MAT 212

LINEAR ALGEBRA 1

FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA
CENTRE FOR OPEN DISTANCE AND e-LEARNING

FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA
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CENTRE FOR OPEN DISTANCE AND e-LEARNING (CODeL)

B.TECH. COMPUTER SCIENCE PROGRAMME

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LINEAR ALGEBRA I

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MAT 212: LINEAR ALGEBRA I

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MAT 212 Study Guide

Introduction

MAT212 Linear Algebra I is a one-semester 3-credit unit 200 level course designed to teach the university mathematics student the basics of the subject of linear algebra. The prerequisites for this course are MAT 111, MAT 112 and MAT 113.

The course consists of 3 modules and 14 study units of basic knowledge of Linear Algebra. The units are Vector space over the real field, Subspace, Linear combination and spanning sets, Linear dependence and independence, Bases and dimension, Introduction to matrices, Matrix representation, Types of matrices, Operations on matrices, Determinants, Introduction to linear transformation, Matrix transformations, Kernels and images of a linear transformation, Nullity and rank.

Recommended Study Time

This course is a 3-credit unit course having 14 study units. You are therefore enjoined to spend at least 2 hours in studying the content of each study unit.

What You Are About to Learn in This Course

The overall aim of this course, MAT 212 is to introduce you to the study of Linear Algebra. At the end of this course you will:

i. Define vector space
ii. Define matrix
iii. Know the areas of application of matrix formulation
iv. Solve problems using determinant
v. Explain the meaning of kernel and range of a linear transformation
vi. Determine the nullity and rank of a given linear transformation

Course Aims

This course aims to introduce students to Linear Algebra. It is expected that the knowledge will enable the reader to effectively use the knowledge of Linear Algebra in his/her profession.

Course Learning Outcome

The main Learning Outcome of this course is to give you a good foundation in Linear Algebra. The course has two goals: to teach the fundamental concepts and techniques of matrix algebra and abstract vector spaces, and to teach the techniques associated with understanding the definitions and theorems forming a coherent area of mathematics. So there is an emphasis on worked examples of nontrivial size and on proving theorems carefully.

Therefore, at the end of this course you should be able to: -

i. Define vector space
ii. Show that a given set of vectors spans a given vector space
iii. Define Linear Dependence and Independence
iv. Define matrix;
v. Know the areas of application of matrix formulation
vi. Solve problems using determinant
vii. Know the definition of determinants
viii. Find the inverse of a given linear transformation, or show that it does not exist
ix. Explain the meaning of kernel and range of a linear transformation;
x. List the properties of kernel and range of a linear transformation
xi. Determine the nullity and rank of a given linear transformation

Working Through This Course Material.

The course is written in Units. Each unit should take you 3 hours to work through. The course consists of 14 units of 4 modules:

Module 1 -Units 1 - 5
Module 2-Units 1- 5
Module 3-Units 1- 4

Course Materials

The major components of the course are:

1. Course Guide
2. Study Units
3. Text Books
4. Assignment File

5. Presentation Schedule

Study Units

There are 14 units in this course as follows: -

MODULE ONE

UNIT 1: Vector Space over the Real Field

UNIT 2: Subspace

UNIT 3: Linear Combination and Spanning Sets

UNIT 4: Linear Dependence and Independence

UNIT 5: Bases and Dimension

MODULE TWO

UNIT 1: Introduction to Matrices

UNIT 2: Matrix Representation

UNIT 3: Types of Matrices

UNIT 4: Operations on Matrices

UNIT 5: Determinants

MODULE THREE

UNIT 1: Introduction to Linear Transformation

UNIT 2: Matrix Transformations

UNIT 3: Kernels and Images of a Linear Transformation

UNIT 4: Nullity and Rank

References/Further Reading and Other Resources

You have the References/Further Reading used for each unit as 7.0 of the unit. Generally, they are

listed together here below:

1. Brookes, Mike (2005), *The Matrix Reference Manual*, London: Imperial College.
2. Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.
3. Hazewinkel, Michiel, (2001), "Determinant", *Encyclopedia of Mathematics*, Springer, ISBN978-1-55608-010-4.
4. Nicholson, W. K (1995): Linear Algebra with Applications. P. W. S Publishing company.

5. Odili,G. A. (2000): Algebra for Colleges and Universities: An Integral Approach. Anachuna Educational Books. ISBN978-2897-37-x. Robert, A. Beezer (2006): A First Course in Linear Algebra. http://linear.ups.edu/.

Assignment File

You will find all details of the work you must submit to your tutor, for scoring, in this file.

The marks you obtained for these assignments will count towards the final mark you obtain for this course. Further information on assignment will be found in the Assignment file. There are assignments on each Unit in the Course.

Presentation Schedule

The presentation schedule included in this course guide provides you with important dates for completion of each Tutor Marked Assignment (TMA). You are required to submit all materials for your Tutor-Marked Assignment (TMA)s as at when due.

Exercises and Solutions

You are advised to attempt each exercise before turning to the solutions, as these exercises are meant to serve as self-assessment questions.

Assessment

There are two aspects to the assessment of this course. First, there are Tutor Marked Assignment (TMA)s; and second, the written examination. Therefore, you are expected to take note of the facts, information and problem solving gathered during the course. The Tutor Marked Assignment (TMA)s must be submitted to your tutor for formal assessment, in accordance to the deadline given. The work submitted will count for 40% of your total course mark.

At the end of the course, you will need to sit for a final written examination. This examination will account for 60% of your total score. You will be required to submit some assignments by uploading them to MAT 212 page on the u-learn portal.

Tutor-Marked Assignment (TMA)s (TMA)

There are two aspects to the assessment of the course. First is the Tutor-Marked Assignment (TMA). Second, there is a written examination. You are to use the information and the exercises in the course to solve the Tutor Marked Assignment (TMA).

Final Examination and Grading

The final examination for MAT 212 will be of three hours duration and have a value of 50% of the total course grade. The examination will consist of questions which reflect the self-assessment questions and Tutor Marked Assignment (TMA)s that you have previously encountered. Furthermore, All areas of the course will be assessed, so revise the entire course before the examinations. You might find it useful to review your TMAs and comment on them before the examination. The final examination covers information from all parts of the course.

Practical Strategies for Working Through This Course

1. Read this course guide carefully.
2. Decide when it is convenient for you to study, the time you are expected, to spend on each Unit, and submission dates for assignments.
3. Keep your chosen schedule time to avoid lagging behind in your studies.
4. Work through your units in a hierarchical order, as one Unit will lead to the next, for you to understand the whole concepts in the course.
5. Do and submit all assignments well before the prescribed deadline.
6. Commence the study of the next Unit as soon as you have finished the one before it. Endeavor to keep strictly to your schedule
7. On completing the course Units, review the course; check the Learning Outcome of the course guide, to prepare you for the final examinations.

Tutors and Tutorials

There are 20 hours of tutorials (10 x 2-hour session) provided in support of this course. You will be notified of the dates, times and location of these tutorials, together with the name and phone number of your tutor, as soon as you are allocated a tutorial group. Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter and provide assistance to you during the course.

You must mail your Tutor-Marked Assignment (TMA) to your tutor well before the due date (at least two working days). They will be marked by your tutor and returned to you as soon as possible. Do not hesitate to contact your tutor by telephone, e-mail, or discussion board if you need help.

The following might be circumstances in which you would find help necessary. Contact your tutor if:

- You do not understand any part of the study Units or the assigned readings.
- You have difficulty with the exercises or examples.
- You have a question or problem with an assignment, with your tutor's comment on an assignment or with the grading of an assignment.
- You should try as much as possible to attend the tutorials. This is the only chance to have a one on one encounter with your tutor and to ask questions which will be answered instantly.
- You can raise any problem encountered in the course of your study.

To gain the maximum benefit from course tutorials, prepare a question list before attending them. You will learn a lot from actively participating in discussions.

GOODLUCK!

Table of Contents

Module 1

Vector Space

Unit 1

Vector Space Over The Real Field

Contents

1.0 Introduction

Linear algebra is the study of two fundamental objects, vector spaces and linear transformations. In this unit you will learn about vector space and subspace, which will lead to an extra increment of abstraction. The power of mathematics is often derived from generalizing many different situations into one abstract formulation, and that is exactly what we will be doing right now.

2.0 Learning Outcome

At the end of this unit, you should be able to

1. Define vector space.
2. Show if a given set forms a vector space with respect to the two defined binary operations or not.

3.0 Learning Content

3.1 Definition of Vector Space

Suppose that **V** is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of **V** and (2) **scalar multiplication,** which combines a complex number with an element of **V**. Then **V**, along with the two operations, is a **vector space** if the following ten properties hold.

1. **AC Additive Closure**

   If $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, then $\mathbf{u} + \mathbf{v} \in \mathbf{V}$.

2. **SC Scalar Closure**

   If $k \in K$ and $\mathbf{u} \in \mathbf{V}$, then $k\mathbf{u} \in \mathbf{V}$.

3. **C Commutativity**

   If $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

4. **AA Additive Associativity**

   If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

5. **Z Zero Vector**

There is a vector, **0**, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{V}$

6. **AI Additive Inverses**

   If $\mathbf{u} \in \mathbf{V}$, then there exists a vector $-\mathbf{u} \in \mathbf{V}$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

7. **SMA Scalar Multiplication Associativity**

   If a, b $\in$ K and $\mathbf{u} \in \mathbf{V}$, then $a(b\mathbf{u}) = (ab)\mathbf{u}$.

8. **DVA Distributivity across Vector Addition**

   If a $\in$ K and $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, then $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

9. **DSA Distributivity across Scalar Addition**

   If a, b $\in$ K and $\mathbf{u} \in \mathbf{V}$, then $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

10. **O One**

   If $\mathbf{u} \in \mathbf{V}$, then $1\mathbf{u} = \mathbf{u}$.

The objects in **V** are called vectors, no matter what else they might really be, simply by virtue of being elements of a vector space.

3.2 Examples of vector Space

Example 1

The vector space of polynomials, P_n.

Set: P_n, the set of all polynomials of degree n or less in the variable x with coefficients

from K.

Equality:

$a_0+a_1x+a_2x^2+\ldots+a_nx^n = b_0+b_1x+b_2x^2+\cdot\cdot\cdot+b_nx^n$ if and only if $a_i = b_i$ for $0 \leq i \leq n$

Vector Addition:

$(a_0+a_1x+a_2x^2+\cdot\cdot\cdot+a_nx^n) + (b_0+b_1x+b_2x^2+\cdot\cdot\cdot+b_nx^n) =$
$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdot\cdot\cdot + (a_n + b_n)x^n$

Scalar Multiplication:

$k(a_0+a_1x+a_2x^2+\cdot\cdot\cdot+a_nx^n) = (ka_0) + (ka_1)x + (ka_2)x^2 + \cdot\cdot\cdot + (ka_n)x_n$

This set, with these operations, will fulfil the ten properties, though we will not work all the details here. However, we will make a few comments and prove some of the properties. First, the zero vector (Property **Z**]) is what you might expect, and you can check that it has the required property.

$$0 = 0 + 0x + 0x^2 + \cdots + 0x^n$$

The additive inverse (Property **AI**) is also no surprise, though consider how we have chosen to write it.

$$-(a_0+a_1x+a_2x^2+\cdots+a_nx^n) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n$$

Now let's prove the associativity of vector addition (Property **AA**). This is a bit tedious, though necessary. Throughout, the plus sign ("**+**") does triple-duty. You might ask yourself what each plus sign represents as you work through this proof.

$\mathbf{u}+(\mathbf{v} + \mathbf{w})$

$=(a_0+a_1x+a_2x^2+\cdots+a_nx^n)+((b_0+b_1x+b_2x^2+\ldots+b_nx^n)+(c_0 + c_1x + c_2x^2+\ldots+ c_nx^n))$

$= (a_0 + a_1x + \cdots + {}^{anxn}) + ((b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n)$

$= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + \cdots + (a_n + (b_n + c_n))x^n$

$= ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + \cdots + ((a_n + b_n) + c_n)x^n$

$= ((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n) + (c_0 + c_1x + \cdots + c_nx^n)$

$= ((a_0 + b_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)) + (c_0 + c_1x + \cdots + c_nx^n)$

$= (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

Notice how it is the application of the associativity of the (old) addition of complex numbers in the middle of this chain of equalities that makes the whole proof happen.

The remainder is successive applications of our (new) definition of vector (polynomial) addition. Proving the remainder of the ten properties is similar in style and tedium. You might try proving the commutativity of vector addition (Property **C**), or one of the distributive properties (Property **DVA**], Property **DSA**).

Example 2

The crazy vector space

Set: $\boldsymbol{C} = \{(x_1, x_2) \mid x_1, x_2 \in \boldsymbol{C}\}$.

Vector Addition: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$.

Scalar Multiplication: $a(x_1, x_2) = (ax_1 + a - 1, ax_2 + a - 1)$.

Now, the first thing I hear you say is "You can't do that!" And my response is, "Oh yes, I can!" I am free to define my set and my operations any way I please. They may not look natural, or even useful, but we will now verify that they provide us with another example of a vector space. And that is enough. If you are adventurous, you might try first checking some of the properties yourself. What is the zero vector? Additive inverses? Can you prove associativity? Ready, here we go.

Property **AC**, Property **SC**: The result of each operation is a pair of complex numbers, so these two closure propertiess are fulfilled

Property **C**:

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)\\ &= (y_1 + x_1 + 1, y_2 + x_2 + 1) = (y_1, y_2) + (x_1, x_2)\\ &= \mathbf{v} + \mathbf{u}\end{aligned}$$

Property **AA**:

$$\begin{aligned}&\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (x_1, x_2) + (y_1, y_2) + (z_1, z_2))\\ &= (x_1, x_2) + (y_1 + z_1 + 1, y_2 + z_2 + 1)\\ &= (x_1 + (y_1 + z_1 + 1) + 1, x_2 + (y_2 + z_2 + 1) + 1)\\ &= (x_1 + y_1 + z_1 + 2, x_2 + y_2 + z_2 + 2)\\ &= ((x_1 + y_1 + 1) + z_1 + 1, (x_2 + y_2 + 1) + z_2 + 1)\\ &= (x_1 + y_1 + 1, x_2 + y_2 + 1) + (z_1, z_2)\\ &= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2)\\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}\end{aligned}$$

Property **Z**: The zero vector is . . . $\mathbf{0} = (-1, -1)$. Now I hear you say, "No, no, that can't be, it must be (0, 0)!" Indulge me for a moment and let us check my proposal.

$\mathbf{u} + \mathbf{0} = (x_1, x_2) + (-1, -1) = (x_1 + (-1) + 1, x_2 + (-1) + 1) = (x_1, x_2) = u$

Feeling better? Or worse?

Property **AI**: For each vector, $\mathbf{u}$, we must locate an additive inverse, $-\mathbf{u}$. Here it is, $-(x_1, x_2) = (-x_1 - 2, -x_2 - 2)$. As odd as it may look, I hope you are withholding judgment. Check:

$\mathbf{u}+(-\mathbf{u}) = (x_1, x_2)+(-x_1-2, -x_2-2) = (x_1+(-x_1-2)+1, -x_2+(x_2-2)+1) = (-1, -1) = \mathbf{0}$

Property **SMA**:

$$
\begin{aligned}
a(b\mathbf{u}) &= a(b(x_1, x_2))\\
&= a(bx_1 + b - 1, bx_2 + b - 1))\\
&= (a(bx_1 + b - 1) +a - 1, a(bx_2 + b - 1) +a - 1))\\
&= ((abx_1 + ab - a) + a - 1, (abx_2 +ab -a) + a - 1))\\
&= (abx_1 + ab - 1, abx_2 + ab - 1))\\
&= (ab)(x_1, x_2)\\
&= (ab)\mathbf{u}
\end{aligned}
$$

Property **DVA** If you have hung on so far, here’s where it gets even wilder. In the next two properties we mix and mash the two operations.

$$
\begin{aligned}
a(\mathbf{u} + \mathbf{v}) &= a((x_1, x_2) + (y_1, y_2))\\
&= a(x_1 + y_1 + 1, x_2 + y_2 + 1)\\
&= (a(x_1 + y_1 + 1) + a - 1, a(x_2 + y_2 + 1) + a - 1)\\
&= (ax_1 + ay_1 + a + a - 1, ax_2 + ay_2 + a + a- 1)\\
&= (ax_1 + a - 1 + ay_1 + a - 1 + 1, ax_2 + a - 1 + ay_2 + a - 1 + 1)\\
&= ((ax_1 + a - 1) + (ay_1 + a - 1) + 1, (ax_2 + a - 1) + (ay_2 + a - 1) + 1)\\
&= (ax_1 + a - 1, ax_2 + a - 1) + (ay_1 + a - 1, ay_2 + a - 1)\\
&= a(x_1, x_2) + a(y_1, y_2)\\
&= a\mathbf{u} + a\mathbf{v}
\end{aligned}
$$

Property **DSA**:

$$
\begin{aligned}
(a+b)\mathbf{u} &= (a+b)(x_1, x_2)\\
&= ((a + b)x_1 + (a + b) - 1, (a+ b)x_2 + (a+ b) - 1)\\
&= (ax_1 +bx_1 + a + b - 1, ax_2 + bx_2 + a + b - 1)\\
&= (ax_1 + a - 1 + bx_1 + b - 1 + 1, ax_2 + a - 1 + bx_2 + b - 1 + 1)\\
&= ((ax_1 + a - 1) + (bx_1 + b - 1) + 1, (ax_2 + a - 1) + (bx_2 + b - 1) + 1)\\
&= (ax_1 + a - 1, ax_2 + a - 1) + (bx_1 + b - 1, bx_2 + b - 1)\\
&= a(x_1, x_2) + b(x_1, x_2)\\
&= a\mathbf{u} + b\mathbf{u}
\end{aligned}
$$

Property **O**: After all that, this one is easy, but no less pleasing.

$$1\mathbf{u} = 1(x_1, x_2) = (x_1 + 1 - 1, x_2 + 1 - 1) = (x_1, x_2) = \mathbf{u}$$

That's it, ***C*** is a vector space, as crazy as that may seem.

Notice that in the case of the zero vector and additive inverses, we only had to propose possibilities and then verify that they were the correct choices. You might try to discover how you would arrive at these choices, though you should understand why the process of discovering them is not a necessary component of the proof itself.

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

You have learnt in this unit the concept of vector spaces; effort was also made to explain the properties of vector space.

5.0 Summary

For a set **V** upon which two binary operations **(vector addition and scalar multiplication)** were defined to be called a vector space, the following ten properties must hold.

If **u, v,w** ∈ **V** , and a, b ∈ K then

1. **u** + **v** ∈ **V** .
2. k**u** ∈ **V**.
3. **u + v** = **v + u**.
4. **u + (v + w)** = **(u + v) + w**.
5. There is a vector, **0**, called the zero vector, such that **u** + **0** = **u** for all **u** ∈ **V** .
6. There exists a vector **−u** ∈ **V** so that **u** + (**−u**) = **0**.
7. a(b**u**) = (ab)**u**.
8. a(**u** + **v**) = a**u** + a**v**.
9. (a + b)**u** = a**u** + b**u**.

10. $1\mathbf{u} = \mathbf{u}$.

6.0 Tutor Marked Assignment (TMA)

Prove each of the ten properties of Definition of Vector Space for each of the following examples of a vector space:

1. The vector space of infinite sequences

   Set: $C^{\infty} = \{(c_0, c_1, c_2, c_3, \ldots) \,/\, c_i \in \boldsymbol{C}, i \in N\}$.

2. The vector space of functions

   Set: $F = \{f \,/\, f : \boldsymbol{C} \rightarrow \boldsymbol{C}\}$.

3. The singleton vector space

   Set: $Z = \{z\}$.

7.0 References/Further Reading

Odili,G. A. (2000): Algebra for Colleges and Universities: An Integral Approach. Anachuna Educational Books. ISBN978-2897-37-x.

Robert, A. Beezer (2006): A First Course in Linear Algebra. http://linear.ups.edu/.

Unit 2

Subspace

Content

1.0 Introduction

Certain applications involve the use of subsets of vector spaces which are vector space also. It will be convenient to have a name for such a subset. In this unit you will learn about **subspace** of a vector space.

2.0 Learning Outcome

At the end of this unit you should be able to

1. Define subspace
2. Show that a given set is a subspace of a vector space or not.

3.0 Subspace

3.1 Definition Subspace

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V , W $\subseteq$ V . Then W is a subspace of V.

Let's look at an example of a vector space inside another vector space.

3.2 Examples of Subspace

Example 1

A subspace of C^3

We know that C^3 is a vector space. Consider the subset,

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} / 2x_1 - 5x_2 + 7x_3 = 0 \right\}$$

It is clear that W $\subseteq$ $\mathbf{C^3}$, since the objects in ***W*** are column vectors of size 3. But is W a vector space? Does it satisfy the ten properties of Definition **VS** when we use the same operations? That is the main question.

Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors from W. Then we know that these vectors cannot be totally arbitrary, they must have gained membership in W by virtue of meeting the membership test. For example, we know that x must satisfy $2x_1-5x_2+7x_3 = 0$ while y must satisfy $2y_1-5y_2+7y_3 = 0$.

Our first property (Property **AC**) asks the question, is $x + y \in W$? When our set of vectors was $\mathbf{C^3}$, this was an easy question to answer. Now it is not so obvious. Notice first that

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and we can test this vector for membership in W as follows,

$2(x_1 + y_1) - 5(x_2 + y_2) + 7(x_3 + y_3) = 2x_1 + 2y_1 - 5x_2 - 5y_2 + 7x_3 + 7y_3$

$= (2x_1 - 5x_2 + 7x_3) + (2y_1 - 5y_2 + 7y_3)$

$= 0 + 0$ $\qquad\qquad x \in W, y \in W$

$= 0$

and by this computation we see that $\mathbf{x} + \mathbf{y} \in W$. One property down, nine to go.

If k is a scalar and $\mathbf{x} \in W$, is it always true that $k\mathbf{x} \in W$? This is what we need to establish Property **SC**. Again, the answer is not as obvious as it was when our set of vectors was all of $\boldsymbol{C^3}$. Let's see.

$$kx = k\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}$$

and we can test this vector for membership in W with

$2(kx_1) - 5(kx_2) + 7(kx_3) = k\,(2x_1 - 5x_2 + 7x_3)$

$= k0$ $\qquad\qquad \mathbf{x} \in \boldsymbol{W}$

$= 0$

and we see that indeed $k\mathbf{x} \in W$. Always.

If W has a zero vector, it will be unique. The zero vector for $\boldsymbol{C^3}$ should also perform the required duties when added to elements of ***W***. So the likely candidate for a zero vector in ***W*** is the same zero vector that we know $\boldsymbol{C^3}$ has.

You can check that $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a zero vector in W too (Property **Z**).

With a zero vector, we can now ask about additive inverses (Property **AI**). As you might suspect, the natural candidate for an additive inverse in ***W*** is the same as the additive inverse from $\boldsymbol{C^3}$. However, we must insure that these additive inverses actually are elements of ***W***. Given **x** $\in$ W, is **−x** $\in$ W?

$$-x = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}$$

and we can test this vector for membership in ***W*** with

$$\begin{aligned} 2(-x_1) - 5(-x_2) + 7(-x_3) &= -(2x_1 - 5x_2 + 7x_3) \\ &= -0 \qquad\qquad \mathbf{x} \in \boldsymbol{W} \\ &= 0 \end{aligned}$$

and we now believe that **−x** $\in$ ***W***.

Is the vector addition in ***W*** commutative (Property ***C***)? Is **x + y = y + x**? Of course! Nothing about restricting the scope of our set of vectors will prevent the operation from still being commutative. Indeed, the remaining five properties are unaffected by the transition to a smaller set of vectors, and so remain true. That was convenient.

So W satisfies all ten properties, hence ***W*** is therefore a vector space, and thus earns the title of being a subspace of $\boldsymbol{C^3}$.

Self-Assessment Exercise(s)

Self-Assessment Answer

3.2 Testing Subspaces

In the last example, we proceeded through all ten of the vector space properties before believing that a subset was a subspace. But six of the properties were easy to prove, and we can lean on some of the properties of the vector space (the superset) to make the other four easier. Here is a theorem that will make it easier to test if a subset is a vector space. A shortcut if there ever was one.

Theorem 1

Testing Subsets for Subspaces

Suppose that **V** is a vector space and **W** is a subset of **V**, **W** $\subseteq$ **V**. Endow **W** with the same operations as **V**. Then **W** is a subspace if and only if three conditions are met

1. **W** is non-empty, **W** $\neq \phi$.
2. If **x** $\in$ **W** and **y** $\in$ **W**, then **x + y** $\in$ **W**.
3. If ***k*** $\in$ **C** and **x** $\in$ **W**, then ***k*** x $\in$ W.

Proof ($\Rightarrow$) We have the hypothesis that **W** is a subspace, so by Definition of vector space we know that **W** contains a zero vector. This is enough to show that **W** $\neq \phi$. Also, since **W** is a vector space it satisfies the additive and scalar multiplication closure properties, and so exactly meets the second and third conditions. If that was easy, then the other direction might require a bit more work.

($\Leftarrow$) We have three properties for our hypothesis, and from this we should conclude that **W** has the ten defining properties of a vector space. The second and third conditions of our hypothesis are exactly Property **AC** and Property **SC**. Our hypothesis that **V** is a vector space implies that Property **C**, Property **AA**, Property **SMA**, Property **DVA**, Property **DSA** and Property **O** all hold. They continue to be true for vectors from **W** since passing to a subset, and keeping the operation the same, leaves their statements unchanged. Eight down, two to go.

Suppose $\mathbf{x} \in \mathbf{W}$. Then by the third part of our hypothesis (scalar closure), we know that $(-1)\mathbf{x} \in \mathbf{W}$. But $(-1)\mathbf{x} = -\mathbf{x}$, so together these statements show us that $-\mathbf{x} \in \mathbf{W}$. $-\mathbf{x}$ is the additive inverse of $\mathbf{x}$ in $\mathbf{V}$, but will continue in this role when viewed as element of the subset **W**. So every element of **W** has an additive inverse that is an element of **W** and Property **AI** is completed. Just one property left.

While we have implicitly discussed the zero vector in the previous paragraph, we need to be certain that the zero vector (of **V**) really lives in **W**. Since **W** is non-empty, we can choose some vector $\mathbf{z} \in \mathbf{W}$. Then by the argument in the previous paragraph, we know $-\mathbf{z} \in \mathbf{W}$. Now by Property **AI** for **V** and then by the second part of our hypothesis (additive closure) we see that $\mathbf{0} = \mathbf{z} + (-\mathbf{z}) \in \mathbf{W}$

So **W** contain the zero vector from **V** . Since this vector performs the required duties of a zero vector in **V** , it will continue in that role as an element of **W**. This gives us, Property **Z**, the final property of the ten required.

Three conditions, plus being a subset of a known vector space, gets us all ten properties. Fabulous!

This theorem can be paraphrased by saying that a subspace is "a non-empty subset (of a vector space) that is closed under vector addition and scalar multiplication."

Example 2

A subspace of P4

P_4 is the vector space of polynomials with degree at most 4. Define a subset W as

$$W = \{p(x) \mid p \in P_4,\ p(2) = 0\}$$

so W is the collection of those polynomials (with degree 4 or less) whose graphs cross the x-axis at $x = 2$. Whenever we encounter a new set it is a good idea to gain a better understanding of the set by finding a few elements in the set, and a few outside it. For example $x^2 - x - 2 \in W$, while $x^4 + x^3 - 7 \notin W$.

Is W nonempty? Yes, $x - 2 \in W$.

Additive closure? Suppose $p \in \boldsymbol{W}$ and $q \in \boldsymbol{W}$. Is $p + q \in \boldsymbol{W}$? p and q are not totally arbitrary, we know that $p(2) = 0$ and $q(2) = 0$. Then we can check $p+q$ for membership in W,

$$
\begin{aligned}
(p + q)(2) &= p(2) + q(2) && \text{Addition in } P_4 \\
&= 0 + 0 && p \in W,\ q \in W \\
&= 0
\end{aligned}
$$

so we see that p + q qualifies for membership in W.

Scalar multiplication closure? Suppose that $k \in \boldsymbol{C}$ and $p \in \boldsymbol{W}$. Then we know that $p(2) = 0$. Testing _p for membership,

$$
\begin{aligned}
(kp)(2) &= kp(2) && \text{Scalar multiplication in } P_4 \\
&= k0 && p \in \boldsymbol{W} \\
&= 0
\end{aligned}
$$

so $kp \in \boldsymbol{W}$.

We have shown that ***W*** meets the three conditions of **Theorem TSS** and so qualifies as a subspace of P_4. Notice that by Definition of Subspaces we now know that ***W*** is also a vector space. So all the properties of a vector space apply in full.

Much of the power of **Theorem TSS** is that we can easily establish new vector spaces if we can locate them as subsets of other vector spaces.

It can be as instructive to consider some subsets that are not subspaces. Since **Theorem TSS** is an equivalence. We can be assured that a subset is not a subspace if it violates one of the three conditions, and in any example of interest this will not be the "non-empty" condition. However, since a subspace has to be a vector space in its own right, we can also search for a violation of any one of the ten defining properties in Definition of Vector Space or any inherent property of a vector space, Notice also that a violation need only be for a specific vector or pair of vectors.

Example 3

A non-subspace in C^2, zero vector

Consider the subset ***W*** below as a candidate for being a subspace of $\boldsymbol{C}^2$

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} / 3x_1 - 5x_2 = 12 \right\}$$

The zero vector of ***C*²**, $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ will need to be the zero vector in ***W*** also. However, $0 \notin$ ***W*** since 3(0) − 5(0) = 0 $\neq 12$. So ***W*** has no zero vector and fails Property **Z** of Definition of Vector Space. This subspace also fails to be closed under addition and scalar multiplication. Can you find examples of this?

Example 4

A non-subspace in C², additive closure

Consider the subset ***X*** below as a candidate for being a subspace of ***C*²**

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} / x_1 x_2 = 0 \right\}$$

You can check that $0 \in$ ***X***, so the approach of the last example will not get us anywhere.

However, notice that

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in X \text{ and } y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in X \text{ . Yet}$$

$$x + y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin X$$

So ***X*** fails the additive closure requirement of either Property **AC** or **Theorem TSS**, and is therefore not a subspace.

Example 5

A non-subspace in *C*², scalar multiplication closure

Consider the subset ***Y*** below as a candidate for being a subspace of **C²**

$$Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} / x_1 \in Z, x_2 \in Z \right\}$$

Z is the set of integers, so we are only allowing "whole numbers" as the constituents of our vectors. Now, 0 $\in$ **Y** , and additive closure also holds (can you prove these claims?).

So we will have to try something different. Note that $k = \frac{1}{2} \in C$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in Y$ but

$$kx = \frac{1}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} \notin Y$$

So ***Y*** fails the scalar multiplication closure requirement of either Property **SC** or **Theorem TSS**

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

You have learnt in this unit subspaces with examples and tests for subspace of a given vector space.

5.0 Summary

Suppose that **V** is a vector space and **W** is a subset of **V** , **W** $\subseteq$ **V** . Endow **W** with the same operations as **V** . Then **W** is a subspace if and only if three conditions are met

1. **W** is non-empty, **W** $\neq \phi$.

2. If **x** $\in$ **W** and **y** $\in$ **W**, then **x + y** $\in$ **W**.

3. If ***k*** $\in$ **C** and **x** $\in$ **W**, then ***k*** x $\in$ W.

6.0 Tutor Marked Assignment (TMA)

1. What is a subspace?

7.0 References/Further Reading

Odili,G. A. (2000): Algebra for Colleges and Universities: An Integral Approach. Anachuna Educational Books. ISBN978-2897-37-x.

Robert, A. Beezer (2006): A First Course in Linear Algebra. http://linear.ups.edu/.

Unit 3

Linear Combination and Spanning Sets

Contents

1.0 Introduction

In this unit you will learn that each vector space V studied here has a finite number of vectors that completely describes *V.*

2.0 Learning Outcome

At the end of this unit, you should be able to:

1. Define Linear combination.
2. Write a given vector as a linear combination of other vectors.
3. Define Spanning sets
4. Show that a given set of vectors spans a given vector space

3.0 Learning Content

3.1 Definition of Linear Combination

Let V be a vector space over the field *F* and let $v_1, v_2, \ldots, v_n \in \boldsymbol{V}$ Then if $a_1, a_2, \ldots, a_n \in \boldsymbol{F}$, the vector

$$\boldsymbol{V}= a_1v_1+ a_2v_2+ \ldots + a_nv_n$$

is called a **linear combination over F of** $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ or simply a linear combination of

$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$

3. 2 Examples of linear combination

Example 1.

Write (2, 1, 5) in $\boldsymbol{E^3}$ as a linear combination of(l, 2,1), (1,0, 2), and (1,1,0):

Solution

We want to find a_1, a_2, a_3 so that

$(2, 1, 5) = a_1(l, 2, 1) + a_2(1, 0, 2) + a_3(l, 1,0)$

or

$(2, 1, 5)' = (a_1 + a_2 + 0_3, 2a_1 + a_3, a_1 + 2a_2)$

which yields equations

$a_1 + a_2 + a_3 = 2$

$2a_1 \quad + a_3 = 1$

$a_1 + 2a_2 \quad = 5$

Solving these equations gives

$a_1 = l$, $a_2 = 2$, $a_3 = -l$, so

(2,1,5) = (1,2,1) + 2(1,0,2) -(1,1,0)

Example 2.

Can (3, -1, 4) be written as a linear combination of (1, -1, 0), (0,1,1) and (3, -5, -2)?

Solution

We check to see whether the equation (3, -1, 4) = a_1(l, -1, 0) + a_2(0, 1, 1) + a_3(3, -5, -2) has a solution. This is equivalent to

$a_1 \quad + 3a_3 = 3$

$-a_1 + a_2 - 5a_3 = -1$

$\quad a_2 - 2a_3 = 4$

In reduced form this is

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

and there is no solution. Hence (3, -1, 4) cannot be written as a linear combination of the given vectors

Self-Assessment Exercise(s)

Self-Assessment Answer

3.3 Definition of Spanning Set.

Let S = {v_1, v_2, ..., v_n} be a set of vectors in vector space ***V***. The set S **spans *V*, or *V* is spanned** by *S,* if every vector in ***V*** is a linear combination of the vectors in S.

3.4 Examples of Spanning Sets.

Example 1.

Let ***V*** be the vector space E^3. Show that {(1, 2,1), (1,0, 2), (1,1,0)} spans E^3.

Solution

We must show that any vector (a, *b, c)* in E^3 can be written as a linear combination of the three given vectors. That is, we must show that there are numbers a_1, a_2, a_3 so that

$$(a, b, c) = a_1(1, 2,1) + a_2 (1,0, 2) + a_3(1,1,0),$$

regardless of what real values *a, b,* and *c* take.

Equivalently, we have

$$a_1 + a_2 + a_3 = a$$

$$2a_1 \qquad + a_3 = b$$

$$a_1 + 2a_2 \qquad = c$$

which has solutions

$$a_1 = \frac{-2a + 2b + c}{3}, \quad a_2 = \frac{a - b + c}{3}, \quad a_3 = \frac{4a - b - 2c}{3}$$

Thus {(1,2, 1), (1,0,2), (1,1,0)} spans E^3

Example 2.

Let ***V*** be $\boldsymbol{P_2}$, the vector space consisting of polynomials of degree $\leq$ 2 and the zero polynomial.

Let $v_1 = t^2 + 2t + 1$ and $v_2 = t^2 + 2$.

Determine whether {v_1, v_2} span ***V***?

Solution

Again let $\boldsymbol{v} = at^2 + bt + c$ be any vector in $\boldsymbol{V}$, where a, b, and c are any real numbers. We must find a_1 and a_2 so that

$$at^2 + bt + c = a_1(t^2 + 2t + 1) + a_2(t2 + 2)$$

$$= (a, + a_2)t^2 + (2a_2)t + (a, + 2a_2)$$

Now two polynomials agree for all values of t only if the coefficients of respective powers of t agree. Thus we get the equations

$$a_1 \quad + \quad a_2 \quad = a$$

$$2a_1 \qquad\quad =b$$

$$A_1 + 2a_2 \quad = c$$

Putting the augmented matrix into reduced row echelon form we have

$$\left(\begin{array}{cc|c} 1 & 0 & 2a - c \\ 0 & 1 & c - a \\ 0 & 0 & b - 4a + 2c \end{array}\right)$$

Therefore, a solution exists only if $b - 4a + 2c = 0$'. but this places a restriction on *(a, b, c* and so me very first equation cannot be solved for an arbitrary vector *(a, b, c).* Therefore $\{v_{\}}\ v_2\}$ does not span $\boldsymbol{V}$.

Example 3

Let v, = $t^2 + 2t + 1$ and $v_2 = t^2 + 2$. Describe Span $\boldsymbol{S}$ where $\boldsymbol{S} = \{v_1, v_2\}$

Solution

Suppose (a, *b, c)* is span $\boldsymbol{S}$. Then the equation

$(a, b, c) = a_1(t^2 + 2t + 1) + a_2(t^2 + 2)$ must be solvable. Working as in **the previous example**, we conclude that $b - 4a + 2c = 0$. Thus

$$S = \left\{ (a,b,c)/a = \frac{b}{4} + \frac{c}{2} \right\}.$$

span of $\boldsymbol{S}$ is all vectors whose first component is the sum of $\frac{1}{4}$ of second component and $\frac{1}{2}$ of third component. So, for example, $(5, 4, 9) \notin$ span $\boldsymbol{S}$ and $(5, 4, 8) \in$ span $\boldsymbol{S}$.

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

At the end of this unit, you learnt how to define Linear Combination, write a given vector as a linear combination of other vectors, define spanning sets and how to show that a given set of vectors spans a given vector space.

5.0 Summary

Let $v_1, v_2, \ldots, v_n \in \boldsymbol{V}$ Then if $a_1, a_2, \ldots, a_n \in \boldsymbol{F}$, the vector

$\boldsymbol{V} = a_1v_1 + a_2v_2 + \ldots + a_nv_n$ is called a **linear combination over F of** $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$

The set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ $\boldsymbol{V}$ **spans** $\boldsymbol{V}$**, or** $\boldsymbol{V}$ **is spanned** by *S,* if every vector in $\boldsymbol{V}$ is a linear combination of the vectors in S.

6.0 Tutor Marked Assignment (TMA)

1. What is Linear Combination?

7.0 References/Further Reading

Odili,G. A. (2000): Algebra for Colleges and Universities: An Integral Approach. Anachuna Educational Books. ISBN978-2897-37-x.

Robert, A. Beezer (2006): A First Course in Linear Algebra. http://linear.ups.edu/.

Unit 4

Linear Dependence and Independence

Contents

1.0 Introduction

Suppose that $\boldsymbol{v}$ is a vector in a given vector space $\boldsymbol{V}$ and that A is a subspace of $\boldsymbol{V}$. It is natural to ask, is $\boldsymbol{v}$ an element of A? This is equivalent to:

If A is the subspace spanned by $\boldsymbol{S}$, then is $\boldsymbol{v}$ a linear combination of a finite subset of $\boldsymbol{S}$? In order to develop methods to answer such a question, we need the concept of linear dependence.

2.0 Learning Outcome

At the end of this unit, you should be able to

1. Define Linear Dependence and Independence
2. Determine whether vector space is linearly independent or not.

3.0 Learning Content

3.1 Definition of Linear Dependence and Independence

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a nonempty set of (distinct) vectors of the vector space $\boldsymbol{V}$. We say that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a **linearly dependent set** if there are scalars $a_1 \ldots, a_n \in \boldsymbol{F}$ and not all equal to the zero of $\boldsymbol{F}$ such that

$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n = \phi$ the zero vector. Otherwise $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is called **linearly independent.**

To determine whether a set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is linearly independent or linearly dependent, we need to find out about the solution of

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n = \phi$$

If we find (by actually solving the resulting system or by any other technique) that only the trivial solution $a_1 = a_2 = \ldots = a_n = 0$ exists, the $\boldsymbol{S}$ is linearly independent. However, if one or more of the a_n is non zero, then the set $\boldsymbol{S}$ is linearly dependent.

3.2 Examples of Linear Dependence and Independence

Example 1

Determine whether S = {(1, 0), (0, 1), (1, -1)} is linearly independent.

Solution

Consider

$$a_1(1, 0) + a_2(0, 1)+a_3(1, -1) = \phi = (0,0)$$

Which is equivalent to

$$a_1 + a_3 = 0$$

$$a_2 - a_3 = 0$$

This system has solution $a_3 = k$, $a_1 = -k$, $a_2 = k$, if $k \neq 0$, then we have a non-trivial solution, and so ***S*** is not linearly independent - it is linearly dependent.

Example 2

Let S ={(1,0,1), (0,1,2), (-2,1,1)} be a set of vectors in $\boldsymbol{E}^3$. Is ***S*** linearly independent?

Solution

Consider

$a_1(l,0,l) + a_2(0,1,2) + a_3(-2,l,l) = \theta = (0,0,0)$ which is equivalent to

$$a_1 \quad -2a_3 = 0$$

$$a_2 + a_3 = 0$$

$$a_1 + 2a_2 + a_3 = 0$$

We can easily solve this system by eliminating one variable at a time. Thus subtracting (1) from (3), we get $2a_2 + 3a_3 = 0$, and subtracting twice (2) from this, we get $a_3 = 0$. Substituting into (1) and (2), this gives $a_1 = 0$ and $a_2 = 0$. Hence since $a_1 = a_2 = a_3 = 0$, ***S*** is linearly independent.

Self-Assessment Exercise(s)

Self-Assessment Answer

3.3 Theorem L1

Let $\boldsymbol{S} = \{\boldsymbol{v_1, v_2, \ldots, v_n}\}$ be a set of at least two vectors $(n \geq 2)$ in a vector space $\boldsymbol{V}$. Then $\boldsymbol{S}$ is linearly dependent if and only if one of the vectors in $\boldsymbol{S}$ can be written as a linear combination of the rest.

Proof

If $\boldsymbol{S}$ is linearly dependent, then there are constants $a_1, a_2, \ldots, a_n$ some of which are non zero, such that

$\boldsymbol{a_1v_1 + a_2v_2 + \ldots + a_n v_n} = \phi$ Suppose $\boldsymbol{a_k}(\boldsymbol{l} \leq \boldsymbol{k} \leq \boldsymbol{n})$ is a non zero coefficient in the linear ombination. Then

$$\boldsymbol{a_kv_k = - a_1v_1 - a_2v_2 - \ldots - a_{k-1} v_{k-1} - a_{k+1} v_{k+1} - \ldots - a_n v_n}$$

and since $\boldsymbol{a_k \neq 0}$

$$\mathrm{v}_k = -\frac{\mathrm{a}_1}{a_k}\mathrm{v}_1 - \frac{\mathrm{a}_2}{a_k}\mathrm{v}_2 - \ldots - \frac{\mathrm{a}_{k-1}}{a_k}\mathrm{v}_{k-1} - \frac{\mathrm{a}_{k+1}}{a_k}\mathrm{v}_{k+1} - \ldots - \frac{\mathrm{a}_n}{a_k}\mathrm{v}_n$$

Therefore, $\boldsymbol{v_k}$ is a linear combination of the othet vectors in $\boldsymbol{S}$.

Conversely, suppose that vk is a linear combination of the other vectors of $\boldsymbol{S}$

In particular, let

$$\boldsymbol{v_k = - d_1v_1 - d_2v_2 - \ldots - d_{k-1} v_{k-1} - d_{k+1} v_{k+1} - \ldots - d_n v_n}$$

Then, adding (-l)$\boldsymbol{v_k}$ to both sides, we have

$$\phi \boldsymbol{= - d_1v_1 + \ldots + (-1)\, v_k + \ldots + d_n v_n}$$

Because the coefficient of $\boldsymbol{v_k}$ is non zero, the set $\boldsymbol{S}$ is linearly dependent.

3.4 Example

Show that

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 6 & 9 \end{pmatrix} \right\}$$

is linearly dependent in $\boldsymbol{M_{22}}$. Write one of the matrices as a linear combination of the others.

Solution

Consider

$$a_1\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}+a_2\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}+a_3\begin{pmatrix}-1 & 2\\ 6 & 9\end{pmatrix}=\phi=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

This equation is equivalent to

$$\begin{array}{llll} a_1 & -a_2 & -a_3 & =0 \\ a_1 & & +2a_3 & =0 \\ 3a_1 & +a_2 & +9a_3 & =0 \end{array}$$

which reduces to

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

Therefore, $a_1 = -2k$, $a_2 = -3k$, $a_3 = k$ is a solution, where k is arbitrary. Thus the set S is linearly dependent. Choosing ***k=1*** we have

$$-2\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}-3\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}+\begin{pmatrix}-1 & 2\\ 6 & 9\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

We can write

$$\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}=\frac{-3}{2}\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1 & 2\\ 6 & 9\end{pmatrix}$$

$$\begin{pmatrix}-1 & 2\\ 6 & 9\end{pmatrix}=2\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}+3\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}=-\frac{2}{3}\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-1 & 2\\ 6 & 9\end{pmatrix}$$

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

At the end of this unit, you have learnt how to define a Linear Dependence and Independence and also how to determine whether vector space is linearly independent or not.

5.0 Summary

A nonempty set of (distinct) vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$of the vector space $\boldsymbol{V}$ is a **linearly dependent set** if there are scalars $a_1 \ldots, a_n \in \boldsymbol{F}$ and not all equal to the zero of $\boldsymbol{F}$ such that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n = \phi$ the zero vector. Otherwise $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is called **linearly independent.**

To determine whether a set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is linearly independent or linearly dependent, we need to find out about the solution of $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n = \phi$

If we find that only the trivial solution $a_1 = a_2 = \ldots = a_n = 0$ exists, the $\boldsymbol{S}$ is linearly independent. However, if one or more of the a_n is non zero, then the set $\boldsymbol{S}$ is linearly dependent.

The set $\boldsymbol{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, $n \geq 2$ in a vector space $\boldsymbol{V}$ is linearly dependent if and only if one of the vectors in $\boldsymbol{S}$ can be written as a linear combination of the rest.

6.0 Tutor Marked Assignment (TMA)

7.0 References/Further Reading

Odili,G. A. (2000): Algebra for Colleges and Universities: An Integral Approach. Anachuna Educational Books. ISBN978-2897-37-x.

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Unit 5

Bases and Dimension

Contents

1.0 Introduction

You have seen in the preceding Units that linearly independent sets of vectors often play a special role in describing vector spaces. In this Unit, you will study sets of vectors that play a role in an arbitrary vector space *V* similar to that of the set $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ in E^n. If a set ***S*** of vectors spans ***V*** and ***S*** is linearly dependent, then representation of a vector ***x*** in terms of vectors in ***S*** is not unique. If we want uniqueness, the spanning set must also be linearly independent. Such a set is called a **basis** for *V.* Bases are very useful in coding theory.

2.0. Learning Outcome

At the end of this unit you should be able to

1. Define Basis for a vector space
2. Construct a Basis for ***V*** by choosing vectors from ***V***
3. Given a Set ***S*** of vectors in ***V***, Construct a Basis for ***V*** by enlarging or deleting some (but not all) vectors from ***S***.
4. Show if a vector is a Basis for a given vector space or not.
5. Define dimension of vector space
6. Determine the dimension of any given vector space.

3.0 Learning Content

3.1 Definition of Basis

Let *V* be a vector space over the field *F.* A **basis** for ***V*** is a subset ***B*** of ***V*** such that:

(a) ***B*** is a linearly independent set and

(b) ***B*** spans ***V.***

3.2 Examples of Bases

Example 1

Show that $\boldsymbol{E^n}$ has the basis $\boldsymbol{E} = \{\boldsymbol{e_1}\ \boldsymbol{e_2}, \dots, \boldsymbol{e_n})$, where $e_j = (0,0,\dots,0,1,0,\dots,0)$

Where 1 is the j^{th} component

Solution

For the span consider any $\boldsymbol{x} = (\boldsymbol{x_1, x_2, \ldots, x_n}) \in \boldsymbol{E^n}$ and note that

$$\boldsymbol{x = x_1 (1,0,\ldots 0,) + x_2(0,1,0\ldots 0) + \ldots + x_n(0,\ldots,0,1)}$$
$$\boldsymbol{= x_1 e_1 + x_2 e_{2}, + \ldots + x_n e_n}$$

For linear independence consider

$a_1e_1 + a_2 e_2 + \ldots + a_n e_n = \phi = (0,0, \ldots, 0)$

So that $a_1 = a_2 = \ldots = a_n = 0$. Therefore, $\boldsymbol{E}$ is a basis for $\boldsymbol{E^n}$.

The vector e_1, e_2, ..., e_n are called the **usual or standard** basis for $\boldsymbol{E}^n$.

Self-Assessment Exercise(s)

Self-Assessment Answer

Example 2

Show that the set $S = \{t^2 + 1, t-1, 2t + 2\}$ is a basis for the vector space P_2.

Solution

To show this we must show that S span P_2 and V is linearly independent. To show that S spans P_2

we take any vector in P_2, that is, a polynomial $at^2 + bt + c$, and wish to find a_1, a_2 and a_3 so that

$$at^2 + bt + c = a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2)$$
$$= \quad a_1t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3)$$

Since two polynomials agree for all values of t only if the coefficients of respective powers of t agree, we get

$$\begin{aligned} a_1 \qquad\qquad\qquad &= a \\ a_2 + 2a_3 &= b \\ a_1 - a_2 + 2a_3 &= c \end{aligned}$$

Then

$$a_1 = a, a_2 \frac{a+b-c}{2}, a_3 = \frac{c+b-a}{4}$$

Hence, S spans P_2. To show that S is linearly independent, we form,

$a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2) \quad = 0$

$\Rightarrow a_1t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3) = 0$

Again, this can hold for all values of ***t*** only if $a_1 = 0$, $a_2 + 2a_3 = 0$ and $a_1 - a_2 + 2a_3 = 0$. We get $a_1 = a_2 = a_3 = 0$, which implies that S is linearly independent.

Example 3

The set

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$ is a basis for the vector space $\boldsymbol{M_{22}}$

Solution

To verify that S is linearly independent, we form

$$a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Which implies that

$a_1 = a_2 = a_3 = a_4 = 0$, Hence S is linearly independent.

To verify that S spans $\boldsymbol{M}_{22}$, we take any vector

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and we must find scalars a_1, a_2, a_3, a_4 such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we find that $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$, so that S spans $\boldsymbol{M}_{22}$

In the last Example, the coefficients in the linear combination of basis elements were unique for any given vector,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ This is true in general.

Theorem 1.

Let S ={v_1, ..., v_n} be a basis for a vector space *V.* Let v be in ***V***. The coefficients in the representation v = a_1v_1 + ... + a_nv_n are unique.

Proof

Suppose we have two representations

$\boldsymbol{v = a_1v_1 + \dots + a_nv_n}$

$\boldsymbol{v = b_1v_1 + \dots + b_nv_n}$

for ***v***, we will show that the coefficients are actually equal. To do this form ***v + (-v)***, which equals ϕ and combine terms to obtain.

ϕ $\boldsymbol{= (a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n}$

Since S is a basis, it is a linearly independent set. Thus the coefficients in the last linear combination must all be zero. That is, $a_1 = b_1, \dots, a_n = b_n$ and the original linear combinations are the same.

Example 4

Show that the sets *S*= {(1,2), (3,-l), (1,0)} is not a basis for E^2.

Solution

The set S is linearly dependent because, for example,

$$(1,2) + 2(3,-1) - 7(1,0) = (0,0)$$

So *S* cannot be a basis for E^2

Self-Assessment Exercise(s)

Self-Assessment Answer

Definition of Minimal set of generators

Let ***V*** be a vector space over ***F.*** A **Minimal set of generators** is a set S of vectors such that

(a) $V = \langle S \rangle$

(b) If $T \subset S$, $\langle T \rangle$ is a proper subspace of V.

Theorem 2

Let ***V*** be a vector space over the field ***F***. Let S be a minimal set of generators for V. Then S is a basis for V.

Proof

Clearly, $V = \langle S \rangle$, so we need only prove that S is linearly independent.

Suppose S is linearly dependent, then there exists a nonempty subset $T = \{V_1, ..., V_r\}$ of S such that T is linearly dependent. By **Theorem 16.4** there is a $V_i \in T$ such that V_i is a linear combination of the elements of $T - \{V_i\}$.

Thus $S - \{V_i\}$ spans the same vector space as ***S***, This contradicts that S is a minimal set of, generators, and so S is linearly independent. Thus ***S*** is a basis for V.

Example 5

Let V = E^3, is the set S = {(1,0,0), (0,1,0), (0,0,1),(1,-1,1)} minimal for V.

Solution

Clearly S as a generating set for V is not minimal, since S - {(1,-1,1)} also generates E^3. The subset B = {(1,0,0), (0,1,0), (0,0,1)} is clearly a minimal set of generators for E^3 and is a basis for E^3 since its three elements are the standard unit vectors of E^3.

Self-Assessment Exercise(s)

Self-Assessment Answer

3.4 Definition of maximal linearly independent subset of V

Let $\boldsymbol{V}$ be a vector space over the field $\boldsymbol{F}$. A subset S of V is called a **maximal linearly independent** subset of V if

(a) $\boldsymbol{S}$ is linearly independent, and

(b) If $\boldsymbol{S} \subset$ T, then T is linearly dependent.

Note that the three notions of basis, minimal set of generators, and maximal linearly independent set are all equivalent.

Theorem 3

Let $\boldsymbol{V}$ be a vector space over F with a basis $\boldsymbol{B}$. If $\boldsymbol{B}$ has n elements, then every basis of $\boldsymbol{V}$ has n elements.

Before we can prove this theorem, we need the following lemma.

Lemma 1

Let A = $\{v_1, \ldots, v_n\}$ be a linearly independent set of n vectors of the vector space $\boldsymbol{V}$ over $\boldsymbol{F}$. Then any set B of linearly independent vectors in the subspace <A>

spanned by A has at most n elements.

Proof

Let B = $\{u_1, \ldots, u_r\}$ be a linearly independent set of vectors in the subspace <A>. Then u_i is dependent on the set A; hence by **Theorem 16.4** the set $\{v_1 \ldots, v_n, u_1\}$ is a linearly dependent set. Hence there are field elements $a_1, \ldots, a_{1+n}$, not all zero, such that

$$a_1v_1 + \ldots + a_nv + a_{n+1}u_1 = 0$$

Now if all of $a_1 \ldots, a_n$ are zero, then $a_{n+1} \neq 0$ and therefore $u_1 = 0$, a contradiction. Thus, at least one of $a_1 \ldots, a_n$ is not zero. Hence assume by reindexing if necessary, that $a_1 \neq 0$. Thus v_1 is dependent on $A_1 = \{v_2, v_3, \ldots, v_n, u_1\}$, and A_1 spans the same space as A.

Next, we see that u_2 is in the space **$<A_1>$** spanned by A_1, and so $\{v_2, \ldots, v_n, u_1, u_2\}$ is a linearly dependent set. As before there are field elements $b_1, \ldots, b_{n+1}$ not all zero, such that

$$b_1v_2 + \ldots + b_{n-1}v_n + b_nu_1 + b_{n+1}u_2 = 0$$

Again, if $b_1 = \ldots = b_{n-1} = 0$, then $b_nu_1 + b_{n+1}u_2 = 0$ and at least one of the b_n or b_{n+1} is not zero. This contradicts that the subset $\{u_1, u_2\}$ of B is a linearly independent set. Thus, we may assume, by reindexing if necessary, that $b_1 \neq 0$, and so v_2 is dependent on $A_2 = \{v_3, \ldots, v_n, u_1, u_2\}$,

whence **$<A_2>$** $= <A_1> = <A>$

Now suppose that $r > n$. Then, continuing in this way for a total of n steps, we get $A_n = \{u_1, \ldots, u_2\}$ spans the same subspace as $<A>$. But then u_{n+1} is in the space $<A_n>$, and therefore the set $\{u_1, \ldots, u_n, u_{n+1}\} \subseteq B$ is linearly dependent- a contradiction, since a non empty subset of a linearly independent set is linearly independent Therefore, $r \leq n$.

Proof (of Theorem 3)

Let B be a basis for V with m elements. Let A be another basis for V with n elements. From

Lemma 16.8 $m \leq n$. But now, reversing the roles of A and B, it follows that $n \leq m$. Thus, $m=n$

Theorem 4

If $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is a set of non zero vectors which spans a subspace W of a vector space V, then some subset of S is a basis for W

Proof

If S is a linearly independent set, then by definition S is a basis for W. If S is linearly dependent, then one of the vectors can be written as a linear combination of the others. Suppose v_m is such a vector (if not, shift the vectors in S around and relabel so that this is true). We claim that $S^1 - \{v_1 \ldots, v_{m-1}\}$ still spans W. To see this, let x be in W with

$x = a_1 v_1 + \ldots + a_{m-1} v_{m-1} + a_m v_m$

Now $v_m = d_1 v_1 + \ldots + d_{m-1} v_{m-1}$, so we can substitute this expression into the former linear combination to obtain

$x = (a_1 + a_m d_1) v_1 + \ldots + (a_{m-1} + a_m d_{m-1}) v_{m-1}$

Thus S^1 spans W. If S^1 is linearly independent, S^1 is a basis for W. If S^1 is linearly dependent one of the vectors in S^1 is a linear combination of the others. Now we argue as before. In this way we must arrive eventually at a linearly independent set which spans W. (If we reduce to a set with a single vector, that set is linearly independent because S was a set of non zero vectors). The resulting set is a basis for W.

Self-Assessment Exercise(s)

Self-Assessment Answer

3.5 Definition of Dimension

Let V be a vector space over the field F. If V has a basis with n elements, then we say that V is an **n-dimensional vector space** or that V has **dimension n** over F. We denote this by $\dim_F V = n$, or more simply by $\dim V = n$ when the field F is clear from context. If V does not have a finite basis, then we say that V is infinite - dimensional over F, and we denote this by $\dim_F V = \infty$. The trivial vector space $V = 0$ is said to have dimension 0.

The inclusion of the phrase "over F" in the definition of dimensionality is no mere pedantic adornment. For if we change the field of scalars to a proper subfield, then the dimension of the space may change.

Example 6

Let V be the subspace of E^3 spanned by ***S*** = {vi, v_2, v_3} where v_1 = (0,1,1), v_2 = (1,0,1) and v_3 = (1,1,2). Determine dim V.

Solution

Since S is linearly dependent, and $v_3 = v_1 + v_2$, a basis for ***V*** is {v_1, v_2}. Hence, dim V = 2

Theorem 5

Let dim V = n, and let ***S* = {*v_1 v_2*, ..., *v_n*}** be a subset of V. The following are equivalent:

1. Set ***S*** is a basis for ***V***
2. Set ***S*** is linearly independent
3. Set ***S*** spans ***V***

Proof

S is a basis for ***V*** implies ***S*** is linearly independent follows from the definition of basis.

Suppose ***S*** is linearly independent and ***S*** does not span V. Then there is a vector V_{n+1} $\in$ V which is not in span ***S***. That is

$$T = \{V_1, V_2, \ldots, V_n, V_{n+1}\}$$

is a linearly independent set from ***V***. But then dim V $\geq$ n + 1, which contradicts the hypothesis that **dim V = n**. Thus ***S*** spans ***V***.

Suppose ***S*** spans ***V*** and is not a basis for ***V***. However, this subset must have less than n vectors in it, which implies that dim ***V*** < n, a contradiction. Hence, ***S*** spans ***V*** implies that ***S*** is a basis for ***V***.

Example 7

Show that M_{23} has dimension 6

Solution

A basis is (infact, this is the standard basis)"

$$\left\{ \begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \right\}$$

so dim $\boldsymbol{M}_{23}$ = number of vectors in $\boldsymbol{S}$ = 6

Example 8

The vector space P_n has dimension n + 1

Solution

A basis is $\boldsymbol{S} = \{1, \boldsymbol{x}, \ldots, \boldsymbol{x}^n\}$.

To verify this, we check first for linear independence. The equation

$$a_1 1 + a_2 x + a_3 x^2 + \ldots + a_{n+1} x^{n+l} = 0$$

holds only if the polynomial on the left is zero for all real *x.* From algebra this occurs only if all the coefficients are zero, that is, only if $a_1 = a_2 = \ldots = a_{n+1} = 0$. Therefore, $\boldsymbol{S}$ is linearly independent. That S spans P_n follows from the fact that any polynomial in P_n is of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Self-Assessment Exercise(s)

Self-Assessment Answer

3.6 Definition

Let **A** be an **m x n** matrix. The **row rank** of a matrix is the number of non zero rows in the reduced row echelon form of **A**. The column rank of a matrix is the number of non zero rows in the reduced row echelon form of $\mathbf{A}^T$. The row and column ranks of the zero matrix are defined to be zero.

Example 9

Calculate the row and column ranks of

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix}$$

Solution

Row reducing **A** we have

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix} \begin{matrix} \xrightarrow{R_2 \to R_2 + 2R_1} \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & -3 & -4 & 1 & -3 \\ 0 & -3 & -4 & 1 & -3 \\ 0 & -3 & -4 & 1 & -3 \end{pmatrix}$$

$$\begin{matrix} \xrightarrow{R_3 \to R_3 - R_2} \\ R_4 \to R_4 - R_2 \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & -3 & -4 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{-1}{3} R_2} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & \frac{4}{3} & \frac{-1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} & 0 \\ 0 & 1 & \frac{4}{3} & \frac{-1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A_R$$

The final matrix $\boldsymbol{A_R}$ has row rank 2, so row rank of $\boldsymbol{A}$ = 2.

For the column rank we can do column operations or form $\boldsymbol{A^T}$, do row operations, and transpose. We will use column operations

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix} \begin{matrix} C_2 \to C_2 + C_1 \\ \xrightarrow{C_3 \to C_3 + C_1} \\ C_4 \to C_4 - 2C_1 \\ C_5 \to C_5 - C_1 \end{matrix} \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 3 & 4 & -1 & 3 \\ 3 & 6 & 8 & -2 & 6 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_3 \to C_3 - \frac{4}{3}C_2} \begin{array}{l} \\ C_4 \to C_4 + \frac{1}{2}C_2 \\ C_5 \to C_5 - C_2 \end{array} \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_2 \to \frac{1}{3}C_2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_2 \to C_2 - C_1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_1 \to C_1 + 2C_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$C_2 \to -1C_2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix} = A_C$$

The column rank of ***A*** is 2 also

The column rank and row rank ***A*** in the last Example were equal. This is always true.

Theorem 6

For any matrix A,

row rank A = column rank A

Proof

Let $\boldsymbol{v_1, \ldots, v_m}$ be the rows of $\boldsymbol{A_{mxn}}$. Suppose that after reduction a basis $\{\boldsymbol{w_1, w_2, \ldots, w_k}\}$, $\boldsymbol{k} \le \boldsymbol{m}$, is found for span $\{\boldsymbol{v_1, \ldots, v_m}\}$. Thus

$$\boldsymbol{v_1 = a_{11}w_1 + a_{12}w_2 + \ldots + a_{1k}w_k}$$

$$\boldsymbol{v_2 = a_{21}w_1 + a_{22}w_2 + \ldots + a_{2k}w_k}$$

$$\ldots \quad . \; . \; . \quad . \; . \; . \quad \ldots \quad \ldots(1)$$

$$\boldsymbol{v_m = a_{m1}w_1 + \quad + a_{m2}w_2 \quad + a_{mk}w_k}$$

Now writing

$$\boldsymbol{v_j = (b_{j1}, b_{j2}, \ldots b_{jn})} \qquad \boldsymbol{1 \le j \le n}$$

and

$$\boldsymbol{w_j = c_{i1}, c_{i2}, \ldots c_{in})} \qquad \boldsymbol{1 \le i \le n}$$

we find after substitution into (1) that

$$\begin{pmatrix} b_{1j} \\ b_{2j} \\ \cdot \\ \cdot \\ \cdot \\ b_{mj} \end{pmatrix} = C_{1j} \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{pmatrix} + \ldots + C_{kj} \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{mk} \end{pmatrix}$$

This means that the transpose of each column is a linear combination of ***k*** vectors; therefore, the column rank of ***A*** is less than or equal to ***k***. That is, column rank ***A*** ≤ row rank ***A*** In the same way, we find row rank ***A*** ≤ column rank ***A***. Therefore, the ranks are equal.

Theorem 7

If **S** = {v_1, ..., v_m} is a vector subset of $\mathbf{E}^n$ and **A** is the matrix formed by putting v_1 in row 1, v_2 in row 2, and so on, and if **B** is the reduced row echelon form of **A**, then the nonzero rows of **B** form a basis for the row space of **A**. That is, the nonzero rows of **B** form a basis for span **S**.

Proof

Let the matrix be

$$A_{mxn} = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_m \end{pmatrix}$$

By definition of row operations, if a row of zeros is obtained, that row was equal to a linear combination of other vectors in the set. The remaining rows are therefore all linear combinations of the independent vectors from the original set. Thus the span of the nonzero rows is equal to span ***S****.* Thus the non zero rows, being independent, form a basis for span ***S*** and dim (span ***S****)* = number of non zero rows.

Self-Assessment Exercise(s)

Self-Assessment Answer

3.7 Basic Problem

Now that we know what a basis of a vector space is, we can state one of the fundamental problem of linear algebra.

Given a vector space *V,* the basis problem may take one of the following forms

Problem 1: Construct a basis for ***V***, by choosing vectors from *V*

Problem 2: Given a set *S* of vectors in ***V***, construct a basis for *V* by enlarging ***S***, or deleting some (but not all) vectors from *S,* or both

Before we try to solve this problem, one pertinent question arises, Is a solution even possible? We shall rely on the last **Theorem** which tells us to "throw out dependent vectors from a spanning set" to get a basis.

Example 10

Find a basis for the solution space of

$$x_1 + 2x_2 + 3x_4 + x_5 = 0$$
$$2x_1 + 3x_2 + 3x_4 + x_5 = 0$$
$$x_1 + x_2 + 2x_3 + 2x_4 + x_5 = 0$$
$$3x_1 + 5x_2 + 6x_4 + 2x_5 = 0$$
$$2x_1 + 3x_2 + 2x_3 + 5x_4 + 2x_5 = 0$$

Solution

The equations in augmented matrix form are

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 1 & . & 0 \\ 2 & 3 & 0 & 3 & 1 & . & 0 \\ 1 & 1 & 2 & 2 & 1. & . & 0 \\ 3 & 5 & 0 & 6 & 2 & . & 0 \\ 2 & 3 & 2 & 5 & 2 & . & 0 \end{pmatrix} \xrightarrow{\text{Row reduction}} \begin{pmatrix} 1 & 0 & 0 & -3 & -1 & . & 0 \\ 0 & 1 & 0 & 3 & 1 & . & 0 \\ 0 & 0 & 1 & 1 & 2. & . & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 \end{pmatrix}$$

And the general solution is

$$(x_1, x_2, x_4, x_5) = (3k+j, -3k-j, -k-\tfrac{1}{2}j, k, j)$$

$=(3k, -3k, -k, k, 0)+(j, -j, -\tfrac{1}{2}j, k, j)$

$=k(3, -3, -1, 1, 0)+j(1, -1, -\tfrac{1}{2}, 0, 1)$

Since k and j can take on any values, letting them first be 1 and 0, and then 0 and 1 , we get as solutions

$v_1=(3,-3,-l,l,0)$ $v_2 = (l,-l,-'/2,0,l)$

Clearly $S= \{v_1, v_2\}$ spans the solution space and is linearly independent, S is a basis for the solution space. Therefore, the dimension of the solution space is 2.

Example 11:

The set spans S =n{(0,0,2,0), (1,1,4,2), (1,2,1,3), (2,1,2,3) } is a vector space. Find a basis for it.

Solution

Form A and row-reduce.

$$A=\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 4 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 1 & 4 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & -6 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 4 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -6 & -1 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 + R_2]{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -9 & 0 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -9 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 + 3R_3 \\ R_4 \to R_4 + 9R_3}]{R_1 \to R_1 - 7R_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Conclusion

At the end of this unit, you have learnt how to define basis for a vector space, construct a basis for V by choosing vectors from V, given a Set S of Vectors in V, you know how to construct a basis for V by enlarging or deleting some (but not all) vectors from S. You have also learnt how to know if a Vector is a basis for a given vector space or not, how to define a dimension of vector space and how to determine the dimension of any given vector space.

Summary

Let *V* be a vector space over the field *F.* A **basis** for ***V*** is a subset ***B*** of ***V*** such that:

(a) ***B*** is a linearly independent set and

(b) ***B*** spans ***V.***

Let V be a vector space over the field F. If V has a basis with n elements, then we say that V is an **n-dimensional vector space** or that V has **dimension n** over F. We denote this by $\dim_F V = n$, or more simply by dim V = n when the field F is clear from context. If V does not have a finite basis, then we say that V is infinite - dimensional over F, and we denote this by $\dim_F V = \infty$. The trivial vector space V = 0 is said to have dimension 0.

Let **A** be an **m x n** matrix. The **row rank** of a matrix is the number of non zero rows in the reduced row echelon form of **A**. The column rank of a matrix is the number of non zero rows in the reduced row echelon form of $\mathbf{A}^T$. The row and column ranks of the zero matrix are defined to be zero.

6.0 Tutor-Marked Assignment (TMA)

1. What is Basis?

2. Give some examples of Bases

7.0 References/Further Reading

By the last theorem, T = {(1,0,0,1), (0,1,0,1), (0,0,1,0)} is a basis for span S and dim (span S) = 3.

Odili,G. A. (2000): Algebra for Colleges and Universities: An Integral Approach. Anachuna Educational Books. ISBN978-2897-37-x.

Robert, A. Beezer (2006): A First Course in Linear Algebra. http://linear.ups.edu/.

Module 2

Matrices

Unit 1

Introduction to Matrices

Contents

1.0 Introduction

The history of matrices goes back to ancient times! But the term "matrix" was not applied to the concept until 1850.

We have come across system of linear equations, and the various solution techniques. In this unit we are going to discuss the origin of matrix, its definition, and areas of its application.

2.0 Learning Outcome

Upon completion of this unit, students should be able to:

1. Know the origin of matrix;
2. Define matrix;
3. Know the areas of application of matrix formulation.

3.0 Learning Content

3.1 Definition (Matrix)

"Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed or produced. The origin of mathematical matrices lie with the study of systems of simultaneous linear equations. An important Chinese text from between 300 BC and AD 200, *Nine Chapters of the Mathematical Art* (*Chiu Chang Suan Shu*), gives the first known example of the use of matrix methods to solve simultaneous equations.

A matrix is an array of numbers arranged in a rectangular form. The numbers in the array are called "entries" in the matrix.

3.2 Areas of Application of Matrix

In mathematics, a matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions, arranged in *rows* and *columns*. The individual items in a matrix are called its *elements* or *entries.* An example of a matrix with 3 rows and 3 columns is given as below,

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 9 & 8 \end{bmatrix}$$

Matrices find applications in most scientific fields. In physics, matrices are used to study electrical circuits, optics, and quantum mechanics. In computer graphics, matrices are used to project a 3-dimensional image onto a 2-dimensional screen, and to create realistic-seeming motion.

A major branch of numerical analysis is devoted to the development of efficient algorithms for matrix computations, a subject that is centuries old and is today an expanding area of research.

Algorithms that are tailored to the structure of particular matrix structures, e.g. sparse matrices and near-diagonal matrices, expedite computations in finite element method and other computations. Infinite matrices occur in planetary theory and in atomic theory.

4.0 Conclusion

At the end of this unit, you have learnt about the origin of matrix and it definition and how to apply matrix formulation.

5.0 Summary

At the end of this unit we were able to discuss the meaning of a meaning of a matrix, the historical development of matrix and the areas of its application.

6.0 Tutor Marked Mark Assignment

(Q) Define a matrix and explain its application in the areas of Agriculture

7.0 References/Further Reading

Brookes, Mike (2005), The Matrix Reference Manual, London: Imperial College.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Hazewinkel, Michiel, (2001), "Determinant", Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4.

Unit 2

Matrix Representation

Content

1.0 Introduction

In this section we are going to study various ways through which we can represent matrix from linear equations, we are to going to consider various examples of matrices in the form of rows and columns.

2.0 Learning Outcome

Upon completion of this unit, students should be able to:

1. List the defining properties of a linear transformation;
2. List a number of different examples of linear transformation;
3. Verify whether or not a transformation is linear or not.

3.0 Representation of a Matrix from Linear equations

We consider the following system m linear equations in n unknowns

$$\begin{array}{lll}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\
. & . & . \\
. & . & . \\
. & . & . \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{nn}x_n & = & bm
\end{array}$$

(2.1)

Equation (2.1) can be represented in a matrix form as given in equation (2.2)

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ bm \end{pmatrix} \tag{2.2}$$

The left hand side matrix can be written as a product of two matrices; as shown in equation (2.3)

$$\begin{pmatrix} a_{11} + a_{12} + \ldots + a_{1n} \\ a_{21} + a_{22} + \ldots + a_{2n} \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ a_{m1} + a_{m2} + \ldots + a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ bm \end{pmatrix} \tag{2.3}$$

Equation (2.3) can be represented by a simple matrix equation as shown in equation (2.4)

$$A\vec{x} = \vec{b}$$

(2.4)

where,

A	=	Coefficient matrix
$\vec{x}$	=	Column vector
$\vec{b}$	=	Column Constant

Example 2.1 (2 x 3 Matrix)

A 2 x 3 matrix, is a type of matrix with 2 rows and 3 columns

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & -4 \end{pmatrix}$$

A 2 x 3 matrix

Example 2.2 (3 x 1 Matrix)

A 3 x 1 matrix, is a type of matrix with 3 rows and 1 column

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

A 3 x 1 matrix

Example 2.3 (3 x 3 Matrix)

A 3 x 3 matrix, is a type of matrix with 3 rows and 3 columns

$$\begin{pmatrix} 4 & 1 & 2 \\ 5 & 0 & 1 \\ -6 & 1 & -2 \end{pmatrix}$$

A 3 x 3 matrix

Example 2.4 (1 x 1 Matrix)

A 1 x 1 matrix, is a type of matrix with 1 row and 1 column(5)

A 1 x 1 matrix

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

At the end of this unit we were able to discuss the various representation of matrices in terms of the entries.

5.0 Summary

6.0 Tutor Marked Assignment (TMA)

(Q) Represent the system of equations below in a matrix form

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
. \qquad\qquad\qquad\qquad . \quad . \\
. \qquad\qquad\qquad\qquad . \quad . \\
. \qquad\qquad\qquad\qquad . \quad . \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{nn}x_n = bm
\end{array}
$$

7.0 References/Further Reading

Brookes, Mike (2005), *The Matrix Reference Manual*, London: Imperial College.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Hazewinkel, Michiel, (2001), "Determinant", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

Unit 3

Types of Matrices

Contents

1.0 Introduction

In this section we are going to treat different types of matrices and their examples

2.0 Learning Outcome

Upon completion of this unit, students should be able to:

1. List the defining properties of a linear transformation;
2. List a number of different examples of linear transformation;
3. Verify whether or not a transformation is linear or not.

3.0 Learning Content

3.1 Major Types of Matrix

(i) **Row Matrix (vector):** is a 1 x n array of entries in a row,

For example,

A = $(a_1, a_{12}, \ldots a_n)$

(ii) **Column Matrix (vector):** is an n x 1 array of entries in a column,

For example,

$$A = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{pmatrix}$$

(iii) **Null Matrix**

A Null matrix is an m x n matrix where all entries are zero

Example,

$$A = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ 0 & 0 & . & . & . & 0 \end{pmatrix}$$

(iv) **Square Matrix**

A square matrix is a matrix with n rows and n columns

Example,

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

$a_{11}, a_{22}, ..., a_{nn}$ are on the main diagonal

(v) **Diagonal Matrix**

A diagonal matrix is a type of matrix where all entries other than the main diagonal entries are zero.

Example,

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(vi) **Upper Triangular Matrix**

Is a square matrix where the entries along the main diagonal as well as entries below the main diagonal are zeros.

Example,

$$U = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(vii) **Lower Triangular Matrix**

Is a square matrix where the entries, except entries below the main diagonal, are zeros

Example,

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \end{pmatrix}$$

(viii) **Unity (Identity) Matrix**

Is a diagonal matrix where every entry is a unity

Example

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ix) **Symmetric Matrix**

Is a square matrix where all entries above the main diagonal are mirror images of entries below the main diagonal

Example

$$\Delta = \begin{pmatrix} 1 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 3 \end{pmatrix}$$

(x) **Skew-Symmetric Matrix**

A square matrix is skew-symmetric if every entry is such that;

$a_{ij} = -a_{ji}$; $a_{ii} = 0$

where all entries above the main diagonal are mirror images of entries below the main diagonal

Example

$$\Lambda = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$$

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

At the end of this unit we were able to discuss the various types of matrices with respect to the entry type.

5.0 Summary

6.0 Tutor Marked Assignment (TMA)

Give examples of the following types of marices

(i) A column matrix

(ii) A row matrix

(iii) Upper triangular matrix

(iv) Lower triangular matrix

7.0 References/Further Reading

Brookes, Mike (2005), The Matrix Reference Manual, London: Imperial College.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Hazewinkel, Michiel, (2001), "Determinant", Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4

Unit 4

Operations on Matrices

Contents

1.0 Introduction

We are going to consider some basic operations that are relevant in matrix operations that will lead to the finding of an inverse of a matrix.

2.0 Learning Outcome

By the end of this unit, student should be able to;

1. know the minor of a matrix
2. know the cofactor of a matrix
3. solve problems using determinant

3.0 Learning Content

3.1 The Minor of an Entry a_{ij}

The minor of an entry a_{ij} of a matrix A is the determinant of the submatrix obtained by eliminating the ith row and the jth column of A, and is denoted by $|M_{ij}|$

3.2 Cofactor of a_{ij}

The cofactor of a_{ij} is the minor of a_{ij} multiplied by the prescribed sign of a_{ij},

For example the cofactor of C_{ij} is defined by $|C_{ij}| = (-1)^{i+j}|M_{ij}|$

3.2.1 Cofactor Matrix of A

The cofactor matrix of A is the matrix obtained by replacing the entries a_{ij} of A by their cofactors

For example, Cofactor matrix of A is given by;

$$A = \begin{pmatrix} C_{11} & C_{12} & . & . & . & C_{1n} \\ C_{21} & C_{22} & . & . & . & C_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ C_{n1} & C_{n2} & . & . & . & C_{nn} \end{pmatrix}$$

3.3 Adjoint of a Matrix

The adjoint matrix of A is the transpose of the cofactor matrix of A, for example

$$Adj(A) = \begin{pmatrix} C_{11} & C_{21} & . & . & . & C_{n1} \\ C_{12} & C_{22} & . & . & . & C_{n2} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ C_{1n} & C_{2n} & . & . & . & C_{nn} \end{pmatrix}$$

The inverse of a matrix A denoted by A^{-1} is given by;

$$A^{-1} = \frac{Adj(A)}{\det(A)}$$

The inverse A^{-1} is defined by the relation,

$$AA^{-1} = A^{-1}A = I$$

Example

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

Solution

$$\begin{aligned} |A| &= 1(3+2) + (2-0) \\ &= 7 \end{aligned}$$

Cofactor matrix of A

$$= \begin{pmatrix} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -2 & 4 \\ 1 & 1 & -2 \\ 1 & 1 & 5 \end{pmatrix}$$

$$\text{Adj(A)} = \begin{pmatrix} 5 & 1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{Adj(A)}{\det(A)}$$

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 5 & 1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{5}{7} & \frac{1}{7} & \frac{1}{7} \\ \frac{-2}{7} & \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-2}{7} & \frac{5}{7} \end{pmatrix}$$

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

At the end of this unit we were able to discuss the operations that are perform on matrices including determining the adjoint of a matrix.

5.0 Summary

6.0 Tutor Marked Assignment (TMA)

(Q) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

7.0 References/Further Reading

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Hazewinkel, Michiel, (2001), "Determinant", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

Unit 5

Determinants

Content

1.0 Introduction

In linear algebra, the determinant is a value associated with a square matrix. It can be computed from the entries of the matrix by a specific arithmetic expression, while other ways to determine its value exist as well. The determinant provides important information when the matrix is that of the coefficients of a system of linear equations, or when it corresponds to a linear transformation of a vector space: in the first case the system has a unique solution if and only if the determinant is nonzero, while in the second case that same condition means that the transformation has an inverse operation.

In this unit we are going to look at various ways of representing linear equations using the concept of matrix, we shall also consider the solution techniques of determinants in solving system of linear equations.

2.0 Learning Outcome

By the end of this unit the student should be able to

- (i) know the definition of determinants
- (ii) know the order of determinants
- (iii) solve system of linear equations using determinants

3.0 Learning Content

3.1 Concepts of Determinants

Determinants arise naturally from the solution of systems of linear equations.

We consider the system of equations below,

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Solving by elimination method we have,

$$a_1b_2x + b_1b_2y = b_2c_1 \qquad (1)$$

$$a_2b_1x + b_1b_2y = b_1c_2 \qquad (2)$$

Subtracting equation (2) from equation (1), we obtain;

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \tag{3}$$

Similarly,

$$a_1a_2x + a_2b_1y = a_2c_1 \tag{4}$$

$$a_1a_2x + a_1b_2y = a_1c_2 \tag{5}$$

Subtracting equation (5) from equation (4)

$$(a_2b_1 - a_1b_2)y = a_2c_1 - a_1c_2 \tag{6}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \tag{7}$$

Equation (3) can be written as;

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Similarly, equation (7) can also be written as;

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Self-Assessment Exercise(s)

Self-Assessment Answer

3.2 Order of a Matrix:

The number of rows and number of columns in a matrix, is called order of the matrix, denoted by $m\,x\,n$ or (m,n).

Where;

m = number of rows,

n = number of columns.

Example 1

(1) Solve using determinants the system of equation below,

$$2x - y = 1$$
$$x + 2y = 8$$

Solution

$$x = \frac{\begin{vmatrix} 1 & -1 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}}$$

$$= \frac{10}{5} = -2$$

Similarly,

$$y = \frac{\begin{vmatrix} 2 & 1 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}}$$

$$= \frac{15}{5} = 3$$

Self-Assessment Exercise(s)

Please insert SAQ

Self-Assessment Answer

Please provide Self-Assessment Answers in SAQ

Example 2

Solve the system of linear equations below using the principle of determinants

$$x + y - z = 4$$
$$x - y - 2z = 3$$
$$3x + y + 2z = 5$$

Solution

$$x = \frac{\begin{vmatrix} 4 & 1 & -1 \\ 3 & -1 & -2 \\ 5 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 2 \end{vmatrix}}$$

$$= \quad \frac{-24}{-12} = 2$$

$$y = \frac{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 3 & -2 \\ 3 & 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 2 \end{vmatrix}}$$

$$= \quad \frac{-12}{-12} = 1$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 3 \\ 3 & 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 2 \end{vmatrix}}$$

$$= \frac{-12}{-12} = -1$$

4.0 Conclusion

5.0 Summary

At the end of this unit we were able to discuss the meaning of determinants, the order of a matrix and the solution techniques for solving the system of linear equation.

6.0 Tutor Marked Assignment (TMA)

(Q) Solve using determinant, the system of linear equations below

$$x + y - z = 4$$
$$x - y - 2z = 3$$
$$3x + y + 2z = 5$$

7.0 References/Further Reading

Brookes, Mike (2005), *The Matrix Reference Manual*, London: Imperial College.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Hazewinkel, Michiel, (2001), "Determinant", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

Module 3

Transformation in Matrices

Unit 1

Introduction to Linear Transformation

Contents

1.0 Introduction

You have already encountered real-valued functions that play an incredibly important role in calculus, mathematics generally and numerous other applied fields such as Engineering and physics. Linear functions, which you studied in MAT 111, are also examples of linear transformations that you study in detail in this unit. You recall that a function of one variable, y = f(x), transforms each number x into the domain of f (a subset of R or R itself) exactly one number f(x) in R. The same function can also be said to map or transform a vector x (since R is a vector space) into another vector y = f(x) in the range of f, and this meaning is reflected by the alternative function notation: f: x→f(x).

2.0 Learning Outcome

Upon completion of this unit, students should be able to:

1. List the defining properties of a linear transformation;
2. List a number of different examples of linear transformation;
3. Verify whether or not a transformation is linear or not.

3.0 Learning Content

3.1 Definitions and basic properties of linear transformations.

Definition

If V and W are two vector spaces, a function T: V→ W is called a linear transformation if it satisfies the following axioms:

T_1 : $T(v + v_1) = T(v) + T(v_1)$ for all v and $v_1 \in V$

T_2: $T(rv) = rT(v)$ for all v∈ V and $r \in R$.

A linear transformation T: $V \to V$ is called a linear operator on V.

Axiom T_1 is just the requirement that T preserves Vectors addition while T_2 means that T preserves scalar multiplication.

Definition

If V and W are vector spaces, the following are linear transformations:

Zero transformation

A mapping T: $V \to W$, which maps the vector space V into the vector space W is called zero transformation if T(v)=0 for every v∈ V.

Identity transformation

The mapping I: $V \to V$, defined b linear transformation that maps a vector space V into itself is called a linear **OPERATOR** on V, thus the identity transformation is a linear operator on V.

Scalar operator V→V

a: $v \to v$ where a(v)=av for all v∈V.

Here, a is a real number. The zero (0) will be used to denote zero transformation from V→W for any spaces V and W.

Theorem 3.1.1:

Let T: V→W be a linear transformation then

i. T(0)=0
ii. T(-v)=-T(v) for all v∈V
iii. $T(r_1v_1+r_2v_2 + \ldots+r_kv_k)= r_1T(v_1)+r_2T(v_2)+ \ldots+r_kT(v_k)$
For all $v_1 \in V$ and r_1 in

Proof:

i. T(0) = T(0v) = 0T(v) for all v in V
ii. T(-v) = T[(-1)v] = (-1)T(v) = -T(v) for all v in V.
iii. If k=1, this is $T(r_1v_1) = r_1T(v_1)$ by axiom T_2. The general result is proved by induction on k. if it holds for a particular $k\geq 1$ then using axiom T_1 and the induction assumptions, we have
$$T(r_1v_1+\ldots+r_kv_k + r_{k+1}v_{k+1}) = T(r_1v_1+\ldots+r_kv_k)+ T(r_{k+1}v_{k+1})$$
$$= r_1T(v_1)+ \ldots+r_2T(v_2)+ r_{k+1}T(v_{k+1}).$$

This completes the induction and so proves property (iii).

Self-Assessment Exercise(s)

Self-Assessment Answer

Example 1

Show that each of the transformations, T , is a linear transformation.

(a) T: $P_3 \rightarrow P_2$ defined by $T(a_0 + a_1x + a_2x^2) = a_2x^2$.

(b) Let C be the set of all real valued functions of one variable that are defined and continuous for all x in R. Define T: C→C by T(f(x)) = 2f(x) for each f in C and x in R.

Solution

(a) P_2 is a space . suppose $P(x) = a_0 + a_1x + a_2x^2$ and $Q(x) = b_0 + b_1x + b_2x^2$ are any two polynomials in P_2 then to verify part (ii) of the definition:

$T(P(X)+Q(x)) = T((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2))$

$= T((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2)$

$= (a_2 + b_2)x^2)$ by definition of T

$= a_2x^2 + b_2x^2$

$= T(P(X)) + T(Q(x))$ by the definition of T.

Thus proving (ii) of the definitions. Similarly, to verify part (iii) of the definition:

$T(cP(x)) = T(c(a_0 + a_1x + a_2x^2))$.

$= T(ca_0 + ca_1x + ca_2x^2)$

$= (ca_2x^2)$

$= cP(x)$ by the definition of T, thus proving (iii) of the definition.

(b) To show T is a linear transformation, we prove properties (ii) and (iii) as follows, where f and g are any two functions in C and c in R:

$T(f(x)+g(x)) = 2(f(x)+g(x))$

$= 2f(x)+2g(x) = T(f(x))+T(g(x))$.

Next, $T(cf(x)) = 2(cf(x)) = c(2f(x)) = cT(f(x))$.

Example 2

Let T: $\mathrm{R}^3 \rightarrow R^2$ be the function defined by the formula, T(x, y, z) = (x, x + y + z). Determine whether T is a linear transformation.

Solution

If u= $(\mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1)$ and v= $(\mathrm{x}_2, \mathrm{y}_2, \mathrm{z}_2)$, then

u + v= $(\mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1)$ + $(\mathrm{x}_2, \mathrm{y}_2, \mathrm{z}_2)$

=$(\mathrm{x}_1 + \mathrm{x}_2$, $\mathrm{y}_1 + \mathrm{y}_2$,$\mathrm{z}_1 + \mathrm{z}_2)$

T(u + v)=T$(\mathrm{x}_1 + \mathrm{x}_2$, $\mathrm{y}_1 + \mathrm{y}_2$,$\mathrm{z}_1 + \mathrm{z}_2)$

=($(\mathrm{x}_1 + \mathrm{x}_2)$, $(\mathrm{x}_1 + \mathrm{x}_2$+$\mathrm{y}_1$+$\mathrm{y}_2$+$\mathrm{z}_1$+$\mathrm{z}_2)$)

=$(\mathrm{x}_1, \mathrm{x}_1 + \mathrm{y}_1 + \mathrm{z}_1) + (\mathrm{x}_2, \mathrm{x}_2 + \mathrm{y}_2 + \mathrm{z}_2)$

=T$(\mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1)$+T $(\mathrm{x}_2, \mathrm{y}_2, \mathrm{z}_2)$

=T(u)+T(v).

Furthermore, if k is any scalar, then

k u =k$(\mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1)$ = $(\mathrm{kx}_1, \mathrm{ky}_1, \mathrm{kz}_1)$, i.e.

T(k u)=T$(\mathrm{kx}_1, \mathrm{ky}_1, \mathrm{kz}_1)$ or T(k(x, y, z))

= (k$\mathrm{x}_1, \mathrm{kx}_1 + \mathrm{ky}_1 + \mathrm{kz}_1)$

= (k$\mathrm{x}_1, \mathrm{k}(\mathrm{x}_1 + \mathrm{y}_1 + \mathrm{z}_1))$

= k $(\mathrm{x}_1, \mathrm{x}_1 + \mathrm{y}_1 + \mathrm{z}_1)$

=k T$(\mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1)$

=k T(u).

Thus, T is a linear transformation.

Example 3

Let T: R→ R be the function defined by the formula, T(x) = x+y+5. Verify whether T is a linear transformation or not.

Solution

Let x, y $\in R$.

T(x + y) = x+y+5……………………………………………………………… (1)

But T(x) = x+5 and T(y) = y+5.

Therefore, T(x) + T(y) = x+y+10... (2)

From (1) and (2), we conclude T(x + y) ≠ T(x) + T(y).

Hence, T is not a linear transformation.

Or, T(k x) = kx+5

k T(x) = k(x+5) = k x + 5k ≠ T(k x).

Self-Assessment Exercise(s)

Self-Assessment Answer

Example 4

If the mapping T:$\mathrm{R}^2 \rightarrow \mathrm{R}^2$ is defined by the formula, T(x,y)=(-x,y). Show that T is a linear transformation.

Solution

T(0,0)=(0,0).

Hence, T is linear because, it takes the zero vector into the zero vector.

Example 5

Define a function T: $\mathrm{R}^2 \rightarrow \mathrm{R}^3$ by T$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-2y \\ 3x \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ in R^2. Show that T is a linear transformation

Solution

We verify the axioms. Given $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ in R^2, compute T($\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$)= T $\begin{bmatrix} x+x_1 \\ y+y_1 \end{bmatrix}$

$$= \begin{bmatrix} (x+x_1)+(y+y_1) \\ (x+x_1)-2(y+y_1) \\ 3(x+x_1) \end{bmatrix}$$

$$= \begin{bmatrix} x+y \\ x-2y \\ 3x \end{bmatrix} + \begin{bmatrix} x_1+y_1 \\ x_1-2y_1 \\ 3x_1 \end{bmatrix} = \mathrm{T} \begin{bmatrix} x \\ y \end{bmatrix} + \mathrm{T} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

This proves axiomT_1, and T_2 is proved as follows:

T(r $\begin{bmatrix} x \\ y \end{bmatrix}$)=T $\begin{bmatrix} rx \\ ry \end{bmatrix}$ = $\begin{bmatrix} rx+ry \\ rx-2ry \\ 3rx \end{bmatrix} = r \begin{bmatrix} x+y \\ x-2y \\ 3x \end{bmatrix} = rT \begin{bmatrix} x \\ y \end{bmatrix}$.

Hence, T preserves addition and scalar multiplication and so, is a linear transformation. The linear transformation in the above example can be described using matrix multiplication:

$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ in R^2.

Example 6

The function T: $\mathrm{R}^2 \to \mathrm{R}^2$ defined by $T\begin{pmatrix} \mathrm{a} \\ \mathrm{b} \end{pmatrix} = \begin{pmatrix} \mathrm{a} \\ \mathrm{b}+1 \end{pmatrix}$ is not a linear transformation, since, for example

$$\mathrm{T}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \mathrm{T}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

But $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mathrm{T}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Example 7

Determine whether the function $\mathrm{R}^2 \to \mathrm{R}^2$ defined by $\mathrm{T(x,y) = (x^2, y)}$ is linear?

Solution

We have

$$\mathrm{T\big((x,y)+(z,y)\big) = T(x+z, y+w) = ((x+y)^2, y+w)}$$

$$\mathrm{\neq (x^2, y) + (z^2, w) = T(x,y) + T(z,w)}.$$

So, T does not preserve additivity. So, T is not linear.

Alternatively, you could check that T does not preserve scalar multiplication. Also, alternatively you could check this failure(s) numerically. For example,

$$\mathrm{T\big((1,1)+(2,0)\big) = T(3,1) = (9,1) \neq T(1,1) + T(2,0)}.$$

Self-Assessment Exercise(s)

Please insert SAQ

Self-Assessment Answer

3.2 Range, Domain and Co domain

Definition

Let T: v→T(v), the set of all values *T* (v) is called the **range** of *T:*

Definition.

The vector spaces *V* and *W* are sometimes called, respectively, the **domain** and **co domain** of *T:*

Note: We sometimes write this $T : v \to T(v)$ and as $T : V \to W$ in order to stress the transformation from one vector or vector space to another. We also write that *T* **maps** v into *T* (v) and *T* is a **mapping** from *V* to *W:*

Definition.

A linear transformation *T* is sometimes called a **linear operator**, especially when $V = W$:

Theorem 3.2.1 *The range of the linear transformation* $T: V \to W$ *is a vector space.*

Proof. The values of *T* are all in the vector space *W*. Recall that when a set *S* of vectors is already in a vector space then to show *S* is a subspace (i.e., a vector space) it is only necessary to show that it is closed under addition and scalar multiplication. Prove this yourself. That is, show the range is closed under addition and scalar multiplication. Hint: make use of the linearity properties (b) and (c) above.

Definition.

The dimension of the vector space that is the range of a linear transformation *T* is called the rank of *T* and is sometimes denoted rank (*T*).

4.0 Conclusion

In this unit you have learnt the concept of linearity in transformation, nonlinear transformation and valuable theorems on linear transformation with examples.

5.0 Summary

Conditions for Linearity of a Linear Transformation: Let T: V→T(v), then

1. T_1 : $T(v + v_1) = T(v) + T(v_1)$ for all v and $v_1 \in V$
2. T_2: $T(rv) = rT(v)$ for all v$\in V$ and $r \in R$.

Types of linear transformations

If V and W are vector spaces, the following are linear transformations, then

1. A mapping T: $V \to W$, which maps the vector space V into the vector space W is called zero transformation if T(v)=0 for every v$\in V$.
2. The mapping I: $V \to V$, defined b linear transformation that maps a vector space V into itself is called a linear **OPERATOR** on V
3. a: $v \to v$ where $a(v) = av$ for all $v \in V$ is called a **Scalar operator .**
4. Let T: v→T(v), the set of all values *T* (v) is called the **range** of T
5. The vector spaces v and T(v) are sometimes called, respectively, the **domain** and **co domain** of T
6. A linear transformation *T* is sometimes called a **linear operator**, especially when v = T(v):
7. The dimension of the vector space that is the range of a linear transformation *T* is called the rank of *T* and is sometimes denoted rank (*T*).

6.0 Tutor-Marked Assignment (TMA)

Investigate whether or not the following transformations are linear.

1. T: $p_2 \to p_1$ defined by T($a_0 + a_1x + a_2x^2$) = a_1x.
2. T: $p_1 \to p_1$ defined by T($a_0 + a_1x$)=$a_0x + a_1$.
3. T: $p_2 \to p_2$ defined by T($a_0x + a_1x + a_2x^2$) = 2+$a_1x + a_2x^2$.
4. T: $R^2 \to R^2$defined by T(x,y)=(x^2, y).

5. Show that each of the following functions is a linear transformation:

(a) $T: R^2 \to R^3$; $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + 2y \\ 3y \end{bmatrix}$.

(b) $T: R^3 \to R^2$; $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - 3y + 5z \\ 0 \end{bmatrix}$.

(c) $T: R^2 \to R^2$; $T(x, y) = (x, -y)$, reflection in the x-axis.

(d) $T: R^3 \to R^3$; $T(x, y, z) = (x, y, -z)$, reflection in the x-y plane.

(e) $T: C \to C$; $T(Z) = \bar{Z}$ (conjugation).

7.0 References/Further Reading

Nicholson, W. K (1995): linear Algebra with applications. P. W. S Publishing company.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Unit 2

Matrix Transformations

Contents

1.0 Introduction

In this unit, this students learn matrix transformation in R^2 and R^3, standard basis, matrix of transformation with examples. Efforts are also made to give examples on inverse transformations.

2.0 Learning Outcome

Upon completion of this unit, you should be able to:

1. Find and use matrices of linear transformations from $\mathrm{R}^{\mathrm{n}} \to \mathrm{R}^{\mathrm{m}}$;
2. List and work with the linear transformations R^2 and R^3;
3. Find the inverse of a given linear transformation, or show that it does not exist;
4. Explain the meaning of a matrix of a general linear transformation; and
5. Find matrix of a general linear transformation.

3.0 Learning Content

3.1 Matrix Transformation

Definition

If A is any m$\times n$ matrix, the matrix transformation $\mathrm{T_A}$: $\mathrm{R^n} \to \mathrm{R^m}$ is defined by $\mathrm{T_A(\,X)\ in\ (X) = AX}$ for all columns X in $\mathrm{R^n}$.

Theorem 3.1.1

$\mathrm{T_A}\colon \mathrm{R^n} \to \mathrm{R^m}$ is a linear transformation for each m$\times n$ matrix A.
Proof:
Given X and $\mathrm{X_1}$ in $\mathrm{R^n}$ and r in R, matrix arithmetic gives
$\mathrm{T_A(\,X + X_1) = A(\,X + X_1) = AX + AX_1}$
$\mathrm{T_A(X) + T_A}(X_1)$.
$\mathrm{T_A(rX) = A(rX) = r(AX) = rT_A(X)}$.
Hence $\mathrm{T_A}$ is a linear transformation.

Example 1

Consider the matrix operators onR^2 given by the three types of elementary matrices:

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then

$T_E\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}$, $T_F\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\y\end{bmatrix}$, $T_G\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+y\\y\end{bmatrix}$.

Self-Assessment Exercise(s)

Self-Assessment Answer

Example 2

If T: $R^3 \to R$ is a linear transformation with T(3, -1, 2) = 5 and T(1, 0, 1) = 2, compute T(-1, 1, 0).

Solution

This can be done by theorem 2 provided that (-1, 1, 0) can be expressed as a linear combination of T(3, -1, 2) and T(1, 0, 1). This is indeed possible:

(-1, 1, 0) = -(3, -1, 2)+2(1, 0, 1).

So, T(-1, 1, 0) = -T(3, -1, 2)+2T(1, 0, 1)

= -5+4 = -1.

Definition

If $A = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$ and X=$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

If a vector in R^n expressed in matrix , then AX is a vector in R^m and the function T: $R^n \to R^m$ defined by the formula:

T(x)=$AX = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a linear transformation called multiplication by

A. Such linear transformations are called matrix transformations. To establish that T(X)=AX is a linear transformation, we let u and v be $n\times 1$ matrices and let k be a scalar. Using properties of matrix multiplication, we have

A(u + v)=Au +Av and A(ku) = k(Au)

Or, T(u + v)= T(u)+T(v) and T(k u) = k T(u).

Hence, T(X) = AX is a linear transformation.

Example 3

Let T: $\mathrm{R}^2 \to \mathrm{R}^2$ be the linear transformation such that T(1,1)=(0,2) and T(1,-1)=(2,0). Then compute

(a) T(1,4) (b) T(-2,1).

Solution

(a) We write (1,4)=a(1,1)+b(1,-1).
Solving, (1,4)=2.5(1,1)-1.5(1,-1).
So, T(1,4)=2.5T(1,4)-1.5T(1,-1)
=2.5(0,2)-1.5(2,0)
=(-3,5).

(b) We write (-2,1)=a(1,1)+b(1,-1).
Solving (-2,1)= -0.5(1,1)-1.5(1,-1).
So, T(-2,1)=-0.5T(1,1)-1.5T(1,-1)
=-0.5(0,2)-1.5(2,0)
=(-3,-1).

Example 4

Let A be a 2×3 matrix and T: $\mathrm{R}^3 \to \mathrm{R}^2$ be the matrix transformation T(X) = AX, Given that

$T(\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}1\\1\end{pmatrix}$, $T(\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}3\\0\end{pmatrix}$, $T(\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}4\\-7\end{pmatrix}$. Find the following:

(a) $T(\begin{pmatrix}1\\3\\8\end{pmatrix})$ (b) $T(\begin{pmatrix}x\\y\\z\end{pmatrix})$ (c) A.

Solution

(a) T(X)=AX

$$T(\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

T$\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

T$\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

To find,

T$\left(\begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}\right) = AX = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 42 \\ -55 \end{pmatrix}$

(b) T$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = AX = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 3y + 4z \\ x - 7z \end{pmatrix}$

(c) A= $\begin{pmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{pmatrix}$

Self-Assessment Exercise(s)

Self-Assessment Answer

Example 5

Let T: $\mathrm{R}^3 \to \mathrm{R}^3$ be a linear transformation such that

$\mathrm{T}(1,0,0) = (2,4,-1)$, $T(0,1,0) = (1,3,-2)$, $T(0,0,1) = (0,-2,2)$. Compute T(-2,4,-1).

Solution

We have

$$(-2,4,-1) = -2(1,0,0) + 4(0,1,0) - (0,0,1).$$

So, T(-2,4,-1)=$-2T(1,0,0) + 4T(0,1,0) - T(0,0,1)$

$= (2,4,-1) + (1,3,-2) + (0,-2,2)$

$= (3,5,-1).$

Theorem 3.1.2:

Let V and W be vector spaces and let $\{e_1, e_2, \ldots e_n\}$ be basis of V. Given any vectors $w_{1,} w_{2,} \ldots, w_n$ in W (they needn't be distinct), there exists a unique linear transformation T : V→W satisfying $T(e_i) = w_i$ for each $i == 1, 2, \ldots, n$. In fact, the action of T is as follows:

Given $v = v_1 e_1 + v_2 e_{2+\cdots+} v_n e_n$ in V, then

$T(v) = T(v_1 e_1 + v_2 e_{2+\cdots+} v_n e_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$

Proof:

If such a transformation T does not exist, and if S is any other such transformation then $T(e_i) = w_i$ holds for each i , so S=T. Hence, T is unique if it is exists and it remains to show that there really is such a linear transformation. Given v∈ V , we must specify T(v) in W. Because $\{e_{1,} e_{2,} \ldots, e_n\}$ is a basis of V, we have v= $(v_1 e_1 + v_2 e_{2+\cdots+} v_n e_n)$, where $v_{1,} v_{2,\ldots,} v_n$ are uniquely determined by V. hence we can define T: V→W by

$T(v) = T(v_1 e_1 + v_2 e_{2+\cdots+} v_n e_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$ for all $v = v_1 e_1 + v_2 e_{2+\cdots+} v_n e_n$ in V. This satisfies $T(e_i) = w_i$ for each ; the verification that T is linear is left as exercise to the reader.

Example 6

Find a linear transformation T: $R^3 \to R^2$ such that :

$T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$, $T\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$, $T\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

Solution

The set $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is a basis of R^3, so then theorem 3 applies. The expansion of an arbitrary vector in R^3 as a linear combination of these vectors is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}(x+y-z)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}(x-y+z)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}(-x+y+z)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, the transformation T must be given by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}(x+y-z)T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}(x-y+z)T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}(-x+y+z)T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$=\frac{1}{2}(x+y-z)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2}(x-y+z)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}(-x+y+z)\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 3x+y-z \\ 2(y-z) \end{bmatrix}.$$

Theorem 3.1.3:

Let T: $\mathrm{R}^n \to \mathrm{R}^m$ be a linear transformation. Write vectors in R^n as columns:

1 . There exists an m$\times n$ matrix A such that T(X) = AX for all columns X in R^n; that is T = $\mathrm{T_A}$.

2 . the columns of A are respectively $T(\mathrm{E_1}), \mathrm{T(E_2)}, \dots, \mathrm{T(E_n)}$, where $\{\mathrm{E_1}, \mathrm{E_2}, \dots, \mathrm{E_n}\}$ is the standard basis of R^n. hence, A can be written in terms of its columns as

$A = [T(\mathrm{E_1})\ \ \mathrm{T(E_2)}\ \ \ \dots\ \ \ \mathrm{T(E_n)}]$.

Proof:

Write $(\mathrm{E_1}) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{m1} \end{bmatrix}$, $T(\mathrm{E_2}) = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{m2} \end{bmatrix}$, …, $T(\mathrm{E_n}) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{mn} \end{bmatrix}$.

Then $A = [\mathrm{a_{ij}}]$ is an $\mathrm{m} \times \mathrm{n}$ matrix whose $\mathrm{j^{th}}$ column is $T(\mathrm{E_j})$. Given X$\in R^n$, write

$\mathrm{X} = \mathrm{x_1 E_1} + \mathrm{x_2 E_2} + \dots + \mathrm{x_n E_n}$, $\mathrm{x_i} \in R$. Now, compute T(X) using theorem 2.

$$\mathrm{X} = \mathrm{x_1 T(E_1)} + \mathrm{x_2 T(E_2)} + \dots + \mathrm{x_n T(E_n)}$$

$$= \mathrm{x_1}\begin{bmatrix} \mathrm{a_{11}} \\ \mathrm{a_{21}} \\ \vdots \\ \mathrm{a_{m1}} \end{bmatrix} + \mathrm{x_2}\begin{bmatrix} \mathrm{a_{12}} \\ \mathrm{a_{22}} \\ \vdots \\ \mathrm{a_{m2}} \end{bmatrix} + \dots + \mathrm{x_n}\begin{bmatrix} \mathrm{a_{1n}} \\ \mathrm{a_{2n}} \\ \vdots \\ \mathrm{a_{mn}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n} \\ \mathrm{a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n} \\ \vdots \\ \mathrm{a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n} \end{bmatrix}$$

= AX

= $\mathrm{T_A}$X.

Because is holds for all x∈ R^n, it follows that $T = \mathrm{T_A}$.

Note: the matrix A in theorem 4 is called the standard matrix of T.

Example 7

Find the standard matrix of T: $\mathrm{R}^3 \rightarrow \mathrm{R}^2$ when $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y + z \\ x - z \end{bmatrix}$.

Solution

The desired matrix can be observed directly: $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

However, the second part of theorem 4 also gives the matrix. If $\{E_1, E_2, E_3\}$ is the standard basis of R^3, then the columns are indeed $T(\mathrm{E}_1), \mathrm{T}(\mathrm{E}_2)$ and $\mathrm{T}(\mathrm{E}_3)$, as the reader may verify.

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

The work done in this unit is detailed enough to give students quick understanding on matrix transformations. Efforts are also made to give examples on inverse transformations and matrices of transformations.

5.0 Summary

1. If A is any $\mathrm{m} \times n$ matrix, the matrix transformation$\mathrm{T_A}$: $\mathrm{R^n} \rightarrow \mathrm{R^m}$ is defined by $\mathrm{T_A}(\mathrm{X})$ in $(\mathrm{X}) = \mathrm{AX}$ for all columns X in $\mathrm{R^n}$.
2. If $A = \begin{pmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$ and X=$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

   If a vector in $\mathrm{R^n}$ expressed in matrix, then AX is a vector in $\mathrm{R^m}$ and the function T: $\mathrm{R^n} \rightarrow \mathrm{R^m}$ defined by the formula:

T(x)=$AX=\begin{pmatrix} a_{11} & a_{12}\dots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ a_{m1} & a_{m2}\cdots & a_{mn}\end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}$ is a linear transformation called multiplication by A.

6.0 Tutor-Marked Assignment (TMA)

1. In each of the following cases, find a linear transformation with the given properties and compute T(v).

(a) $T: R^2 \to R^3$; $T(1,2)=(1,0,1)$, $T(-1,0)=(0,1,1)$; $v=(2,1)$.
(b) $T: R^2 \to R^3$; $T(2,-1)=(1,-1,1)$, $T(1,1)=(0,1,0)$; $v=(-1,2)$
(c) $T: P_2 \to P_3$; $T(x^2)=x^3$, $T(x+1)=0$, $T(x-1)=x$; $v=x^2+x+1$

2. Let T $(v_1, v_2, v_3) = (2v_1+v_2,\ 2v_2-3v_1, v_1-v_3)$. Find

(a) T (-4, 5, 1) (b) the preimage of w = (4, 1, -1).

3. Let T: $R^3 \to R^3$ be a linear transformation such that T(1, 0, 0)=(2, 4, -1), T(0, 1, 0)=(1, 3, -2), T(0, 0, 1)=(0, -2, 2). Compute T(-2, 4, -1).

4. Let A=$\begin{pmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0\end{pmatrix}$ be the linear transformation T(X)=AX.

(a) Compute T(1, 0, -1, 3, 0).
(b) Compute the preimage under T, of (-1, 8).

Hint: solve $\begin{pmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5\end{pmatrix}=\begin{pmatrix} -1 \\ 8\end{pmatrix}$.

5. in each case, assume that T is a linear transformation,

(a) If $T: V \to R$, and $T(v_1)=1, T(v_2)=1$, find $T(3v_1-5v_2)$
(b) If $T: R^2 \to R^2$ and $T\begin{bmatrix} 1 \\ 3\end{bmatrix}=\begin{bmatrix} 1 \\ 1\end{bmatrix}$, $T\begin{bmatrix} 1 \\ 1\end{bmatrix}=\begin{bmatrix} 0 \\ 1\end{bmatrix}$; find $T\begin{bmatrix} -1 \\ 3\end{bmatrix}$.

7.0 References/Further Reading

Nicholson, W. K (1995): linear Algebra with applications. P. W. S Publishing company.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company

Unit 3

Kernels and Images of a Linear Transformation

Contents

1.0 Introduction

This unit explains the meaning of kernels and images of linear transformations with relevant theorems and examples on how to find them.

2.0 Learning Outcome

Upon completion of this unit, students should be able to:

1. Explain the meaning of kernel and range of a linear transformation;
2. List the properties of kernel and range of a linear transformation;
3. Find the kernel and range of a linear transformation.

3.0 Learning Content

3.1 Kernels and Images of a Linear Transformation

Definition

Let T: V→ W denotes a linear transformation. The kernel of T (denoted kerT) and image of T (denoted imT or T(v) are defined by

(a) KerT= $\{v \in V: T(v) = 0\}$

(b) imT=$\{T(v): v \in V\}$.

the kernel of T is often called the null space of T. it consists of all vectors v$\in V$ satisfying the condition that T(v)=0. The image of T is often called the range of T and consists of all vectors w$\in W$ of the form w=T(v) for some v$\in V$.

Theorem 3.1.1

If T: V→W is a linear transformation, kerT is a subspace of V, and imT is a subspace of W.

Proof:

The fact that T(0)=0 shows that both kerT and imT contain the zero vector. If v and v_1 lie in kerT, then T(0)=0=T(v_1), so

T(v+v_1) = T(v)+T(v_1) = 0+0=0,

T(rv) = rT(v) = r0=0 for all r$\in R$. Here v+v_1 and rv lie in kerT (they satiafy the required condition), so kerT is a subspace of V. if w and w_1 lie in imt, write w=T(v) and w_1 =T(v_1) where v and v_1 lie in V. then

$w+w_1$= T(v)+T(v_1)= T(v+v_1),

rw=rT(v)=T(rv) for all r$\in R$.

Hence w+w_1 and rw both lie in imT (they have the required form), so imT is a subspace of W.

Example 1

If $T: \mathrm{R}^3 \to R^3$ is defined by $T(x, y, z) = (x - y, z, y - x)$, find kerT and imT, and compute their dimensions.

Solution

We use the definitions: KerT= $\{(\mathrm{x}, \mathrm{y}, \mathrm{z}): (\mathrm{x}-\mathrm{y}, \mathrm{z}, \mathrm{y}-\mathrm{x}) = (0,0,0)\}$

$= \{(\mathrm{t}, \mathrm{t}, 0): \mathrm{t} \in \mathrm{R}\}$

imT= $\{(\mathrm{x}-\mathrm{y}, \mathrm{z}, \mathrm{y}-\mathrm{x}): \mathrm{x}, \mathrm{y}, \mathrm{z} \ \in \mathrm{R}\}$

$= \{\mathrm{s}, \mathrm{t}, -\mathrm{s} : \mathrm{s}, \mathrm{t} \ \in \mathrm{R}\}$.

Hence dim(kerT)=1 and dim(imT)=2.

Example 2

Suppose that T: $\mathrm{R}^4 \to \mathrm{R}^3$ is a linear transformation, with ker.T= W. Let

$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix}$ be a matrix representation of T.

(a) Find a basis for image of T=U, say.

(b) Find a basis for ker.T=W.

(c) Confirm that dim. R^4 = rank(T) + nullity(T) in this case.

Solution

(a) Let T: $\mathrm{R}^4 \to \mathrm{R}^3$, with A= a matrix representation of T. So, the images of generators of R^4 under T, generate the image U under T. i.e. the column space of A is A^c

$= \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 0 & 2 & 2 \\ 1 & -1 & -2 \end{pmatrix}$.

Row reduce A^c to echelon form, we have

$$\begin{pmatrix}1 & 2 & 1\\0 & 2 & 2\\2 & -1 & -3\\1 & -1 & -2\end{pmatrix} \to \begin{pmatrix}1 & 2 & 1\\0 & 2 & 2\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} \to \begin{pmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

i.e. (1,2,1), (0, 1, 1) is a basis for U. S, dimU =2.

(b) For kerT=W, to establish a basis for W, we want a set (x,y,s,t); x,y,s.t $\in R$, such that T(x,y,s,t)=(0,0,0) $\in R^3$. i.e.

x+2y+t=0

2x-y+2s-t=0

x-3y+2s-2t=0.

The solution space of this homogeneous system of linear equation is the kernel W, of T.

So,

x-3y+2s-2t=0

2x-y+2s-t=0

x+2y+t=0

this system reduces to

x-3y+2s-2t=0

5y-2s+3t=0

5y-2s+3t=0

i.e. x-3y+2s-2t=0

5y-2s+3t=0

So, dimW=2.

Thus, we can use s and t as free variables to generate any two vectors in R^4 as basis for ker T.

(b) Clearly, dimR^4=rank(T)+ nullity of T.

=2+2=4.

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

A detail insight into kernel and images of linear transformations has been covered in this units with detailed examples.

5.0 Summary

1. Let T: V→ W denote a linear transformation.

   KerT = $\{v \in V: T(v) = 0\}$

   imT = $\{T(v): v \in V\}$.

2. If T: V→W is a linear transformation, kerT is a subspace of V, and imT is a subspace of W.

6.0 Tutor-Marked Assignment (TMA)

(1) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y - z \\ x + 3y + z \\ 3x - y - 2z \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ 3 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Find the basis for the range of T.

Hint: Range of T =span of the columns of A.

Solution: $\left\{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right\}$

2. Suppose that T: $R^4 \to R^3$ is a linear transformation, with ker.T= W. Let

$A = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 4 & -2 & 0 & -2 \\ 0 & 1 & 1 & -1 \end{pmatrix}$ be a matrix representation of T.

(a) Find a basis for image of T=U, say.

(b) Find a basis for ker.T=W.

(c) Confirm that dim. R^4 = rank(T) + nullity(T) in this case.

3 . For each of the following matrix A, find a basis for the kernel and image of $\mathrm{T_A}$.

(a) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$

4 . In each of the cases, (i)find a basis of kerT, and (ii) find a basis of imT.

(a) $\mathrm{T}: \mathrm{R}^3 \rightarrow \mathrm{R}^2;\ T(a + bx + c\mathrm{x}^2) = (a, b)$

(b) $\mathrm{T}: \mathrm{R}^3 \rightarrow \mathrm{R}^3; T(x, y, z) = (x + y, x + y, 0)$.

7.0 References/Further Reading

Nicholson, W. K (1995): linear Algebra with applications. P. W. S Publishing company.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.

Unit 4

Nullity and Rank

Content

1.0 Introduction

In this unit the students learn the concepts of nullity and rank, relationship between dim(kerT) and nullity(T), dim(imT) and rank(T) with relevant examples. Tutor-Marked Assignment (TMA) are provided at the end of the unit to exercise the students on the topic.

2.0 Learning Outcome

Upon completion of this unit, students should be able to:

1. explain the meaning of nullity and rank of a given linear transformation;
2. determine the nullity and rank of a given linear transformation;
3. explain the meaning of one to one linear transformation;
4. explain the meaning of inverse linear transformation; and
5. Determine whether a linear transformation is also an isomorphism.

3.0 Learning Content

3.1 Nullity and Rank

Definition

Given a linear transformation T: V→W,

1 . dim(kerT) is called the nullity of T and denoted as nullity(T).

2 . dim(imT) is called the rank of T and denoted as rank(T).

The rank of a matrix was defined earlier on to be the dimension of the columns of A, the column space of A. The two usages of the word rank are consistent in the following sense.

Example 1

Given an $m\times n$ matrix, show that $\mathrm{imT_A}$ = Col A, so $\mathrm{rankT_A}$= rank of A.

Solution

Write A= $[C_{1,\dots,}C_n]$ in terms of its columns, then

$$\mathrm{imT_A} = \{AX: X \in R^n\} = \left\{ [C_{1,\dots,}C_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in R \right\}$$

$$=\{x_1C_1 + \cdots + x_nC_n : x_i \in R\}.$$

Hence $\mathrm{imT_A}$ =span$\{C_{1,\dots,}C_n\}$ is the column space of A.

Example 2

Given the 4×3 matrix A=$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 2 & -3 \\ -2 & -1 & 1 \end{bmatrix}$, compute the kernel and image of the corresponding matrix transformation$T_A : R^3 \to R^4$, and determine the rank and nullity of T_A.

Solution

Bring A to reduced row-echelon form:

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 2 & -3 \\ -2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 3 & -5 \\ 0 & -3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So rank$T_A = \mathrm{rankA} = 2$. Moreover, the solutions of AX=0 are$[-t\ 5t\ 3t]^T$, where T is a parameter. Because kerT = $\{X \in R^3 : AX = 0\}$, this means that nullity T_A= dim(kerT_A)=1.

Self-Assessment Exercise(s)

Self-Assessment Answer

Definition

If it is Let T: V→W be a linear transformation.

1 . T is said to be onto if imT= W

2 . T is said to be one to one if T(v)=T(v_1) implies v=v_1.

Theorem 3.1.1

If T: V→W is a linear transformation, then T is one to one if and only if kerT=0.

Proof:

If it is one to one, let V be any vector in kerT. Then T(v)=0, so T(v)=T(0). Hence v=0 because T is one to one. Conversely , assume that kerT=0 and let T(v)=T(v_1) with v and v_1 in, then T(v-v_1)= T(v)-T(v_1)=0, so

v-v_1 lies in kerT=0. This means that v-v_1 =0, so v=v_1. This proves that T is one to one.

Example 3

The identity transformation 1_v : V→V is both one to one and onto for any vector space V.

Example 4

Consider the linear transformations:

S: $R^3 \to R^2$ given by S(x, y, z)= (x+y, x-y)

T: $R^2 \to R^3$ given by T(x, y)=(x+y, x-y, x).

Show that T is one to one but not onto, where as S is onto but one to one.

Solution

The verification that they are linear is omitted. T is one to one because

KerT=$\{(x, y): x + y = x - y = x = 0\} = \{(0,0)\} = 0$.

However, it is not onto. For example (0, 0, 1) does not lie in imT because if

(0, 0, 1)= (x+y, x-y, x) for some x and y, then

x+y=0=x-y and x=1, an impossibility. Turning to S, it is not one to one by theorem 2 because (0, 0, 1) lies in kerS. But every (s,t) in R^2 lies in imS because (s, t) = (x+y, x-y) for some x and y (in fact x=$\frac{1}{2}(s+t)$ and y= $\frac{1}{2}(s-t)$. Hence S is onto.

Self-Assessment Exercise(s)

Self-Assessment Answer

Definition.

The nullity of a linear transformation T is the dimension of the vector space that is the kernel of T:

Example 5

A linear transformation $T: V \rightarrow W$ is one-to-one if and only if the kernel of T contains only the single Vector, the zero vector of V (equivalently, the nullity is zero).

Solution.

If the kernel has a non-zero vector v (so T (v) = 0) then for any other vector u $\in$ V it follows

that:

T (u + v) = T (u) + T (v) = T (u) + 0 = T (u) and so T is not one-to-one. That is: nullity (T) > 0 which implies T is not one-to-one or in the contra positive form:

T is one-to-one implying that nullity (T) = 0. Conversely, if nullity (T) = 0 then T must be one-to-one, because:

T (u) = T (v) => T (u -v) = 0

=> u - v $\in$ ker (T).

Hence, u - v = 0 is the zero vector and so u = v.

Self-Assessment Exercise(s)

Self-Assessment Answer

Theorem 3.1.2: *If $T : V \to W$ is a linear transformation and the vector space V has dimension n; then:*

rank (T) + nullity (T) = n*. That is, the difference between the dimension V and the dimension of its kernel subspace, namely* n - ker (T) *is equal to the dimension, nullity (T), of the range T; (a subspace of W).*

Theorem 3.1.3 (dimension theorem)

Let T: V→W be any linear transformation and assume that kerT and imT are both finite dimensional. Then V is also finite dimensional and

dimV=dim(kerT) + dim(imT).

In other words, dimV=nullity(T)+rank(T).

Definition.

If $T : V \to W$ is a linear transformation then:

a. T is said to be **one-to-one** if for every pair of distinct vectors v_1; v_2 ($v_1 \neq v_2$) in V, the vectors $T(v_1)$, $T(v_2)$ are also distinct vectors in W.
b. If T is one-to-one then the **inverse transformation** T^{-1} is defined from $W' \to V$; where W' is the range of T (a subspace of W) as follows. For each pair $v \in V$ and $w \in W'$ with $W = T(v)$; define

$T^{-1}(w) = v$.

3.2 Singular and Non-Singular Linear Transformations

Definition

Singular Linear Transformations:

Let T: V→U be a linear transformation as usual. If there exists v$\in V$ such that v $\neq 0$, T(v) = 0 then T is called a singular linear transformation. Any n×n matrix A is called singular if det(A)=0, and so a singular matrix is not invertible. We called a linear transformation $T_A: R^n \to R^n$ singular if matrix A is singular.

Non Singular Linear Transformations

T is a nonsingular linear transformation if only 0$\in V$ is linearly transformed to 0$\in U$ which implies that kerF $= \{0\}$. Similarly, just like in the case of singular matrix, any n×n matrix A is called nonsingular if det(A)≠0, and so a nonsingular matrix is invertible. We called a linear transformation $T_A: R^n \to R^n$ nonsingular if matrix A is nonsingular.

Remarks:

i . A one-one linear transformation is called an isomorphism.

ii .T: V→ U defines an isomorphism if and only if T is nonsingular.

iii .T: V→ U is a nonsingular linear transformation iff the image of an independent set of vectors is also a linearly independent set of vectors.

Example 6

Let T be a finite dimensional vector space over R. let T: V→ U be a linear transformation. Show that V and the image of T have the same dimension iff T is nonsingular.

Solution

We know that dimV= dim(Im.T)+dim(ker.T). Hence, V and im.T have the same dimension iff dim(kerF) =0 or kerF=$\{0\}$. i.e. iff T is nonsingular.

Self-Assessment Exercise(s)

Self-Assessment Answer

Isomorphism

A linear transformation T: V→W is called an isomorphism if it is both onto and one to one. The vector spaces V and W are called isomorphic there exists an isomorphism T: V→W.

Definition

The identity transformation 1_v : V→V is an isomorphism for any vector spece V. the word isomorphomism comes from two Greek roots: iso, meaning "same" and morphos, meaning "form". The isomorphism T induces a pairing v↔T(v) between vectors v in V and vectors T(v) in W that preserves vectors addition and scalar multiplication.

Theorem 3.1.4

If V and W are finite dimensional spaces, the following conditions are equivalent for a linear transformation T: V→W.

1 . T is isomorphic

2 . If $\{e_1, e_2, \ldots, e_n\}$ is any basis of V, then $\{T(e_{1)}, T(e_2), \ldots, T(e_n)\}$ is a basis of W.

3 . There exists a basis $\{e_1, e_2, \ldots, e_n\}$ of V such that $\{T(e_{1)}, T(e_2), \ldots, T(e_n)\}$ is a basis of W.

Theorem 3.1.5

Two finite dimensional vectors V and W are isomorphic if and only if dimV =dimW.

Proof:

It remains to show that if T: V→W is an isomorphism, then dimV=dimW. But if $\{e_1, e_2, \ldots, e_n\}$ is a basis of V, then $\{T(e_{1)}, T(e_2), \ldots, T(e_n)\}$ is a basis of W.

So dimW =n= dimV.

Self-Assessment Exercise(s)

Self-Assessment Answer

4.0 Conclusion

The concepts of nullity and rank of linear transformations has been discussed intensively in this unit with enough examples and theorems for illustrations.

5.0 Summary

1. Given a linear transformation T: V→W,
   dim(kerT) is called the nullity of T and denoted as nullity(T).
   dim(imT) is called the rank of T and denoted as rank(T)
3. The identity transformation 1_v : V→V is both one to one and onto for any vector space V.
4. The nullity of a linear transformation *T* is the dimension of the vector space that is the kernel of *T:*
5. *T* is said to be **one-to-one** if for every pair of distinct vectors v_1; v_2 ($v_1 \neq v_2$) in *V,* the vectors *T*(v_1), *T* (v_2) are also distinct vectors in *W.*
6. a linear transformation $T_A: R^n \to R^n$ nonsingular if matrix A is nonsingular
7. a linear transformation $T_A: R^n \to R^n$ nonsingular if matrix A is nonsingular.
8. dimV= dim(Im.T)+dim(ker.T)

9. A linear transformation T: V→W is called an isomorphism if it is both onto and one to one.

6.0 Tutor-Marked Assignment (TMA)

1 . For each of the following matrix A, find a basis for the kernel and image of T_A, and find the rank and nullity of T_A.

(a) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$

7.0 References/Further Reading

Nicholson, W. K (1995): linear Algebra with applications. P. W. S Publishing Company.

Carl, Meyer (2000): Matrix analysis and Linear Algebra. Siam publishing Company.