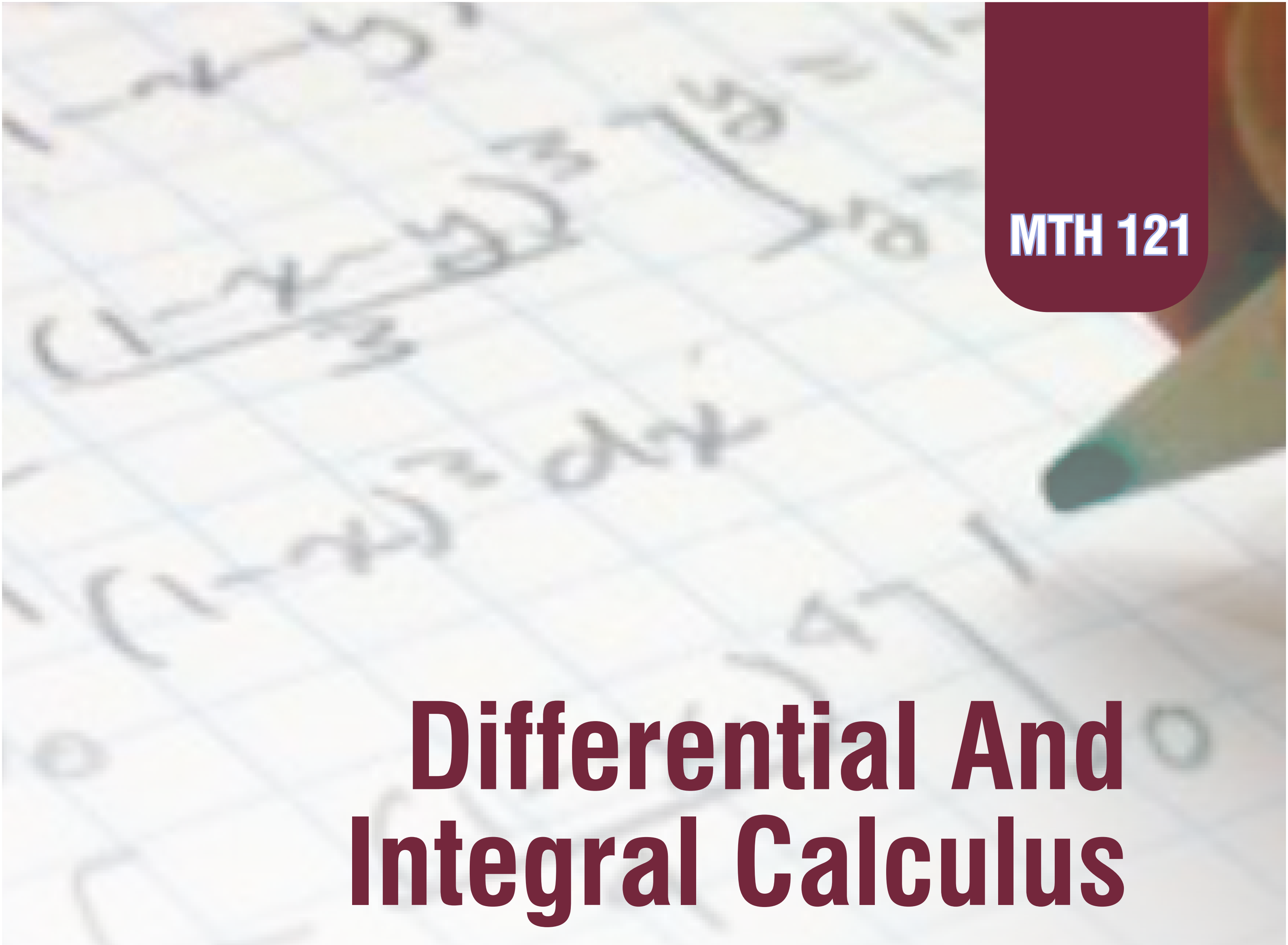

MTH 121

Differential And Integral Calculus

FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA
CENTRE FOR OPEN DISTANCE AND e-LEARNING

FEDERAL UNIVERSITY OF TECHNOLOGY MINNA, NIGER STATE, NIGERIA.

CENTRE FOR OPEN DISTANCE AND e-LEARNING (CODeL)

B.TECH. COMPUTER SCIENCE PROGRAMME

COURSE TITLE

DIFFERENTIAL AND INTEGRAL CALCULUS

COURSE CODE

MAT 121

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MAT 121

COURSE UNIT

2

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MAT 121 Study Guide

Introduction

MAT 121 Differential and Integral Calculus is a 3- credit unit course for students studying towards acquiring a Bachelor of Technology in Mathematics and Statistics and other related disciplines. The course is divided into 5 modules and 11 study units. It will first take a brief review of Function of a real variable, limits and idea of continuity. This course will then go ahead to deal with the derivative, as limit of change and techniques of differentiation. The course went further to deal with extreme curve sketching, Integration as an inverse of differentiation and methods of integration. The course concluded by dealing with definite integrals, application to area and volume.

The course guide therefore gives you an overview of what the course; MAT 121 is all about, the textbooks and other materials to be referenced, what you expect to know in each unit, and how to work through the course materials.

Recommended Study Time

This course is a 3-credit unit course having 15 study units. You are therefore enjoined to spend at least 3 hours in studying the content of each study unit.

What you are about to learn in this Course

The overall aim of this course, MAT 121 is to introduce you to the basic concepts of differential and integral calculus and to enable students to have basic knowledge of differential and integral calculus as it applied to their disciplines. This course highlights different methods of solving differential and integral calculus problems:

The differentiation of inverse trigonometric, curve sketching method, the substitution methods, method of integration by parts and further integration by parts, the use of reduction formula, integration using method of trigonometric substitution, integration using partial fractions and application of integration such as area under a curve, length of a curve and volume of revolution.

Course Aims

The aim of this course is to introduce students to the basic concepts of differential and integral calculus systems. It is believed that the knowledge will enable students understand the functionalities and capabilities of differential and integral calculus because calculus is a versatile branch of Mathematics employed as a very useful tool in the study of functions. Several information about functions and the quantity they represent can be obtained by techniques in calculus. The application of calculus to physical problems depends very much on expressing physical quantities in terms of functions whose analysis gives the required information about the quantities of interest. This makes the study of the theory of functions essential in calculus. This subject of calculus itself is classified into two distinct parts;

(a) Differential calculus
(b) Integral calculus

Differential calculus is the study of rate of change of functions with respect to change in the independent variable while Integral calculus is associated with summation of aggregate value of functions as in the study of area and volume.

Course Objectives

It is important to note that each unit has specific objectives. Students should study them carefully before proceeding to subsequent units. Therefore, it may be useful to refer to these objectives in the course of your study of the unit to assess your progress. You should always look at the unit objectives after completing a unit. In this way, you can be sure that you have done what is required of you by the end of the unit. However, below are overall objectives of this course. On completing this course, you should be able to:

(i) Give the definition of a function.
(ii) Know when a function is continuous or otherwise.
(iii) Understand the concept of derivatives
(iv) Carryout the derivative of a function using the first principle
(v) How to apply the product rule
(vi) How to apply the quotient rule
(vii) See integration as reverse process of differentiation;
(viii) Find a function whose derivative we already know

Working through this course

To complete this course, you are required to study all the units, the recommended textbooks, and other relevant materials. Each unit contains some self-assessment exercises and tutor marked assignments, and at some point, in this course, you are required to submit the tutor marked assignments. There is also a final examination at the end of this course. Stated below are the components of this course and what you have to do.

Course Materials

The major components of the course are:

1. Course Guide
2. Study Units
3. Text Books s
4. Assignment File
5. Presentation Schedule

Study Units

There are 11 study units and 6 modules in this course. They are:

Module One		
	Unit 1	Function Theory
	Unit 2	Graphs
Module Two		
	Unit 1	Limit of a Functions
	Unit 2	Differential Calculus
Module Three		
	Unit 1	Further Problems in Differentiation
	Unit 2	Inverse and Parametric Functions
Module Four		
	Unit 1	Extreme Curve Sketching

	Unit 2	Integration as an Inverse of Differentiation
Module Five		
	Unit 1	Method of Integration
Module Six		
	Unit 1	Definite Integrals
	Unit 2	Application to Area and Volume

Recommended Texts

The following texts and Internet resource links will be of enormous benefit to you in learning this course:

1. BLAKEY, J Intermediate Pure Mathematics, 5th Edition. Macmillan Press Limited.1977 London
2. BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London
3. CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London
4. (4) STROUD, K.A Engineering Mathematics, 4th Edition. Macmillan Press Limited, 1995. London
5. STROUD, K.A Further Engineering Mathematics, 3rd Edition. Macmillan Press Limited, 1995. London
6. (TRANTER, C.J And LAMBE, C.G Advanced Level Mathematics, Pure and Applied, 4th Edition Holder & Stoughton, 1979. Great Britain.

Assignment File

The assignment file will be given to you in due course. In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain for these assignments will count towards the final mark for the course. Altogether, there are tutor marked assignments for this course.

Presentation Schedule

The presentation schedule included in this course guide provides you with important dates for completion of each tutor marked assignment. You should therefore endeavour to meet the deadlines.

Assessment

There are two aspects to the assessment of this course. First, there are tutor marked assignments; and second, the written examination. Therefore, you are expected to take note of the facts, information and problem solving gathered during the course. The tutor marked assignments must be submitted to your tutor for formal assessment, in accordance to the deadline given. The work submitted and an online test will count for 30% of your total course mark.

At the end of the course, you will need to sit for a final written examination. This examination will account for 70% of your total score. TUTOR-MARKED ASSIGNMENT (TMA)

There are TMAs in this course. You need to submit all the TMAs. When you have completed each assignment, send them to your tutor as soon as possible and make certain that it gets to your tutor on or before the stipulated deadline. If for any reason you cannot complete your assignment on time, contact your tutor before the assignment is due to discuss the possibility of extension. Extension will not be granted after the deadline, unless on extraordinary cases.

Final Examination and Grading

The final examination for MAT 121 will last for a period of 3 hours and have a value of 70% of the total course grade. The examination will consist of questions which reflect the Self-Assessment Exercises and tutor marked assignments that you have previously encountered. Furthermore, all areas of the course will be examined. It would be better to use the time between finishing the last unit and sitting for the examination, to revise the entire course. You might find it useful to review your TMAs and comment on them before the examination. The final examination covers information from all parts of the course.

Practical Strategies for Working through this Course

1. Read the course guide thoroughly
2. Organize a study schedule. Refer to the course overview for more details. Note the time you are expected to spend on each unit and how the assignment relates to the units. Important details e.g. details of your tutorials and the date of the first day of the semester are available. You need to gather together all this information in one place such as a diary, a wall chart calendar or an organizer. Whatever method you choose, you should decide on and write in your own dates for working on each unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course works. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the table of content at the beginning of each unit. You will almost always need both the study unit you are working on and one of the materials recommended for further readings, on your desk at the same time.
6. Work through the unit, the content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit, you will be encouraged to read from your set books
7. Keep in mind that you will learn a lot by doing all your assignments carefully. They have been designed to help you meet the objectives of the course and will help you pass the examination.
8. Review the objectives of each study unit to confirm that you have achieved them.

If you are not certain about any of the objectives, review the study material and consult your tutor.

9. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you can keep yourself on schedule.
10. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the assignment is

returned, pay particular attention to your tutor's comments, both on the tutor marked assignment form and also written on the assignment. Consult you tutor as soon as possible if you have any questions or problems.

11. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this course guide).

Tutors and Tutorials

There are 6 hours of tutorials provided in support of this course. You will be notified of the dates, time and location together with the name and phone number of your tutor as soon as you are allocated a tutorial group. Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter and provide assistance to you during the course. You must mail your tutor marked assignment to your tutor well before the due date. At least two working days are required for this purpose. They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail or discussion board if you need help. The following might be circumstances in which you would find help necessary: contact your tutor if:

1. You do not understand any part of the study units or the assigned readings.
2. You have difficulty with the self-test or exercise.
3. You have questions or problems with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should endeavor to attend the tutorials. This is the only opportunity to have face to face contact with your tutor and ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from the course tutorials, have some questions handy before attending them. You will learn a lot from participating actively in discussions.
GOODLUCK!

Table of Contents

Module 1

Unit 1

Function Theory

Contents

1.0 Introduction

It is very important to have the idea of what a function is in Mathematics before we can undertake the study of calculus.

2.0 Learning Outcomes

At the end of this unit you should be able to:

1. Give the definition of a function.
2. Determine if a function is continuous or otherwise.
3. Know when a function tends to a certain limit say L.

3.0 Learning Content

3.1 Definition of a Function

Consider any two set X and Y. If there exist a mapping that mapped the set X to any subset A of Y in which every element in X has an image in A and every element in A has a pre-image in X, then the set X is the Domain of the map and Y is the co-domain. The set A constitute the range of the function. This functional relation is formally represented as

y=f(x)..*1.1*

That is, *y* is a function of the variable *x* called the independent variable and *y* the dependent variable. If the function is a bijective mapping in which every element in Y has a pre-image in X then there exists a unique mapping called the inverse mapping (function) given as

g(y)=x..*1.2*

3.2 Continuous Function and Its Properties

The function $f(x)$ defined in any domain D is said to be continuous at any point x_0 in D if given ε >0 then $\exists\delta(\varepsilon)$ such that

$|f(x)-f(x_0)|\leq\varepsilon$ Whenever $|x-x_0|<\delta$ 1.3

The following are some of the common properties exhibited by continuous functions:

Suppose $f(x)$ and $g(x)$ are any two functions of x that are defined and continuous in any domain D then the following are true.

$(i)\ (f+g)(x)=f(x)+g(x)$

(ii) For any two arbitrary constants α and β

$(\alpha f+\beta g)(x)=\alpha f(x)\times\beta g(x)$

$(iii)\ (f\times g)(x)=f(x)\times g(x)$

$(iv)\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

$(v)\ (f \bullet g)(x) = f(g(x))$

NOTE:

In the case of property (iv) above the function is undefined at the zero of g(x). And so a restriction is always placed in the domain of definitions for all rational functions so as to isolate the singularities from the domain of the resulting quotient function.

3.3 Limit of a Function

A function f(x) is said to tend to the limit L as x tends to a point x_0 if given $\varepsilon > 0\ \exists\ \delta(\varepsilon)$ such that

$|f(x) - L| \le \varepsilon$ Whenever $|x - x_0| \le \delta$

It is instructive from the above definitions that for any function $f(x)$ to be continuous at appoint x_0 the function must tend to $f(x)$ as x tends to the point x_0 .it must however be noted that the existence of a limit at appoint does not imply continuity at that point. If the point x_o is a point of discontinuity of the function $f(x)$ then the limit of the function ceases to be unique.

The limit thus becomes directional in the sense that the value of the limit now depends on the direction we take it. The limit as we move from left to right differs from that obtained while moving in the opposite direction. Hence, we have the left-hand limits and the right-hand limits.

NOTE:

If in a domain D a function is continuous at every point throughout the domain we say that the function is continuous in D.

3.4 Examples of Functions

Below we consider some few examples to illustrate the above concepts:

Let $f(u) = u^3 + 1$ and $g(u) = \cos 5u$, then we have the following

$(i)\ (f \pm g)(u) = f(u) \pm g(u) = u^3 + 1 \pm \cos 5u.$

$(ii)\ (f \times g)(u) = f(u)g(u) = (u^3 + 1)\cos 5u$

$(iii)\ \left(\frac{f}{g}\right)(u) = \frac{f(u)}{g(u)} = \frac{(u^3+1)}{(\cos 5u)} = (u^3 + 1)\sec 5u$ (Valid for cos5u $\neq$ 0, i.e. u $\neq$ $\frac{4p+1}{10}\pi, p \in Z$)

$(iv)\ (f \bullet g)(u) = f(g(u)) = (\cos 5u)^3 + 1 = \cos^3 u + 1$

$(v)\ (g \bullet f)(u) = f(g(u)) = \cos 5(u^3 + 1)$

The examples in (iv) and (v) above could be used to establish that composition of functions as earlier defined is not in general commutative.

Recall that a binary operator $\bullet$ defined over a set ψ is said to be commutative if given any two elements Γ and λ of ψ such that $\lambda \bullet \Gamma = \Gamma \bullet \lambda \in \psi$

From the above example, we conclude that composition of function is not in general commutative.

Self-Assessment Exercise(s) 1

1. Given $f(x) = x^2 + 1$ and $g(x) = \sin 5x$ then we have the following find
   (a) $(f \mp g)(x)$
   (b) $(f \bullet g)(x)$

4.0 Conclusion

It is believed that by now you already know what a function is, when a function is said to be continuous, properties of a continuous function and the limit of a functions. The applications of what u have learnt here will come in a later time.

5.0 Summary

You have learnt in this unit that: -

a) The idea of a function as a mapping of two set say, X and Y.

b) A function f(x) is continuous at any point x_0 in its domain of definition if as x tends to x_0 f(x) tends to $f(x_0)$.

c) A function f(x) tends to a limit L as x tends to a point x_0 if given $\varepsilon > 0 \ \exists \ \delta(\varepsilon)$ such that

$|f(x) - L| \le \varepsilon$ Whenever $|x - x_0| \le \delta$

6.0 Tutor Marked-Assignment

1. When is a function said to be bijective?
2. When is a function said to be continuous?
3. Give the properties of continuous functions.

7.0 Reference /Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B. D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Graphs

Contents

1.0 Introduction

In other to undergo the study of calculus in Mathematics, possession of the knowledge of graphs such as that of linear and quadratic is something that can never be over emphasize. So therefore, in this unit, you will be introduced to graphs and some of its components such as slope and intercept.

2.0 Learning Outcomes

At the end of this unit you should be able to identify:

1. the graph of a linear function.
2. the graph of a quadratic function.

3.0 Learning Content

3.1 Graphs

This is a diagram showing the relationship between a dependent variable y and independent variable x.

3.2 Linear Functions

The general expression for a linear function of a variable x is give $y = \alpha x + \beta$ Where α is a constant referred to as the gradient (slope) of the function y and β the intercept of the graph on the y axis.

We note here that β corresponds with the value of y when $x = 0$. This gives the intercept of the graph on y-axis. Hence the equation of y-axis is given as $x = 0$. whereas the equation of x-axis is given as $y = 0$.

Now for any given point (x_0, y_0) to lie on the straight line we must have that $\frac{y - y_0}{x - x_0} = \alpha$.

Hence the equation of the line that passes through the point (x_0, y_0) with gradient m is given as $y = \alpha x + (\mathrm{y}_0 - \alpha x_0)$

Hence the intercept of the line on the y-axis is given as $y = (y_o - \alpha x_o)$

We note here that the intercept of any graph on the x-axis corresponds with the zero of the function that is the roots of the equation obtained by setting the function to zero.

3.2 Quadratic Functions

The function $f(x)$ is said to be a quadratic function of x if $f(x)$ is of the form $f(x) = \alpha x^2 + \beta x + \gamma$.

The graph is symmetrical about y-axis if $\beta = 0$ and has either a minimum or a maximum turning point depending on whether the constant $\alpha < 0$ or $\beta > 0$.

More shall be discussed on the sketching of graphs of function as the course progresses as a full discussion of the concept requires some knowledge of the calculus which shall be developed later in the programme. But for the meantime we shall go over the following that are basic requirement for the sketching of any graph.

3.4 Intercepts

For any given graph, we should have the idea of the intercepts on the coordinate axis.
In general, the function is represented as $y = f(x)$

The intercept on the y-axis is obtained by setting $x = 0$. That is the solution of $f(0)$.

From the above we have that for a linear function, we can have at most one intercept on the x-axis and a maximum of two x-axis intercepts in the case of the quadratic functions.

3.5 Slope (Gradient/Tangent)

In sketching the graph of a function, it is often helpful to know the slope of the function. This is especially useful in the case of a linear function. As can be seen from the representation of a linear function we have only two parameters (the slope and the intercept) the linear function can be sketched uniquely if we know any point on the graph together with its slope.

3.6 Symmetry

This property is a very invaluable idea about the nature of the graph. We note that the graph of a function $f(x)$ is symmetrical about the y-axis if the function is an even function of x. If however the function is odd function of x then the graph will be symmetrical about the origin.

At this point it is therefore necessary to explain the concept of even and odd functions of x. the function $f(x)$ is said to be an even function of x if $f(-x) = f(x)$ and odd function if $f(-x) = -f(x)$.

3.7 Limiting Values

This gives an approximation or estimation of the function $f(x)$ as x tends to some prescribed value. The knowledge of the above attribute is therefore the requirements for sketching the graph of any function $f(x)$.

4.0 Conclusion

It is believed that by now you already know what a graph is, and its components such as the slope and intercept.

5.0 Summary

You have learnt in this unit that:

a) A Graph- is a diagram showing the relationship between a dependent variable y and independent variable x.
b) A Linear function is $y = \alpha x + \beta$ Where α is a constant referred to as the gradient (slope) of the function y and β the intercept of the graph on the y axis.
c) Quadratic functions as $f(x) = \alpha x^2 + \beta x + \gamma$.
d) Intercepts.
e) Slope.
f) Symmetry.
g) Limiting value.

6.0 Tutor-Marked Assignments

1. Check if the following function are symmetry or not:
   a. $y = x^2$ b. $y = -x^2$
2. Define the following in your own way:
   a. Graph
   b. Gradient
   c. Symmetry
   d. Limiting values.

7.0 Reference/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 2

Unit 1

Limit of a Function

Contents

1.0 Introduction

The study of calculus will never be completed without discussing limit of a function in detail. So therefore, you are expected to carefully pay attention to this topic (limit).

2.0 Learning Outcomes

At the end of this unit you should be able:

1. State the limit of a function.
2. State the calculate the limit of a given function.
3. State the properties of limit.

3.0 Learning Content

3.1 Limit of a Function

Suppose $y = f(x)$ is a continuous function of x defined in some real $j = [\alpha, \beta]$. Suppose there exist a point $x_o \in (\alpha, \beta)$, then as we approach the point x_o along x-axis if $f(x)$ tends a particular value I as x tends to x_o. Mathematically this statement can be stated in an equivalent form thus,

Given an $\varepsilon > 0 \; \exists \delta(\varepsilon)$ such that

$|f(x) - I| < \varepsilon$ Whenever $|x - x_o| < \delta$

i.e $\lim \; f(x)_{x \longrightarrow x_0} = I$

For any continuous function $f(x)$ we have that the limit as x tends to a point x_o is

a. Zero
b. Constant k (say)
c. Infinity

3.2 Properties of Limits

Suppose $f(x)$ and $g(x)$ are functions defined in an interval $[\alpha, \beta]$ and x_0 any point in (α, β) Let $f(x)$ and $g(x)$ have limits F and G respectively as x tends to the point x_0 then we have the following:

a. $\lim_{x \longrightarrow x_o} [f(x) \pm g(x)] = \lim_{x \longrightarrow x_0} f(x) \pm \lim_{x \longrightarrow x_o} g(x) = F \pm G$

b. $\lim_{x \longrightarrow x_o} (f(x) \times g(x)) = \lim_{x \longrightarrow x_o} f(x) \times \lim_{x \longrightarrow x_o} g(x) = F \times G$

c. $\lim_{x \longrightarrow x_o} \left(\frac{f(x)}{g(x)} \right) = \left(\frac{\lim f(x)}{\lim g(x)} \right)_{x = x_o} = \frac{F}{G}$

d. $\lim_{x \longrightarrow xo} [kf(x)] = k \lim_{x \longrightarrow x_o} f(x) = kF$ (k=constant)

3.2 Examples

Compute the limit of the following functions at the specified points

a. $$\lim\left(\frac{x^2-1}{x-1}\right)_{x\longrightarrow 1}$$

Solution

To take the limit of m a rational (quotient) functions we first clear out the common factors between the numerator and the denominator where they exist making sure that the denominator is completely factorized.

Now $f(x)=\left(\frac{x^2-1}{x-1}\right)=\frac{(x-1)(x+1)}{(x-1)}=(x+1)$

Hence $\lim\left(\frac{x^2-1}{x-1}\right)_{x\longrightarrow x_o}=\lim_{x\longrightarrow x_0}(x+1)=1+1=2$

2. $$\lim_{u\longrightarrow 0}\left(\frac{u}{(2u-1)(2u+1)(2u+3)}\right)$$

Solution

In situations where our function is a rational function in which the denominator is resolvable into factors not in the numerator we resolve the function into partial fraction before taking the limits of the resulting sum functions.

Now $f(u)=\frac{u}{(2u-1)(2u+1)(2u+3)} = \frac{\alpha}{2u-1}+\frac{\beta}{2u+1}+\frac{\gamma}{2u+3}$

$$\Rightarrow \alpha(2u+1)(2u+3)+\beta(2u-1)(2u+3)+\gamma(2u-1)(2u+1)=u$$

On equating coefficients of like powers of u on both sides of the equation we have the following algebraic equations in the constants $\alpha,\beta, and\gamma$

$$4\alpha+4\beta+4\gamma=0$$
$$8\alpha+4\beta=1$$
$$3\alpha-3\beta-\gamma=0$$

Solving these three equations simultaneously we have that

$$\alpha=\frac{1}{16},\beta=\frac{1}{8}, and\gamma=\frac{3}{16}$$

Hence, $f(u)=\frac{1}{16}\left(\frac{1}{2u-1}+\frac{2}{2u+1}+\frac{3}{2u+3}\right)$

$$\Rightarrow \lim_{u\longrightarrow o} f(u)=\frac{1}{16}\lim_{u\longrightarrow 0}\left(\frac{1}{2u-1}+\frac{2}{2u+1}-\frac{3}{2u+3}\right)$$

$$= \frac{1}{16}\left\{\lim_{u \longrightarrow 0} \frac{1}{2u-1} + 2\lim_{u \longrightarrow 0} \frac{1}{2u+1} - 3\lim_{u \longrightarrow 0} \frac{1}{2u+3}\right\}$$

$$= \frac{1}{16}(-1+2-1) = 0$$

3. $\lim_{t \longrightarrow 0}\left(\frac{te^t + 2\cos t - \sin t - 2}{t^3}\right)$

Solution

Taking the limit directly we observe that we the indeterminate form i.e

$$\lim_{t \longrightarrow 0} f(t) = \frac{0}{0}$$

We shall therefore use the expansion method. This method requires us to expand the numerator in the series of t to degree of the denominator. That is

$$f(t) = \left(\frac{t\left(1+t+\frac{t^2}{2}+\frac{t^3}{6}\dots\dots\dots\right)+2\left(1-\frac{t^2}{2}+\frac{t^4}{24}\dots\dots\dots\right)-\left(t-\frac{t^3}{6}+\frac{t^5}{120}\dots\dots\dots\right)-2}{t^3}\right)$$

$$= \frac{7t^3}{6t^3} = \frac{7}{6} \Rightarrow \lim_{t \longrightarrow 0} f(t) = \frac{7}{6}$$

4. $\lim_{z \longrightarrow \infty}\left(\frac{1-2z^{-1}-3z^{-2}}{1-3z^{-1}-28z^{-2}}\right)^z$

Solution

$$f(z) = \left(\frac{(1-3z^{-1})(1+z^{-1})}{(1+4z^{-1})(1-7z^{-1})}\right)^z$$

On setting $-3z^{-1} = u^{-1}, 4z^{-1} = v^{-1} and -7z^{-1} = w^{-1},$ we have that

$$f(z) = \frac{(1+z^{-1})^z(1+u^{-1})^{-3u}}{(1+w^{-1})^{-7w}(1+v^{-1})^{4v}}$$

Now $\lim_{z \longrightarrow \infty} f(z) = \lim_{z \longrightarrow \infty} \frac{(1+z^{-1})^z(1+u^{-1})^{-3u}}{(1+v^{-1})^{-7w}(1+v^{-1})^{4v}}$

$$= \frac{\lim_{z \longrightarrow \infty}(1+z^{-1})^z \lim_{u \longrightarrow \infty}(1+u^{-1})^{-3u}}{\lim_{v \longrightarrow \infty}(1+v^{-1})^{4v} \lim_{w \longrightarrow \infty}(1+w^{-1})^{-7w}}$$

$$= \frac{e^1e^{-3}}{e^4e^{-7}} = e^1 = 2.718282$$

Note: it should be noted from the above transformations that $\lim_{y \longrightarrow \infty} [u, v, w] \longrightarrow \infty$

Self-Assessment Exercise(s) 1

1. $\lim \left(\frac{x^2 - 1}{x + 1} \right)_{x \longrightarrow 1}$
2. $\left(\frac{\sin z}{z} \right)$ as z $\longrightarrow 180$

4.0 Conclusion

At this juncture, we conclude that you have adequate knowledge of limit of a function and so therefore you are set for the next unit which is Differential Calculus.

5.0 Summary

You have learnt in this unit:

1. The Meaning of limit of a given function as, given an $\varepsilon > 0 \; \exists \delta(\varepsilon)$ such that $|f(x) - I| < \varepsilon$ Whenever $|x - x_o| < \delta$
2. The properties of limit of a function as given in the note
3. how to compute the limit of a given function by direct substitution, series method, and by factorization?

6.0 Tutor-Marked Assignments

Determine the limits of the following functions at the specified point on the abiscas

1. $\left(\frac{\sin z}{z} \right)$ as z $\longrightarrow 0$
2. $\frac{z^m - 1}{z - 1}$ as $z \longrightarrow 1$
3. $\frac{(u+2)^3 - \alpha^2}{u}$ as u tends to 0
4. $\frac{\sec u - \cos u}{\sin u}$ as $u \longrightarrow 0$

7.0 Reference/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B. D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Differential Calculus

Contents

1.0 Introduction

One of the most important topics in Mathematics is Differential Calculus. It is widely used in the field of Science and Technology. Therefore, you are expected to pay serious attention as we proceed.

2.0 Learning Outcomes

At the end of this unit, you should be able to:

1. Discuss the concept of derivatives
2. Evaluate the derivative of a function using the first principle

3.0 Learning Content

3.1 Differential Calculus

Let $f(x)$ be a function of an independent variable x defined in some real interval $J = [\alpha, \beta]$ then the derivative (differential coefficient or slope) of $f(x)$ with respect to x is the rate of change of $f(x)$ with respect to change in the variable x.

Differential calculus therefore is the mathematics of rate of change.

A function $f(x)$ is said to be differentiable with respect to x if the following limit called the Newton's quotient exists:

$$\lim_{h \longrightarrow o} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

This limit if it exists is called the derivatives of $f(x)$ with respect to x denoted as $f'(x)$ or $\frac{df}{dx}$.

Hence the derivative of $f(x)$ at any point x_o is given by

$$f'(x) = \lim_{x \longrightarrow x_0} \left(\frac{f(x) - f(x_o)}{x - x_o} \right) = \left(\frac{df}{dx} \right)_{x = x_o}$$

The above definition of differentiation by the used of infinitesimal increment is referred to as differentiation from first principles.

Examples

Obtain the derivative of the following functions from first principle

(1) $y(x) = \alpha x^n \; (\alpha = constant, n \in N)$

Solution

$y(x) = \alpha x$

$\Rightarrow$

$y(x+h) = \alpha (x+h)^n$

$$=\alpha\left(x^{n} + nhx^{n-1} + n(n-1)\frac{h^{2}}{2!}x^{n-2} + \dots\dots\dots + h^{n} \right)$$

i.e $y(x+h) - y(x) = \alpha\left(x^{n} + nhx^{n-1} + n(n-1)\frac{h^{2}}{2!}x^{n-2} + \dots\dots\dots + h^{n} \right) - \alpha x^{n}$

$$= \alpha\left(nhx^{n-1} + \frac{n(n-1)h^{2}x^{n-2}}{2!} + \dots\dots\dots\dots\dots + h^{n} \right)$$

But $\frac{dy}{dx} = \lim_{h \longrightarrow o}\left[\frac{y(x+h) - y(x)}{h} \right]$

i.e $\frac{dy}{dx} = \alpha \lim_{h \longrightarrow 0}\left(\frac{nhx^{n-1} + \frac{n(n-1)h^{2}x^{n-2}}{2!} + \dots\dots\dots\dots\dots\dots\dots\dots\dots + h^{n}}{h} \right)$

$$= \alpha \lim_{h \longrightarrow 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}hx^{n-2} + \dots\dots\dots\dots\dots\dots h^{n-1} \right]$$

$$= \alpha nx^{n-1}$$

2 $y(x) = x^{-r}$

Then $y(x+h) = (x+h)^{-r}$

$$y(x+h) - y(x) = (x+h)^{-r} - x^{-r}$$

$$= \left(\frac{1}{(x+h)^{r}} - \frac{1}{x^{r}} \right)$$

$$\frac{dy}{dx} = \lim_{h \longrightarrow 0}\left(\frac{1}{(x+h)^{r}} - \frac{1}{x^{r}} \right)$$

$$= \lim_{h \longrightarrow 0}\left(\frac{1}{h} \right)\left[\frac{x^{r} - (x+h)^{r}}{x^{r}(x+h)^{r}} \right]$$

$$= \lim_{h \longrightarrow 0}\left[x^{r} - \left(x^{r} + rhx^{r-1} + r(r-1)\frac{h^{2}}{2!}x^{r-2} + \dots\dots\dots\dots\dots\dots h^{r} \right) \right] \Big/ h$$

$$= \lim_{h \longrightarrow 0}\left[\left(-rx^{-r-1} - r(r-1)\frac{h}{2!}x^{r-2} + \dots\dots\dots\dots\dots\dots h^{r-1} \right) \right]$$

$$= -rx^{-r-1}$$

3. $y = uv$ Where u and v are both functions of x.

Solution

Note that $y(x+h)=u(x+h).v(x+h)$

$y(x+h)-y(x)=[u(x+h).v(x+h)-u(x)v(x)]$

$$\lim_{h\longrightarrow o}\left(\frac{y(x+h)-y(x)}{h}\right)$$

$$=\lim_{h\longrightarrow 0}[(u(x+h).v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x))]\left(\frac{1}{h}\right)$$

$$=\lim_{h\longrightarrow 0}[(u(x+h)-u(x)).v(x+h)]\left(\frac{1}{h}\right)+\lim_{h\longrightarrow o}[v(x+h)-v(x)]u(x)\left(\frac{1}{h}\right)$$

$$=v(x).u'(x)+u(x).v'(x)$$

4. $y(x)=u(x)+v(x)$

Then $y(x+h)=u(x+h)+v(x+h)$

$$y(x+h)-y(x)=[u(x+h)+v(x+h)]-[u(x)+v(x)]$$

$$\lim_{h\longrightarrow o}\left[\frac{y(x+h)-y(x)}{h}\right]$$

$$=\lim_{h\longrightarrow o}\left[(u(x+h)+v(x+h)-(u(x)+v(x)))\frac{1}{h}\right]$$

$$=\lim_{h\longrightarrow o}\left[\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\right]$$

$$=\lim_{h\longrightarrow o}\left[\frac{u(x+h)-u(x)}{h}\right]+\lim_{h\longrightarrow o}\left[\frac{v(x+h)-v(x)}{h}\right]$$

$$=u'(x)+v'(x)$$

5. $y(x)=\dfrac{u(x)}{v(x)}$

$$y(x+h)=\frac{u(x+h)}{v(x+h)}$$

$$y(x+h)-y(x)=\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}$$

$$= \frac{u(x+h).v(x) - u(x)v(x+h)}{v(x)v(x+h)}$$

$$= \frac{u(x+h)v(x) + u(x)v(x) - u(x)v(x+h)}{v(x)v(x+h)}$$

$$\lim_{h \longrightarrow o} \left(\frac{y(x+h) - y(x)}{h} \right)$$

$$= \lim_{h \longrightarrow o} \left(\frac{u(x+h).v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x)v(x+h)} \right)$$

$$\lim_{h \longrightarrow o} \left(\frac{\frac{[u(x+h) - u(x)]v(x)}{h} - \left[\frac{v(x+h) - v(x)}{h} \right] u(x)}{v(x)v(x+h)} \right)$$

$$= \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{v^2}$$

6. $y(x) = \sin kx,$

Where k is a constant

Solution

Given that $y(x) = \sin kx, \Rightarrow y(x+h) = \sin k(x+h)$

Hence, $y(x+h) - y(x) = \sin k(x+h) - \sin kx$

But $\frac{dy}{dx} = \lim_{h \longrightarrow o} \left(\frac{y(x+h) - y(x)}{h} \right) = \lim_{h \longrightarrow o} \left(\frac{\sin k(x+h) - \sin kx}{h} \right)$

Note that : $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$\frac{dy}{dx} = \lim_{h \longrightarrow o} \left(\frac{s\cos\left[k\left(x + \frac{h}{2}\right)\right]\sin\left[\frac{kh}{2}\right]}{h} \right)$$

$$= 2\lim\left(\cos\left[k\left(x + \frac{h}{2}\right)\right]\right)_{h \longrightarrow o} \times \lim\left(\frac{\sin k\left(\frac{h}{2}\right)}{h} \right)_{h \longrightarrow o}$$

$$=2 \lim\left(\cos\left[k\left(x + \frac{h}{2}\right)\right]\right)_{h \longrightarrow o} \times \frac{k}{2}$$

$=k \cos kx$

Self-Assessment Exercise(s) 1

1. Using the fist principle calculate the derivative of the following functions.
   a. $y(x) = \sin x + 2x$
   b. $y(u) = 6u^3$

4.0 Conclusion

At this juncture, we hereby conclude that you have acquired adequate knowledge on how to compute the derivative of a given function using the first principle and that you are set to build on this skill.

5.0 Summary

In this unit, you have that:

A function $f(x)$ is said to be differentiable with respect to x if the following limit called the Newton's quotient exists:

$$\lim_{h \longrightarrow o} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

This limit if it exists is called the derivatives of $f(x)$ with respect to x denoted as $f'(x)$ or $\frac{df}{dx}$.

Hence the derivative of $f(x)$ at any point x_o is given by $f'(x) = \lim_{x \longrightarrow x_0} \left(\frac{f(x) - f(x_o)}{x - x_o} \right) = \left(\frac{df}{dx} \right)_{x = x_o}$

- How to compute the derivative of a given function using the first principle.

6.0 Tutor-Marked Assignments

1. Compute the first derivative of the following functions:
   a. $y(x) = 3x^3$
   b. $y(x) = \tan x$
   c. $y(x) = \cos x$

7.0 Reference/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

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CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 3

Unit 1	Further Problem in Differentiation
Unit 2	Inverse and Parametric Functions

Unit 1

Further Problems in Differentiation

Contents

1.0 Introduction

In this unit, we shall introduce you to different techniques of carrying out the derivative of a function depending on how the function is.

2.0 Learning Outcomes

At the end of this unit, you should be able to:

1. apply the product rule
2. apply the quotient rule
3. apply the sum of function.

3.0 Learning Content

3.0 Differentiation of Sum, Product and Quotient Functions

Determine the derivatives of the following functions:

1. $y = \frac{3Sinx - Sin^3 x}{3}$

Solution

Note that this is an example of a sum function.

Hence

$$\frac{dy}{dx} = \frac{d}{dx}(Sinx) - \frac{1}{3}\frac{d}{dx}(Sin^3 x) = Cosx - \frac{1}{3}(3CosxSin^2 x)$$

$$= Cosx - (CosxSin^2 x) = Cosx(1 - Sin^2 x) = CosxCos^2 x$$

i.e. $\frac{dy}{dx} = Cos^3 x$

2. $y = x^3\sqrt{(1 + Cot^2 x)}$

Note that this is an example of Product function.

$$y = x^3\sqrt{1 + \frac{Cos^2 x}{\sin^2 x}} = x^3\sqrt{\frac{Sin^2 x + Cos^2 x}{Sin^2 x}} = x^3\sqrt{\frac{1}{Sin^2 x}} = x^3\frac{\sqrt{1}}{\sqrt{Sin^2 x}} = \frac{x^3}{Sinx}$$

$$\Rightarrow y(x) = \frac{x^3}{Sinx}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 Sinx - x^3 Cosx}{Sin^2 x}$$

$$= x^2(3Cosec\,x - xCotxCosecx)$$

$$= x^2 Cosec\,x(3 - xCotx)]$$

3 Given that $y=\sqrt{\frac{1+Sinx}{1-Sinx}}$ show that $\frac{dy}{dx}=\frac{1}{1-Sinx}$

Solution

Let $y=\sqrt{\frac{1+Sinx}{1-Sinx}}=\sqrt{u}$ where u$=\frac{1+Sinx}{1-Sinx}$

Now $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{u}}\times\frac{du}{dx}$

Using the quotient rule already mentioned earlier we note that $\frac{du}{dx}$ is given as;

$$\frac{dy}{dx}=\frac{Cosx(1-Sinx)+Cosx(1+Sinx)}{(1-Sinx)^2}=\frac{2Cosx}{(1-Sinx)^2}$$

Therefore, the differential coefficient of the function y is given as;

$$\frac{dy}{dx}=\frac{2Cosx}{(1-Sinx)^2}\times\frac{1}{2}\sqrt{\frac{(1-Sinx)}{(1+Sinx)}}$$

$$=\frac{Cosx}{(1-Sinx)^2}\times\frac{\sqrt{1-Sinx}}{1+Sinx}=\frac{\sqrt{1-Sin^2x}}{(1-Sinx)^2}\times\frac{\sqrt{1-Sinx}}{\sqrt{1+Sinx}}$$

$$=\frac{\sqrt{(1+Sinx)(1-Sinx)}}{(1-Sinx)^2}\times\frac{\sqrt{1-Sinx}}{\sqrt{1+Sinx}}$$

$$=\frac{\sqrt{(1-Sin^2x)}}{(1-Sinx)^2}\times\frac{\sqrt{(1-\sin x)}}{\sqrt{(1+Sinx)}}=\frac{1-Sinx}{(1-Sinx)^2}=\frac{1}{1-Sinx}$$

3. If $y(x)=Tanx(1+2Sec^2x)-3xSec^2x,$ Show that the differential coefficient of y is given as $\frac{dy}{dx}=6Sec^2xTanx(Tanx-x)$

Solution

We note that;

$$y(x)=Tanx(1+2Sec^2x)-3xSec^2x,$$

$$=Sec^2x(2Tanx-3x)+Tanx$$

Therefore $\frac{dy}{dx}$ is given by the expression;

$$\frac{dy}{dx}=2Secx\frac{d}{dx}[Secx]\times(2Tanx-3x)+Sec^2x(2Sec^2x-3)+Sec^2x$$

$$=2Sec^2xTanx(2Tanx-3x)+Sec^2x(2Sec^2x-3)+Sec^2x$$

$= Sec^2x[4Tan^2x - 6xTanx + 2Sec^2x - 3 + 1]$

$= Sec^2x[4Tan^2x - 6xTanx + 2(Sec^2x - 1)]$

$= Sec^2x[4Tan^2x - 6xTanx + 2Tan^2x]$

$= 6Sec^2xTanx(Tanx - x)$

4 Given that $y(x) = \frac{Sinx}{x^2}$ show that ; = $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2 + 2)y = 0$

Solution

Given that $y(x) = \frac{Sinx}{x^2}$, then

$$\frac{dy}{dx} = \frac{x^2Cosx - 2xSinx}{x^4} \equiv \frac{xCosx - 2Sinx}{x^3}$$

$$\frac{d^2y}{dx^2} = \frac{(Cosx - xSinx - 2Cosx)x^3 - 3x^2(xCosx - 2Sinx)}{x^4}$$

$$= \frac{6Sinx - 4xCosx - x^2Sinx}{x^4}$$

$\therefore$ the expression $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2 + 2x)y$ is given as

$$x^2\left[\frac{6Sinx - 4xCosx - x^2Sinx}{x^4}\right] + 4x\left[\frac{xCosx - 2Sinx}{x^3}\right] + (x^2 + 2)\frac{Sinx}{x^2}$$

$$= \frac{6Sinx - 4xCosx - x^2Sinx + 4xCosx - 8Sinx + x^2Sinx + 2Sinx}{x^2}$$

$$= \frac{(6 - 8 + 2)Sinx - (4x - 4x)Cosx - (x^2 - x^2)Sinx}{x^2} = \frac{0}{x^2} = 0$$

Self-Assessment Exercise(s) 1

1. Find the differential coefficient of: y =$3x^2$sin 2x
2. Find the rate of change of y with respect to x given $y = 3\sqrt{x}\,ln2x$
3. Find the derivative of y given that $y = \frac{4sin5x}{5x^2}$

4.0 Conclusion

At this juncture, we therefore conclude that you have acquired adequate knowledge on how to carry out the derivatives of a function using the various principles of differentiation depending on the kind of function given.

5.0 Summary

In this unit, you have learnt that when function y is of the form:

1. $y = uv$, where u and v are each functions of a variable. We apply the product rule.
2. $y = \frac{u}{v}$ we apply the quotient rule.

You have also learnt that the solution to a given differential equation satisfied the differential equation.

6.0 Tutor-Marked Assignments

Differentiate the following function:

1. $y = \frac{x^4 - 3x^3 - 4x^2 + 5}{x^2}$
2. $y = (2x+3)^3(4x^2 - 1)^2$
3. $y = (1+x)(2+x)(3+x)$
4. $y = \frac{x+1}{x-1}, x \neq 1.$

7.0 Reference /Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Inverse and Parametric Functions

Contents

1.0 Introduction

In other for you to be well grounded in the knowledge of differential calculus, we shall treat the above name topic in this unit. We therefore advise you to pay serious attention to this topic.

2.0 Learning Outcomes

At the end of this unit you should be able to:

1. Find the derivative of an inverse function.
2. Find the derivative of a parametric function.

3 Learning Content

3.1 Inverse and Parametric Function

3.1.2 Inverse Functions

To any function $y(x)$ defined in a domain D there may exists an inverse function $g(y)$ whose domain of definition D is the co-domain of $y(x)$. The function $y(x)$ has an inverse iff in a given interval $[a,b]$ $y(x)$ is a strictly increasing or decreasing function.

A function $f(x)$ defined in an interval $[a,b]$ is said to be strictly increasing in $[a,b]$ if given some $\varepsilon > 0$ then $f(x+\varepsilon) > f(x)$. It is said to be strictly decreasing if $f(x+\varepsilon) < f(x)$

Examples

1. If $x = yTany$; compute y'

Solution

We consider the inverse function $u = Tan^{-1}x$

Given that $u = Tan^{-1}x$

Then $x = Tanu$

$$\frac{dx}{du} = Sec^2u = 1 + Tan^2u = 1 + x^2$$

But $\frac{du}{dx} = \left(\frac{dx}{du}\right)^{-1} = \frac{1}{1+x^2}$

Therefore, $\frac{du}{dx} = \frac{1}{1+x^2}$

Hence given that $x = Tany$, then $\frac{dx}{dy} = Tany + ySec^2y$

But $Tany = \frac{x}{y} and Sec^2 y = \frac{y(x^2 + y^2)}{y^2}$

$\therefore \frac{dy}{dx} = \frac{y}{x + x^2 + y^2}$

2 If $y = Sin^{-1}(3x - 4x^3)$ then show that $\sqrt{(1 - x^2)} \frac{dy}{dx} = 3$

Solution

Given that $y = Sin^{-1}(3x - 4x^3)$ we thus have that ;

$3x - 4x^3 = Siny$ and $(3 - 12x^2)dx = Cosydy$

i.e.

$$\frac{dy}{dx} = \frac{3 - 12x^2}{Cosy} = \frac{3(1 - 4x^2)}{Cosy}$$

Recall that $Cos\theta \equiv \sqrt{1 - Sin^2\theta}$ and by our problem definition we have $Siny = 3x - 4x^3$,

We therefore have;

$$\frac{dy}{dx} = \frac{3(1 - 4x^2)}{\sqrt{1 - 9x^2 + 24x^4 - 16x^6}} = \frac{3(1 - 4x^2)}{\sqrt{(1 - x^2)(16x^4 - 8x^2 + 1)}}$$

$$= \frac{3(1 - 4x^2)}{\sqrt{(1 - x^2)(1 - 4x^2)}} \equiv \frac{3(1 - 4x^2)}{(1 - 4x^2)\sqrt{1 - x^2}} = \frac{3}{\sqrt{1 - x^2}}$$

Hence; $\sqrt{1 - x^2} \frac{dy}{dx} = \sqrt{1 - x^2} \times \frac{3}{\sqrt{1 - x^2}} = 3$

3 If $\sqrt{y} = \tan^{-1} x,$ show that y satisfies the following differential equation:

$$(1 + x^2)\frac{d}{dx}\left\{(1 + x^2)\frac{dy}{dx}\right\} = 2$$

Solution

It suffices to show that:

Given that $\sqrt{y} = \tan^{-1} x;$ we thus have that $x = \tan\sqrt{y}$

i.e.

$$dx = \frac{Sec^2\sqrt{y}dy}{2\sqrt{y}}$$

$$\frac{dy}{dx}=\frac{2\sqrt{y}}{Sec^2\sqrt{y}}\equiv\frac{2\sqrt{y}}{1+\tan^2\sqrt{y}}=\frac{2\sqrt{y}}{1+x^2}\equiv\frac{2\tan^{-1}x}{1+x^2}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{2\left(\frac{1}{1+x^2}\right)(1+x^2)-4x\tan^{-1}x}{(1+x^2)^2}\equiv\frac{2}{(1+x^2)^2}-\frac{4x\tan^{-1}x}{(1+x^2)^2}$$

Hence,

$$(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}-2=(1+x^2)^2\left[\frac{2}{(1+x^2)^2}-\frac{4x\tan^{-1}x}{(1+x^2)^2}\right]+\frac{2x(1+x^2)\times2\tan^{-1}x}{1+x^2}-2$$

$$=2-4x\tan^{-1}x+4x\tan^{-1}x-2=0$$

3.1.2 Parametric Functions

In some cases, it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable, e.g. $y=\cos 2t, x=\sin t.$ in this case, any value we give to t will produce a pair of values for x and y. The third variable, t is called a parameter, and the two expressions for x and y are called parametric equations.

Examples

1 if the variables x and y are defined parametrically as;

$$x=a(\theta-Sin\theta); y=a(1-Cos\theta)$$

Show that $1+\left(\frac{dy}{dx}\right)^2=Cosec^2\frac{\theta}{2}$

Solution

From the definitions above we have that;

$$\frac{dy}{d\theta}=a(1-Cos\theta); \frac{dy}{d\theta}=aSin\theta$$

Now; $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{dy}{dx}\div\frac{dx}{d\theta}=\frac{aSin\theta}{a(1-Cos\theta)}=\frac{Sin\theta}{1-Cos\theta}$

$$=\frac{Sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right)}{1-\left(Cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right)\right)}=\frac{2Cos\frac{\theta}{2}Sin\frac{\theta}{2}}{1-Cos^2\frac{\theta}{2}+Sin^2\frac{\theta}{2}}$$

Therefore;

$$1+\left(\frac{dy}{dx}\right)^2\equiv1+\left(Cot\frac{\theta}{2}\right)^2$$

$$=1+\frac{Cos^2\frac{\theta}{2}}{Sin^2\frac{\theta}{2}}=\frac{Cos^2\frac{\theta}{2}+Sin^2\frac{\theta}{2}}{Sin^2\frac{\theta}{2}}=\frac{1}{Sin^2\frac{\theta}{2}}$$

$$=Co\sec^2\frac{\theta}{2}$$

2 if $x=3t+t^3; y=3-t^{\frac{5}{2}}$, prove that when $\frac{d^2y}{dx^2}=0$ then x has one of the values $0,\pm 6\sqrt{3}$

Solution

From the above definitions;

$$\therefore \frac{dy}{dt}=3\left(1+t^2\right), \frac{dy}{dt}=-\frac{5}{2}t^{\frac{3}{2}}$$

$$\therefore \frac{dy}{dx}=-\frac{5}{2}\left(\frac{t^{\frac{3}{2}}}{3\left(1+t^2\right)}\right)=-\frac{5}{6}\left(\frac{t^{\frac{3}{2}}}{1+t^2}\right)$$

$$\frac{d^2y}{dx^2}=-\frac{5}{6}\left(\frac{\frac{3}{2}t^{\frac{1}{2}}\left(1+t^2\right)-2t^2}{\left(1+t^2\right)^2}\right)=-\frac{5}{12}\left(\frac{3t^{\frac{1}{2}}+3t^{\frac{5}{2}}-4t^{\frac{5}{2}}}{\left(1+t^2\right)^2}\right)$$

$$=-\frac{5}{12}\left(\frac{3t^{\frac{1}{2}}-t^{\frac{5}{2}}}{\left(1+t^2\right)^2}\right)$$

Hence the condition that $\frac{d^2y}{dx^2}=0$ implies that;

$$-\frac{5}{2}\left(\frac{3t^{\frac{1}{2}}-t^{\frac{5}{2}}}{\left(1+t^2\right)^2}\right)=0$$

i.e.

$$5\left(3t^{\frac{1}{2}}-t^{\frac{5}{2}}\right)=0=5t^{\frac{1}{2}}\left(3-t^2\right)$$

This gives the corresponding values of t as $t=0,\pm\sqrt{3}$

Recalling that $x=3t+t^3$, the corresponding values of x that satisfy the condition $\frac{d^2x}{dx^2}=0$ are therefore $x=0,\pm 6\sqrt{3}$

3 if the variables x and y are defined parametrically as :

$x = a(\theta + Sin\theta); y = a(1 - Cos\theta)$, obtain the expressions for $\frac{dy}{dx}$ in term of the half angle $\frac{\theta}{2}$ (a is a constant).

Solution

From the defining equation, $x = a(\theta + Sin\theta); y = a(1 - Cos\theta)$. We therefore have the following:

$$\frac{dx}{d\theta} = a(1 + Cos\,\theta); \frac{dy}{d\theta} = aSin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{aSin\theta}{a(1 + Cos\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{aSin\theta}{a(1 + Cos\theta)} \equiv \frac{Sin\theta}{1 + Cos\theta} = \frac{Sin\left(\frac{\theta}{2} + \frac{\theta}{2}\right)}{1 + Cos\left(\frac{\theta}{2} + \frac{\theta}{2}\right)} = \frac{2Cos\frac{\theta}{2}Sin\frac{\theta}{2}}{1 + Cos^2\frac{\theta}{2} - Sin^2\frac{\theta}{2}}$$

$$= \frac{2Cos\frac{\theta}{2}Sin\frac{\theta}{2}}{Cos^2\frac{\theta}{2} + Sin^2\frac{\theta}{2} + Cos^2\frac{\theta}{2} - Sin^2\frac{\theta}{2}} = \frac{Cos\frac{\theta}{2}Sin\frac{\theta}{2}}{Cos^2\frac{\theta}{2}} = \frac{Sin\frac{\theta}{2}}{Cos\frac{\theta}{2}} = Tan\frac{\theta}{2}$$

$$\Rightarrow \frac{dy}{dx} = Tan\frac{\theta}{2}$$

Self-Assessment Exercise(s) 1

1. The parametric equations of a function are given by y =3 cos 2t, x =2 sin t. Determine expressions for (a)$\frac{dy}{dx}$ $(b)\frac{d^2y}{dx^2}$
2. Given that $x = 5\theta - 1$ and $y = 2\theta(\theta - 1)$ determine $\frac{dy}{dx}$ in terms of θ

4.0 Conclusion

At this juncture we therefore conclude that, you have acquired adequate knowledge on how to compute the derivatives of an inverse function and parametric functions. We therefore advise that you consult the reference materials below for more knowledge.

5.0 Summary

In this unit, you have learnt that:

1. To any function $y(x)$ defined in a domain D there may exists an inverse function $g(y)$ whose domain of definition D is the co-domain of $y(x)$. The function $y(x)$ has an inverse if in a given interval $[a,b]$ $y(x)$ is a strictly increasing or decreasing function.

2. In some cases, it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable, e.g. $y = \cos 2t, x = \sin t.$ in this case, any value we give to t will produce a pair of values for x and y. The third variable, t is called a parameter, and the two expressions for x and y are called parametric equations.

6.0 Tutor-Marked Assignments

1. Compute $\frac{dy}{dx}$ of the following parametric equations:
   a. $x = \frac{2-3t}{1+t}$, $y = \frac{3+2t}{1+t}$
   b. $y = 3Sin\theta - Sin^3\theta, x = Cos^3\theta$
2. Find $\frac{dy}{dx}$ of $\sqrt{y} = Sin^{-1}x$

7.0 Reference/Further Readings

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B. D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 4

Unit 1 Extreme Curve Sketching
Unit 2 Integration as an Inverse of Differentiation

Unit 1

Extreme Curve Sketching

Contents

1.0 Introduction

As was pointed out at the beginning of lecture note in order to use calculus to solve practical problems arising from real life situation, the problem must first be translated into mathematical expression involving real variables the relationship of which must be firmly established. In investigating some representative procedures there is often need for sketching of the graphs for such physical quantities. This sketch is very useful for illustrating some salient properties of the physical quantity represented by the functions. A sketch which is not a detail drawing is nonetheless to indicate key points and a general characteristic which is the main objective of this section are things related to the sign of the function, sign of the derivative and sign of the second-order derivative of the function. These key points and general characteristics that are illustrated however depend on the properties of interest of the physical quantity the function represents. Since curve sketching is a mere representation of the curve without going through the troubles of computing the ordinates then a sketch to be a true representation of the actual graph certain very important characteristics of the function must be put into consideration while attempting to sketch the graph. Some of the most important of such points to consider are explained below:

2.0 Learning Outcomes

At the end of this unit you should be able to determine the following on a curve sketch:

1. Intercept of a graph
2. Symmetry
3. Turning Points
4. Discontinuity
5. Imaginary Points
6. The Origin

3.0 Learning Contents

3.1 Intercepts

One of the most important points to be consider in sketching any graph is the intercept of the graph on the co-ordinates axes. If for instance the curve of interest is given as $y = f(x)$.. Then since the y-axis is the straight line given as $x = 0$

Then $f(0)$ gives the intercept of the graph on the ordinate. Similarly, we recall that $x-$axis is the straight line given as $y = 0$.The intercept on this axis is therefore given as $f(x) = 0$.

3.2 Symmetry

It is again very important to have the knowledge of this property of a function to facilitate sketching the graph. There are certain class of polynomial functions that comprise of only even powers of the independent variable and other still comprising of only odd powers of the independent variable. In the former case the function is symmetric about the ordinate and the latter exhibit symmetry about the origin.

It should be noted that when a polynomial function $f(x)$ comprise of only even powers of the independent variable then $f(-x) = f(x)$. This class of function is referred to as even functions. Such functions are symmetrical about the ordinate while there is yet the other class of polynomial that comprise of only odd powers of the independent variable functions in which $f(-x) = -f(x)$ .This functions like these are generally symmetrical about the origin and are referred to as odd function of the independent variable X.

It should be noted however that not all functions belong to this special class of functions but certain functions can be resolved into the sum of odd and even functions. Therefore, once it is established that a function is symmetrical about an axis (origin) we need just sketch the half plane and induce it into the other half plane.

3.3 Turning Points

As it was already established in the previous section of this lecture note on extremer problems, the knowledge of the turning points of a function is very important in sketching the graph of the function. It is already known that points where the first-order derivative $\frac{dy}{dx}$ of the function $y(x)$ vanishes constitute the turning points of $y(x)$.Hence the pre-knowledge of this provides is with the needed information about the convexity of the function $y(x)$.

3.4 Discontinuity

Let $y(x)$ be our function of interest. If in the domain of $y(x)$ there is a point x_0 at which this magnitude of the function suddenly becomes infinitely large then such points much be given special attention while sketching the graph as at such points the right-hand limits and the left-hand limits become distinct from each other. Such points like these are referred to as the points of the discontinuity of the function $y(x)$.

3.5 Imaginary Points

Suppose $y(x)$ is a function we want to sketch. Then any value x_0 of the independent variable that makes $y^2 < 0$ corresponds with the imaginary points of the function $y(x)$. In this case no real points occur on the curve.

Self-Assessment Exercise

1. locate the turning point on the curve $y = 3x^2 - 6x$and determine its nature by examining the sign of the gradient on either side

3.6 The Origin

To investigate whether a curve y passes through the origin we need to study the value of the first-order derivative $\frac{dy}{dx}$ of the curve for small values of x and y.

We understand that the first-order derivative $\frac{dy}{dx}$ is a measure of the slope of the tangent to the curve. Therefore, a small value of this quantity points to the fact that the curve lies near the x-axis while a large value indicates its nearness to y-axis. A value of $\frac{dy}{dx}$ close to infinity implies that the tangent to the curve at the origin approximately bisects the angle between the axes.

Examples

(1). Sketch the graph of the function given by $y = x(x+1)(x-2)$

Solution

(i). Intercepts:

From the given function $y = x(x+1)(x-2)$

$y(0) = 0$

Indicating that the curve passes through the origin. That is the curve passes through the point (0,0)

(a) $x-$axis

The intercepts on the $x-$axis as was discussed previously are the points that satisfy the equation:

$$y = x(x+1)(x-2) = 0$$

These points are clearly the points $x = 0, -1, 2$

(b) $y-$axis

The intercepts of the function on the ordinate are obtained by setting $x = 0$
In the defining equation. The only point therefore that the curve intercept the intercepts the y =0

(ii) Symmetry

The function $y = x(x+1)(x-2)$ contains both even and odd powers of x and so is neither even nor odd function. Therefore, the curve exhibits no symmetry about the ordinate or the origin.

(c) Discontinuity

Clearly, the function $y = x(x+1)(x-2)$ is continuous for all finite values of the independent variable x

(d) Turning Point s

You now turn your attention to the bounds of the function by considering the behavior of the function at the intercept and at the turning points.
You recall that the turning points of the function are indicated by the points where the first-order derivative $\frac{dy}{dx}$ of the function y vanishes.

$$y = x(x+1)(x-2)$$

Now, $$\frac{dy}{dx} = (x+1)(x-2) + x(x+1) + (x-2)$$

$$= 3x^2 + 2x - 2$$

But, at the turning points are the points that satisfy the quadratic equation;

$$3x^2 - 2x - 2 = 0$$

i.e, $$x = \frac{2 \pm \sqrt{4+24}}{6} = \frac{2 \pm 2\sqrt{7}}{6}$$

Hence, the turning points of the functions are the points;

$$x_1 = \frac{1-\sqrt{7}}{3} \text{ and } x_2 = \frac{1+\sqrt{7}}{3}$$

We observe here that -1< x_1<0 and that 1< x_2 <2

We now investigate the nature of the turning points by investigating the sigh of the second-order derivative $\frac{d^2y}{dx^2}$ of the function y at its turning points x_1 and x_2 .

Recall that $\frac{dy}{dx} = 3x^2 - 2x - 2$

Therefore $\frac{d^2y}{dx^2} = 6x - 2$

Similarly, $y_{\min} = y(x_2)$

i.e,

$$y_{\min} = \left(\frac{1+\sqrt{7}}{3}\right)^3 - \left(\frac{1+\sqrt{7}}{3}\right)^2 - 2\left(\frac{1+\sqrt{7}}{3}\right)$$

$$\approx -2.1126$$

Finally we investigate the bounds of the function $y(x)$.This is be set done by investigating the behavior of $y(x)$ as;

(i) $x \to \infty$ from the intercept at $x = 2$

(ii) $x \to -\infty$ from the intercept at $x = -1$

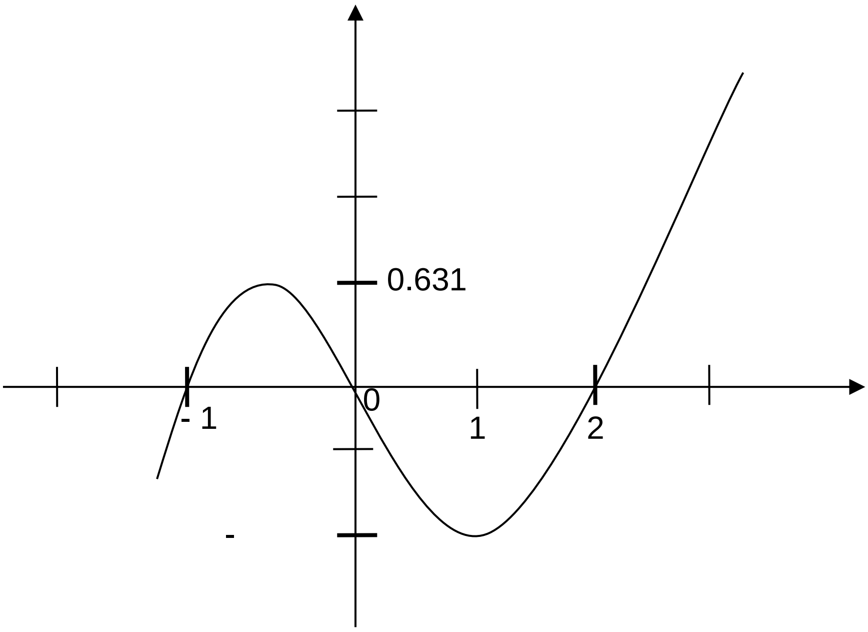
In the first case recall that $y = x^3 - x^2 - 2x$

$\therefore$ Lim ($(y(x))_{x\to\infty} = \lim(x^3 - x^2 - 2x)_{x\to\infty} = \infty$)

On the other hand the limit of the function at the other intercept is obtained as follows:

$$\lim(\ y(x))_{x\to\infty} = \lim(\ x^3 - x^2 - 2x)_{x\to\infty} = -\infty$$

$y = x(x+1)(x-2)$



(2) Sketch the graph of the function $y = \dfrac{x^2 - 5x + 4}{x^2 - 5x + 6}$

Solution

To sketch the graph of any rational function $y = \dfrac{p(x)}{q(x)}$ it is always necessary to investigate the point(s) x_0 within the domain of the $x = x_k$ constitute the vertical asymptotes of the function y.

Then from the function $y = \dfrac{(x-1)(x-4)}{(x-2)(x-3)}$

In this function it is clear to see that the lines $x = 2$ and $x = 3$ are the vertical asymptotes of the graph.

Intercept

We recall that the intercepts on the $x - axis$ are obtained from the equation $y = 0$.

Hence the intercepts of the function y and $x - axis$ is given as; $x = 1{,}4$

Similarly, the intercept on the ordinate is given by $y(0)$. From $y = \dfrac{(x-1)(x-4)}{(x-2)(x-3)}$

We then obtain as the intercept on this axis as $\dfrac{2}{3}$

Turning Points

Given that $y = \frac{x^2 - 5x + 4}{x^2 - 5x + 6}$

Then,

$$\frac{dy}{dx} = \frac{(2x-5)(x^2 - 5x + 6) - (2x-5)(x^2 - 5x + 4)}{(x^2 - 5x + 6)^2} = \frac{2(2x-5)}{(x^2 - 5x + 6)^2}$$

But at the turning points of y the first-order derivative $\frac{dy}{dx}$ of the function vanishes.

Hence, the turning point of the function is the point $x = \frac{5}{2}$

Nature of turning point

=-4$\left[\frac{(3x^2 - 15x + 19)}{(x^2 - 5x + 6)^3}\right]$

Hence,

$$\left(\frac{d^2 y}{dx^2}\right)_{x=\frac{5}{2}} = -4\left[\frac{(3x^2 - 15x + 19)}{(x^2 - 5x + 6)^3}\right]_{x=\frac{5}{2}} = 64 > 0$$

Hence, the turning point is a minimum turning point.

$$y_{\min} = y\left(\frac{5}{2}\right) = 9$$

Symmetry/Bounds

Consider the implicit expression $y(x^2 - 5x + 6) = (x^2 - 5x + 4)$

i.e.

$$(y-1)x^2 - 5(y-1) + 6y - 4 = 0$$

This expression contains both evens and odd powers of the variable the x and hence the function is either and even and odd function the graph therefore exhibit no symmetry either about nor symmetrical about the origin. Again, from the implicit equation above we recall that for real values of the

25$(y-1)^2 \geq 4(y-1)(6y-4)$

Hence, the bounds of function y is given as

$(y-1)(y-9) < 0$

Showing that the function y cannot assume values between 1 and 9.

We then investigate the behavior of the function at (i). as $x \to \infty$ from 4

(ii). the right and left-hand limits at $x = 3$

(iii). the right and left-hand limits at $x = 2$

(iv). As $x \to -\infty$ from 1

(i) Recall that $y = \dfrac{x^2 - 5x + 4}{x^2 - 5x + 6}$

Hence,

(ii) $$(\lim y)_{x \to 3^+} = \left[\lim\left(\frac{(3+\varepsilon)^2 - 5(2+\varepsilon) + 4}{(2+\varepsilon)^2 - 5(2-\varepsilon) + 6}\right)\right]_{\varepsilon \to 0} = \lim\left[1 - 2\left(\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon}\right)\right]_{\varepsilon \to 0} = -\infty$$

$$(\lim y)_{x \to 3^-} = \left[\lim\left(\frac{(3-\varepsilon)^2 - 5(3-\varepsilon) + 4}{(3-\varepsilon)^2 - 5(3-\varepsilon) + 6}\right)\right]_{\varepsilon \to 0} = \lim\left[1 + 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon - 1}\right)\right]_{\varepsilon \to 0} = +\infty$$

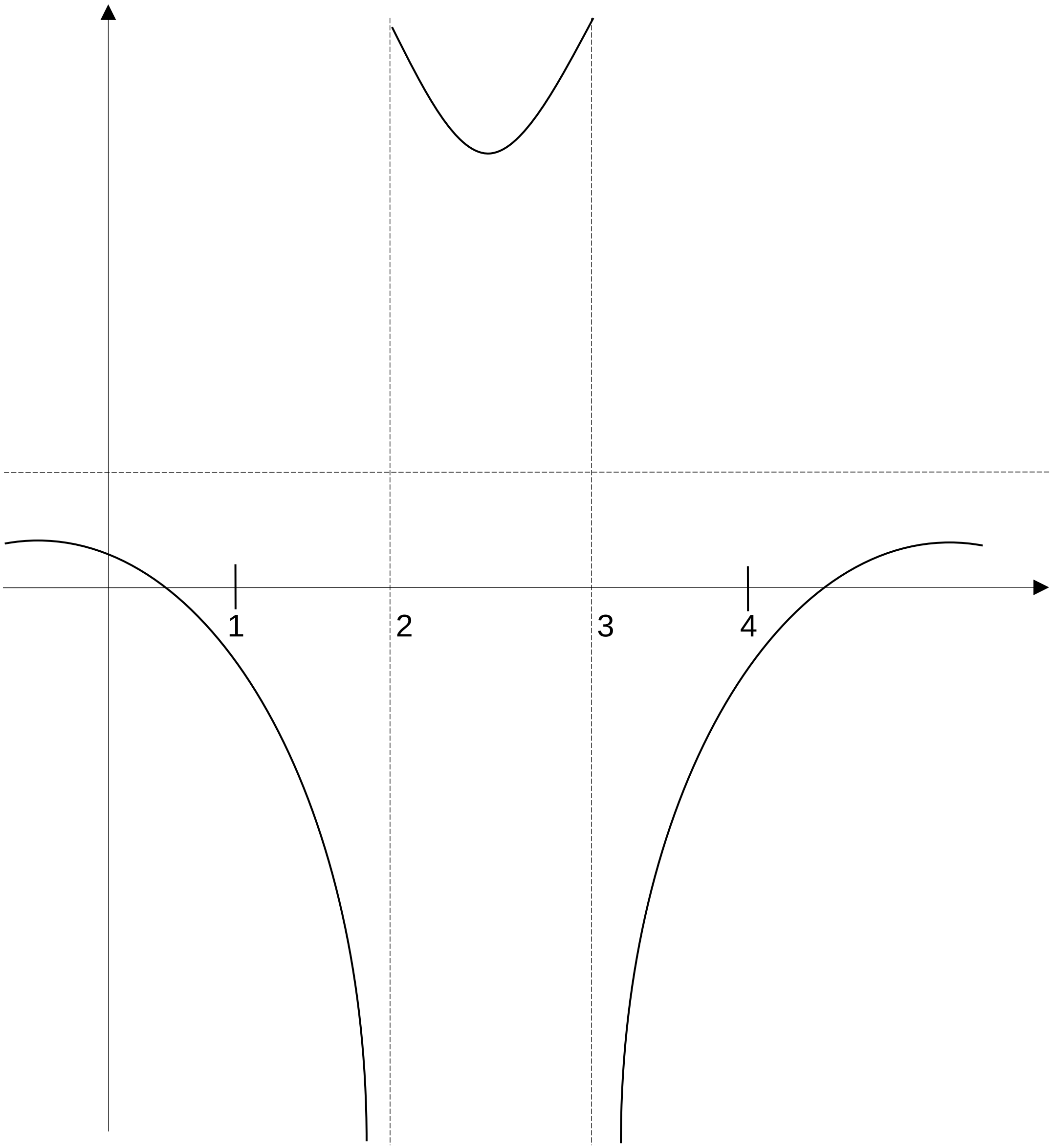
(iii). $$(\lim y)_{x \to 2^+} = \left[\lim\left(\frac{(2+\varepsilon)^2 - 5(2+\varepsilon) + 4}{[(2+\varepsilon)^2 - 5(2+\varepsilon) + 6}\right)\right]_{\varepsilon \to 0} = \lim\left[1 + 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon - 1}\right)\right]_{\varepsilon \to 0} = +\infty$$

$$(\lim y)_{x \to 2^-} = \left[\lim\left(\frac{(2-\varepsilon)^2 - 5(2-\varepsilon) + 4}{(2-\varepsilon)^2 - 5(2-\varepsilon) + 6}\right)\right]_{\varepsilon \to 0} = \lim\left[1 - 2\left(\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon}\right)\right]_{\varepsilon \to 0} = -\infty$$

(iv). $$(\lim y)_{x \to -\infty} = \left[\lim\left(\frac{x^2 - 5x + 4}{x^2 - 5x + 6}\right)\right]_{x \to -\infty} = \lim\left(\frac{1 - \frac{5}{x} + \frac{4}{x^2}}{1 - \frac{5}{x} + \frac{6}{x^2}}\right)_{x \to -\infty} = 1$$

We now use the various information obtained above about the turning points, the limiting values and the bounds of the function to obtain the required sketch of the function y as seen overleaf.

$$y = \frac{x^2 - 5x + 4}{x^2 - 5x + 6}$$



(3). Sketch the curve of the function given $y = \dfrac{x^2 + 1}{x^2 + x + 1}$

a. **Solution**

In order to sketch the graph of the function above it will be necessary to investigate the behavior of the function from the following properties of the function:

Intercepts.

We recall that the intercepts of the function y on the $x-axis$ are the solution of the equation $y = 0$. clearly, from the function $y = \dfrac{x^2 + 1}{x^2 + x + 1}$, the equation $y = 0$ has no real solution. This therefore implies that the function $y = \frac{x^2+1}{x^2+x+1}$ has no intercept on the $x-axis.$ Now with the

definition of the function y we thus have that $y(0)=1$. This thus indicates that the function has an intercept 1 on the $y-axis.$

Asymptotes

For any function y we recall that the vertical asymptotes are the points $x = x$

Where the function suddenly becomes infinitely large. Now from the given function

We find that $y = \frac{x^2+1}{x^2+x+1}$ is always finite for all values of the independent variable x .

Hence there does not exist any vertical asymptote for the function.

Symmetry

The rational function $y = \frac{x^2+1}{x^2+x+1}$ is clearly not an even function nor an odd function and so exhibits no symmetry either about the ordinate of the origin.

Turning Points

Given that; $$y = \frac{x^2+1}{x^2+x+1}$$

Then,

$$\frac{dy}{dx} = \frac{2x\left(x^2+x+1\right)-\left(x^2+1\right)\left((2x+1)\right)}{\left(x^2+x+1\right)^2} = \frac{x-1}{(x^2+x+1)^2}$$

But at the turning point $\frac{dy}{dx} = 0$

The turning points of the function are therefore the solution of the quadratic equation;

$$x^2 - 1 = 0$$

i.e,

The turning points are the points $x = 1,-1$

Nature of Turning Points

Recalling that $$\frac{dy}{dx} = \frac{x^2-1}{\left(x^2+x+1\right)^2}$$

Thus, $$\frac{d^2y}{dx^2} = \frac{2x(x^2+x+1)^2 - \left(2x+1\right)\left(x^2-1\right)\left(x^2+x+1\right)}{\left(x^2+x+1\right)^4} = \frac{x^2+4x+1}{\left(x^2+x+1\right)^3}$$

Hence, $\left\langle \frac{d^2y}{dx^2} \right\rangle_{x=1} = \frac{2}{9} \phi 0$

This therefore shows that the point $x = 1$ is a minimum turning point with $y_{\min} = \frac{2}{3}$

Similarly, $\left(\frac{d^2y}{dx^2} \right)_{x=1} = \frac{-2}{1} = -2\pi 0$

Hence, the point $x = -1$ is a maximum turning point with the corresponding maximum value of $y_{\max} = 2$

Bounds

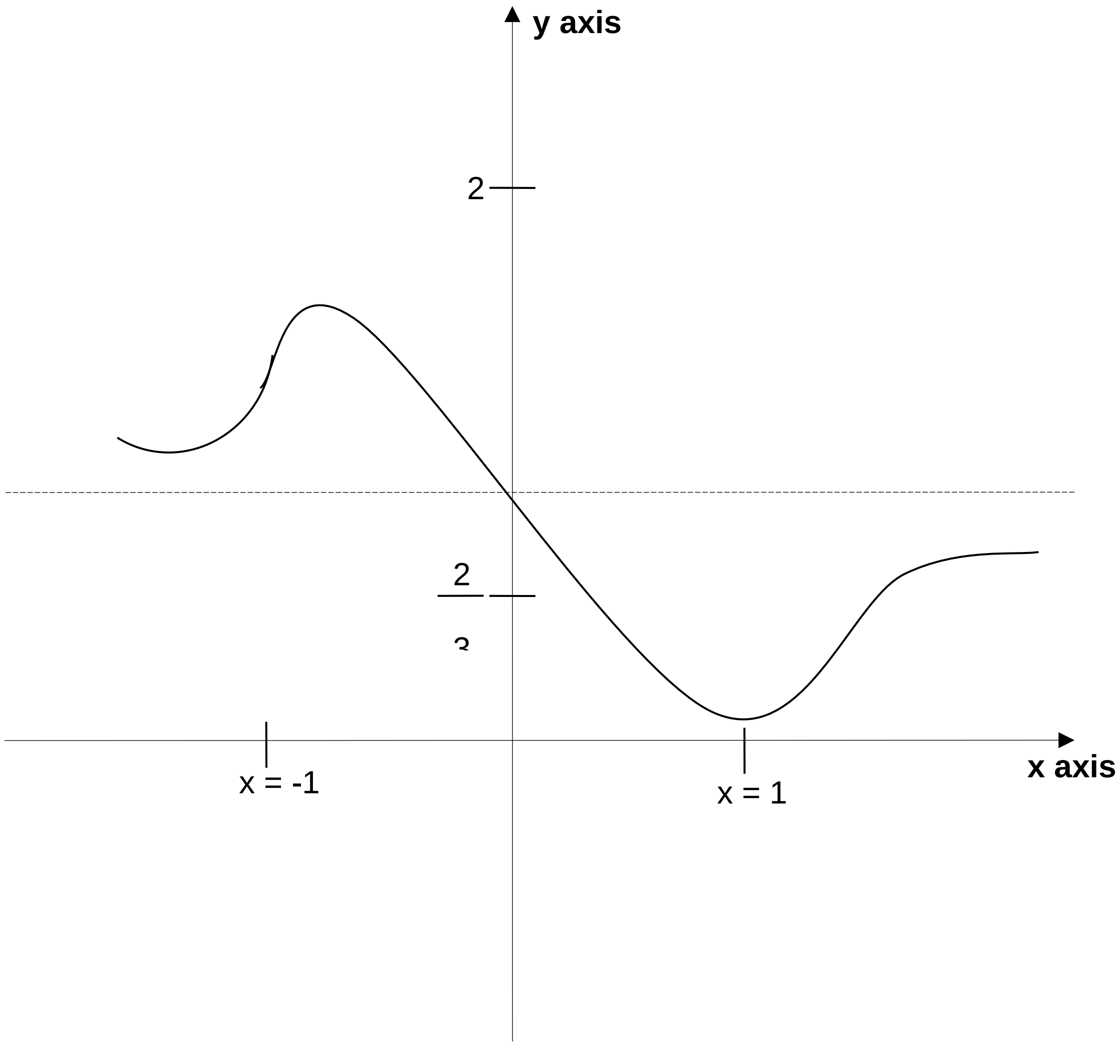
Recall that $y = \frac{x^2 + 1}{x^2 + x + 1}$

Hence, $x \to \infty$ we have $(\lim\ y)_{x\to\infty} = \left[\lim \left(\frac{1 + \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}} \right) \right]_{x\to-\infty} = 1$

In the same way as $x \to -\infty$ we have;

$$(\lim\ y)_{x\to-\infty} = \left[\lim \left(\frac{1 + \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}} \right) \right]_{x\to-\infty} = 1$$

$y = \frac{x^2 + 1}{x^2 + x + 1}$



(4). Sketching the graph of the function $y = \frac{x}{1+x^2}$ determine the minimum and maximum values of the function y .Prove also that the graph lies between the region $y = \pm\frac{1}{2}$.

Solution

(i) Symmetry

Given that $y = \frac{x}{1+x^2}$, we observe that $y(-x) = -\left(\frac{x}{1+x^2}\right) = -y$

This therefore shows that y is an odd function of x. Hence the graph is symmetrical about the origin.

(ii). Intercepts

On the $y - axis$ where $x = 0$ we have $y = \frac{0}{1+0} = 0$

Similarly, on the abiscas we have $x = 0$. Therefore the graph passes through the origin $(0,0)$

(iii). Recall that the turning points of a function y are indicated by the solution of the equation

$$\frac{dy}{dx} = 0$$

Given $y = \frac{x}{1+x^2}$ we have $\frac{dy}{dx} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

Since $\frac{dy}{dx} = 0,$ we therefore have that $\frac{1-x^2}{(1+x^2)^2} = 0$

$\Rightarrow 1 - x^2 = 0$

i.e,

$(1+x)(1-x) = 0$

Thus, the turning points of the function are $x = -1,1$

Bounds

From the function y as defined above we have;

$$y(1+x^2) - x = 0$$

i.e.

$$yx^2 - x + y = 0$$

Solving the quadratic equation for values of the variable x we have

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$$

Now x has real values only when $1 - 4y^2 \geq 0$

i.e.

$$(1-2y)(1+2y) \geq 0$$

The solution set of the inequality is $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$

Minimum and Maximum values

We recall that the derivative of the function is $\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$

Hence,

$$\frac{d^2y}{dx^2} = \frac{-2x(1+x^2) - 4x(1-x^2)(1+x^2)}{(1+x^2)^4} = -\frac{2x(1+x^2) + 4x(1-x^2)}{(1+x^2)^3}$$

i.e.

$$\frac{d^2y}{dx^2} = \frac{2x(x^2-4)}{(1+x^2)^3}$$

Clearly, the sign of the function $\frac{2x(x^2-4)}{(1+x^2)^3}$ is indicated by the sign of $2x(x^2-4)$ since the denominator is always positive for all real values of x. We therefore verify for the sign of the function $2x(x^2-4)$ at the turning points of y. Suppose $N(x) = 2x(x^2-4)$, then $N(-1) = 6\phi 0$ and $N(1) = -6$. Therefore we conclude that

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = \pi 0 \text{ and } \left(\frac{d^2y}{dx^2}\right)_{x=-1} = \varphi 0$$

Thus, the points $x = 1$ and $x = -1$ are respectively the maximum and minimum points of y.

Therefore, we have, $y_{max} = \frac{1}{2}$, $y_{min} = -\frac{1}{2}$

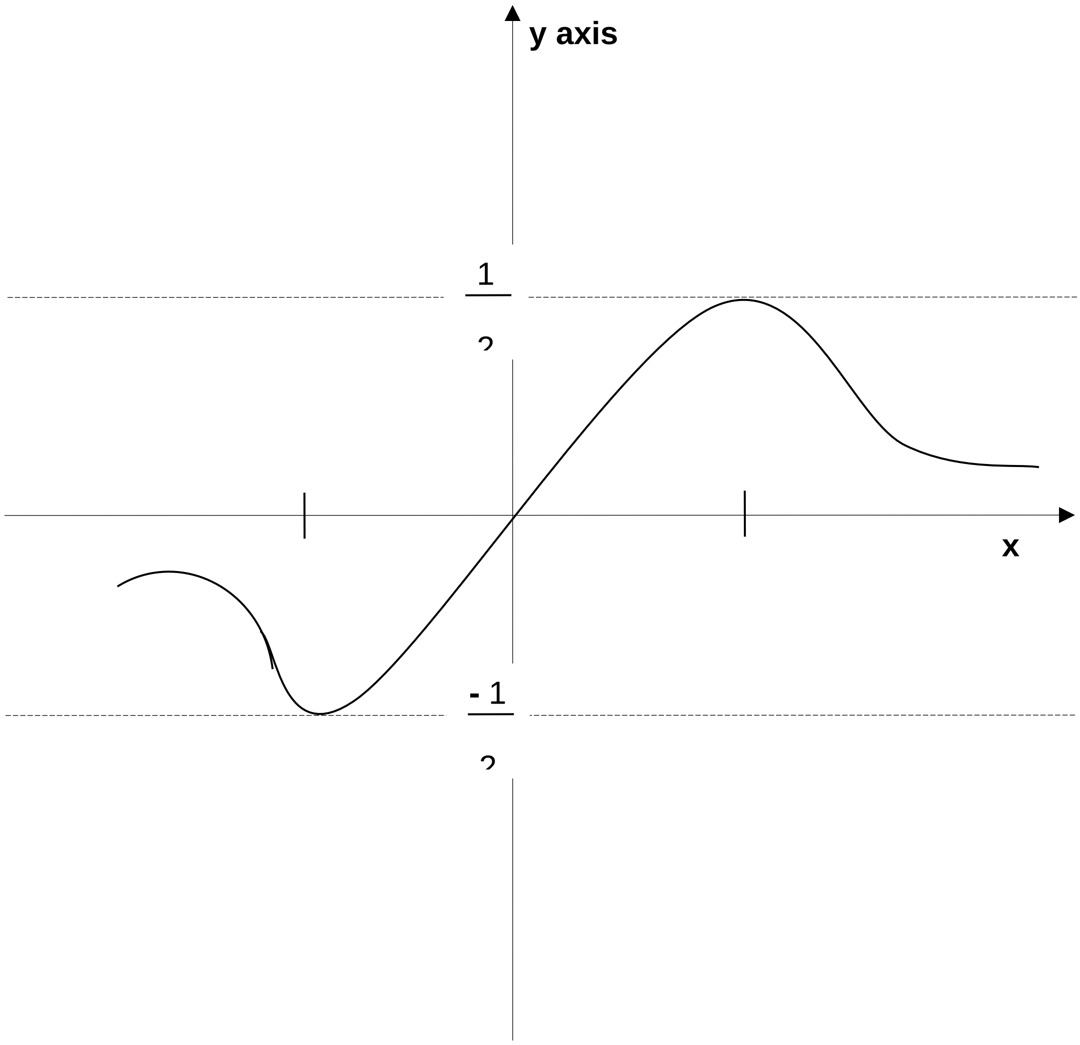
The function $y = \frac{x}{1+x^2}$, is continuous for all real values of x. Clearly, $y(x)$ has no vertical asymptotes.

Finally,

$$(\lim\ y)_{x\to\infty} = \lim\left(\frac{x}{1+x^2}\right)_{x\to\infty} = \left[\lim\left(\frac{\frac{1}{x}}{\frac{1}{x^2}+1}\right)\right]_{x\to\infty} = \frac{0}{1} = 0$$

$$(\lim\ y)_{x\to-\infty} = \lim\left(\frac{x}{1+x^2}\right)_{x\to-\infty} = \left[\lim\left(\frac{\frac{1}{x}}{\frac{1}{x^2}+1}\right)\right]_{x\to-\infty} = \frac{0}{1} = 0$$

$y = \frac{x}{1+x^2}$



Self-Assessment Exercise(s) 2

1. Find the equation of the normal to the curve $y = x^2 - x - 2$ at the point (1, -2)
2. A rectangular area is formed having a perimeter of 40 cm. Determine the length and breadth of the rectangle if it is to enclose the maximum possible area

4.0 Conclusion

At this juncture, we hereby conclude that you have acquired adequate knowledge on how to get your Intercept, Symmetry, turning points, Discontinuity, Imaginary points to enable you sketch your graph

5.0 Summary

In this unit, you have learnt that:

(a) To sketch a curve in polar coordinates substitute $(r,-\theta)$ for (r,θ) in $r = f(\theta)$, if $f(\theta) = f(-\theta)$ then the curve is symmetric with respect to x-axis.

(b) Substitute $(r,\pi-\theta)$ for (r,θ), if $f(\theta) = f(\pi-\theta)$ then the curve is symmetric with respect to y-axis.

(c) Determine the maximum or minimum points.

6.0 Tutor-Marked Assignments

(1). It known that the graph of the function $y = \frac{\alpha x + \beta}{(x+1)(x-2)}$ has a stationary value of -1, at the point $x = 0$. Determine the values of the constants α and β hence the second turning point of the function.

(2). Sketch the graph of the function $y = \left(x - 1^{1}(12x^{2} - 9x - 43)\right)$

(3) Given the function $y = 2\sin x - x$ determine the optimal values of the function in the interval $0 \le x \le 2\pi$. Hence sketch the graph.

(4). Sketch the graph of $y = x^{3} - 15x + 7 - \frac{12}{x}$.

(5). Indicate on a sketch the main feature of the function $x = \frac{15 + 10t}{4 + t^{2}}$

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B. D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Integration as an Inverse of Differentiation

Contents

1.0 Introduction

Integration is the reverse process of differentiation.

i.e. Given $g(x)$, find $f(x)$

where $g(x) = f'(x)$

$$\frac{d}{dx}(x^n) = n(x^{n-1})$$

So the Integration of nx^{n-1} should give x^n.

E.g. x^3 is the integral of $3x^2$.

The symbols $\int f(x)dx$ denote the integral of $f(x)$ with respect to the variable x.

2.0 Learning Outcomes

At the end of this unit, you should be able to:

1. See integration as reverse process of differentiation;
2. Find a function whose derivative we already know.

3.0 Learning Contents

3.1 Arbitrary Constant

For the functions $x^3; x^3 - 3;$ and $x^3 + 1,$ the derivatives are given as

$$\frac{d}{dx}(x^3) = 3x^2, \ \frac{d}{dx}(x^3 - 3) = 3x^2, \ \frac{d}{dx}(x^3 + 1) = 3x^2$$

With all giving the same solution $3x^2$.

i.e. $\int 3x^2 dx = x^3 + C$

C is called the constant of integration. Such an integral is called an indefinite integral.

In general, the derivatives of $x^3 + C,$ C = constant, is given by

$$\frac{d}{dx}(x^3 + C) = 3x^2.$$

So when $3x^2$ is integrated, the arbitrary constant C is added to the result, and so

$$\int 3x^2 dx = x^3 + C$$

The integration of certain function can be deduced from the knowledge of differentiation as shown below.

3.2 Standard Forms

Since

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

Replacing n by $n+1$, we have

$$\frac{d}{dx}\left(x^{n+1}\right)=(n+1)x^n$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$$

And integration is inverse of differentiation. Integrating x^n therefore gives:

i) $\int x^n dx=\frac{x^{n+1}}{n+1}+C$ provided $n\neq -1$

Similarly

ii) $\frac{d}{dx}(\sin x)=\cos x$ and so $\int \cos x dx=\sin x+C$

iii) $\frac{d}{dx}(\cos x)=-\sin x$

$$\Rightarrow \int \sin x dx=-\cos x+C$$

iv) $\frac{d}{dx}\left(e^x\right)=e^x$

$$\Rightarrow \int e^x dx=e^x+C$$

v) $\frac{d}{dx}(\log_e x)=\frac{1}{x}$

$$\Rightarrow \int \frac{1}{x}dx=\log_e x+C$$

this gives a table here

Table 1.1

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}+C$ provided $n \neq 1$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
e^x	$e^x + C$
$\frac{1}{x}$	$\ln x + C$

Self-Assessment Exercise(s) 1

1. Determine $\int \frac{3}{x^2}\,dx$
2. $(a) \int 4\cos 3x dx$ $(b) \int 5 sin2\theta d\theta$

3.3 Some Properties of Integration

What are indefinite integrals?

Let $f(x)$ and $g(x)$ be functions and k a constant.

Then

i) $kf(x)dx = k\int f(x)dx$

ii) $[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

iii) $[f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

Example

Integrate the following with respect to x.

(a) $2x^5$ (b) $\frac{1}{\sqrt[3]{x^2}}$ (c) $\frac{1}{x}+3x^3$

Solution

(a) $\int 2x^5 dx = 2\int x^5 dx$

$$= \frac{2x^{5+1}}{5+1} + C = \frac{2x^6}{6} + C$$

$$= \frac{x^6}{3} + C$$

(b) $\int \frac{dx}{\sqrt[3]{x^2}} = \int \frac{dx}{x^{\frac{2}{3}}} = \int x^{-\frac{2}{3}} dx$

$$= \frac{x^{\frac{2}{3}+1}}{-\frac{2}{3}+1} + C$$

$$= \frac{x^{\frac{2}{3}}}{\frac{1}{3}} + C = 3x^{\frac{1}{3}} + C$$

(c) $\int \left(\frac{1}{x} + 3x^3 \right) dx = \int \frac{dx}{x} + \int 3x^2 dx$

$$= \log_e x + 3\int x^3 dx$$

$$= \log_e x + \frac{3}{4}x^4 + C$$

4.0 Conclusion

At this juncture, we hereby conclude that you have acquired adequate knowledge on how to apply standard forms of integration to solve problems and also use different methods to solve integral problem.

5.0 Summary

In this unit, you have learnt that:

1. Integration is the reverse of differentiation.
2. The symbol for integration is $\int$.

(1) If $\frac{dy}{dx} = f'(x)$

Then $dy = f(x)dx$

$\int dy = \int f(x)dx$

$y = F(x)dx + c$

(2) Given the derivative of a function, we can find the function by appropriate integration.

(3) Some methods of integration are, (a) by substitution, (b) by parts, (c) by partial fraction.

(4) We recall the following standard integral

S/No.	$f(x)$	$\int f(x)dx$
1.	ax^n	$\frac{ax^{n+1}}{n+1}+C(n\neq -1)$
2.	$\cos x$	$\sin x + C$
3.	$\sin x$	$-\cos x + C$
4.	$\sec^2 x$	$\tan x + C$
5.	$\cos ec^2 x$	$-\cot x + C$
6.	secxtanx	secx+C
7.	$\cos ecx \cot x$	-cosecx+C
8.	e^x	$e^x + C$
9.	$\frac{1}{x}$	$linx + C$

(5) $\int [f(x_1)+f(x_2)]dx = \int f(x_1)dx + \int f'(x_2)dx$

6.0 Tutor-Marked Assignments

Integrate the following with respect to x

(1) $2x^7$ (2) $\frac{3}{\sqrt{x^4}}$ (3) $\frac{1}{x^2}+2x^3$

(4) $4x^3 - 3x^2 + 1$ (5) $x^6 + \frac{2}{\sqrt[3]{x^2}} + \frac{1}{3x^3}$

7.0 References /Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 5

Unit 1	Method of Integration
Unit 2	Definite Integrals
Unit 3	Applications to Area and Volume

Unit 1

Method of Integration

Contents

1.0 Introduction

Two functions f and F may be related as follow:

$$\frac{d}{dx}F(x) = f(x) \text{...(1)}$$

If this is the case, f is called the derivative of F and F an anti-derivative of f. It has been shown in the previous unit how the derivative of a given function can be obtained through differentiation.

In this unit, we shall be concern with the converse.

Given $f(x)$, find $F(X)$ such that $\frac{d}{dx}F(x) = f(x)$ is satisfied. For instance, if $f(x) = 2x$, we may choose $F(x) = x^2$ and note that

$$\frac{d}{dx}F(x) = \frac{d}{dx}x^2 = 2x = f(x).$$

If $f(x) = \sin x$, we then choose $F(x) = \frac{d}{dx}(-\cos x)$

$$= -(-\sin x)$$
$$= \sin x$$
$$= f(x)$$

2.0 Learning Outcomes

At the end of this unit, you should be able to:

(i) Define an indefinite integral as an anti-derivative and hence apply the definition in integrating given functions.

(ii) Integrate given functions by suitable integration methods e.g. by substitution, by parts and by partial fractions.

3.0 Learning Content

3.1 Substitution Method

The method of substitution involves changing the variable of integration such that the integrand takes familiar form. This is illustrated with the following example:

Example

(1) Evaluate $(3x+4)^8\,dx$

Let $U = 3x + 4$

$$\frac{du}{dx} = 3$$

$$\Rightarrow dx = \frac{du}{3}$$

$$\therefore \int (3x+4)^8 dx = \int u^8 . \frac{du}{3}$$

$$= \frac{1}{3}\int u^8 du = \frac{1}{3}\left[\frac{u^8}{9}\right] + C$$

$$= \frac{1}{27}u^9 + C$$

Substitution back $u = 3x + 4$

$$\int (3x+4)^8 dx = \frac{1}{27}(3x+4)^9 + C$$

(2) Evaluate $\int e^{3x} dx$

Solution

Let $u = 3x$

$$\frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$$

$$\therefore \int e^{3x} dx = \int e^u . \frac{du}{3} = \frac{1}{3}\int e^u du$$

$$= \frac{1}{3}e^u + C$$

Since $u = 3x$

We have $\int e^{3x} dx = \frac{1}{3}e^{3x} + C$

In general

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + C$$

Where a is any constant.

(3) Evaluate $\int \cos 4\theta d\theta$

Solution

Let $u = 4\theta$

$$\frac{du}{d\theta} = 4 \Rightarrow d\theta = \frac{du}{4}$$

$$\therefore \int \cos 4\theta d\theta = \int \cos u.\frac{du}{4}$$

$$= \frac{1}{4}\int \cos u du = \frac{1}{4}\sin u + C$$

$$= \frac{1}{4}\sin 4\theta + C$$

In general

$$\int \cos ax dx = \frac{1}{a}\sin ax + C$$

also

$$\int \sin ax dx = -\frac{1}{a}\cos ax + C$$

Where a is any constant.

(4) Evaluate $\int x^2 \sin\left(x^3\right)dx$

Solution

Let $y = x^3$

$$\frac{dy}{dx} = 3x^2 \Rightarrow dx = \frac{dy}{3x^2}$$

$$\therefore \int x^2 \sin\left(x^3\right)dx = \int x^2 \sin y.\frac{dy}{3x^2}$$

$$= \frac{1}{3}\int \sin y dy = -\frac{1}{3}\cos y + C$$

$$= -\frac{1}{3}\cos\left(x^3\right) + C$$

(5) Evaluate $\int \sin^6 x \cos x dx$

Solution

Let $u = \sin x$

$$\frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$$

$$\therefore \int \sin^6 x \cos x dx = \int u^6 \cos x.\frac{du}{\cos x}$$

$$= \int u^6 du = \frac{u^7}{7} + C$$

$$= \frac{\sin^7 x}{7} + C$$

(6) Evaluate $\int x^2 e^{-x^3} dx$

Solution

Let $u = -x^3$

$$\frac{du}{dx} = -3x^2 \Rightarrow dx = \frac{du}{-3x^2}$$

$$\therefore \int x^2 e^{-x^3} dx = \int x^2 e^u . \frac{du}{-3x^2}$$

$$= -\frac{1}{3}\int e^u du = -\frac{1}{3} e^u + C$$

$$= -\frac{1}{3} e^{-x^3} + C$$

(7) Evaluate $\frac{x^2 + 1}{\sqrt{x^3 + 3x + 3}} dx$

Solution

Let $u = x^3 + 3x + 3$

$$\frac{du}{dx} = 3x^2 + 3 \Rightarrow dx = \frac{du}{3x^2 + 3}$$

$$\therefore \int \frac{x^2 + 1}{\sqrt{x^3 + 3x + 3}} dx = \int \frac{x^2 + 1}{\sqrt{u}} . \frac{du}{3(x^2 + 1)}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{u}} du = \frac{1}{3}\int u^{-\frac{1}{2}} du$$

$$= \frac{1}{3} . \frac{u^{\frac{1}{2}}}{1/2} + C$$

$$= \frac{2}{3}\sqrt{x^3 + 3x + 3} + C$$

(8) Evaluate $I = \int_0^{\frac{\Pi}{2}} \sin(2x + \Pi) dx$

Solution

Let $u = 2x + \Pi$

$$\frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$$

We also change the limit

When $x = 0$, $u = 2(0) + \Pi = \Pi$

When $x = \Pi/2$, $u = 2(\Pi/2) + \Pi = 2\Pi$

$$I = \int_0^{\Pi/2} \sin(2x + \Pi)dx = \frac{1}{2}\int_{\Pi}^{2x} \sin u du$$

$$= -\left[\frac{1}{2}\cos u\right]_{\Pi}^{2\Pi} = -\frac{1}{2}[\cos 2\Pi - \cos \Pi]$$

$$= -\frac{1}{2}[1 - (-1)]$$

$$= -\frac{1}{2}(2) = -1$$

Self-Assessment Exercise(s) 1

1. Find integral of $\int (2x-5)^7\, dx$
2. $3tan^2 4xdx$

3.2 Method of Integration by Parts

If $U(x)$ and $V(x)$ are functions of x ,the derivative of their product $U(x)V(x)$ is given by the product rule

i.e. $\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}$

Integrating, we get

$$\int \frac{d}{dx}(UV)dx = \int U\frac{dV}{dx}dx + \int V\frac{dU}{dx}dx$$

i.e. $UV = \int U\frac{dV}{dx}dx + \int V\frac{dU}{dx}dx$

This gives

$$\int U\frac{dV}{dx}dx = UV - \int V\frac{dU}{dx}dx$$

Or

$$\int UdV = UV - \int VdU$$

The formula above is called the integration by parts formula.

Example

(1) $\int x \sin x dx$

$$\int U \frac{dV}{dx} dx = UV - \int V \frac{dU}{dx} dx$$

From the integrand

Let $U = x,$ then $\frac{dU}{dx} = 1$ or $dU = dx$

and $dV = \sin x dx,$ then $\int dV = \int \sin x dx$

$V = -\cos x$

from $\int UdV = UV - \int VdU$

$$\Rightarrow \int x \sin x dx = -x \cos x - \int (-\cos x).1dx$$

$$= -x \cos x + \int \cos x dx$$

$$\therefore \int x \sin x dx = -x \cos x + \sin x + C$$

(2) $\int x^2 \ln x dx$

Solution

Let $U = \ln x, dV = x^2 dx$

$$dU = \frac{1}{x} dx \; V = \int x^2 dx = \frac{x^3}{3}$$

using $\int UdV = UV - \int VdU$

we have

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{1}{x} . \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

(3) $\int x^2 e^{-2x} dx$

Solution

Let $U = x^2$, $dV = e^{-2x} dx$

$dU = 2x$, $V = \int e^{-2x} dx = -\frac{1}{2} e^{-2x}$

Note

We chose $U = x^2$ since differentiating x^2 will reduce the power.

using $\int U dV = UV - \int V dU$

We have

$$\int x^2 e^{-2x} dx = -\frac{x^2}{2} e^{-2x} - \int \left(-xe^{-2x}\right) dx$$

$$= \frac{x^2}{2} e^{-2x} + \int xe^{-2x} dx$$

We still use the integration by part formula again to complete the integration.

i.e.

$$\int xe^{-3x} dx = -\frac{x}{2} e^{-2x} + \frac{1}{2}\int e^{-2x} dx$$

$$= -\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C$$

$$\therefore \int x^2 e^{-2x} dx = -\frac{x^2}{2} e^{-2x} - \frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C$$

(4) $\int \ln x dx$

Solution

We rewrite the integral as

$$\int 1.\ln x dx$$

Let $U = \ln x$, $dV = 1dx$

$$dU = \frac{1}{x}, V = \int dx = x$$

using $\int UdV = UV - \int VdU$

we have

$$\int 1.\ln xdx = x\ln x - \int \frac{1}{x}.xdx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C$$

Self-Assessment Exercise(s) 2

1. Determine $\int xcosx\ dx$
2. Determine $\int 2ln3x\ dx$

3.3 Further Integration by Parts

Example

(1) $\int e^{-2x} \sin 3xdx$

Solution

Let $U = e^{-2x}$, $dV = \sin 3xdx$

$$dU = -2e^{-2x} \quad , \quad V = \int \sin 3xdx = -\frac{1}{3}\cos 3x$$

$$\int e^{-2x} \sin 3xdx = -\frac{1}{3}e^{-2x}\cos 3x - \int \frac{2}{3}e^{-2x}\cos 3xdx$$

$$= -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{3}\left[e^{-2x}\int \cos 3xdx - \int\left[-2e^{-2x}\int \cos 3xdx\right]dx\right]$$

$$= -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{3}\left[\frac{1}{3}e^{-2x}\sin 3x + \int \frac{2}{3}e^{-2x}\sin 3xdx\right]$$

let $I = \int e^{-2x}\sin 3xdx$

$$\therefore I = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x - \frac{4}{9}I$$

$$I + \frac{4}{9}I = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x$$

$$I = \frac{9}{13}\left[-\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x\right]$$

$$\therefore \int e^{-2x} \sin 3x dx = \frac{1}{13}\left[-3e^{-2x} \cos 3x - 2e^{-2x} \sin 3x\right]$$

Self-Assessment Exercise(s) 3

1. Determine $\int 5xe^{4x}\ dx$
2. Find the integral of $\int \frac{dx}{1+cosx}$

3.4 The Reduction Formula

The method will be illustrated with examples:

(1) Use integration by part formula to evaluate the integral below.

$$\int e^{ax} \cos bx dx$$

Solution

Using the formula

$$\int U dV = UV - \int V dU$$

$$let\ u = e^{ax}\ , \qquad du = ae^{ax}$$

$$and\ \ dv = \cos bx \Rightarrow v = \int \cos bx = \frac{1}{b} \sin bx$$

$$\therefore \int e^{ax} \cos bx dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx dx$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx dx \qquad \text{(i)}$$

To integrate $\int e^{ax} \sin bx dx$

$$let\ u = e^{ax}, \quad du = ae^{ax}$$

$$and\ \ dv = \sin bx \Rightarrow v = -\frac{1}{b} \cos bx$$

$$\therefore \int e^{ax} \sin bx dx = -\frac{1}{b} e^{ax} coxbx + \int \frac{a}{b} e^{ax} \cos bx dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx dx \qquad \text{(ii)}$$

Substitute equation (2) into (1), we have

$$\int e^{ax}\cos bx dx = -\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx dx$$

$$let\ \ I = \int e^{ax}\cos bx dx$$

$$\therefore I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - - \ \frac{a^2}{b^2} I$$

$$\therefore I = \frac{1}{a^2+b^2}\left[ae^{ax}\cos bx + be^{ax}\sin bx\right]$$

$$\therefore \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

To evaluate:

$$\int e^{3x}\cos 5x dx$$

we set

$a = 3$, $b = 5$

$$\therefore \int e^{3x}\cos 5x dx = \frac{e^{3x}}{3^2+5^2}\left[3\cos 5x + 5\sin 5x\right]$$

$$= \frac{e^{3x}}{34}\left[3\cos 5x + 5\sin 5x\right]$$

(2) Use integration by part formula to evaluate the integral below.

$$\int x^p e^{mx} dx$$

Solution

$$let\ \ I_p = \int x^p e^{mx} dx$$

using $\int u dv = uv - \int v du$

$$let\ \ u = x^p \Rightarrow du = px^{p-1}$$

$$and\ \ dv = \int e^{mx} dx \Rightarrow v = \frac{e^{mx}}{m}$$

$$\therefore I_p = \frac{x^p e^{mx}}{m} - \int \frac{e^{mx}}{m} px^{p-1} dx$$

$$\therefore I_p = \frac{x^p e^{mx}}{m} - \frac{p}{m} I_{p-1}$$

$$I_p = \frac{xe^{mx}}{m} - \frac{1}{m} I_0$$

but we recall that

$$I_p = \int x^p e^{mx} dx$$

for each $p = 1, 2, 3, 4, \ldots..$

when $p = 1$

$\therefore$ *for* $p = 0$

$$I_0 = \int x^0 e^{mx} dx = \int e^{mx} dx = \frac{e^{mx}}{m}$$

substituting in to equation (ii)

$$I_1 = \frac{xe^{mx}}{m} - \frac{1}{m} I_0$$

$$= \frac{xe^{mx}}{m} - \frac{e^{mx}}{m^2}$$

$$= \left[\frac{x}{m} - \frac{1}{m^2}\right] e^{mx}$$

when $p = 2$

$$\text{from } I_p = \frac{x^p e^{mx}}{m} - \frac{p}{m} I_{p-1}$$

$$I_2 = \frac{x^2 e^{mx}}{m} - \frac{2}{m} I_1$$

$$\text{but} \quad I_1 = \left[\frac{x}{m} - \frac{1}{m^2}\right] e^{mx}$$

$$\therefore I = \frac{x^2 e^{mx}}{m} - \frac{2}{m}\left[\frac{x}{m} - \frac{1}{m^2}\right] e^{mx}$$

$$= \left[\frac{x^2}{m} - \frac{2x}{m^2} + \frac{2}{m^3}\right] e^{mx}$$

when $p = 3$

$$\text{from } I_p = \frac{x^p e^{mx}}{m} - \frac{p}{m} I_{p-1}$$

$$I_3 = \frac{x^3 e^{mx}}{m} - \frac{3}{m} I_2$$

$$but\ I_2 = \left[\frac{x^2}{m} - \frac{2x}{m^2} + \frac{2}{m^3}\right]e^{mx}$$

$$\therefore I_3 = \frac{x^3 e^{mx}}{m} - \frac{3}{m}\left[\frac{x^2}{m} - \frac{2x}{m^2} + \frac{2}{m^3}\right]e^{mx}$$

$$= \left[\frac{x^3}{m} - \frac{3x^2}{m^2} + \frac{6x}{m^3} - \frac{6}{m^4}\right]e^{mx}$$

$when\ p = 4$

$$from\ I_p = \frac{x^p e^{mx}}{m} - \frac{p}{m} I_{p-1}$$

$$I_4 = \frac{x^4 e^{mx}}{m} - \frac{4}{m} I_3$$

$$but\ I_3 = \left[\frac{x^3}{m} - \frac{3x^2}{m^2} + \frac{6x}{m^3} - \frac{6}{m^4}\right]e^{mx}$$

$$\therefore I_4 = \frac{x^4 e^{mx}}{m} - \frac{4}{m}\left[\frac{x^3}{m} - \frac{3x^2}{m^2} + \frac{6x}{m^3} - \frac{6}{m^4}\right]e^{mx}$$

$$= \left[\frac{x^4}{m} - \frac{4x^3}{m^2} + \frac{12x^2}{m^3} - \frac{24x}{m^4} + \frac{24}{m^5}\right]e^{mx}$$

Deductively, integration by part is given by

$$I_p = \left[\frac{x^p}{m} - \frac{px^{p-1}}{m^2} + \frac{p(p-1)x^{p-2}}{m^3} - \frac{p(p-1)(p-2)x^{p-1}}{m^4} + K + \frac{(-1)^p p!}{m^{p+1}}\right]e^{mx}$$

3. Use integration by part formula to evaluate the integral below.

$$\int x^p \cos rsdx$$

Solution

$$Let\ I_p = \int x^p \cos rxdx$$

$$let\ u = x^p \Rightarrow du = px^{p-1}$$

$$and\ dv = \cos rxdx \Rightarrow v = \frac{\sin rx}{r}$$

$$I_p = \frac{x^p \sin rx}{r} - \int \frac{p}{r} x^{p-1} \sin rxdx$$

$$= \frac{x^p \sin rx}{r} - \frac{p}{r}\int x^{p-1} \sin rxdx$$

$$= \frac{x^p \sin rx}{r} - \frac{p}{r}\left[-\frac{x^{p-1}}{r}\cos rx - \int -\left[\frac{(p-1)}{r}x^{p-2}\cos rx\right]dx\right]$$

$$= \frac{x^p \sin rx}{r} + \frac{p}{r^2}x^{p-1}\cos rx - \frac{p(p-1)}{r^2}\int x^{p-2}\cos rx dx$$

$$I_p = \frac{x^p \sin rx}{r} + \frac{p}{r^2}x^{p-1}\cos rx - \frac{p(p-1)}{r^2}I_{p-2}$$

$take\ p = 2,\ 3,4,K$

$for\ \ p = 2$

$$I_2 = \frac{x^2 \sin rx}{r} + \frac{2}{r^2}x\cos rx - \frac{2(2-1)}{r^2}I_{2-2}$$

$$= \frac{x^2 \sin rx}{r} + \frac{2}{r^2}x\cos rx - \frac{2}{r^2}I_0$$

To obtain l_0, *we recall*

$$I_p = \int x^p \cos rx dx$$

$$\therefore I_0 = \int \cos rx dx = \frac{\sin rx}{r}$$

$$\therefore I_2 = \frac{x^2 \sin rx}{r} + \frac{2}{r^2}x\cos rx - \frac{2}{r^2}\left[\frac{\sin rx}{r}\right]$$

$$= \frac{x^2}{r}\sin rx + \frac{2}{r^2}x\cos rx - \frac{2}{r^3}\sin rx$$

$For\quad p = 3$

$$from\ I_p = \frac{x^p \sin rx}{r} + \frac{p}{r^2}x^{p-1}\cos rx - \frac{p(p-1)}{r^2}I_{p-2}$$

$$I_3 = \frac{x^3 \sin rx}{r} + \frac{3}{r^2}x^2\cos rx - \frac{6}{r^2}I_1$$

To obtain I_1, *we recall*

$$I_1 = \int x\cos rx dx$$

Using integration by part formula we have

$$I_1 = \int x\cos rx dx = \frac{x}{r}\sin rx - \int \frac{1}{r}\sin rx dx$$

$$= \frac{x}{r}\sin rx + \frac{1}{r^2}\cos rx$$

$$I_3 = \frac{x^3}{r}\sin rx + \frac{3}{r^3}x^2\cos rx - \frac{6}{r^2}\left[\frac{x}{r}\sin rx + \frac{1}{r^2}\cos rx\right]$$

$$= \frac{x^3}{r}\sin rx + \frac{3}{r^2}x^2\cos rx - \frac{6x}{r^2}\sin rx - \frac{6}{r^4}\cos rx$$

$when \;\; p = 4$

$$from \; I_p = \frac{x^p}{r}\sin rx + \frac{p}{r^2}x^{p-1}\cos rx - \frac{p(p-1)}{r^2}I_{p-2}$$

$$I_4 = \frac{x^4}{r}\sin rx + \frac{4}{r^2}x^3\cos rx - \frac{12}{r^2}I_2$$

$$= \frac{x^4}{r}\sin rx + \frac{4}{r^2}x^3\cos rx - \frac{12}{r^2}\left[\frac{x^2}{r}\sin rx + \frac{2}{r^2}x\cos rx - \frac{2}{r^3}\sin rx\right]$$

$$= \frac{x^4}{r}\sin rx + \frac{4}{r^2}x^3\cos rx - \frac{12}{r^3}\sin rx - \frac{24}{r^4}x\cos rx + \frac{24}{r^5}\sin rx$$

$when \;\; p = 5$

$$from \; I_p = \frac{x^p}{r}\sin rx + \frac{p}{r^2}x^{p-1}\cos rx - \frac{p(p-1)}{r^2}I_{p-2}$$

$$I_5 = \frac{x^5}{r}\sin rx + \frac{5}{r^2}x^4\cos rx - \frac{20}{r^2}I_3$$

$$= \frac{x^5}{r}\sin rx + \frac{5}{r^2}x^4\cos rx - \frac{20}{r^2}\left[\frac{x^3}{r}\sin rx + \frac{3}{r^2}x^2\cos rx - \frac{6}{r^3}x\sin rx - \frac{6}{r^4}\cos rx\right]$$

$$I_5 = \frac{1}{r}x^5\sin rx + \frac{5}{r^2}x^4\cos rx - \frac{20}{r^3}x^3\sin rx - \frac{60}{r^4}x^2\cos rx + \frac{120}{r^5}x\sin rx + \frac{120}{r^6}\cos rx$$

$\therefore$ The reduction formula can be obtained from the above to be

$$I_p = \frac{1}{r}x^p\sin rx + \frac{p}{r^2}x^{p-1}\cos rx - \frac{p(p-1)}{r^3}x^{p-2}\sin rx - \frac{p(p-1)(p-2)}{r^4}x^{p-3}\cos rx$$

$$+ \frac{p(p-1)(p-2)(p-3)}{r^5}x^{p-4}\sin rx + \frac{p(p-1)(p-2)(p-3)(p-4)}{r^6}x^{p-5}\cos rx.......$$

4.0 Conclusion

At this juncture, we hereby conclude that you have acquired adequate knowledge on how to evaluate integrals problems using different methods of integration.

5.0 Summary

In this unit, you have learnt that:

(1) $\frac{d}{dx}[F(x)+c] = f(x) = \int f(x)dx = F(x)+c$

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq 1)$

(b) $\int kf(x)dx = k\int (x)dx(k\text{E}R)$

(c) $\int [f(x)+g(x)]dx = \int f(x)dx + g(x)dx$

(d) $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$ then

$\int f(x)dx = \int f(x)\frac{dx}{dt}.dt$ integration by substitution

(2) Integration by parts:

$\int udv = uv - \int vdu$

u and v are fractions of x.

(3) Integration of rational functions $\frac{P(x)}{Q(x)}$ involves integration by partial fractions.

If p(x) has degree less than 2, then

$\int \frac{p(x)}{(x-a)(x-b)}dx = \int \frac{rdx}{x-a} + \int \frac{sdx}{x-b}$

Where r and s are to be determined by partial fraction rule. If degree of p(x)=2, then

$\int \frac{p(x)dx}{(x-a)(x-b)} = \int kdx + \int \frac{rdx}{x-a} + \int \frac{sdx}{x-b},$

If degree of p(x) >2, then

$\int \frac{p(x)dr}{(x-a)(x-b)} = \int q(x)dx + \int \frac{rds}{x-a} + \int \frac{sdx}{x-6} + C$

6.0 Tutor-Marked Assignment

Evaluate the following integrals

(1) $\int (2x^2-3)^9 dx$ (2) $\int e^{2-3y}dy$

(2) $\int x^3 \cos(x^4)dx$ (4) $\int \frac{x^2+2}{\sqrt{x^3+6x+5}}dx$

(3) $\int_0^{\Pi/4} \cos(2x + \Pi/2)dx$ (6) $\int x^3 \ln x dx$

(4) $\int e^{-3x}\cos 2x dx$

(5) Use integration by part to show that

$$\int e^{ax}\cos\beta x\,dx = \frac{e^{ax}}{\alpha^2+\beta^2}\left[\alpha\cos\beta x + \beta\sin\beta x\right]$$

Hence evaluate $\int e^{7x}\cos 9x\,dx$

Use reduction formula to evaluate the following integrals

6 $\int x^6 e^{3x}dx$ $\int x^7\cos 3x\,dx$

7.0 Reference/Further Reading

STROUD, K. A Engineering Mathematics, 4th Edition. MacMillan Press Limited, 1995. London

STROUD, K.A Further Engineering Mathematics, 3rd Edition. MacMillan Press Limited, 1995. London

TRANTER, C.J and LAMBE, C.G Advanced Level Mathematics, Pure and Applied, 4th Edition. Holder & Stoughton, 1979. Great Britain.

Unit 2

Definite Integrals

Contents

1.0 Introduction

Integral calculus is a study of two basic problems:

(i) Given a function f , find a function F such that

$$\frac{d}{dx}F(x) = f(x)$$

(ii) Given a curve $y = f(x)$ as shown below, find the area of the shaded region.

The first problem leads to the study of indefinite integral as we have seen in the previous unit. The second problem leads to the investigation of definite integral, which will be the subject matter of this unit. We shall see how the area above can be expressed as a limit of a certain sum and how the indefinite integrals are related. Applications and approximation of the definite integral will also be discussed

2.0 Learning Outcomes

At the end of this unit, you should be able to:

1. Associate the area under a curve with an integral;
2. Calculate area under a curve using the definite integral;
3. Determine definite integrals using different techniques of integration.
4. State the mean value theory on integrals and use same to the mean value of functions with given interval.

3.0 Learning Content

3.1 Definite Integral

If $\int f(x)dx = F(x) + C$

The integral $\int_a^b f(x)dx$ is called the definite integral, where a and b are called the

The upper limit.

And $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

OR

$$\int_a^b f(x)dx = F(x)_a^b = F(b) - F(a)$$

Note that in definite integral the choice of constant of integration cancels out.

Example

1) $\int_{1}^{3} 3x^2 dx$

$= 3\int_{1}^{3} x^2 dx$

$= \left[\frac{3x^3}{3}\right]_1^3 = \left[x^3\right]_1^3$

$= 3^3 - 1^1 = 27 - 1 = 26$

2) $\int_{0}^{\pi/2} \sin\theta d\theta$

$= -\cos\theta |_0^{\frac{\pi}{2}} = -\left[\cos \pi/2 - \cos 0\right]$

$= -[0 - 1] = 1$

Self-Assessment Exercise(s) 1

1. $\int_0^{\frac{\pi}{4}} 4cos^2\theta \, d\theta$
2. $\int_3^4 \frac{x^2-3x+6}{x(x-2)(x-1)} dx$

3.2 Properties of the Definite Integrals

$(i) \int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$

$(ii) \int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx = \int_{a}^{c} f(x)dx$

Example

Use the integral below to verify the two properties of definite integral

$\int_{2}^{5} 5x^4 dx$

$(i) \int_{2}^{5} 5x^4 dx = x^5 |_2^5 = 5^5 - 2^5$

$= 3125 - 32 = 3093$

$\int_{5}^{2} 5x^4 dx = x^5 |_5^2 = 2^5 - 5^5$

$= -3093$

so

$$\int_2^5 5x^4dx = -\int_5^2 5x^4dx$$

(ii) We prove that

$$\int_2^5 5x^4dx = \int_2^3 5x^4dx + \int_3^5 5x^4dx$$

now

$$\int_2^5 5x^4dx = x^5\Big|_2^3 = 3^5 - 2^5 = 243 - 32 = 211$$

and

$$\int_3^5 5x^4dx = x^5\Big|_3^5 = 5^5 - 3^5 = 3125 - 243 = 2882$$

$$\therefore \int_2^3 5x^4dx + \int_3^5 5x^4dx = 211 + 2882 = 3093$$

$$\Rightarrow \int_2^5 5x^4dx = \int_2^3 5x^4dx + \int_3^5 5x^4dx$$

4.0 Conclusion

At this juncture, we believed you are now familiar with some properties of Definite Integrals and can also use the knowledge to solve problems.

5.0 Summary

In this unit, you have learnt that:

(1) The definite integral,

$$\int_a^b f(x)dx$$

Geometrically represents the area bounded by the curve $y = f(x)$, the lines, $x = a,\ x = b$ and the x-axis.

(2) Let

$$A_1 = \int_a^c f(x)dx$$

$$A_2 = \int_c^b f(x)dx$$

if $A = \int_a^b f(x)dx$ then

$$A = A_1 + A_2$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(3) We can change the limit of an integral thus;

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(4) Let A_1 the area bounded by the curve $y = f_1(x)$ the lines $x = a,\ x = b$ and the x-axis. Let A_2 the area bounded by the curve $y = f_2(x)$ the lines $x = a,\ x = b$ and the x-axis. If we denote the area common to the two curves by A then

$$A = A_1 - A_2 = \int_a^b f_1(x)dx - \int_a^b f_2(x)dx.$$

(5) The area bounded by the curve $x = f(y)$ the lines $y = c, y = d$ and the $y - axis$ is given by

$$, A = \int_c^d f(y)dy$$

6.0 Tutor-Marked Assignments

Evaluate the following definite integrals.

$(1)\ \int_1^2 (3x^2 + 1)dx\ (2)\ \int_0^{\pi} (\sin x + \cos x)dx$

$(3)\ \int_1^2 \left(\frac{1}{x} + e^x\right)dx$

Use the integrals below to verify the two properties of definite integral.

$(4)\ \int_1^3 (4x^3 + 2x)dx\ (5)\ \int_0^{\pi/2} \sin \theta d\theta$

7.0 References /Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L.H Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.Lo

Unit 3

Applications to Area and Volume

Contents

1.0 Introduction

We can generate the volume under the curve $y = f(x)$ is revolved about the x-axis through 2π radians is called volume of a solid of revolution. Similarly, the volume generated when $x_1 = f(y)$ is revolved about the y-axis through 2π radians is given by

$$\int_c^d \pi x^2 dy$$

2.0 Learning Outcomes

At the end of this unit, you should be able to:

1. Find plan areas under given curves through evaluation of definite integrals using integration techniques;
2. Apply integration to solve problems relating to displacement, velocity and acceleration;
3. Find volumes of solids generated by revolution using integrals;
4. Estimate the integrals of functions using the trapezium rule.

3.0 Learning Content

3.1 Area under a Curve

Consider the graph of $y = f(x)$ below

To estimate the area bounded by the curve $y = f(x)$ and line $y = 0$ $the\ x - axis$

Let ABCD = $\delta A,$ element of area.

OA = x, AB = δx

$$AB = DE = FC = \delta x.$$

$$AD = y, CE = DF = \delta y$$

$$area\ ADEB < \delta A < area\ AFCB$$

$$i.e. y\delta x < \delta A < (y + \delta y)\delta x$$

If the evaluation of the integral is from the point $x = a$ $x = b$ on the $x - axis$, then the Area under the curve is given by the definite integral

$$A = \int_a^b y dx$$

$$= \int_a^b f(x) dx$$

We shall illustrate this in the following examples.

Example

(1) Find the area enclosed between the curve $y = x^2 + 2x - 3$ and the x-axis.

Solution

$$y = x^2 + 2x - 3$$
$$= (x-1)(x+3)$$

The curve cuts the x-axis when $y = 0$

$$0 = (x-1)(x+3)$$
$$x = 1, -3$$

Let these two points be $P(-3,0)$ *and* $Q(1,0)$

$$A = \int_{-3}^{1} y dx = \int_{-3}^{1} (x^2 + 2x - 3) dx$$

$$= \left[\frac{x^3}{3} + x^2 - 3x\right]_{-3}^{1} = \left[\frac{(1)^3}{3} + (1)^2 - 3(1) - \frac{(-3)^3}{3} - (-3)^3 + 3(-3)\right]$$

$$= \frac{1}{3} + 1 - 3 + 9 - 9 - 9 = -\frac{32}{3}$$

$$= -10\frac{2}{3}$$

(2) Find the area bounded by the curve $y = x^2 + 2$, the x-axis and the ordinates $x = 1, x = 3$.

Solution

$$y = x^2 + 2$$

$$A = \int_{1}^{3} (x^2 + 2) dx = \left[\frac{x^3}{3} + 2x\right]_{1}^{3}$$

$$= \left(\frac{27}{3} + 2(3)\right) - \left(\frac{1}{3} + 2\right)$$

$$= 15 - 2\frac{1}{3}$$

$$= 12\frac{2}{3}$$

3.2 Length of a Curve

To calculate the length of the curve described by the equation $y = f(x)$ between $x = a$ $x = b$, we subdivide the interval $[a, b]$ into n small parts, and take a look at the graph on one of these parts. This is to enable us obtain an approximation and take the limit as n tends to infinity in order to produce an integral which gives the true length.

In the figure above, the length of the curve $PQ = \delta s$ is called the element of distance. If δx and δy are very small, the curve PQ and line PQ will be almost the same. Using Pythagoras Theorem, this length is expressed as

$$\delta s^2 = \delta x^2 + \delta y^2$$
$$i.e.\ \delta r \approx \delta s \ \ as \ \ \delta x \to 0, \delta y \to 0.$$
$$\frac{\delta s^2}{\delta x^2} = \frac{\delta x^2}{\delta x^2} + \frac{\delta y^2}{\delta x^2}$$

Or

$$\frac{\delta s}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

Then

$$\delta s = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\, \delta x$$

Then total length is then given by

$$s = \int_a^b ds = \int_a^b \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

Note:

$\frac{\delta y}{\delta x}$ is called the Newton Quotient and tends to $\frac{dy}{dx}$ $as \ \delta x \to 0.$

So we have obtained the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

for the length of the graph of $y = f(x)$ between $x = a \ and \ x = b$.

Remarks:

If the curve $y = f(x)$ is parameterized such that

$x = x(t),\ y = y(t),$

Then $dx = x(t)dt,\ dy = y(t)dt$

And

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{dx^2 + dy^2}$$

$$=\left(\sqrt{x(t)^2 + y(t)^2}\right)dt$$

And the length is given by

$$s = \int_{t_1}^{t_2}\left(\sqrt{x(t)^2 + y(t)^2}\right)dt$$

Where $t = t_1$ at $x = a$

$and\ t = t_2$ at $x = b$

Examples

(1) Determine the length of the graph $y = \sin x$ between $x = 0$ and $x = a$

Solution

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$\left(\frac{dy}{dx}\right)^2 = \cos^2 x$$

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$= \int_a^b \sqrt{1+\cos^2 x dx} = \int_0^a \sqrt{\sin^2 x}dx$$

$$= \int_0^a \sin x dx = -\cos x\Big|_0^a$$

$$= -\cos a - (-\cos 0) = -\cos a + 1$$

$$= 1 - \cos a$$

The length is $s = 1 - \cos a$

If $a = \pi,$ then $\cos\pi = -1,$ $s = 1-(-1) = 2.$

(2) Let $y = mx + c$ be a straight line. Find its length between $x = a$ and $x = b$

Solution

$y = mx + c$

$\frac{dy}{dx} = m$

$\left(\frac{dy}{dx}\right)^2 = m^2$

$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2 dxs}$

$s = \int_a^b \sqrt{1+m^2}\,dx = (b-a)\sqrt{1+m^2}$

$= (b-a)\sqrt{1+\tan^2\theta} = (b-a)\sqrt{\sec^2\theta} = (b-a)\sec\theta$

$= \frac{(b-a)}{\cos\theta}$

If θ is the angle that the line makes with the $x = axis$ then $(m = \tan\theta)$

(3) Given a circle of radius r, find its circumference.

Solution

Let $x^2 + y^2 = r^2$ be the circle.

$y^2 = r^2 - x^2$

Differentiating wrt x gives

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{y}$

But $y\sqrt{r^2 - x^2}$

So,

$\frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2}$

$= \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \pm\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx$$

The $+$ is for the upper half semicircle while the $-$ is the lower semicircle. The circumference will be

$$C = 2\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx$$

Using change of variable,

Let $x = r\cos\theta$

$$x^2 = r^2\cos^2\theta$$

$$r^2 - x^2 = r^2 - r^2\cos^2\theta = r^2(1-\cos^2\theta)$$

$$= r^2\sin^2\theta$$

$$\sqrt{r^2 - x^2} = r\sin\theta$$

$$x = r\cos\theta$$

$$dx = -r\sin\theta d\theta$$

Or

$$\frac{dx}{d\theta} = -r\sin\theta$$

$$dx = -r\sin\theta d\theta$$

$$\therefore \frac{r}{\sqrt{r^2 - x^2}}dx = \frac{-r^2\sin\theta d\theta}{r\sin\theta} = -rd\theta$$

$$s = -2\int_a^{\pi} rd\theta = -2r\int_{\pi}^{0} d\theta = -2r[\theta]_{\pi}^{0}$$

$$= 2\pi r$$

Self-Assessment Exercise(s) 1

1. Determine the area enclosed by y =2x +3, the x-axis and ordinates x =1 and x =4
2. Sketch the graph $y = x^3 + 2x^2 - 5x - 6$ between x =-3 and x =2 and determine the area enclosed by the curve and the x-axis

3.3 Volume of Revolution

If we take the graph of $y = f(x)$ on the interval $[a,b]$, spin it round the $x-axis$ this produces what is known as a solid of revolution as shown in the figure below. We like to get a formula for the volume of this solid.

We subdivided the interval $[a,b]$ into many small bits, take a look at one of the bits and try to get an approximation for the volume of the 'thin slice' of the solid obtained by rotating the piece of the graph on this interval.

$AB = \delta x$
$OA = x$
$AP = AR = BT = BU = TQ = \delta y$

The volume of revolution element δV is given by

$\pi(y+\delta y)^2 \delta x < \delta V < \pi y \delta x.$

In the notation of the diagram, the thin slice of then solid is virtually a cylinder of radius y and thickness δx. Since the volume of a cylinder is the product of its height and the base area, we obtain the approximation

$$\delta V = \pi y^2 \delta x$$

for the volume of the slice.

The approximation to the total volume is,

$$V = \sum \pi y^2 \delta x$$

Now, taking the limit as $n \to \infty$ gives

$$Volume = \int_a^b \pi y^2 dx$$

Example

1) Let $y = mx$ be the line on $[0, h]$, and spin it around the $x - axis$ to produce a cone of height h and semi-angle θ, where $\tan\theta = m.$

$(CONE)$

By our formula, the volume of this cone is

$$V = \int_0^h \pi m^2 x^2 dx = \left[\pi m^2 \frac{1}{3} x^3\right]_0^h$$

$$= \frac{1}{3}\pi m^2 h^3$$

If R is the radius of the base of the cone then $m = \tan\theta = R/h$. so, we get

$$V = \frac{1}{3}\pi R^2 h = \frac{1}{3} base \times height$$

2 If we take the semicircle $y = \sqrt{r^2 - x^2}$ on $[-r, r]$ and spin it round the $x - axis$, we get a SPHERE of radius r.(SPHERE)

From our formula, the volume of this sphere is

$$V = \int_{-r}^{r} \pi\left(r^2 - x^2\right) dx = \left[\pi r^2 x - \frac{1}{3}\pi x^3\right]_{-r}^{r}$$

$$= \frac{4}{3}\pi r^3$$

So the volume of a sphere of radius r is

$$V = \frac{4}{3}\pi r^3$$

Self-Assessment Exercise(s) 2

1. The curve $y = x^2 + 4$ is rotated one revolution about the x-axis between the limits x =1 and x =4. Determine the volume of the solid of revolution produced\
2. Determine the volume generated when the area above the x-axis bounded by the curve $x^2 + y^2 = 9$ and the ordinates x =3 and x =-3 is rotated one revolution about the x-axis

4.0 Conclusion

At this juncture we hereby conclude that you have acquired adequate knowledge on how to find the area of a functions, length of a curve and volume of revolution (SPHERE).

5.0 Summary

In this unit, you have learnt that:

(a) The area bounded by the curve $x = f(y)$ the lines $y = c, y = d$ and the $y - axis$ is given by ,

$$A = \int_c^d f(y)dy$$

(b) Integration could be applied to kinematics. We can find the velocity distance of a body moving in a straight line with a given acceleration after time t by integration.

(c) If the gradient of a curve is given, its equation can be found by integration.

(d) The volume generated when the curve $y = f(x)$ is revolved about the $x - aixs$ through 2π radians is called volume of solid of revolution and is given by

$$V = \int_a^b \pi y^2 dx$$

a and x are the ordinates.

(e) The volume generated when $x = f(y)$ is revolved about the $y - axis$ through 2π radians is given by

$$V = \int_c^d \pi x^2 d\,\mathrm{y}$$

c and d are the abscissa

(f) The trapezium rule can be used to find the approximate value of an integral.

$$\int_{x_1}^{x_n} f(x)dx = \tfrac{1}{2}h[f(x_1) + f(x_n) + 2(f(x_2) + f(x_1) + ... + f(x_{n-1}))]$$

6.0 Tutor-Marked Assignments

Find the area enclosed by the graphs of the given functions and lines.

1) $f(x) = x^2,\ g(x) = 2x$
2) $f(x) = 2x^2 + 2,\ g(x) = x + 1$
3) $y = \sqrt{16 - 2x},\ y = \sqrt{16 - 4x},\ x - axis$
4) $f(x) = \sin(x/2),\ g(x) = \cos(x/2),\ x = 0,\ x = \pi$
5) Consider the area enclosed between the graph of $y = 1 - x^2$ and the $x - axis$. Which line is parallel to the $x - axis$ divides this area into two equal parts?
6) Determine the length of the graph $y = \cos x$ between $x = 0$ and $x = a$

Determine the volume of the solid of revolution generated by revolving the area enclosed by the graph of each function about the indicated axis.

7) $y = x^3$, the $x - axis,\ x = 0,$ and $x = 2;$ about the $x - axis$
8) $y = \sec x,\ y = 1,\ x = -1,$ and $x = 1;$ about the $x - axis$
9) $y = 2\sqrt{x}$, the $y - axis$, and $y = 4;$ about the $x - axis$
10) Consider the curve $(ellipse)$ given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Find the volume of the solid produced by rotating this about the $x - axis$.

7.0 References/Further Reading

STROUD, K. A. Engineering Mathematics, 4th Edition. MacMillan Press Limited 1995. London

STROUD, K. A. Further Engineering Mathematics, 3rd Edition. MacMillan Press Limited, 1995. London

TRANTER, C.J. and LAMBE, C.G Advanced Level Mathematics, Pure and Applied, 4th Edition. Holder & Stoughton, 1979.Great Britain.

Answers to Self-Assessment Exercises

MODULE ONE

Unit 1

SAE 1

(a) $x^2 + 1 \mp sin5x$
(b) $sin5(x^2 + 1)$

MODULE TWO

Unit 1

SAE 1

1. 0

2. 0

Unit 2

SAE 1

a. cosx + 2

b. $18u^2$

MODULE THREE

Unit 1

SAE 1

1. 6x(x cos 2x + sin 2x)
2. $\frac{dy}{dx} = \frac{3}{\sqrt{x}}\left(1 + \frac{1}{2}ln2x\right)$
3. $\frac{dy}{dx} = \frac{4}{5x^2}(5xcos5x - 4sin5x)$

Unit 2

SAE 1

1. (a)-6sint (b)-3
2. $\frac{dy}{dx} = \frac{2}{5}(\theta - 1)$

MODULE FOUR

Unit 1

SAE 1

1. (1, -3)

SAE 2

1. $y = -x - 1$
2. The length and breadth of the rectangle are each 10 cm, i.e. a square gives the maximum possible area. When the perimeter of a rectangle is 40 cm, the maximum possible area is 10 ×10 =100cm^2

Unit 2

SAE 1

1. $\frac{-3}{x}+c$
2. $(a)\frac{4}{3}sin3x+c$ $(b)-\frac{5}{2}cos2\theta+c$

MODULE FIVE

Unit 1

SAE 1

1. $\frac{1}{16}(2x-5)^8$+c

2. $3\left(\frac{tan4x}{4}-x\right)+c$

SAE 2

1. $xsinx+cosx+c$
2. $[2x(ln3x-1)+c]$

SAE 3

1. $\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)+c$
2. $tan\frac{x}{2}+c$

Unit 2

SAE 1

1. 2.178
2. 0.6275

Unit 3

SAE 1

1. 24 square units
2. 21.08 square units

SAE 2

1. 420.6πcubic units
2. 36$\pi cubic\ units$