

**COMPARING DIFFERENT
NUMERICAL METHODS OF
SOLVING
ORDINARY FIRST ORDER
DIFFERENTIAL EQUATIONS**

BY

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CERTIFICATION

This project have been read and approved as meeting the requirement of the Department of Mathematics/Computer Science, federal University of Technology, Minna for the award of Postgraduate Diploma in Computer Science.

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DEDICATION

TO THE GLORY OF GOD

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ABSTRACT

This project is based on some numerical methods of solving ordinary first order differential equations.

comparisons between simple Euler, Modified Euler, Improved Eulers and Runge kutta methods are considered and based on accuracy, memory space and computing time.

The need to solve differential equation was one of the original and primary motivations to the development of analog and digital computers. The numerical solution of such problems still requires a substantial fraction of the available computing time on existing computers. Our goal is to introduce numerical methods for solving ordinary differential equations especially boundary value problem.

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CHAPTER ONE

INTRODUCTION

1.1 Introduction

A wide variety of application in science, engineering and other fields lead to differential equations. So few of these equations can be solved analytically. Hence it is one of the major problems of numerical analysis. The solutions of some practically important classes of differential equations can be represented in closed form i.e in terms of finite combination of elementary functions such as polynomials, logarithms and exponential functions.

But it is known that inspite of an excellent collection of explicit solutions, even relatively easy looking equations such as

$$\frac{dy}{dx} = x^2 + y^2 \quad \dots\dots(1.1.0)$$

can not be solved interms of elementary function, hence the use of numerical methods to solve differential equation.

1.2 Definitions

An ordinary differential equation is an equation containing one independent and one dependent variable and at least one of its derivative with respect to the the independent variable, no one of the two variables need enter the equation explicitly. It is one of the most important mathematical tools used in modelling problems in the

physical sciences. Thus a differential equation is a relationship between an independent variable X, a dependent variable Y and one or more differential co-efficients of Y with respect to X. Consider the following examples:-

$$x^2 \frac{dy}{dx} + y \sin x = 0 \quad \dots\dots (1.2.0)$$

$$xy \frac{d^2y}{dx^2} + y \frac{dy}{dx} + e^{3x} = 0 \quad \dots\dots (1.2.1)$$

Differential equations represent dynamic relationship i.e quantities that change and are thus frequently occurring in scientific and engineering problems.

1.3 Formation of Differential Equation

Differential equation may be formed in practice from a consideration of physical problems to which they refer. Mathematically, they can occur when arbitrary constants are eliminated from a given function.

Consider

$$y = A \sin x + B \cos x \quad \dots\dots (1.3.0)$$

where A and B are two arbitrary constants, if we differentiate , we get

$$\frac{dy}{dx} = A \cos x - B \sin x \quad \dots\dots (1.3.1)$$

which is identical to the original equation but with the sign changed

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x \quad \dots\dots (1.3.2)$$

i.e

$$\frac{d^2y}{dx^2} = -y \quad \dots\dots (1.3.3)$$

Therefore

$$\frac{d^2y}{dx^2} + y = 0 \quad \dots\dots (1.3.4)$$

1.4 Order and Degree of Defferential Equation

The Order of a differential equation is given by the highest derivative involved in the equation i.e If the highest derivative that occurs in an equation is

$$\frac{d^n y}{dx^n},$$

the equation is said to be of order n. For example

$$x \frac{dy}{dx} - y^2 = 0 \quad \dots\dots (1.4.0)$$

is an equation of the first order.

$$\frac{d^3y}{dx^3} - y \frac{dy}{dx} + e^{4x} = 0 \quad \dots\dots (1.4.1)$$

is an equation of the third order.

$$xy \frac{d^2y}{dx^2} - y^2 \sin x = 0 \quad \dots\dots (1.4.2)$$

is an equation of the second order.

The Degree of differential equation is the power to which the highest derivative is raised. For example

$$x \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^4 = 0 \quad \dots\dots\dots (1.4.3)$$

is a differential equation of the third order and of the fourth degree.

NOTE that the degree of the equation is determined by the power of the highest derivative.

Therefore, a differential equation which involves derivatives or differentials of a function or functions has as an order the largest positive integer n for which the n^{th} derivative occurs in the differential equation. If the differential equation is written as a polynomial, then the highest power of the highest derivative is called the degree of the equation. It should also be noted that order is defined for all differential equations but degree is not defined for some. Take for example

$$\frac{d^2y}{dx^2} = \sin y \quad \dots\dots\dots (1.4.4)$$

is a second order differential equation, however degree is not defined, since the equation is not a polynomial.

1.5 Solution of Differential Equation

To solve a differential equation, we have to find the function for which the equation is true, This means that we have to manipulate the equation so as to eliminate all the differential co-efficients and

leave a relationship between y and x , that is the differential equation which normally arise from physical situations and its often required to obtain a functional relationship between x and y alone, having eliminated the derivatives.

This relation is referred to as the solution of differential equation. It can be shown that solution of order n involves n arbitrary constants. For example

$$\frac{dy}{dx} = x \quad \dots\dots (1.5.0)$$

straight forward integration yields the solution

$$y = \frac{1}{2}x^2 + A \quad \dots\dots (1.5.1)$$

also

$$\frac{d^2y}{dx^2} = 0 \quad \dots\dots (1.5.2)$$

By two successive integration i.e

$$\int \frac{d^2y}{dx^2} dx \Rightarrow \frac{dy}{dx} = A \quad \dots\dots (1.5.3)$$

$$\int \frac{dy}{dx} dx = Ax + B \quad \dots\dots (1.5.4)$$

where A and B are arbitrary constants. The general methods of solution of differential equations (Analytical, Graphical and Numerical) are discussed in chapter two.

1.6 Initial Value Problem.

The first order differential equation is given by

$$\frac{dy}{dx} = f(x, y) \quad \dots\dots (1.6.1)$$

The equation possess an infinite number of solutions. For example, the function

$$y_{(x)} = ce^x \quad \dots\dots (1.6.2)$$

is for any value of the constant c, a solution of the differential equation

$$\frac{dy}{dx} = Ay \quad \dots\dots (1.6.3)$$

where A is a given constant.

We can pick out a particular condition, $y_{(a)} = n$. For the above example, the particular solution satisfying this initial condition is

$$y_{(x)} = ne^{2(x-a)} \quad \dots\dots (1.6.4)$$

Hence, the differential equation (1.6.1) with a given initial condition constitutes the classical initial value problem.

$$\frac{dy}{dx} = f(x, y), \quad y(a) = n \quad \dots\dots (1.6.5)$$

However, the differential equation may be given together with a set of n constraints or boundary conditions, satisfying values of the function y and/or its derivative at two or more distinct values of the independent variable. This type of problem is called the boundary value problem. The theory for the initial value problems is well established and relatively simple.

CHAPTER TWO
GENERAL METHODS OF SOLUTION

2.1 There are variety of methods for solving ordinary differential equations; Some of the methods are discussed below:-

2.2 Analytical Method

Most of ordinary differential equations either cannot be solved by analytical methods or they have too laborious analytical solutions because of large number of integrals involved. There are four main types of first order first degree differential equations, they are:

2.2.1 Variable Variable

These are differential equation with the terms containing x and dx can be separated from the terms containing y and dy by means of simple algebraic transformations and after this, the integration is carried out immediately. They have the general form.

$$\frac{dy}{dx} = f(x) \Phi(y) \quad \dots\dots\dots (2.2.0)$$

where $f(x)$ and $\Phi(y)$ are function of x and y alone respectively. we may then "separate the variable" and write

$$\frac{dy}{\Phi(y)} = f(x) dx \quad \dots\dots\dots (2.2.1)$$

and their general solution is expressed by the formula

$$\int \frac{dy}{\Phi(y)} = \int f(x) dx + c \quad \dots\dots\dots (2.2.2)$$

where c is arbitrary constant (which is usually regarded as being included into the sign of indefinite integral) in order to stress that the constant enters into the general solution. Thus we see that equation (2.2.1) has been integrated by quadratures.

Consider the following example

$$\frac{dy}{dx} = x^2y + y \quad \dots\dots\dots (2.2.3)$$

Expressing the right hand side in the form $f(x)\Phi(x)$ and separate the variables, we have

$$\frac{dy}{dx} = (x^2+1)y \quad \dots\dots\dots (2.2.4)$$

$$\frac{dy}{y} = (x^2+1) dx \quad \dots\dots\dots (2.2.5)$$

To complete the solution of this type of equation, we simply integrate the separated result i.e

$$\int \frac{dy}{y} = \int (x^2+1) dx + c \quad \dots\dots\dots (2.2.6)$$

$$\log_e y = \frac{1}{3}x^3 + x + c \quad \dots\dots\dots (2.2.7)$$

which is the general solution to the differential equation.

Equation (2.2.7) above could be written as

$$\log_e y = \frac{1}{3}x^3 + x + \log_e B \quad \dots\dots\dots (2.2.8)$$

thus

$$\log_e y - \log_e B = x + \frac{1}{3}x^3 \quad \dots\dots\dots (2.2.9)$$

which implies that

$$\log_e y/B = x + 1/3x^3 \quad \dots\dots\dots (2.2.10)$$

Taking the anti-logarithm of equation (2.2.10), we have

$$\frac{y}{B} = e^{x + \frac{x^3}{3}} \quad \dots\dots\dots (2.2.11)$$

Therefore,

$$y = Be^{x + \frac{x^3}{3}} \quad \dots\dots\dots (2.2.12)$$

is an alternative and possibly more convenient form of general solution.

2.2.2 Homogenous Equation

$M(x,y)$ is said to be a homogeneous function of degree n if the sum of the powers of x and y in each term of M is n . Take for example

$$x^2y - 3xy^2 + 2y^3 \quad \dots\dots\dots (2.2.13)$$

is a homogenous equation of degree 3. if the first order differential equation is written in the form:

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \quad \dots\dots\dots (2.2.14)$$

where M and N are homogenous functions of the same degree, then the equation is said to be homogenous. Considering equation (2.2.14)

where M and N are homogenous of degree n, we may divide them by x^n and express the RHS as a function of the simple variable V, where $V = y/x$

Consider the following example

$$(x^2+y^2) \frac{dy}{dx} = xy \quad \dots\dots (2.2.15)$$

This implies that

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1+(\frac{y}{x})^2} = \frac{V}{1+V^2} \quad \dots\dots (2.2.16)$$

M and N both being divided by x^2 .

This suggests the use of the substitution $V=y/x$ or $y=vx$ as the standard method of solving homogenous differential equation, Hence

$$\frac{dy}{dx} = V + x \frac{dv}{dx} \quad \dots\dots (2.2.17)$$

and substituting this in the differential equation, we will see that the result is a new differential equation in which the variables v and x can then integrate to obtain relation between v and x and finally, replace v by y/x. Therefore equation (2.2.15) becomes

$$V + x \frac{dv}{dx} = \frac{V}{1+V^2} \quad \dots\dots (2.2.18)$$

that is

$$x \frac{dv}{dx} = \frac{V}{1+V^2} - V = -\frac{V^3}{1+V^2} \quad \dots\dots (2.2.19)$$

in which v and x are separables. We now integrate and substitute for v which gives the general solution

$$\frac{dx}{x} = \left[-\frac{1}{V^3} - \frac{1}{V}\right] dV \quad \dots\dots\dots (2.2.20)$$

which implies that

$$\log_e x = \frac{1}{2V^2} - \log_e V + \log_e A \quad \dots\dots\dots (2.2.21)$$

Hence

$$\log_e \frac{y}{A} = \frac{x^2}{2y^2} \quad \dots\dots\dots (2.2.22)$$

2.2.3 Linear differential Equation

If the differential equation can be written in the form

$$\frac{dy}{dx} + Py = Q \quad \dots\dots\dots (2.2.23)$$

where P and Q are functions of x only, the equation is said to be linear of first order, since dy/dx and y occur linearly.

Considering the equation (2.2.23), the presence of the terms dy/dx and y suggests the differentiation of a product involving y. To produce this product, we multiply the equation throughout by a function of x, U say, to be determined later. This gives

$$\frac{Udy}{dx} + UPy = UQ \quad \dots\dots\dots (2.2.24)$$

which may be written as

$$\frac{d}{dx} (Uy) + (UPy - y \frac{du}{dx}) = UQ \quad \dots\dots\dots (2.2.25)$$

This equation could be solved by direct integration if

$$Upy - y \frac{dy}{dx}$$

vanishes since the equation would then reduce to

$$\frac{d}{dx}(UV) = UQ \quad \dots\dots\dots (2.2.26)$$

Hence we impose the condition

$$UPy - y \frac{du}{dx} = 0 \quad \dots\dots\dots (2.2.27)$$

i.e

$$\frac{du}{dx} = Up \quad \dots\dots\dots (2.2.28)$$

Since $y = 0$ is not acceptable as the solution, this is a variable separable differential equation in u and x . now solving to obtain u , we have

$$\frac{du}{u} = Pdx \quad \dots\dots\dots (2.2.29)$$

which implies that

$$\log_e U = \int Pdx \quad \dots\dots\dots (2.2.30)$$

$$U = e^{\int Pdx} \quad \dots\dots\dots (2.2.31)$$

In summary, we could solve linear differential equation with then following steps

1. Write the equation in the standard form

$$\frac{dy}{dx} + Py = Q$$

2. Evaluate

$$\int Pdx$$

and obtain integrating factor

$$U = e^{\int Pdx}$$

(No arbitrary constant is introduced at this stage)

3. Multiply the equation

$$\frac{dy}{dx} + Py = Q$$

by U and check that the result is

$$\frac{d}{dx}(Uy) = UQ$$

4. Integrate to complete the solution as

$$Uy = \int UQdx + A$$

2.2.4 Exact Differential Equation

The first order differential equation

$$P(x, y) dx + Q(x, y) dy = 0 \quad \dots\dots (2.2.32)$$

is said to be exact if a function U(x,y) exists such that

$$du = Pdx + Qdy \quad \dots\dots (2.2.33)$$

If such a function exists, the equation becomes $du = 0$ which leads to $U = A$ where A is a constant.

For instance, consider the differential equation

$$3x^2y^2dx + 2x^3ydy = 0 \quad \dots\dots (2.2.34)$$

By inspection, this can be written as

$$du = d(x^3y^2) = 0 \quad \dots\dots (2.2.35)$$

with solution

$$x^3y^2 = A \quad \dots\dots (2.2.36)$$

Since

$$du = \frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} dy$$

this implies that

$$\frac{\delta u}{\delta x} = P \quad \text{and} \quad \frac{\delta u}{\delta y} = Q \quad \dots\dots (2.2.37)$$

then noting that

$$\frac{\delta^2 U}{\delta y \delta x} = \frac{\delta^2 U}{\delta x \delta y} \quad \dots\dots (2.2.38)$$

we obtain

$$\frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x} \quad \dots\dots (2.2.39)$$

This is the condition for $Pdx + Qdy$ to be an exact differential.

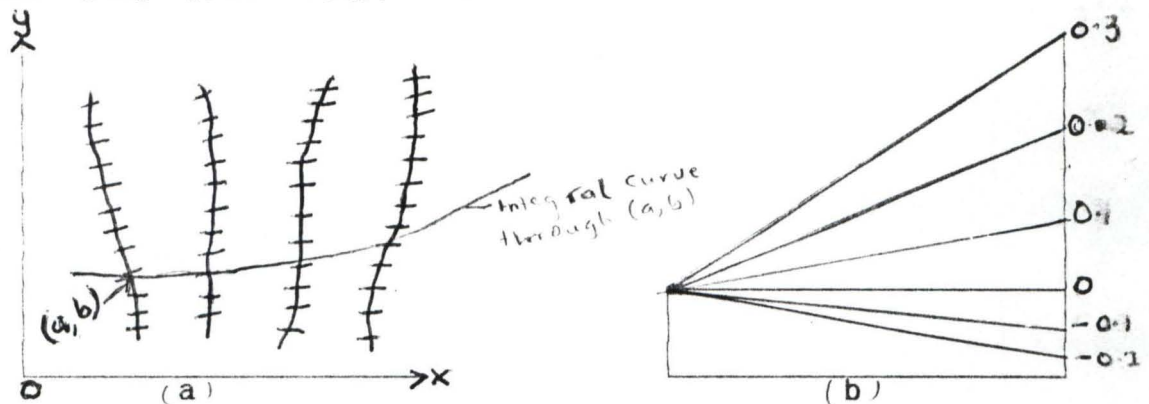
2.3 Graphical Methods

This method enables us to construct the integral curves of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y)$ is a single valued continuous function of x and y

We first draw the family of curve $f(x, y) = K$ where k is a constant. These curves are called ISOCLINAS and connect the points at which the integral curves have the same slope. Each isocline is now crossed at many points by short parallel lines of slope K (where K is the value for the particular isocline crossed), so that these short parallel lines indicate the slope (dy/dx) of the integral curves crossing a particular isocline. By constructing several isoclinals at short intervals, we obtain a general picture of the integral curves satisfying $dy/dx = f(x, y)$ as below



To be effective, the work should be carried out on a drawing board (b) is being used to transfer lines of slopes 0.1, 0.2, 0.3,.... across the page by set square.

Having drawn the isoclinals at reasonably short intervals, the short parallel lines should be made to meet about half way between consecutive isoclines. The link polygon so obtained is finally smoothed off. When a specified integral curve is required, say to pass through (a, b) , only portions of the isoclinals need to be drawn.

This method may give a very useful insight into the nature of the solutions of ordinary differential equations, but they suffer from several serious disadvantages, some of which are:-

- a. Accuracy is limited by the draughtsman's technique
- b. The judgement is subjective
- c. The error is difficult to compute.

2.4 Numerical methods

By numerical method for solving the initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y) \dots\dots (2.4.1)$$

It meant a procedure for finding approximate values y_0, y_1, \dots, y_n of the exact solution $y_{(x)}$ at points $x_0 < x_1 < \dots < x_n \dots$. The first step is to estimate y_1 from the initial conditions and $y'_0 = f(x_0, y_0)$. After knowing y_1 , we determine y_2 and so on.

In general, methods that require only a knowledge of y_n to determine y_{n+1} are called STARTING or SINGLE-STEP METHODS. On the other hand, the methods that make use of data at more than one previous points, say y_n, y_{n-1}, y_{n-2} to determine y_{n+1} are called CONTINUING or MULTISTEP METHODS.

One should be very careful in using numerical methods because they can have inherent difficulties of their own. In the first place, there is the question of convergence i.e as the difference h between the points x_0, x_1, \dots, x_n approaches zero, do the values of the numerical

solutions y_0, y_1, \dots, y_n approaches the values of the exact solution. There is also a serious question of estimating error which arises generally from the following two causes.

- a. The formula used in numerical method is only an approximate one which introduces truncation error.
- b. It is possible to carry only a limited number of digits in any computation which gives rise to rounding error.

2.4.1 Stability of Numerical Procedure

Roughly speaking, stability is the phenomenon of the computed solution behaving like the true solution of differential equation. It is concerned with the way errors propagate through the solution. Much of the study of stability has been confined to the equation

$$\frac{dy}{dx} = \lambda y \dots \dots \dots (2.4.2)$$

for a constant λ

This is because the general problem is so very complicated, thus; Stability may depend upon the differential equation under study, the method used for solution and the step size h . There is the possibility of inherent instability where small variations in the initial conditions attached to the problem give rise to large variation in the true solution. Such ill-conditioning must be reflected in the numerical scheme and will occur whatever method is used.

Consider the differential equation

$$\frac{dy}{dx} = y - 2x \dots (2.4.3)$$

this has the analytical solution

$$y = Ae^x + 2x + 2 \dots (2.4.4)$$

Suppose the initial conditions are $y = 2$ when $x = 0$, then $A = 0$ and the particular solution becomes

$$y = 2x + 2 \dots (2.4.5)$$

The term Ae^x which would dominate for large value of x is absent; However, round-off error and truncation error will ensure that e^x will appear in the solution, albeit with small co-efficient.

This mean that for large value of x , the spurious term will swamp the true solution, since the phenomenon is a feature of the differential equation, the only hope is to reformulate the problem.

2.4.2 Boundary Conditions

We can classify the problems in differential equations to the nature of the boundary conditions. If all the required conditions are given simply at one point, the problem is known as initial value problem or Starting problem or Marching problem as the solution is advanced in steps.

If the boundary conditions are given at two or more points, the problem is known as Boundary value problem or jury problem. the solutions of initial value problem can easily be obtained using direct methods where as the solutions of boundary value problems are in general determined iteratively.

CHAPTER THREE
ERROR ANALYSIS

3.1 Introduction

An error in a numerical computation is the difference between the true (actual) value of a quantity and its computed (approximate) value. if x represents the actual value for which is X^* , then the error say e_r is defined by

$$e_r = x^* - x \quad \dots\dots\dots (3.1.1)$$

In general, e_r may be positive or negative and hence it is considered to be the modulus (absolute) value of the error i.e $|e_r|$

3.2 Sources (Types) of Errors

3.2.1 Gross Error

These are either caused by human mistakes or by the computer. A few of them are as follows:-

- a. Misreading or misquoting of figures, particularly in the interchanges of adjacent digits
- b. Carrying out incorrect sequences of operations
- c. The mathematical description of the problems is inexact e.g the initial data are not reliable
- d. Some of inaccurate mathematical formula to solve the problems.

3.2.2 Rounding error

The error introduced by rounding-off numbers to a limited number of decimal places e.g. it would be impracticable to mention

the distance between 2 points on the earth as 15.2967m. It would be more reasonable if it were to be rounded to the nearest whole number i.e 15m the error introduced by rounding is 0.2967m.

Rounding errors play an important role in numerical analysis, our goal is to eliminate (or minimize) the effects of errors rather trying to obtain a smaller error as a result of rounding-off, we apply the following:

- a. If the first discarded digit is less than 5, the previous digit is unchanged
- b. If the first discarded digit is greater than 5, the previous digit is increased by 1
- c. If the first discarded digit is exactly 5, the previous digit is unchanged if it is odd.

In practice, the following round-off rule is commonly used:-

If the first discarded digit exceeds 5 or equal to 5, we add 1 to the last retained digit. Important feature is that if a number is correct to n decimal places, it has a rounding error

$$|e_r| \leq \frac{1}{2} \times 10^{-n} \quad \dots \dots (3.1.2)$$

3.2.3 Truncation Errors

These are errors introduced when we are forced to approximate the sum of an infinite (or finite) series by the sum of a smaller number of terms i.e truncation error may occur when an infinite series is approximated by a finite one.

3.2.4 Absolute Errors

The true value of a quantity less its rounded value, if X and X^* are respectively the rounded and the true values of a quantity, then the absolute error is defined by

$$AE = |X^* - X| \quad \dots(3.1.4)$$

3.2.5 Relative Errors

This is the ratio of the absolute error to the true value of a quantity and is defined by

$$RE = \left| \frac{AE}{X} \right| \quad X \neq 0 \quad \dots(3.1.5)$$

If the true value is not known, The relative error is defined by

$$RE = \left| \frac{AE}{X^*} \right| \quad X^* \neq 0 \quad \dots(3.1.6)$$

Relative error is more precise than the absolute error, this is particularly so when the true value is either very small or very large. The size of absolute error depends on the unit employed, where as relative error is a dimensionless quantity and does not depend on the unit used.

3.2.6 Percentage Errors

This is a relative error expressed in percentage and is defined as $PE = 100 \times RE$ (3 . 1 . 7)

3.3 PRECISION AND ACCURACY

3.3.1 Precision

The number of digits to which a number is expressed or an answer given, irrespective of the correctness of these digits.

3.3.2 Accuracy

The number of digits to which an answer is correct, accuracy can be quoted by either of the following two ways

- a. To a given number of decimal places
- b. To a given number of significant figures

Precision is a description of maximum absolute error, whereas accuracy is used to describe relative error. Because precision describes absolute measured quantities, expressions of precision usually have dimension, thus when statements about precision are made, the unit involved need to be expressed.

3.4 Types of Numerical Methods

The range of differential equations that can be solved by straight forward analytical methods is relatively restricted. Even solutions in series may not always be satisfactory either because of slow convergence of the resulting series or because of the involved manipulation in repeated stages of differential equation. In such cases where a differential equation and known boundary conditions are given, an approximate solution is often obtainable by the application of numerical methods when the relevant particular

solution is obtained as a set of function values for the range of the independent variable.

Comparatively, few differential equation can be solved in finite form by the methods in standard works and we are now going to consider the numerical solution of ordinary first order differential equation with given initial conditions and having numerical coefficients.

The following methods are known as 'step by step' methods because the values of the dependent variable y are constructed by short steps ahead for equal intervals h of the independent variable x . The problem then is to tabulate the solution of

$$\frac{dy}{dx} = f(x, y) \dots\dots (3.1.8)$$

for which $y = y_0$ $x = x_0$, the values of y , corresponding to the values of $x_0, x_1 = x_0+h, x_2 = x_0+2h, \dots$ of x being found. We shall denote these values by y_0, y_1, y_2, \dots

Generally, numerical methods fall into the following classes:-

- a. A class of methods which produces expression for y in terms of function of x , from which values of y can be obtained by direct substitution, under this, we have
 - i. Pocard's method
 - ii. Taylor's series method.
- b. Another classes of methods find the numerical values of the changes in y due to a given increment in x , under this we have

- i. Eulers method
 - ii. Modified and Improved Eulers methods
 - iii. Runge Kutta methods.
- c. Predictor–Corrector method

Here, we first predict a value for y_{n+1} using a Predictor formula and then use a different formula called the Corrector formula to improve upon this value.

3.4.1 Taylor series Method

We attempt to find the relation between y and x by finding the co-efficients of the Taylor series in which we expand y about the point $x = x_0$

$$Y = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \dots + \dots \quad (3.1.9)$$

where y_0, y_0', \dots are the derivatives and can be calculated from the given equation

$$\begin{aligned} y' &= f(x, y) \\ y_0' &= f(x_0, y_0) \end{aligned} \quad \dots \dots (3.2.0)$$

Let $h = x - x_0$, then (3.1.9) becomes

$$Y = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \dots + \frac{h^k}{k!} y_0^{(k)} \quad \dots \dots (3.2.1)$$

which is a k^{th} order Taylor series formula

Remark

Taylor series method can be very effective but its main disadvantage lies in the calculation of higher order derivations which are sometimes complex to compute. This renders the methods

impracticable. Taylor series method is however often used to provide some values of Y required in a predictor-corrector method.

3.4.2 Simple Euler Method

The oldest and simplest Taylor series method is Euler, it works iteratively and does not require the computation of higher-order derivatives. For solving ordinary differential equation in which Taylor series

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots \quad (3.2.1)$$

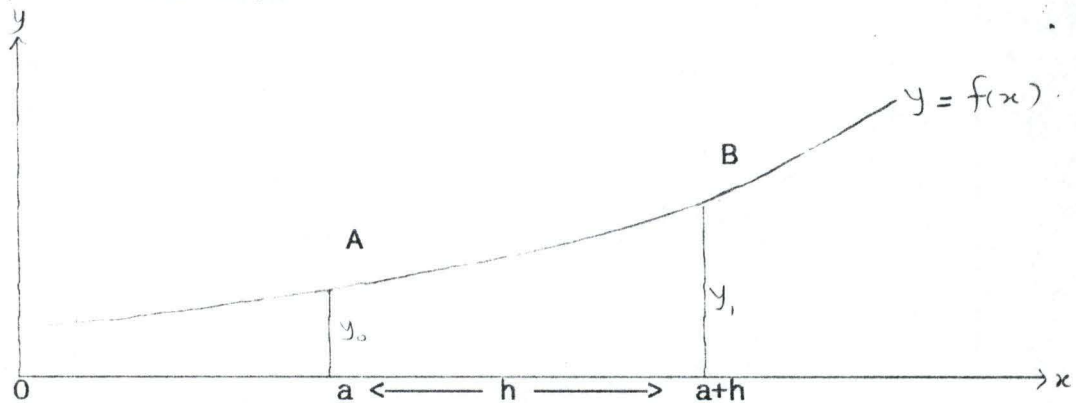
is truncated after the second term to give

$$f(a+h) \approx f(a) + hf'(a) \quad \dots \quad (3.2.2)$$

This truncation is called local truncation error (also called error per step) in Euler method is given by the following relation

$$Ee = \frac{h^2}{2!}y'' \quad x_0 < n < x_0+h \quad \dots \quad (3.2.3)$$

This shows that the "local truncation error" is proportional to the square of the stepwise



If h is the interval between two near ordinates and if we denote $f(a)$

by y_0 , then the relationship

$$f(a+h) \approx f(a) + hf'(a)$$

becomes

$$y_1 = y_0 + h(y')_0$$

$$y_{n+1} = y_n + h(y')_n \quad \dots \dots (3.2.5)$$

Hence knowing y_0 , h , and $(y')_0$, we can compute y_1 , an approximate value for the function value at B.

If we take the value of x , y , y' that we have just found for point B and treat these as new starter values x_0 , y_0 , $(y')_0$, we can repeat the process and find values corresponding to the value C. So we could continue in a step-by-step method. At each stage the determined values of x_0 , y_0 , and $(y')_0$ for the next stage.

Remark

Although the procedure is simple, when h is small, the method is slow and laborious. Once chosen value of h may give sufficiently accurate results for some equations, whereas other equations may require a smaller value of h and in general there is no way of determining how good the approximation will be. By increasing h to cut down numerical work, the method yields crude result which are too inaccurate for practical purposes.

3.4.3 Modified Euler Method

To reduce the inherent error in the simple Euler method, variation is implemented and the new formula becomes:-

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right] \quad \dots\dots (3.2.6)$$

3.4.4 Improved Euler Method (Heun's method)

Also variations is implemented on simple Euler method to obtain

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))] \quad \dots\dots (3.2.6)$$

The local truncation error for the improved Euler method is proportional to the cube of the stepwise.

3.5 Runge Kutta Method

The Runge Kutta methods are a family of methods derived from the Taylor series method for solving first order differential equations is widely used and affords a high degree of accuracy. It is a further step-by-step process where a table of x is accumulated. Several intermediate calculations are required at each stage, but these are straight forward and present little difficulty. In general terms, the method is as follows:-

To solve

$$y' = f(x,y) \quad \dots\dots (3.2.8)$$

with initial condition

$$y = y_0 \quad \text{at } x = x_0$$

for a range of $x = x_0 + (h) x_n$.

starting as usual with $x = x_0$, $y = y_0$, $y' = (y')_0$ and h ,

we have $x_1 = x_0 + h$. Finding y_1 requires four intermediate

calculations:-

$$k_1 = hf(x_0, y_0) = h(y')_0 \quad \dots \dots (3.2.9)$$

$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \quad \dots \dots (3.2.10)$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) \quad \dots \dots (3.2.11)$$

$$k_4 = hf(x_0 + h, y_0 + k_3) \quad \dots \dots (3.2.12)$$

The increment Δy_0 in the y value from $x = x_0$ to $x = x_1$ is then

$$\Delta y_0 = \{k_1 + 2k_2 + 2k_3 + k_4\}/6 \quad \dots \dots (3.2.13)$$

and

$$y_1 = y_0 + \Delta y_0 \quad \dots \dots (3.2.14)$$

i.e

$$y_{n+1} = y_n + \Delta y_n \quad \dots \dots (3.2.15)$$

Remark

Estimation of error in the Runge Kutta method is very laborious, however, the truncation error can be obtained from the next term of the Taylor series and in the simplified form. There is practically no error in the method, the computation involved in the procedure may be rather more tedious but the extra labour involved is rewarded by the higher degree of accuracy of the result.

CHAPTER FOUR
COMPARISONS

4.1 Comparison of Euler (Simple, Modified and Improved) methods

Consider the initial value problem

$$\frac{dy}{dx} = x + y \quad y_{(0)} = 1 \quad \dots\dots(4.1.0)$$

On the interval (0,1) whose exact solution is given by

$$y = 2e^x - x - 1 \quad \dots\dots(4.1.1)$$

Using simple Euler's method to solve the problem, we get the following results

$$y(0.5) = 1.7210200 \quad \text{for } h=0.1 \quad \dots\dots(4.1.2)$$

where as the true solution is

$$y(0.5) = 1.797442541400 \quad \dots\dots(4.1.3)$$

Hence, the total error due to truncation and round-off for h=0.1 is

$$1.7974425414 - 1.7210200 = 0.0764255 \quad \dots\dots(4.1.4)$$

Consider the same problem evaluated using modified Euler method, we obtained the following

$$y(0.5) = 1.7948940 \text{ for } h=0.1 \quad \dots (4.1.5)$$

The total error given is

$$1.7974425414 - 1.7948940 = 0.0025485 \quad \dots (4.1.6)$$

Also considering the same problem evaluated using improved Euler method, we obtained the same result as for modified Eulers method i.e the total error is

$$1.7974425414 - 1.7948940 = 0.0025485 \quad \dots (4.1.7)$$

The greater accuracy of improved modified Euler methods must be paid for by an increase in the number of computation and function evaluation.

4.2 Comparison of Eulers and Runge Kutta Methods

The result obtained when the problem is evaluated using the classical 4th order Runge Kutta methods are

$$y(0.5) = 1.7974410 \text{ for } h=0.1 \text{ (appendix VIII)}$$

A comparison of the four methods using the same stepwise

h 0.1 is given below

S/No	Simple Euler	Modified Euler	improved Euler	Runge Kutta
0.5	1.7210200	1.7948940	1.7948940	1.7974410

It is obvious that the Runge Kutta is the most accurate single step method for this initial value problem. The accuracy of this method is also paid for by the number of computations and the function manipulation at each step.

The Runge Kutta method requires approximately twice the computing time of the improved Eulers method and four (4) times the computing time of the simple Eulers method.

4.3 Accuracy Determination

The accuracy of the solution is not determined by the number of function evaluations alone but also the manner in which these functions are combined together. This is illustrated by evaluating $y(0.5)$ using the four methods and the same number of function evaluation.

Since Simple Eulers method requires only one evaluation of the function per step, while modified and improved Eulers methods require two (2) evaluations and the Runge Kutta method requires four evaluations of the function per step, we can choose $h=0.025$, 0.05 and 0.1 for simple Eulers, improved Eulers and Runge Kutta methods respectively.

This would require approximately the same amount of

computing time since each method would require twenty evaluations of the function. We should therefore expect the methods to have the methods the same accuracy. performing the necessary computations, the result obtained are:

x_n	Simple Euler	Modified Euler	improved Euler	Runge Kutta
0.5	1.7772329	1.79628088	1.7974413	1.7974424

The result shows that it is not only the number of function evaluations which ultimately determines the accuracy of the solution but also the manner in which these functions are combined.

4.4 Error Analysis

We can see that there is practically no error in the Runge Kutta method. The computation involved in the procedure may be rather more tedious than in the other methods considered but the extra labour involved is rewarded by the higher degree of accuracy of the result.

Having obtained the solution to other methods, it is useful to compare the error in each case.

X_n	ABSOLUTE		ERRORS
	Simple Eulers	Modified Eulers	Runge Kutta
0.0	0.000000	0.0000000	0.0000000
0.1	0.010342	0.000342	0.0000000
0.2	0.022806	0.000756	0.0000000
0.3	0.037718	0.001253	0.0000000
0.4	0.055447	0.001845	0.0000000
0.5	0.076423	0.022550	0.0000001

The supremacy of Runge Kutta method is self-evidence and the method is popular where accuracy of result is all important. one disadvantage of Runge Kutta, however is that it does not readily lend itself to any self checking procedure.

CHAPTER FIVE

ANALYSIS OF RESULT

5.1 Summary

From the comparison carried out in chapter four, it is evidently clear that the Runge Kutta method is the most accurate of all the methods considered, although a price has to be paid for the high accuracy and that is the tedious calculation involved. The other methods could be used when one considers time constraints and for instructional purposes.

5.2 Conclusion

Based on our error analysis, it can be seen that the Runge Kutta is more accurate than the Eulers (Simple, modified and improved) methods. It is seen that the Runge Kutta methods are highly accurate near the initial point, but like other methods, the accuracy tends to decrease as we move away from the initial point. Runge Kutta also pay for the higher accuracy in terms of computing time.

5.3 Recommendation

Having concluded that the Runge Kutta is preferred to other methods considered in terms of accuracy, it is therefore

recommended that for a purposeful models requiring the application of numerical methods of solving first order ordinary differential equation, Runge Kutta method is recommended until a more accurate method is discovered.

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APPENDIX I

```
<beginning of file>
PROGRAM EULER
READ(*,*)X,Y,H,XN
WRITE(*,13)
13  FORMAT(2X,4F9.8)
10  YCOMP=F(X,Y)
    YNEW=Y+H*YCOMP
    WRITE(*,*)X,Y,YCOMP,YNEW
    Y=YNEW
    X=X+H
    IF(X .LE. XN)GOTO 10
    STOP
FUNCTION F(X,Y)
F=X+Y
RETURN
END
<end of file>
a:euler.for (0,1,58944) WATFOR-77 V3.0 PC/DOS
```

APPENDIX II

<beginning of file>

```
PROGRAM MODIF
C   MODIFIED EULERS METHOD
   READ(*,*)X,Y,H,XN
   WRITE(*,17)
17  FORMAT(2X, 4F9.8)
15  YCOMP=F(X,Y)
   XMD=X+H/2
   YMD=Y+H/2*YCOMP
   YNEW=Y+H*F(XMD,YMD)
   WRITE(*,*)X,Y,YCOMP,YNEW
   Y=YNEW
   X=X+H
   IF(X .LE. XN)GOTO 15
   STOP
FUNCTION F(X,Y)
F=X+Y
RETURN
END
```

<end of file>

a:modif.for (0,1,59493) WATFOR-77 V3.0 PC/DOS

APPENDIX III

<beginning of file>

```
PROGRAM HEUNS
C   IMPROVED EULERS METHOD (HEUNS METHOD)
   READ(*,*)X,Y,H,XN
   WRITE(*,23)
23  FORMAT(2X,4F9.8)
20  YCOMP=F(X,Y)
   YP=Y+H*YCOMP
   XP=X+H
   YIMP=F(XP,YP)
   YNEW=Y+H/2*(YCOMP+YIMP)
   WRITE(*,*)X,Y,YCOMP,YNEW
   Y=YNEW
   X=X+H
   IF(X .LE. XN)GOTO 20
   STOP
FUNCTION F(X,Y)
F=X+Y
RETURN
END
```

<end of file>

a:heuns.for (0,1,58944) WATFOR-77 V3.0 PC/DOS

APPENDIX IV

<beginning of file>

```
PROGRAM KUTTA
C  RUNGE KUTTA METHOD
  REAL K1, K2, K3, K4, K
  READ(*,*)X,Y,H,XN
  WRITE(*,27)
27  FORMAT(2X,4F9.8)
25  K1=H*F(X,Y)
    K1=H*F(X+H/2,Y+K1/2)
    K3=H*F(X+H/2,Y+K1/2)
    K4=H*F(X+H/2,Y+K1/2)
    K=(K1+2(K2+K3)+K4)/6
    YNEW=Y+K
    WRITE(*,*)X,Y,K,YNEW
    Y=YNEW
    X=X+H
    IF(X .LE. XN)GOTO 25
  STOP
  FUNCTION F(X,Y)
  F=X+Y
  RETURN
  END
```

<end of file>

a:kutta.for (0,1,59157) WATFOR-77 V3.0 PC/DOS

APPENDIX V

<end of file>

a:euler.for (0,1,58944) WATFOR-77 V3.0 PC/DOS

RUN

0 1 0.1 1

X	Y	YCOMP	YNEW
0.00000000	1.00000000	1.00000000	1.10000000
0.10000000	1.10000000	1.20000000	1.22000000
0.20000000	1.22000000	1.42000000	1.36200000
0.30000000	1.36200000	1.66200000	1.52820000
0.40000000	1.52820000	1.92820000	1.72102000
0.50000000	1.72102000	2.22102000	1.94312200
0.60000000	1.94312200	2.54312200	2.19743400
0.70000000	2.19743400	2.89743400	2.48717800
0.80000000	2.48717800	3.28717800	2.81589500
0.90000000	2.81589500	3.71589500	3.18748500

APPENDIX VI

<end of file>

a:modif.for (0,1,59493) WATFOR-77 V3.0 PC/DOS

RUN

0 1 0.1 1

X	Y	YCOMP	YNEW
0.00000000	1.00000000	1.00000000	1.11000000
0.10000000	1.11000000	1.21000000	1.24205000
0.20000000	1.24205000	1.44205000	1.39846500
0.30000000	1.39826500	1.69846500	1.58180400
0.40000000	1.58180400	1.98180400	1.79489400
0.50000000	1.79489400	2.29489400	2.04085800
0.60000000	2.04085800	2.64085800	2.32314800
0.70000000	2.32314800	3.02314800	2.64557800
0.80000000	2.64557800	3.44557800	3.01236400
0.90000000	3.01236400	3.91236400	3.42816200

APPENDIX VII

<end of file>

a:heuns.for (0,1,58944) WATFOR-77 V3.0 PC/DOS
 RUN

0 1 0.1 1

X	Y	YCOMP	YNEW
0.00000000	1.00000000	1.00000000	1.11000000
0.10000000	1.11000000	1.21000000	1.24205000
0.20000000	1.24205000	1.44205000	1.39846500
0.30000000	1.39826500	1.69846500	1.58180400
0.40000000	1.58180400	1.98180400	1.79489400
0.50000000	1.79489400	2.29489400	2.04085800
0.60000000	2.04085800	2.64085800	2.32314800
0.70000000	2.32314800	3.02314800	2.64557800
0.80000000	2.64557800	3.44557800	3.01236400
0.90000000	3.01236400	3.91236400	3.42816200

APPENDIX VIII

<end of file>

a:kutta.for (0,1,59157) WATFOR-77 V3.0 PC/DOS

RUN

0 1 0.1 1

X	Y	K	YNEW
0.00000000	1.00000000	0.11034170	1.11034200
0.10000000	1.11034200	0.13246350	1.24280500
0.20000000	1.24280500	0.15691190	1.39971700
0.30000000	1.39971700	0.18393150	1.58364800
0.40000000	1.58364800	0.21379280	1.79744100
0.50000000	1.79744100	0.24679470	2.04423600
0.60000000	2.04423600	0.28326730	2.32750300
0.70000000	2.32750300	0.32357590	2.65107900
0.80000000	2.65107900	0.36812370	3.01920300
0.90000000	3.01920300	0.41735660	3.43656000