \\
C_2 = \frac{1}{2!}\begin{pmatrix} 0^2\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^2\left(-\frac{892864}{37815}\right) + 1^2\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^2\left(-\frac{1607104}{22689}\right) + (2)^2\left(\frac{578929}{15126}\right) + (3)^2\left(-\frac{669398}{113445}\right)\end{pmatrix} - (4)^1(1) - \left(\frac{1163}{2521}\right) = 0 \\
C_3 = \frac{1}{3!}\begin{pmatrix} 0^3\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^3\left(-\frac{892864}{37815}\right) + 1^3\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^3\left(-\frac{1607104}{22689}\right) + (2)^3\left(\frac{578929}{15126}\right) + (3)^3\left(-\frac{669398}{113445}\right)\end{pmatrix} - \frac{1}{2!}(4)^2(1) - 4^2\left(\frac{1163}{2521}\right) = 0 \\
C_4 = \frac{1}{4!}\begin{pmatrix} 0^4\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^4\left(-\frac{892864}{37815}\right) + 1^4\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^4\left(-\frac{1607104}{22689}\right) + (2)^4\left(\frac{578929}{15126}\right) + (3)^4\left(-\frac{669398}{113445}\right)\end{pmatrix} - \frac{1}{3!}(4)^3(1) - \frac{1}{2!}4^3\left(\frac{1163}{2521}\right) = 0 \\
C_5 = \frac{1}{5!}\begin{pmatrix} 0^3\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^5\left(-\frac{892864}{37815}\right) + 1^5\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^5\left(-\frac{1607104}{22689}\right) + (2)^5\left(\frac{578929}{15126}\right) + (3)^5\left(-\frac{669398}{113445}\right)\end{pmatrix} - \frac{1}{4!}(4)^4(1) - \frac{1}{3!}4^4\left(\frac{1163}{2521}\right) = 0 \\
C_6 = \frac{1}{6!}\begin{pmatrix} 0^6\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^6\left(-\frac{892864}{37815}\right) + 1^6\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^6\left(-\frac{1607104}{22689}\right) + (2)^6\left(\frac{578929}{15126}\right) + (3)^6\left(-\frac{669398}{113445}\right)\end{pmatrix} - \frac{1}{5!}(4)^5(1) - \frac{1}{4!}4^5\left(\frac{1163}{2521}\right) = 0 \\
C_7 = \frac{1}{7!}\begin{pmatrix} 0^5\left(\frac{175261}{45378}\right) + \left(\frac{1}{2}\right)^7\left(-\frac{892864}{37815}\right) + 1^7\left(\frac{146742}{2521}\right) \\ +\left(\frac{3}{2}\right)^7\left(-\frac{1607104}{22689}\right) + (2)^7\left(\frac{578929}{15126}\right) + 3^7\frac{669398}{113445}\end{pmatrix} - \frac{1}{6!}4^6 1 - \frac{1}{5!}4^6\frac{1163}{2521} = -\frac{823159}{1270540}
\end{array}\right\} \quad (3.46)$$

Since $C_{p+2} = C_7$. The $p = 5$ implies the method is of order 5 with error constant $C_7 = -\frac{823159}{12705840}$

For the first discrete method in (3.21)

$$y_{n+3} = \frac{64139}{92191} y_n - \frac{400032}{92191} y_{n+\frac{1}{2}} - \frac{1010475}{92191} y_{n+1} + \frac{679401}{92191} y_{n+2} - \frac{1261792}{92191} y_{n+\frac{3}{2}} - \frac{81}{3179} h^2 f_{n+4} + \frac{22689}{92191} h^2 f_{n+3}$$

where

$\alpha_0 = -\frac{64139}{92191} \alpha_{\frac{1}{2}} = \frac{400032}{92191}, \alpha_1 = \frac{1010475}{92191}, \alpha_{\frac{3}{2}} = \frac{1261792}{92191}, \alpha_2 = -\frac{679401}{92191}, \alpha_3 = 1, \alpha_4 = 0, \beta_4 = -\frac{81}{3179}, \beta_3 = \frac{22689}{92191}$

Applying (3.41), we have

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = -\frac{64139}{92191} + \frac{400032}{92191} + \frac{1010475}{92191} + \frac{1261792}{92191} - \frac{679401}{92191} + 1 = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^1\left(\frac{400032}{92191}\right) + 1^1\left(\frac{1010475}{92191}\right) \\ + \left(\frac{3}{2}\right)^1\left(\frac{1261792}{92191}\right) + (2)^1\left(-\frac{679401}{92191}\right) + (3)^1(1) \end{pmatrix} = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} 0^2\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^2\left(\frac{400032}{92191}\right) + 1^2\left(\frac{1010475}{92191}\right) \\ + \left(\frac{3}{2}\right)^2\left(\frac{1261792}{92191}\right) + (2)^2\left(-\frac{679401}{92191}\right) + (3)^2(1) \end{pmatrix} - \left(-\frac{81}{3179}\right) - \left(\frac{22689}{92191}\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} 0^3\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^3\left(\frac{400032}{92191}\right) + 1^3\left(\frac{1010475}{92191}\right) \\ + \left(\frac{3}{2}\right)^3\left(\frac{1261792}{92191}\right) + (2)^3\left(-\frac{679401}{92191}\right) + 3^3 \end{pmatrix} - 4^2\left(-\frac{81}{3179}\right) - 3^2\left(\frac{22689}{92191}\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} 0^4\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^4\left(\frac{400032}{92191}\right) + 1^4\left(\frac{1010475}{92191}\right) \\ + \left(\frac{3}{2}\right)^4\left(\frac{1261792}{92191}\right) + (2)^4\left(-\frac{679401}{92191}\right) + 3^4 \end{pmatrix} - \frac{1}{2!}4^3\left(-\frac{81}{3179}\right) - \frac{1}{2!}3^3\frac{22689}{92191} = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} 0^3\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^5\left(\frac{400032}{92191}\right) + 1^5\left(\frac{1010475}{92191}\right) \\ + \left(\frac{3}{2}\right)^5\left(\frac{1261792}{92191}\right) + (2)^5\left(-\frac{679401}{92191}\right) + (3)^5(1) \end{pmatrix} - \frac{1}{2!}4^4\left(-\frac{81}{3179}\right) - \frac{1}{2!}3^4\frac{22689}{92191} = 0 \\
C_6 &= \frac{1}{6!}\begin{pmatrix} 0^6\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^6\left(\frac{400032}{92191}\right) + 1^6\left(\frac{1010475}{92191}\right) \\ + \left(\frac{3}{2}\right)^6\left(\frac{1261792}{92191}\right) - 2^6\frac{679401}{92191} + 3^6 \end{pmatrix} - \frac{1}{2!}4^5\left(-\frac{81}{3179}\right) - \frac{1}{2!}3^5\frac{22689}{92191} = 0 \\
C_7 &= \frac{1}{7!}\begin{pmatrix} 0^5\left(-\frac{64139}{92191}\right) + \left(\frac{1}{2}\right)^7\left(\frac{400032}{92191}\right) + 1^7\frac{1010475}{92191} \\ + \left(\frac{3}{2}\right)^7\frac{1261792}{92191} - 2^7\frac{679401}{92191} + 3^7 \end{pmatrix} + \frac{1}{5!}4^6\frac{81}{3179} - \frac{1}{5!}3^6\frac{22689}{92191} = \frac{148311}{14750560}
\end{aligned}\right\} \quad (3.47)$$

Since $C_{p+2} = C_7$. The $p = 5$implies the method is of order 5 with error constant $C_7 = \frac{148311}{14750560}$.

For the second discrete method in (3.21)

$$y_{n+1} = -\frac{785}{24597}y_n + \frac{4384}{8199}y_{n+\frac{1}{2}} - \frac{193}{8199}y_{n+2} - \frac{23}{24597}y_{n+3} + \frac{12832}{24597}y_{n+\frac{3}{2}} + \frac{1}{24597}h^2 f_{n+4} - \frac{2521}{24597}h^2 f_{n+1}$$

where

$$\alpha_0 = \frac{785}{24597}\alpha_{\frac{1}{2}} = -\frac{4384}{8199}, \alpha_1 = 1, \alpha_{\frac{3}{2}} = -\frac{12832}{24597}, \alpha_2 = \frac{193}{8199}, \alpha_3 = \frac{23}{24597}, \alpha_4 = 0, \beta_4 = \frac{1}{24597}, \beta_1 = -\frac{2521}{24597}$$

Applying (3.41), we have

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k}\alpha_j = \frac{785}{24597} - \frac{4384}{8199} + 1 - \frac{12832}{24597} + \frac{193}{8199} + \frac{23}{24597} = 0\\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{785}{24597}\right) + \left(\frac{1}{2}\right)^1\left(-\frac{4384}{8199}\right) + 1^1(1) \\ -\left(\frac{3}{2}\right)^1\frac{12832}{24597} + (2)^1\left(\frac{193}{8199}\right) + (3)^1\left(\frac{23}{24597}\right)\end{pmatrix} = 0\\
C_2 &= \frac{1}{2!}\begin{pmatrix} 0^2\left(\frac{785}{24597}\right) + \left(\frac{1}{2}\right)^2\left(-\frac{4384}{8199}\right) + 1^2(1) \\ -\left(\frac{3}{2}\right)^2\frac{12832}{24597} + (2)^2\left(\frac{193}{8199}\right) + (3)^2\frac{23}{24597}\end{pmatrix} - \left(\frac{1}{24597}\right) + \frac{2521}{24597} = 0\\
C_3 &= \frac{1}{3!}\begin{pmatrix} 0^3\left(\frac{785}{24597}\right) + \left(\frac{1}{2}\right)^3\left(-\frac{4384}{8199}\right) + 1^3(1) \\ +\left(\frac{3}{2}\right)^3\frac{12832}{24597} + (2)^3\frac{193}{8199} + (3)^3\frac{23}{24597}\end{pmatrix} - 4^2\left(\frac{1}{24597}\right) - 1^2\frac{2521}{24597} = 0\\
C_4 &= \frac{1}{4!}\begin{pmatrix} 0^4\left(\frac{785}{24597}\right) + \left(\frac{1}{2}\right)^4\frac{4384}{8199} + 1^4(1) \\ -\left(\frac{3}{2}\right)^4\left(-\frac{12832}{24597}\right) + (2)^4\frac{193}{8199} + (3)^4\frac{23}{24597}\end{pmatrix} - \frac{1}{2!}4^3\frac{1}{24597} - \frac{1}{2!}1^3\frac{2521}{24597} = 0\\
C_5 &= \frac{1}{5!}\begin{pmatrix} 0^3\left(\frac{785}{24597}\right) + \left(\frac{1}{2}\right)^5 - \frac{4384}{8199} + 1^5(1) \\ -\left(\frac{3}{2}\right)^5\left(-\frac{12832}{24597}\right) + (2)^5\frac{193}{8199} + (3)^5\frac{23}{24597}\end{pmatrix} - \frac{1}{2!}4^4\frac{1}{24597} + \frac{1}{2!}1^4\frac{2521}{24597} = 0\\
C_6 &= \frac{1}{6!}\begin{pmatrix} 0^6\left(\frac{785}{24597}\right) - \left(\frac{1}{2}\right)^6\frac{4384}{8199} + 1^6(1) \\ -\left(\frac{3}{2}\right)^6\frac{12832}{24597} + (2)^6\frac{193}{8199} + (3)^6\frac{23}{24597}\end{pmatrix} - \frac{1}{2!}4^5\frac{1}{24597} + \frac{1}{2!}1^5\frac{2521}{24597} = 0\\
C_7 &= \frac{1}{7!}\begin{pmatrix} 0^5\left(\frac{785}{24597}\right) - \left(\frac{1}{2}\right)^7\frac{4384}{8199} + 1^7(1) \\ -\left(\frac{3}{2}\right)^7\frac{12832}{24597} + 2^7\frac{193}{8199} + 3^7\frac{23}{24597}\end{pmatrix} + \frac{1}{5!}4^6\frac{1}{24597} + \frac{1}{5!}1^6\frac{2521}{24597} = -\frac{79}{1311840}
\end{aligned}\right\} \qquad (3.48)$$

Since $C_{p+2} = C_7$. The $p = 5$ implies the method is of order 5 with error constant $C_7 = -\frac{79}{1311840}$

We follow similar procedure for others and even for cases $k = 5$ and $k = 6$ and present the Order and Error constants for the proposed methods as follows;

Table 3.1 Order and Error Constants of the 4SBHBDF

method	Order(p)	**Error Constants** (C_{p+2})
(3.19)	5	$-\frac{1163}{30252}$
(3.20)	5	$-\frac{823159}{12705840}$
(3.20)	5	$-\frac{4847}{564704}$
(3.20)	5	$-\frac{307}{352940}$
(3.20)	5	$-\frac{2909}{3630240}$
(3.20)	5	$\frac{1241}{282352}$
(3.20)	5	$-\frac{17257}{45176320}$
(3.20)	5	$-\frac{1370713}{1626347520}$
(3.21)	5	$-\frac{79}{1311840}$
(3.21)	5	$\frac{991}{275040}$
(3.21)	5	$\frac{1739}{2042880}$
(3.21)	5	$-\frac{148311}{14750560}$

Table 3.2 Order and Error Constants of the 5SBHBDF

Method	Order(p)	**Error Constants** (C_{p+2})
(3.28)	6	$\frac{31668953}{667233000}$
(3.29)	6	$\frac{939263}{556027500}$
(3.29)	6	$\frac{390199}{4448220000}$
(3.29)	6	$\frac{24451283}{125106187}$
(3.29)	6	$\frac{-1503671}{741370000}$
(3.29)	6	$\frac{303803}{26477500}$
(3.29)	6	$\frac{3169035811}{40033980000}$
(3.29)	6	$\frac{-5989}{66193750}$
(3.30)	6	$\frac{-11384321}{28959765625}$
(3.30)	6	$\frac{-56100696}{465541015625}$
(3.30)	6	$\frac{2096458572}{10884958984375}$
(3.30)	6	$\frac{1023157}{133446600}$
(3.30)	6	$\frac{15483281}{266893200}$
(3.30)	6	$-\frac{6722843}{133446600}$

Table 3.3 Order and Error Constants of the 6SBHBDF

Method	Order(p)	Error Constants (C_{p+2})
(3.36)	7	$-\frac{3727}{143680}$
(3.37)	7	$\frac{4381}{517248}$
(3.37)	7	$-\frac{26177}{90518400}$
(3.37)	7	$-\frac{7229}{18103680}$
(3.37)	7	$-\frac{17197}{181036800}$
(3.37)	7	$\frac{12707}{18103680}$
(3.37)	7	$-\frac{913}{172416}$
(3.37)	7	$\frac{89}{1379328}$
(3.37)	7	$\frac{9721}{231727104}$
(3.38)	7	$-\frac{5983}{82759680}$
(3.38)	7	$-\frac{8233129}{181036800}$
(3.38)	7	$-\frac{261523}{965529600}$
(3.38)	7	$\frac{82123}{306396160}$
3.38	7	$\frac{11719}{2814668640}$
3.38	7	$-\frac{47101}{39417280}$
3.38	7	$-\frac{539907687}{33660368}$
3.38	7	$-\frac{13934665}{2408450688}$

3.3.2 Consistency

The sufficient conditions for a linear multistep to be consistent are;

i. $p \geq 1$(i.e. the method has at least order of one).

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $p(r) = \rho'(1) = 0$

iv. $\rho''(1) = 2!\,\sigma(1)$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials respectively.

In section 3.3.1, we have established the axioms.

(i) where $p = 5$, $p = 6$, and $p = 7$ for cases of the 4SBHBDF, 5SBHBDF, and 6SBHBDFrespectively.

(ii) It satisfied $C_0 = \sum_{j=0}^{k} \alpha_j = 0$ in at each cases of 4SBHBDF, 5SBHBDF, and 6SBHBDF.

(iii), we shall consider (3.19) and obtain the first and second characteristic polynomials as;

$$\rho(r) = r^4 - \frac{41608}{7563}r^3 + \frac{66710}{2521}r^2 + \frac{95480}{2521}r - \frac{38144}{2521}r^{\frac{1}{2}} - \frac{356608}{7563}r^{\frac{3}{2}} + \frac{18515}{7563}$$

Then $\rho'(r) = 4r^3 - 3\frac{41608}{7563}r^2 + 2\frac{66710}{2521}r^1 + \frac{95480}{2521} - \frac{1}{2}\frac{38144}{2521}r^{-\frac{1}{2}} - \frac{3}{2}\frac{356608}{7563}r^{\frac{1}{2}}$

$\rho''^{(r)} = 12r^2 - 6\frac{41608}{7563}r + 2\frac{66710}{2521} + \frac{1}{4}\frac{38144}{2521}r^{-\frac{3}{2}} - \frac{3}{4}\frac{356608}{7563}r^{-\frac{1}{2}}$

Therefore,

$\rho''(1) = \frac{840}{2521}$

and

$\sigma(1) = \frac{420}{2521}$

$2!\,\sigma(r) = \frac{840}{2521}$

Hence, $\rho''(1) = 2!\,\sigma(r)$which satisfied the condition (iii). Since the three conditions are satisfied, it follows that (3.19) is consistent.

We follow similar procedure for others and even for cases $k = 5$ and $k = 6$ and present the first and second characteristic polynomials for the other in the table.

Table 3.4 Condition for Consistency of the 4SBHBDF

Method	$\rho''(1)$	$2!\,\sigma(r)$
(3.19)	$-\frac{840}{2521}$	$-\frac{840}{2521}$
(3.20)	$\frac{560}{2733}$	$\frac{560}{2733}$
(3.20)	$-\frac{2540}{1719}$	$-\frac{2540}{1719}$
(3.20)	$-\frac{1445}{3192}$	$-\frac{1445}{3192}$
(3.20)	$\frac{40680}{92191}$	$\frac{40680}{92191}$

Table 3.5: Condition for consistency of the 5SBHBDF

method	$\rho''(1)$	$2!\,\sigma(r)$
(3.28)	$-\frac{3230064864}{12439953125}$	$-\frac{3230064864}{12439953125}$
(3.30)	$-\frac{37600}{34893}$	$-\frac{37600}{34893}$
(3.30)	$-\frac{3746574}{1455767}$	$-\frac{3746574}{1455767}$
(3.30)	$\frac{2050920}{3991321}$	$\frac{2050920}{3991321}$
(3.30)	$-\frac{47682}{265145}$	$-\frac{47682}{265145}$
(3.30)	$-\frac{3230064864}{12439953125}$	$-\frac{3230064864}{12439953125}$

Table 3.6 Condition for Consistency of the 5SBHBDF

method	$\rho''(1)$	$2!\,\sigma(r)$
(3.36)	$\frac{126}{449}$	$\frac{126}{449}$
(3.37)	$-\frac{263925}{1274368}$	$-\frac{263925}{1274368}$
(3.37)	$-\frac{4309515}{14369792}$	$-\frac{4309515}{14369792}$
(3.37)	$-\frac{372925}{896001}$	$-\frac{372925}{896001}$
(3.37)	$-\frac{3248280}{300539}$	$-\frac{3248280}{300539}$
(3.37)	$-\frac{80766}{17597}$	$-\frac{80766}{17597}$
(3.37)	$-\frac{161644}{837699}$	$-\frac{161644}{837699}$
(3.37)	$-\frac{16137}{68392}$	$-\frac{16137}{68392}$

3.3.3 Zero stability

According to Awari (2017), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, if every root with modulus one is simple, i.e. $|r| \leq 1$ and has multiplicity not greater than the order of the differential equation.

To obtain the zero-stability of HBDF, we shall express the proposed methods in matrix difference equation form;

$$A^{(1)}Y_{n+1} = A^{(0)}Y_n + h^2B^{(0)}F_n + h^2B^{(1)}F_{n+1} \tag{3.49}$$

Where

$$Y_{n+1} = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ \vdots \\ y_{n+i} \end{pmatrix} Y_n = \begin{pmatrix} y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{3}{2}} \\ y_{n-2} \\ \vdots \\ y_n \end{pmatrix}$$

$$F_{n+1} = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ \vdots \\ f_{n+i} \end{pmatrix} f_n = \begin{pmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ \vdots \\ f_n \end{pmatrix}$$

$A^{(1)}$,$P^{(0)}$, $B^{(1)}$, and $B^{(0)}$ are $(k+1) \times (k+1)$ matrices obtained from the combined coefficients of the HBDF. The roots of the first characteristics polynomial $\rho(r)$ is obtained from;

$$\rho(r) = \left|rP^{(1)} - P^{(0)}\right| \tag{3.50}$$

3.3.3.1 Zero stability of 4SBHBDF

We express the schemes in 4SBHBDFin the form (3.41) and obtain the$P^{(1)}$,$P^{(0)}$, and $\rho(r)$ as

$$p^{(1)} = \begin{bmatrix} 1 & \frac{1145}{2128} & -\frac{4409}{3591} & \frac{473}{1064} & -\frac{1637}{57456} & 0 \\ -\frac{4384}{8199} & 1 & -\frac{12832}{24597} & \frac{193}{8199} & \frac{23}{24597} & 0 \\ \frac{297}{10705} & -\frac{91611}{171280} & 1 & -\frac{42957}{85640} & \frac{1381}{171280} & 0 \\ -\frac{1312}{573} & \frac{3521}{573} & -\frac{88736}{15471} & 1 & \frac{8269}{15471} & 0 \\ \frac{400032}{92191} & -\frac{1010475}{92191} & \frac{1261792}{92191} & -\frac{679401}{92191} & 1 & 0 \\ -\frac{38144}{2521} & \frac{95480}{2521} & -\frac{356608}{7563} & \frac{66710}{2521} & \frac{41608}{7563} & 1 \end{bmatrix}$$

$$p^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{10433}{14364} \\ 0 & 0 & 0 & 0 & 0 & -\frac{785}{24597} \\ 0 & 0 & 0 & 0 & 0 & -\frac{7}{10705} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5353}{15471} \\ 0 & 0 & 0 & 0 & 0 & \frac{64139}{92191} \\ 0 & 0 & 0 & 0 & 0 & -\frac{18515}{7563} \end{bmatrix}$$

$$\rho(r) = \frac{4101649090816}{34344479374931}r^6 - \frac{19345025348830499072}{3166251898054263821}r^5$$

Then$r = (0,0,0,0,0,1)$, therefore 4SBHBDFis zero-stable since $|r_j| \leq 1$

3.3.3.2 Zero stability of 5SBHBDF

We express the schemes in 5SBHBDFin the form (3.42) and obtain the$P^{(1)}$,$P^{(0)}$, and $\rho(r)$ as

$$p^{(1)} = \begin{bmatrix} 1 & \frac{5163285504}{12439953125} & -\frac{705874}{796157} & \frac{1979742141}{12439953125} & -\frac{344635904}{12439953125} & \frac{42573544}{12439953125} & 0 \\ -\frac{53509375}{132360384} & 1 & -\frac{2815625}{5090784} & \frac{1794717}{16969280} & \frac{149567}{10340655} & -\frac{20551}{12726960} & 0 \\ -\frac{250423}{885326} & \frac{221984256}{532046875} & 1 & -\frac{405735723}{1064093750} & \frac{247228928}{6916609375} & -\frac{1897816}{532046875} & 0 \\ \frac{390625}{453609} & -\frac{404480}{104679} & \frac{2656250}{942111} & 1 & -\frac{8344064}{12247443} & \frac{4712}{104679} & 0 \\ -\frac{509046875}{93169088} & \frac{41370927}{1455767} & -\frac{1343265625}{46584544} & \frac{444206997}{93169088} & 1 & -\frac{24262771}{23292272} & 0 \\ -\frac{1951328125}{207548692} & \frac{197072832}{3991321} & -\frac{405078125}{7982642} & \frac{192559275}{15965284} & -\frac{222129536}{51887173} & 1 & 0 \\ \frac{861828125}{29739528} & -\frac{14402209}{95319} & \frac{10421875}{67284} & -\frac{3094397}{84728} & \frac{51879766}{3717441} & -\frac{2887259}{571914} & 1 \end{bmatrix}$$

$$p^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1650327432}{2487990625} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{213247}{4242320} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5236008}{106409375} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{56632}{314037} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{26878449}{23292272} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{7947519}{3991321} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1175093}{190638} \end{bmatrix}$$

$$\rho(r) = \frac{88631239741749381578565209075712}{9525053801400413472826151066005} r^7 - \frac{15030987135768493855250992132608}{501318621126337551201376371895} r^6$$

Then $r = (0,0,0,0,0,0,1)$, therefore 5SBHBDFis zero-stable since $|r_j| \leq 1$.

3.3.3.3 Zero stability of 6SBHBDF

We express the schemes in 6SBHBDFin the form (3.41) and obtain the $P^{(1)}$, $P^{(0)}$, and $\rho(r)$ as

$$p^{(1)} = \begin{bmatrix} 1 & -\frac{154848}{42745} & \frac{1361421}{273568} & -\frac{749344}{213725} & \frac{41511}{34196} & -\frac{306113}{2735680} & \frac{8883}{854900} & 0 \\ -\frac{16704585}{28739584} & 1 & -\frac{8124165}{57479168} & -\frac{31356}{70165} & \frac{2600795}{14369792} & -\frac{2011131}{114958336} & \frac{234279}{143697920} & 0 \\ \frac{334}{8133} & -\frac{20417536}{37696455} & 1 & -\frac{11236352}{20942475} & \frac{282628}{7539291} & \frac{1393}{8376990} & -\frac{578}{6980825} & 0 \\ \frac{38375}{17841152} & \frac{25}{917} & -\frac{19072125}{35682304} & 1 & -\frac{4511125}{8920576} & \frac{830525}{71364608} & -\frac{12501}{17841152} & 0 \\ \frac{60039}{35194} & -\frac{740864}{87985} & \frac{675243}{35194} & -\frac{6861312}{439925} & 1 & \frac{805401}{351940} & -\frac{107433}{879850} & 0 \\ -\frac{3913540}{300539} & \frac{17954816}{300539} & -\frac{37786230}{300539} & \frac{209471488}{1502695} & -\frac{19696840}{300539} & 1 & \frac{5706396}{1502695} & 0 \\ \frac{4192775}{896001} & -\frac{170256640}{8064009} & \frac{52297525}{1194668} & -\frac{43578112}{896001} & \frac{206983100}{8064009} & -\frac{37584095}{7168008} & 1 & 0 \\ -\frac{8757}{449} & \frac{196096}{2245} & -\frac{161217}{898} & \frac{2216448}{11225} & -\frac{46298}{449} & \frac{195111}{8980} & -\frac{66609}{11225} & 1 \end{bmatrix}$$

$$p^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{543963}{13678400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2496761}{574791680} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{178627}{376964550} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10979}{71364608} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{73411}{1759700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{528391}{1502695} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{10979}{71364608} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{24619}{44900} \end{bmatrix}$$

$$\rho(r) = \frac{630739446852894260045389089193898437 5}{7342722354584605668421124787202974358 4}r^7 - \frac{4743722954584625668421126342787202743 4}{5382826235458460526196832421102477724 02}r^6$$

Then $r = (0,0,0,0,0,0,0,1)$, therefore 6SBHBDFis zero-stable since $|r_j| \le 1$

3.4 Convergence

The necessary and sufficient condition for a Linear Multistep Method to be convergent is the method to be consistent and zero-stable. Since, the proposed Backward Differentiation Formulae are both consistent and zero-stable, we conclude that the proposed methods are convergent.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Results

Problem: 1

Constant Coefficient Linear Type

$$\frac{d^2y(x)}{dx^2} = 8\frac{dy(x)}{dx} - 17y(x)$$

$$y(0) = -4, y'(0) = -1, h = 0.01$$

Exact solution:

$$y(x) = 15e^{4x}\sin(x) - 4e^{4x}\cos(x)$$

Source Hussaini and Muhammad (2021)

Results of Problem 1 is presented in Table 4.1 and Table 4.2

Table 4.1 Numerical Comparison of Exact Solution and the Proposed Methods for Problem 1 at $h = 0.01$

x	Exact	4SBHBDF	5SBHBDF	6SBHDF
0.0	4	4	4	4
0.1	3.703477803016	3.703477800904654	3.70347780303625	3.7034778030095
0.2	2.092512222723	2.092512211056652	2.09252222790784	2.0925122226926
0.3	2.030109209389	2.030109247870297	2.03010920922489	2.0301092094835
0.4	10.68384526906	10.68384536914484	10.6838452687145	10.683845269302
0.5	27.19950587509	27.19950610485733	27.1995058743943	27.199505875615
0.6	56.97102341802	56.97102390216715	56.9710234167154	56.971023419096
0.7	108.5987462326	108.5987471960738	108.598746230267	108.59874623471
0.8	195.6104858887	195.6104877209509	195.610485884562	195.61048589256
0.9	339.0264188636	339.0264222318543	339.026418856481	339.02641887047
1.0	571.1433607200	571.1433667080193	571.143360708019	571.14336073189

The Table 4.1 shows the numerical results of problem 1. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.2 Absolute Error $|Y(t) - y(t)|$ in proposed methods for Problem 1

x	4SBHBDF	5SBHBDF	6SBHBDF	Hussaini and Muhammad (2021)
0.0	0	0	0	0
0.1	2.111386E-09	2.021077E-11	6.511951E-12	1.3159E-07
0.2	1.166701E-08	6.712188E-11	3.104634E-11	6.7720E-07
0.3	3.848114E-08	1.642611E-10	9.438140E-11	2.1628E-06
0.4	1.000783E-07	3.520264E-10	2.363964E-10	5.5378E-06
0.5	2.297648E-07	6.981589E-09	5.226320E-10	1.2576E-05
0.6	4.841329E-07	1.313958E-09	1.067437E-09	5.2072E-05
0.7	9.634269E-07	2.378996E-09	2.065300E-09	9.8616E-05
0.8	1.832210E-06	4.178189E-09	3.833801E-09	2.6315E-05
0.9	3.368216E-06	7.156504E-09	6.833380E-09	1.8063E-04
1.0	6.015070E-06	1.199823E-08	1.187924E-08	3.2162E-04

The results of the errors of problem 1 at h=0.01, proved that as the number of step size k increases, the accuracy increases.

Problem: 2

Constant Coefficient Linear Type

$$y''(x) = 3y' + 8e^{2x}$$

$$y(0) = 1, y'(0) = 1, h = 0.01$$

Exact Solution

$$y(x) = -4e^{2x} + 3e^{3x} + 2$$

Source: Badmus *et al.* (2014)

Results of Problem 1 is presented in Table 4.3 and Table 4.4

Table 4.3 Numerical Comparison of Exact Solution and the Proposed Methods for Problem 2 at $h = 0.01$

x	Exact	4SBHBDF	5SBHBDF	6SBHBDF
0	0	0	0	0
0.1	1.00513852551048	1.005138525510492	1.005138525510484	1.005138525510484
0.2	1.01055824175352	1.010558241753543	1.010558241753527	1.010558241753527
0.3	1.01626544391208	1.016265443912109	1.016265443912083	1.016265443912083
0.4	1.02226654286652	1.022266542866562	1.022266542866525	1.022266542866525
0.5	1.02856806714979	1.028568067149859	1.028568067149798	1.028568067149798
0.6	1.03517666493419	1.035176664934279	1.035176664934192	1.035176664934192
0.7	1.04209910605024	1.042099106050362	1.042099106050249	1.042099106050249
0.8	1.04934228403829	1.049342284038433	1.049342284038292	1.049342284038292
0.9	1.05691321823310	1.056913218233285	1.056913218233101	1.056913218233102
1.0	1.06481905588225	1.064819055882486	1.064819055882258	1.064819055882258

The Table 4.3 shows the numerical results of problem 2. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.4 Absolute Error $|Y(t) - y(t)|$ in proposed methods for Problem 2

x	4SBHBDF	5SBHBDF	6SBHBDF	Badmus *et al*. (2014)
0.0	0	0	0	0
0.1	7.307217E-15	6.163983E-17	5.166811E-18	2.021077E-11
0.2	1.657122E-14	1.344640E-16	1.184919E-17	6.712188E-11
0.3	2.577850E-14	2.103543E-16	1.860079E-17	1.642611E-10
0.4	3.695325E-14	2.911278E-16	2.546709E-17	3.520264E-10
0.5	6.058473E-14	3.468657E-16	3.239023E-17	6.981589E-09
0.6	8.653712E-14	4.193913E-16	3.969403E-17	1.313958E-09
0.7	1.126725E-13	5.010015E-16	5.334440E-17	2.378996E-09
0.8	1.411474E-13	5.859614E-16	6.877556E-17	4.178189E-09
0.9	1.831217E-13	6.762430E-16	8.440513E-17	7.156504E-09
1.0	2.278189E-13	7.395821E-16	1.002848E-16	1.199823E-08

The results show that errors become smaller as the step size k increases. It also observed that there is accuracy as step size k increases.

Problem: 3

Constant Coefficient Linear Type

$$y'' = \frac{6}{x^2}y' - \frac{4}{x}y$$

$$y(1) = -1, y'(1) = 1, h = \frac{1}{320}$$

Exact solution as:

$$y(x) = \frac{5}{3x} - \frac{2}{3x^4}$$

Source: Abada *et al.* (2017)

Results of Problem 3 is presented in Table 4.5 and Table 4.6

Table 4.5 Numerical Comparison of Exact Solution and the Proposed Methods for Problem 3 at $h = \frac{1}{320}$

X	Exact	4SBHBDF	5SBHBDF	6SBHBDF
1.00	1	1	1	1
1.003125	1.00307652585769	1.00307652585771	1.003076525857696	1.003076525857696
1.006250	1.00605750308351	1.00605750308355	1.006057503083516	1.006057503083516
1.009375	1.00894499508883	1.00894499508889	1.008944995088838	1.008944995088837
1.012500	1.01174101816798	1.01174101816806	1.011741018167989	1.011741018167988
1.015625	1.01444754268641	1.01444754268653	1.014447542686415	1.014447542686414
1.018750	1.01706649423567	1.01706649423584	1.017066494235674	1.017066494235673
1.02187 5	1.01959975475628	1.01959975475650	1.019599754756289	1.019599754756288
1.025000	1.02204916362943	1.02204916362969	1.022049163629433	1.022049163629432
1.028125	1.02441651873840	1.02441651873873	1.024416518738405	1.024416518738403
1.031250	1.02670357750080	1.02670357750121	1.026703577500808	1.026703577500806

The Table 4.5 shows the numerical results of problem 3. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.6 Absolute Error $|Y(t) - y(t)|$ in prosed methods for Problem 3

x	4SBHBDF	5SBHBDF	6SBHBDF	Abada *et al.* (2017)
1.000000	1	1	1	1
1.003125	1.584762E-14	2.902001E-16	6.035170E-17	1.0000E-14
1.006250	3.538009E-14	6.233869E-16	1.362264E-16	2.0000E-14
1.009375	5.409330E-14	9.585251E-16	2.105833E-16	3.0000E-14
1.012500	7.645367E-14	1.306081E-15	2.836268E-16	2.0000E-14
1.015625	1.218539E-13	1.526144E-15	3.537719E-16	2.0000E-14
1.018750	1.698903E-13	1.716720E-15	4.242471E-16	2.0000E-14
1.021875	2.166456E-13	1.943949E-15	5.103597E-16	3.0000E-14
1.025000	2.660267E-13	2.173730E-15	6.497260E-16	4.0000E-14
1.028125	3.349836E-13	2.414736E-15	7.865096E-16	4.0000E-14
1.031250	4.057153E-13	2.550703E-15	9.208023E-16	4.0000E-14

The errors of this method at each k-step compared with the abada solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Problem: 4

Linear System of Second Order Initial Value problem (IVP)

$$\frac{d^2y_1}{dx^2} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\frac{d^2y_2}{dx^2} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$y_1(0) = 1, y_1'(0) = 2, y_2(0) = 1, y_2'(0) = 2$, h=0.01

Exact Solution:

$$y_1(x) = e^{2x},\ \ y_2(x) = e^{2x}$$

Source: Hussaini and Muhammad (2021)

Results of Problem 4 is presented in Table 4.7 and Table 4.8

Table 4.7a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 4 at $h = 0.01$ for $y_1(t)$

X	Exact	4 SBHBDF	5 SBHBDF	6 SBHBDF
0.0	1.0000000000000	1.0000000000000	1.0000000000000	1.000000000000
0.1	1.221402758160169	1.2214027581980047	1.221402758160167	1.221402758160167
0.2	1.491824697641270	1.4918246977910167	1.491824697641264	1.491824697641272
0.3	1.822118800390508	1.8221188007610667	1.822118800390498	1.822118800390513
0.4	2.2255409284924676	2.2255409292307309	2.225540928492451	2.225540928492476
0.5	2.7182818284590453	2.7182818297719146	2.718281828459022	2.718281828459060
0.6	3.3201169227365474	3.3201169248974443	3.320116922736516	3.320116922736572
0.7	4.0551999668446745	4.0551999702285565	4.055199966844632	4.05519996684471339
0.8	4.9530324243951148	4.9530324294908558	4.953032424395059	4.953032424395173
0.9	6.0496474644129460	6.0496474718788228	6.049647464412873	6.049647464412931
1.0	7.3890560989306501	7.3890561096167800	7.389056098930556	7.389056098930675

The Table 4.7a shows the numerical results of Problem 4. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.7b Absolute Error $|Y(t) - y(t)|$ in prosed methods for Problem 4

X	4SBHBDF	5SBHBDF	6SBHBDF	Hussani and Muhammad (2021)
0.0	0.0	0.0	0.0	0.0
0.1	1.584762E-14	2.902001E-16	6.035170E-17	3.782E-11
0.2	3.538009E-14	6.233869E-16	1.362264E-16	1.496E-10
0.3	5.409330E-14	9.585251E-16	2.105833E-16	3.704E-10
0.4	7.645367E-14	1.306081E-15	2.836268E-16	7.379E-10
0.5	1.218539E-13	1.526144E-15	3.537719E-16	1.312E-09
0.6	1.698903E-13	1.716720E-15	4.242471E-16	2.160E-09
0.7	2.166456E-13	1.943949E-15	5.103597E-16	3.382E-09
0.8	2.660267E-13	2.173730E-15	6.497260E-16	5.094E-09
0.9	3.349836E-13	2.414736E-15	7.865096E-16	7.464E-09
1.0	4.057153E-13	2.550703E-15	9.208023E-16	1.068E-08

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Table 4.8a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 4 at $h = 0.01$ for $y_2(t)$

X	Exact	4 SBHBDF	5 SBHBDF	6 SBHBDF
0.0	1.000000000000000	1.0000000000000000	1.000000000000000	1.000000000000000
0.1	1.2214027581601698339	1.2214027581980047	1.2214027581601671	1.2214027581601671
0.2	1.4918246976412703178	1.4918246977910167	1.4918246976412642	1.4918246976412721
0.3	1.8221188003905089748	1.8221188007610667	1.8221188003904986	1.8221188003905132
0.4	2.2255409284924676045	2.2255409292307309	2.2255409284924519	2.2255409284924762
0.5	2.7182818284590452353	2.7182818297719146	2.7182818284590227	2.7182818284590603
0.6	3.3201169227365474895	3.3201169248974443	3.3201169227365164	3.3201169227365723
0.7	4.0551999668446745872	4.0551999702285565	4.0551999668446328	4.0551999668447133
0.8	4.9530324243951148036	4.9530324294908558	4.9530324243950594	4.9530324243951731
0.9	6.0496474644129460837	6.0496474718788228	6.0496474644128736	6.0496474644130312
1.0	7.3890560989306502272	7.3890561096167800	7.3890560989305563	7.3890560989307722

The Table 4.8a shows the numerical results of problem 4. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.8b Absolute Error $|Y(t) - y(t)|$ in prosed methods for Problem 4

x	4SBHBDF	5SBHBDF	6SBHBDF	Hussani and Muhammad (2021)
0.0				
0.1	1.584762E-14	2.902001E-16	6.035170E-17	3.782E-11
0.2	3.538009E-14	6.233869E-16	1.362264E-16	1.496E-10
0.3	5.409330E-14	9.585251E-16	2.105833E-16	3.704E-10
0.4	7.645367E-14	1.306081E-15	2.836268E-16	7.379E-10
0.5	1.218539E-13	1.526144E-15	3.537719E-16	1.312E-09
0.6	1.698903E-13	1.716720E-15	4.242471E-16	2.160E-09
0.7	2.166456E-13	1.943949E-15	5.103597E-16	3.382E-09
0.8	2.660267E-13	2.173730E-15	6.497260E-16	5.094E-09
0.9	3.349836E-13	2.414736E-15	7.865096E-16	7.464E-09
1.0	4.057153E-13	2.550703E-15	9.208023E-16	1.068E-08

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Problem: 5

Variable Coefficient Linear Type

$$x^2 \frac{d^2y(x)}{dx^2} + \frac{3xdy(x)}{2dy} - \frac{1y(x)}{2} = 0$$

$$y(1) = 2, y'(1) = 5, h = 0.01$$

Exact solution:

$$y(x) = \frac{14}{3}\sqrt{x} - \frac{8}{3x}$$

Source: Badmus *et al.* (2014)

Results of Problem 5 is presented in Table 4.9

Table 4.9 Numerical Comparison of Exact Solution and the Computed Results from the proposed method.

X	Exact	4SBHBDF	5SBHBDF	6SBHBDF
1.0	1	1	1	1
1.1	2.4701988672182829	2.4701988672608552	2.4701988672187693	2.4701988672182877
1.2	2.8898549811593281	2.8898549812856299	2.8898549811605495	2.8898549811593207
1.3	3.2695365991805926	3.2695365994060417	3.2695365991826082	3.2695365991805986
1.4	3.6169125594644035	3.6169125597911295	3.6169125594672004	3.6169125594644255
1.5	3.9376982887163044	3.9376982891421087	3.9376982887198388	3.9376982887163082
1.6	4.2362516323143080	4.2362516328346191	4.2362516323185326	4.2362516323143180
1.7	4.5159614605420799	4.5159614611519441	4.5159614605469496	4.5159614605420382
1.8	4.7795088555179296	4.7795088562122628	4.7795088555234023	4.7795088555179716
1.9	5.0290473123789455	5.0290473131530200	5.0290473123849836	5.0290473123789457
2.0	5.2663299577411102	5.2663299585905134	5.2663299577476803	5.2663299577410391

The Table 4.9 shows the numerical results of problem 5. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

4.9b Absolute Error $|Y(t) - y(t)|$ in proposed methods for Problem 5

x	4SBHBDF	5SBHBDF	6SBHBDF	Badmus *et al.* (2014)
1.0	0	0	0	0
1.1	4.257245E-11	4.954632E-13	4.794671E-15	2.021077E-11
1.2	1.263052E-10	1.289621E-12	1.118039E-15	6.712188E-11
1.3	2.254480E-10	2.034710E-12	6.058695E-15	1.642611E-10
1.4	3.267263E-10	2.811952E-12	9.758609E-15	3.520264E-10
1.5	4.258054E-10	3.554371E-12	3.835802E-15	6.981589E-09
1.6	5.203131E-10	4.276249E-12	9.720218E-15	1.313958E-09
1.7	6.098615E-10	4.801756E-12	4.168247E-14	2.378996E-09
1.8	6.943314E-10	5.456282E-12	4.202684E-14	4.178189E-09
1.9	7.740732E-10	6.023162E-12	2.180659E-14	7.156504E-09
2.0	8.494079E-10	6.567251E-12	7.110028E-14	1.199823E-08

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Problem: 6

Consider the system of equations of Stiefel and Bettis problem

$$y_1'' + y_1 = 0.001\cos(x)$$

$$y_2'' + y_2 = 0.001\sin(\mathrm{x})$$

$$y_1(0) = 0, y_1'(0) = 0, h = \frac{1}{320}$$

$$y_2(0) = 0, y_2'(0) = 0.9995$$

Exact Solutions are given as;

$$y_1(x) = \cos(x) + 0.0005(x)\sin(x)$$

$$y_2(x) = \sin(x) - 0.0005(x)\cos(x)$$

Source: Yahaya and Tijjani (2015)

Results of Problem 6 is presented in Table 4.10 and Table 4.11.

Table 4.10a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 6 at $h = \frac{1}{320}$for $\boldsymbol{y_1(t)}$

x	Exact	4SBHBDF	5 SBHBDF	6SBHBDF
0.000000	0	0	0	0
0.003125	0.999995122074278	0.999995122074278	0.999995122074278	0.999995122074278
0.006250	0.999980488344701	0.999980488344701	0.999980488344701	0.999980488344701
0.009375	0.999956098954032	0.999956098954032	0.999956098954032	0.999956098954032
0.012500	0.999921954140212	0.999921954140212	0.999921954140212	0.999921954140212
0.015625	0.999878054236352	0.999878054236352	0.999878054236352	0.999878054236352
0.018750	0.999824399670731	0.999824399670731	0.999824399670731	0.999824399670731
0.02187	0.999760990966796	0.999760990966796	0.999760990966796	0.999760990966796
0.025000	0.999687828743151	0.999687828743151	0.999687828743151	0.999687828743151
0.028125	0.999604913713556	0.999604913713556	0.999604913713556	0.999604913713556
0.031250	0.999512246686917	0.999512246686917	0.999512246686917	0.999512246686917

The Table 4.10a shows the numerical results of problem 6. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.10b Absolute Error $|Y(t) - y(t)|$ in proposed methods for Problem 6

x	4SBHBDF	5SBHBDF	6SBHBDF	Error in Yahaya and Tijjani (2015)
0.000000	0	0	0	0
0.003125	2.381376E-22	7.298480E-23	1.041200E-26	5.6685E-22
0.006250	5.405120E-22	1.581604E-22	2.380000E-26	9.2282E-22
0.009375	8.316330E-22	2.454614E-22	3.716060E-26	2.0528E-22
0.012500	1.205856E-21	3.374029E-22	5.041800E-26	3.0176E-21
0.015625	2.552634E-21	3.987581E-22	6.357000E-26	2.6388E-21
0.018750	4.099494E-21	4.649762E-22	7.859600E-26	1.8847E-22
0.02187 5	5.619969E-21	5.433785E-22	1.304450E-25	4.5946E-21
0.025000	7.359366E-21	6.239018E-22	1.917050E-25	3.8066E-21
0.028125	1.094169E-20	7.090608E-22	2.527360E-25	3.9205E-21
0.031250	1.485984E-20	7.636379E-22	3.138200E-25	1.2096E-21

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Table 4.11a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 6 at $h = \frac{1}{320}$ for $y_2(t)$

x	Exact	4SBHBDF	5SBHBDF	6SBHBDF
0.0000	0	0	0	0
0.003125	0.003123432421368	0.003123432421368	0.003123432421368	0.003123432421368
0.006250	0.006246834371010	0.006246834371010	0.006246834371010	0.006246834371010
0.009375	0.009370175377494	0.009370175377494	0.009370175377494	0.009370175377494
0.012500	0.012493424969984	0.012493424969984	0.012493424969984	0.012493424969984
0.015625	0.015616552678538	0.015616552678538	0.015616552678538	0.015616552678538
0.018750	0.018739528034400	0.018739528034400	0.018739528034400	0.018739528034400
0.021875	0.021862320570301	0.021862320570301	0.021862320570301	0.021862320570301
0.025000	0.024984899820758	0.024984899820758	0.024984899820758	0.024984899820758
0.028125	0.028107235322366	0.028107235322366	0.028107235322366	0.028107235322366
0.031250	0.031229296614099	0.031229296614099	0.031229296614099	0.031229296614099

The Table 4.11a shows the numerical results of problem 6. The results show that the proposed methods 4SBHBDF, 5SBHBDF and 6SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.11b Absolute Error $|Y(t) - y(t)|$ in proposed methods for Problem 6

x	4SBHBDF	5SBHBDF	6SBHBDF	Error in Yahaya and Tijjani (2015)
0.000000	0	0	0	0
0.003125	4.312814E-20	3.292285E-22	1.242837E-24	5.6685E-22
0.006250	9.713139E-20	3.185519E-22	2.829327E-24	9.2282E-22
0.009375	1.499253E-19	2.506054E-23	4.408140E-24	2.0528E-22
0.012500	2.135932E-19	1.127375E-21	5.990016E-24	3.0176E-21
0.015625	3.469364E-19	4.921169E-21	7.360440E-24	2.6388E-21
0.018750	4.911499E-19	6.255357E-21	9.197510E-24	1.8847E-22
0.02187 5	6.341507E-19	5.901612E-21	1.221512E-23	4.5946E-21
0.025000	7.880187E-19	4.661617E-21	1.557615E-23	3.8066E-21
0.028125	1.011539E-18	9.136466E-21	1.892938E-23	3.9205E-20
0.031250	1.245920E-18	1.636987E-20	2.228554E-23	1.7865E-20

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this research, block hybrid backward differentiation of order (k+1) have been developed by the interpolation and collocation techniques with off- grids points for the solution of second order ordinary differential equations (ODEs). The power series expansion technique was used as the basis function. Analyses of the basic properties of the methods have also been verified which shows that the methods are of order (k+1). It also showed that the methods are consistent, zero- stable and convergent. Some selected problems showed that second order ordinary differential equations (ODEs) have been considered and agree strongly with the exact solution to determine the efficiency and accuracy of the methods. It was also observed from the error tables and figures that the block hybrid backward differentiation formulae (BHBDF) performed better in solving problems of second order ordinary differential equations (ODEs) as they produce smaller errors.

The accuracy of the methods developed was tested with nine test problems (Real-life problem, stiefel and bettis problem and highly stiff problem) and their corresponding results were compared with other methods develop by researchers. Moreover the outcome of the comparison of the method to the results of exact solution showed that, the proposed methods is more efficient. It should be note that the accuracy and efficiency of the method is dependent on the implementation strategies. If economical computation is required, then the new method is a better choice. The proposed method is therefore recommended for general purpose used. Maple 17 software package was employed to generate the schemes and results.

5.2 Recommendations

It is proposed for further research that;

i. Researchers should try other basis functions different from power series to develop scheme .

ii. The number of k- steps to be increased as the performance of the method is investigated**.**

iii. More off-grid points should be focused in order to enhance global error estimations.

iv. Researchers should consider developing computer software for solution of initial value problems of the proposed method.

5.3 Contributions to Knowledge

The following contributions were made:

i. Formulation of new class of hybrid methods which are based on block hybrid backward differentiation formulae (BHBDF) for the solution of second order ordinary differential equations.

ii. Derivation of some hybrid methods which are self-starting.

iii. The methods are applicable to stiff system, non linear and system of second order ordinary differential equations (ODEs).

iv. The efficiency and accuracy of the proposed method were proven to be relatively high at error analyses ranging between E-14 and E-26, with the lowest error obtained with the method at K = 6. The study introduces new approaches for discretization, interpolation, and collocation, contributing to the evolution of techniques used in computational mathematics.

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