BESSEL FUNCTIONS

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DEDICATION.

I dedicated this work of mine to the Almighty Allah, the Beneficent the Merciful for sparing my life up to this moment.
I also dedicated this work to my entire family especially Mallam Zakari Adamu and Alhaji Hussaini Adamu for their moral and financial support during my course of studies.

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ABSTRACT

The importance of Bessel's function can not be over emphasized, since they are widely used in Engineering, Physics in solving physical problems especially in problems involving cylindrical boundaries which are sometimes called cylinder functions.

This research work covers different types of Bessel functions, i.e. Bessel function of the first and second kinds and also properties of Bessel functions.

Similarly, the application of Bessel functions in determining the temperature in a long cylinder and in solving differential equations are discussed. Finally we write a computer program to test for the analyticity of the Bessel functions.

TABLE OF CONTENTS

CHAPTER THREE.

CHAPTER FOUR

CHAPTER FIVE

BESSEL FUNCTION AND APPLICATIONS

CHAPTER ONE

1.0 INTRODUCTION

Bessel functions can not be discussed in any way without the knowledge of the differential equations.

Differential equations are equations that relate a function to its directives. This type of equation finds wide application in science and technology because it can be used to express natural laws that describe the behaviors and rates of change of quantities.

A well known example in physic concerns radioactivity. A radioactive element changes spontaneously into a stable element, a process that is known as radioactive decay or disintegration. The law describing this behaviour states that the rate of decay with time - that is the amount of a substance changing per second, say - is proportional to the amount of substance present. Initially, when the material is a pure radioactive element, this rate of decay is high. As the radioactive element changes into a stable element, however, the rate of change falls because there is less radioactive material. Therefore, the rate of decay decreases continuously with time.

1.1. HISTORICAL BACKGROUND

Historically, the study of differential equations is almost as old as that of the calculus itself. Sir Isaac Newton discovered a method of infinite Series and calculus in 1665 - 66.

In 1671 he wrote an account of his theory of "fluxions", a fluxion being derivatives of a "fluent", the name Newton gave to his dependent variables. Newton discussed "fluxional equations", or as they are now called, differential equations. These he divided into 3 categories. In modern notation, the first category is that in which dy/dx is a function of x alone or of y alone, the second consists of ordinary differential equations of the first order of the form dy/dx = f(x,y) and the third is made up partial differential equations of the first order. It is in finding solutions to the partial differential equation using the methods of variable separables that gives rise to the use of the Bessel's equations.

1.2 THE IMPORTANCE OF BESSEL FUNCTIONS

Just as the Laplace - equations expressed in Spherical polar coordinates lead to the Legendre equation and Legendre polynomials, its expression in cylindrical polar coordinates leads to the equation known as Bessel's equations. The Bessel's equation of order v is a second order differential equation with variable coefficients and is satisfied by the Bessel function which can be expressed as a power series in power of t/2

Bessel functions are widely used in Engineering, Physics especially in problems in potential theory and diffusion. They frequently arise in problems involving Cylindrical boundaries and are sometimes called Cylinder functions.

1.3 DEFINITION OF BESSEL'S FUNCTIONS

In boundary value problems that involve the Laplacian $\nabla^2 U$ expressed in cylindrical coordinates the process of separating variables often produces an equation of the form

$$(1) \qquad \rho^2 \frac{d^2y}{d\rho^2} + \rho \frac{dy}{d\rho} + (\lambda^2 \rho^2 - v^2)y = 0$$

in the function, y, of the Cylindrical coordinate, ρ. In such problems, as above the parameter, $\chi 2$ is the separation constant whose values are the eigen values associated with equation (1). The parameter, v, is a real number determined by other aspects of the boundary value problem. It is most commonly either Zero or a positive integer.

When we write equation (1) in terms of the variable x, where

$$x = \lambda \rho,$$

equation (1) takes a form free of parameter λ:

$$(2) \qquad x^2y''(x) + xy'(x) + (x^2 - v^2y(x) = 0$$

This linear homogeneous differential equation is Bessel equation. Its solutions are called Bessel functions or, sometimes, cylindrical functions.

1.4 <u>TYPE OF BESSEL FUNCTIONS</u>

We have several kinds of Bessel's functions, but the most common types are the Bessel's function of the first kind and the Bessel's function of the second kind.

1.5 <u>BESSEL'S FUNCTION OF THE FIRST KIND.</u>

For any Bessel function, J_n, given, we can determine its solution from the Bessel equation say;

(1) $$x^2y''(x) + xy'(x) + (x^2-n^2)y(x) = 0 \quad (n=0,1,2,...)$$

in the form of a power series multipled by x^p where p is some constant. That is, we attempt to determine p and the coefficients a_j so that the function

(2) $$y = x^p \Sigma a_j x^j = \Sigma a_j x^{p+j}$$

satisfies equation (1). Here a_o represents the coefficients of the first non-vanishing term in the series, so that $a_o <> 0$. Assuming that we can differentiate the function (2), if we substitutes the function (2) and its derivatives into equation (1) we will obtain the equation, that is,

(3) $$y = \Sigma a_j x^{p+j}$$

$$y' = \sum (p+j) a_j x^{p+j-1}$$

$$y'' = \sum (p+j)(p+j-1) a_j x^{p+j-2}$$

combining equations (1) and (2) we obtain

$$\Sigma[(p+j)(p+j-1)+(p+j) + (x^2-n^2)]a_j x^{p+j} = 0$$

or $$\Sigma[(p+j)^2 - n^2]a_j x^j + \Sigma a_k x^{k+2} = 0$$

The last Sum can be written as

$$\Sigma a_{j-2} x^j;$$

thus the equation becomes.

$$(p-n)(p+n)a_o + (p-n+1)(p+n+1)a_1 x +$$
$$\Sigma[(p-n+j)(p+n+j)a_j + {}_{j-2}]x^j = 0$$

Equation (3) is said to be an identity in x if the coefficient of each power of x vanishes. This condition is satisfied if $p = n$ or $p = -n$ so that the constant term vanishes and if $a_1 = 0$ and

$(p-n+j)(p+n+j)a_j + a_{j-2} = 0 \quad (j=2,3,...)$

we make the choice $p = n$, and we have the recurrence relation

(4) $$a_j = \frac{-1}{j(2n+j)} a_{j-2} \qquad (j=2,3,...)$$

is obtained, giving each coefficient in terms of the second one preceding it in the series. It should be noted that $2n + j <> 0$ in equatun (4). The choice $p = -n$ fails to give a reccurence relation.

Since $a_1=0$, relation (4) requires that $a_3=0$, $a_{2j-1}=0$ $(j=1,2,\ldots)$

For the remaining coefficients the recurrence relation is

(6) $$a_{2j} = \frac{1}{2^2 j(n+j)} a_{2j-2} \qquad (j=1,2,\ldots)$$

Therefore $a_{2j-2} = \dfrac{-1}{2^2 (j-1)(n+j-1)} a_{2j-4}$

and so

$$a_{2j} = \frac{(-1)}{j(j-1)(n+j)(n+j-1)} \frac{1}{2^4} a_{2j-4}$$

continuing this process through j-2 further steps, we find that

$$a_{2j} = \frac{(-1)}{j!} \frac{1}{(n+j)(n+j-1)\ldots(n+1)} \frac{a_o}{2^{2j}}$$

Thus a_o is a common factor in all terms of the series. Simplify the series we select this value for a_o:

$$a_o = 1/2^n n!$$

Then our formula for the non vanishing coefficient becomes

(7) $$a_{2j} = \frac{(-1)^j}{j!(n+j)!} \frac{1}{2^{n+j}} \qquad (j=0,1,2,\ldots)$$

Where we use the convection that $0! = 1$. Therefore our Bessel's equation can now be written, in view of formulas (5) and (7).

That is $y = J_n(x)$

Where J_n is Bessel's function of the first kind of order n, or index n, defined by the equation.

$$J_n(x) = \Sigma \frac{(-1)^j}{j!(n+j)!} (x/2)^{n+2j} \quad (n=0,1,2,..) \tag{8}$$

$$= \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 2!(n+1)(n+2)} \ldots\ldots\right]$$

Since the coefficient of equation (8) satisfy the recurrence relation needed to make its sum satisfy Bessel's equation when the series is differentiable, then $y = J_n(x)$ is a solution of that equation.

1.0 **BESSEL'S FUNCTION OF THE SECOND KIND.**

For integer v = n, the Bessel functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, so that they do not form a basis of solutions. This poses the problem of obtaining a second linearly independent solution when v is an integer n.

Let the general Bessel equation be

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0 \tag{1}$$

If we let n=0, then (1) becomes

(2) $$xy''(x) + y'(x) + xy(x) = 0$$

and the indicial equation has a double root r = o and the desired solution must have the form

(3) $$Y_2(x) = J_o(x)\ln x + \Sigma a_j x^j$$

we substitute y_2 and its derivatives

$$y_2' = J_o'\ln x + J_o/x + \Sigma\, j a_j x^{j-1}$$

$$y_2'' = J_o''\ln x + 2J_o'/x - J_o/x^2 + \Sigma j(j-1)a_j x^{j-2}$$

into (2). Then the logarithmic terms disappear because J_o is a solution of (2), the other two terms containing J_o cancels and we find

$$2J_o' + \Sigma j(j-1)a_j x^{j-1} + \Sigma j a j x^{j-1} + \Sigma a_j x^{j+1} = 0.$$

From the previous section, we obtained the power series of J_o' in the form

$$J_o'(x) = \Sigma \frac{(-1)^j 2j x^{2j-1}}{2^{2j}(j!)^2} = \Sigma \frac{(-1)^j x^{2j-1}}{2^{2j-1} j!(j-1)!}$$

By integrating this series, we have.

$$\Sigma \frac{(-1)^j x^{2j-1}}{2^{2j-2} j!(j-1)!} + \Sigma\, j^2 a_j x^{j-1} + \Sigma\, a_j x^{j+1} = 0$$

We first show that the a_j with odd subscripts are all zero. The coefficients of the power X^o is A_1 and so, $A_1 = 0$. By equating the sum of the coefficients of the power x^{2s} to zero, we have

$$(2s+1)^2 a_{2s+1} + a_{2s-1} = 0, \quad s=1,2,\ldots$$

Since $a_1=0$, we thus obtain $a_3 = 0$, $a_5 = 0,\ldots$ successively. We now equate the sum of the coefficients of x^{2s+1} to zero, For s=0 this gives

$$-1 + 4a_2 = 0 \text{ or } a_2 = \tfrac{1}{4}.$$

For the other values of S we obtain

$$\frac{(-n)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 a_{2s+2} + a_{2s} = 0 \quad (s=1,2,\ldots)$$

for s = 1 this yields

$$1/8 + 16a_4 + a_2 = 0 \text{ or } a_4 = -3/128$$

and in general

(4) $$a_{2j} = \frac{(-1)^{j-1}}{2^{2j}(j!)^2}\left(1 + \tfrac{1}{2} + 1/3 + \ldots + 1/j\right), \quad j=1,2,3,\ldots$$

Using the short notation,

(5) $$h_j = 1 + \tfrac{1}{2} + \ldots +1/j)$$

and inserting (4) and $a_1=a_3=\ldots=0$ into (3),

we obtain the result

$$y_2(x) = J_o(x)\ln x + \Sigma \frac{(-1)^{j-1} h_j \, x^{2j}}{2^{2j}(j!)^2}$$

$$= J_o(x)\ln x + \tfrac{1}{4}x^2 - 3/128x^4 + \ldots \quad (6)$$

Since J_o and Y_2 are linearly independent functions they form a basis of (1). And this standard particular solution thus obtained is known as the Bessel function of the second kind of order zero.

CHAPTER TWO

2.0 PROPERTIES OF BESSEL FUNCTIONS

In this chapter, we look at the various properties of Bessel functions.

2.1 FUNCTIONS LINEARLY INDEPENDENT OF J_n THAT SATISFY BESSEL'S EQUATION

(1) $$x^2y''(x) + xy' + (x^2 - n^2)y(x) = 0, \quad (n=0,1,2,\ldots)$$

can be obtained by various methods. For instance the singular point x=0 of equation (1) belongs to the class known as regular singular points. The series procedure extended so as to give general solutions near regular singular points, applies to Bessel's equation. It gives the solution of $y = Y_n(x)$, where Y_n, which is the Bessel's function of the Second kind, is represented by the sum of $J_n(x)\log x$ and a power series that converges for all x. In particular, when $x > 0$,

(2) $$Y_o(x) = J_o(x)\log x + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2}(1+\tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1 + \tfrac{1}{2} + 1/3) + \ldots$$

Since logx is unbounded and J_n is bounded as x ---> 0, it is clear that Y_n is not a constant times J_n; that is Y_n and J_n are linearly independent solutions of Bessel's equations. If A and B are arbitrary constants, the general solution of that equation is therefore:

(3) $y = AJ_n(x) + BY_n(x) \quad (n = 0,1,2,3...., \ x > 0)$

An integration by parts shows that the gamma function;

(4) $\Gamma(v) = \int_0 e^{-t}t^{v-1}dt \quad (v > 0)$

has the factorial property

(5) $\Gamma(v+1) = v\Gamma(v)$

when $v > 0$. That property is assigned to the function when $v < 0$, so that $\Gamma(v) = \Gamma(v+1)/v$ when $-1 < v < 0$ or when $-2 < v < -1$, etc., thus conditions (4) and (5) together define $\Gamma(v)$ for all v except $v = 0, -1, -2, ...$

We find from equation (4) that $\Gamma(1) = 1$, also it can be shown that Γ is continuous when $v > 0$. Then it follows from property (5) that $\Gamma(+0)= \infty$ and consequently that $|\Gamma(v)|$ becomes infinite as v --> -n (n=1,2,...) where v=2,3,..., $\Gamma(v)$ reduces to a factorial; specifically,

(6) $\Gamma(n+1) = n!$ $(n=1,2,3,\ldots)$.

The proof of property (6) and the further property that

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$ is as follow

$$\Gamma(n+1) = \int_0 t^n e^{-t} dt$$

From property (4)

By Integrating by parts

$$\Gamma(n+1) = [\ t^n(e^{-1}/-1)\ ]_0 + n\int_0 e^{-1}t^{n-1}dt$$

$$= \{0-0\} + n\surd(n)$$

$$\Gamma(n+1) = n\Gamma(n)$$

This is a fundamental recurrence relation for gamma functions. It can be written as:

$\Gamma(n) = (n-1)\Gamma(n-1)$. With this notation;

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$= n(n-1)(n-2)(n-3)\ldots 1\Gamma(1)$

$= \quad n!\Gamma(1).$

For $\Gamma(1) = \int_0 t^0 e^{-t} dt = [\ -e^{-t}\ ]_0 = 0 + 1 = 1$

Therefore, we have $\Gamma(1) = 1$

and $\Gamma(n+1) = n!$ provided n is a positive interger.

For $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have that from (4)

$$\Gamma(n) = \int_0 t^{n-1} e^{-t} dt$$

where n is any positive integer. Let $n=\frac{1}{2}$,

Therefore $\Gamma(\frac{1}{2}) = \int_0 t^{-\frac{1}{2}} e^{-t} dt$

By putting $t=U^2$, $dt = 2udu$, then

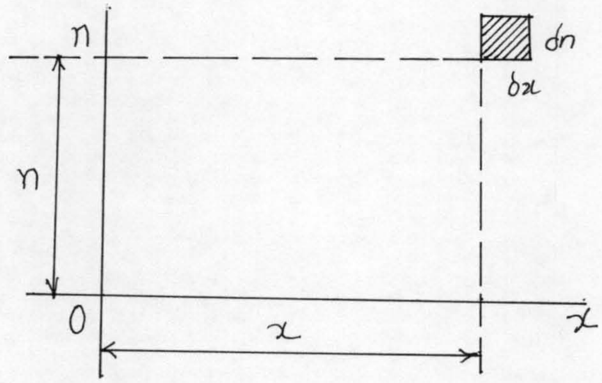
$$\Gamma(\tfrac{1}{2}) = 2\int_0 e^{-u2} du$$

Since the integral $\int_0 e^{-u2} du$ cannot easily be determined by normal means, we try to find a way of solving this problem.

Let $I = \int_0 e^{n2} dn$, then also $I = \int_0 e^{-x2} dx$

$$I^2 = \left(\int_0 e^{-n2} dn \right) \left(\int_0 e^{-x2} dx \right)$$

$$= \int_0 e^{-n2+x2)} dndx$$



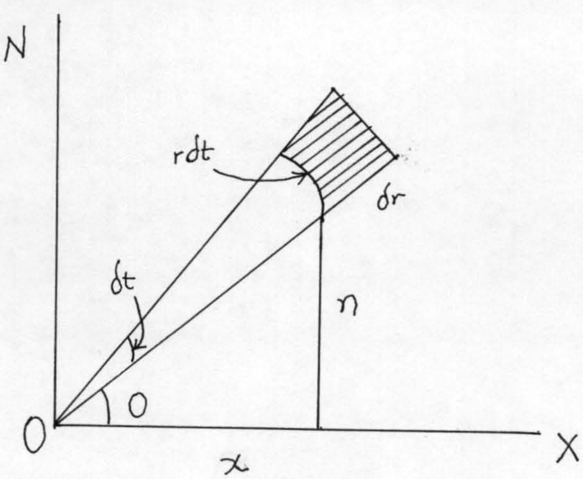
Let $\delta n = \delta n \delta x$ represents an element of area in the xn - plane and the integration with the stated limits covers the whole of the first quadrant.

Converting to polar coordinates, the element of area $\delta n = r\delta\Theta\delta r$

Also, $n^2 + x^2 = r^2$

Therefore $e^{-x2 = n2)} = e^{-r2}$,

For the Integration to cover the same region as



The limits of r are r=0 to r= ∞

The limits of Θ are $\Theta=0$ to $\Theta=\pi/2$

Therefore $I^2 = \int_0^{\pi/2}\int_0 e^{-r2} r dr d\Theta$

$$= \int_0^{\pi/2} [\ -e^{-r2}/2]_0 d\Theta$$

$$= \int_0^{\pi/2} (\tfrac{1}{2}) d\Theta = [\Theta/2]_0^{\pi/2} = \pi/4$$

Therefore $I = \sqrt{\pi}/2$

Therefore $\int_0 e^{-n2} dn = \sqrt{\pi}/2$

Before the diversion, we had established that $\Gamma(\tfrac{1}{2}) = \int_0 e^{-u2} du$

and we know that $\int_0 e^{-u2} du = \sqrt{\pi}/2$

Therefore $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

Now consider the Bessel's equation

(7) $\quad x^2y''(x) + xy'(x) + (x^2 - v^2)y(x) = 0$

in which v is any real number. The functional property of the gamma function can be used to derive the solution

$$y = J_r(x)$$

Where J_r is Bessel's function of the first kind, of index v:.ls1

(8) $$J_n(x) = \frac{(-1)^j}{j!\,\Gamma(v+j+1)} \quad (x/2)^{v+2j}$$

In case v is a negative integer those terms for which the argument $(v+j+1)$ of Γ has values zero or a negative integer are to be dropped from the series.

To verify that J_r satisfies equation (7); we see that the series (8) is a product of x^v by a power series in x that converges for all x. Thus when $v = \pm n$, where $n=0,1,2,\ldots$ either J_r or some of its derivatives fail to exist when $x=0$. When r has none of the values $\pm n$, either J_r or J_{-r} is unbounded as $x \to 0$. Therefore J_r and J_{-r} are linearly independent functions, and the general solution of Bessel's equation (7) can be written as

(9) $$y= AJ_r(x) + BJ_r(x) \quad (v <> 0, \pm 1, \pm 2, \ldots)$$

2.2 DIFFERENTIATION AND RECURRENCE FORMULAS

Since $$x^{-n}J_n(x) = \tfrac{1}{2}n \quad \frac{(-1)^j}{j!\,(n+j)!} \quad (x/2)^{2j}$$

$(n=0,1,2..)$

it follows that

$$d/dx[x^{-n}J_n(x)] = \tfrac{1}{2}n \sum \frac{j(-1)^j}{j(j-1)!(n+j)!} (x/2)^{2j-1}$$

$$= x^{-n}(x/2)^n \sum \frac{(-1)^{k+1}}{k!(n+k+1)!} (x/2)^{2k+1}$$

$$= -x^{-n} \sum \frac{(-1)^k}{k!(n+1+k)!} (x/2)^{n+1+2k}$$

That is,

(1) $\quad d/dx[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x) \quad (n=0,1,2,...)$

As a special case we have the formular

(2) $\quad J_o'(x) = -J_r(x)$

Similarly, from the power series representation if $x^nJ_n(x)$ we can show that

(3) $\quad d/dx[x^nJ_n(x)] = x^nJ_{n-1}(x) \quad (n=1,2,3...)$

From formulas (1) and (2), we have that

$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1)}(x),$$

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x),$$

by eliminating $J_n'(x)$ from these equations we find that

(4) $\quad xJ_{n+1} = 2nJ_n(x) - xJ_{n-1}(x) \quad (n=1,2,...).$

As it can be seen above this recurrence formula expresses J_{n+1} in terms of the functions J_n and J_{n-1} with lower indices.

From formula (3) we get the Integration formula

(5) $$\int_0^r J_{n-1}(x)\,dx = r^n J_n(r) \qquad (n=1,2,\ldots)$$

An important special case is

(6) $$\int_0^r xJ_o(x)\,dx = rJ_1(r)$$

Formulas (1), (3) and (4) are valid when n is replaced by the universal parameter v. Modifications of the derivations simply consist of writing $\Gamma(v+j+1)$ or $(v+j)\Gamma(v+j)$ cin place of $(n+j)!$.

2.3 **INTEGRAL FORMS OF J_n**

Let us consider two absolutely convergent power series with sums $\alpha(x)$ and $\beta(x)$:

(1) $$a_k x^k = \alpha(x), \qquad b_k x^k = \beta(x),$$

Then the Cauchy product of these series converges absoluely, and its sum is the product of their sums, that is,

(2) $$C_m x^m = \alpha(x)\beta(x), \text{ where } C_m = \quad a_k b_{m-k}.$$

In order to associate the coefficient C_j with the coefficients $(-1)^j[j!(n+j)!2^{2j}$ in the power series representing $J_n(x)$, we now let α and β depend also on parameters Θ and n (n=0,1,2,...) as follows:

(3) $\alpha(x,\theta) = \exp(\tfrac{1}{2}ixe^{i\theta})$,

$\beta(x,\theta) = \exp(\tfrac{1}{2}ixe^{-i\theta})$

From the representation of exp z in powers of z, convergent for all complex z, we see that the coefficient of x^k in series (1) are .ls1

(4) $a_k(\theta) = \dfrac{i^k}{2^k k!} e^{ik\theta}$,

$b_k(\theta) = \dfrac{i^k}{2^k k!} e^{i(n-k)\theta}$,

Let $dk = \dfrac{i^k}{2^k k!}$,

the coefficient in the Cauchy product (2) become

(5) $C_m(\theta) = dke^{ik\theta}d_{m-k}e^{i(n-m-k)\theta}$

$= d_k d_{m-k} e^{i(2k-m+n)\theta}$,

Since the

$$\int_{-\pi}^{\pi} e^{ip\theta} d\theta = 0 \text{ if } p=\pm 1, \pm 2, \ldots$$

$$= 2\pi \quad \text{if } p = 0.$$

the integral form $-\pi$ to π if $\exp[i(2k-m+n)\Theta]$ vanishes unless $2k=m-n$. But in formula (5), $2k$ can be equal to $m-n$ only in case $m-n$ is zero or a positive even integer; that is, if $m-n=2j$ $(j=0,1,2,...)$; this includes the condition that $m \geq n$. In that case $k=j$ since $2k=m-n=2j$; also, $m-k=2j+n-j=n+j$ and, according to formula (5),

(6) $$\int_{-\pi}^{\pi} C_{n+2j}(\Theta)\,d\Theta = 2\pi d_j d_{n+j} \quad (j=0,1,2...) \text{ but}$$

(7) $$\int_{-\pi}^{\pi} C_m(\Theta)\,d\Theta = 0 \text{ if } m <> n+2j$$

Now the series in the representation

$$C_m(\Theta)x^m = \alpha(x,\Theta)\beta(x,\Theta)$$

is uniformly convergent with respect to Θ over the interval $(-\pi, \pi)$. To show this, we note that, according to equation (5),

$$|C_m(\Theta)x^m| \leq |x|^m \qquad |dk||d_{m-k}|$$

The numbers represented by the right-hand member are independent of Θ. They are the terms in the Cauchy product if two convergent series of positive terms, namely the series of absolute values of the terms in the power series representations if $\alpha(x,0)$ and

$ß(x,0)$, series that are absolutely convergent. Thus integration of the series is justified and, in view of formulas (6) and (7), we can write,

$$\int_{-\pi}^{\pi}\alpha(x,\Theta)\beta(x,\Theta)d\Theta = \qquad x^m\int_{-\pi}^{\pi}C_m(\Theta)d\Theta$$

$$= R\pi \quad d_j d_{n+j}x^{n+2j}$$

Since $d_j = i^j/(j!2^j)$, the last series becomes

$$i^n \quad \frac{(-1)^j}{j!(n+j)!} \quad (x/2)^{n+2j} \quad = i^n J_n(x)$$

Thus we have the integral representation

$$J_n(x) = i^{-n}\int_{-\pi}^{\pi}\alpha(x,\Theta)\beta(x,\Theta)d\Theta$$

Which becomes, corollary to the definitions (3) if α and $ß$,

(8) $$J_n(x) = \frac{i^{-n}}{2\pi}\int_{-\pi}^{\pi}\exp(ix\cos\Theta)e^{in\Theta}d\Theta$$

$(n=0,1,2,...)$

Now $i=\exp(i\pi/2)$

Therefore

$$J_n(x) = \tfrac{1}{2}\pi\int_{-\pi}^{\pi}\exp(in\cos\Theta)\exp[in(\Theta-\tfrac{1}{2}\pi)]d\Theta$$

Again by substituting for $\Theta=\frac{1}{2}\pi-\Theta$ and noting that the integral is periodic in Θ and Φ with period 2π, we find that

(9) $$J_n(x) = \tfrac{1}{2}\pi\int_{-\pi}^{\pi}\exp[i(x\mathrm{Sin}\Phi - n\Phi]d\Phi.$$

The imaginary point of this Integral vanishes of course, since $J_n(x)$ is real when x is real. Therefore,

$$J_n(x) = \tfrac{1}{2}\pi\int_{-\pi}^{\pi}\mathrm{Cos}(x\mathrm{Sin}\Phi - n\Phi)d\Phi,$$

or since the Integrand here is an even function of Φ,

(10) $$J_n(x) = 1/\pi\int_{0}^{\pi}\mathrm{Cos}(n\Phi - x\mathrm{Sin}\Phi)d\Phi \qquad (n=0,1,2,...)$$

The above equation is the Bessel's integral form of J_n.

2.4 THE ZEROS OF BESSEL FUNCTION, $J_0(x)$

The Integral form

$$\tfrac{1}{2}\pi J_0(x) = \int_{0}^{\frac{1}{2}\pi} \mathrm{Cos}\ (x\mathrm{Sin}\Phi)d\Phi$$

becomes, after the substitution $t=x\mathrm{Sin}\Phi$,

(1) $$\tfrac{1}{2}\pi J_0(x) = \int_{0}^{x}\frac{\mathrm{Cos}\ t\ dt}{\sqrt{x^2-t^2}} \qquad x>0$$

We shall show from the behaviour of the Integrand that this Convergent Improper Integral has the value zero for an Infinite sequence of positive values of x.

Let $C_k = K\pi + \tfrac{1}{2}\pi$ (k=0,1,2,...) and also consider the integral when $x=C_k$. Its integrand

(2) $y_k(t) = \text{Cost}/(C_k{}^2-t^2)^{\frac{1}{2}}$ $(0 \le t \le C_k)$

This has the limit zero as t --> C_k. We define $y_k(C_k$ to be zero;

We define $y_k(C_k)$ to be zero; then y_k is a continuous function of t over the Interval if Integration $0 \le t \le c_k$.

Note: y_o is defined only on the Interval $0 \le t \le \frac{1}{2}\pi$, y_1 on the interval $0 \le t \le 3\pi/2$ etc.

Now $Y_k(t) > 0$ on the interval $0 < t < \frac{1}{2}\pi$ $(k=0,1,2,3...)$ Since Cost>0 there.

Similarly, we see that y_1, y_2... have negative values over the interval $\frac{1}{2}\pi < t < 3/2\pi$, that y_2, y_3,..., have positive values over the next interval of length π, and so on. For a fixed positive integer k, let k_o denote the area under the graph. If $y_k(t)$ over the interval $(0, \frac{1}{2}\pi)$, which is the interval $0 < t < k_o$. Let the positive number k_j denote the area bounded by the t axis and that graph and the lines $t=C_{j-1}$ and $t=C_j$, where $j=1,2,..,k$. Then

(3) $$\tfrac{1}{2}\pi J_o(C_k = \int_0^{C_k}{}_k y_k(t)\,dt$$

$$= K_0 - K_1 + K_2 - \ldots + (-1)^k K_k.$$

Since C_k is fixed, the value of $(C_k{}^2-t^2)^{\frac{1}{2}}$ diminishes when t increases toward C_k. But |Cost| is periodic with period π. It follows that $|y_k(t)|$ increases when t is increased by π and consequently that the areas K_j satisfy the inequalities $K_0<K_1<...<K_t$. If k is an odd integer, then

$\frac{1}{2}\pi J_o(C_k) = -(K_1-K_o) - (K_3-K_2) - ... - (K_k-K_{k-1} < 0;$

but if k is even,

$\frac{1}{2}\pi J_o(C_k) = K_o + (K_2-K_1) +...+(K_k-K_{k-1}) > 0.$

The continuos function $J_o(x)$ therefore takes on positive values at alternate points $x=\frac{1}{2}\pi$, $x=2\pi+\frac{1}{2}\pi$, ... if the infinite set of points $x= k\pi + \frac{1}{2}\pi$ (k=0,1,2,...) and negative values at the other points of that set.Consequently there is at least one point X_k on each interval between consecutive points by that set,

$(K + \frac{1}{2})\pi < X_k < (K +3/2)\pi$ (K=0,1,2,...)

Such that $J_o(X_k) = 0$.

Actually, since J_o is an analysis function of x which is not identically zero, it can have at most n finite number of zeros on any bounded interval. Consequently, the positive roots of the equation $J_o(x_k) = 0$ consist of an infinite set of numbers x_m, (m=1,2,3...) such that $x_m \rightarrow$ as m $\rightarrow$.

2.5 THEOREM: For all x, the analytic function $J_n(x)$ is a particular solution of Bessel's equation, of the form,

$$x^2y''(x) + xy'(x) + (x^2-n^2)y(x) = 0 \quad (n=0,1,2,...).$$

Since Bessel's equation is homogeneous, the function $CJ_n(x)$, where C is any constant, is also a solution.

From the formula

$$J_n(x) = \Sigma \frac{(-1)^j}{j!(n+j)!} (x/2)^{n+2j} \qquad (n=0,1,2,...)$$

$$= x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 2!(n+1)(n+2)} + ...\right]$$

We can see that

$J_n(-x) = (-1)^n J_n(x)$;

that is, J_n is an even function if n=0,2,4...)

but odd if n=1,3,5,...

It easily seen that the series representation of J_o, ie

$$J_o(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + ...,$$

bears some resemblance to the power series for Cos x. Also the series,

$$J_1(x) = t^2/2 - \frac{t^3}{2^3(1!)(2!)} + \frac{t^5}{2^5(2!)(3!)} - \frac{t^7}{2^2(3!)(4!)} + ...$$

CHAPTER THREE

3.1 APPLICATION OF BESSEL FUNCTIONS.

Bessel functions are widely used in Engineering, Physics, especially in problems in potential theory and diffusion. They freqquently arise in problems involving cylindrical functions. Here, we take a look at a particular problem concerning temperature flow in a long cylinder.

3.1 TEMPERATURE IN A LONG CYLINDER

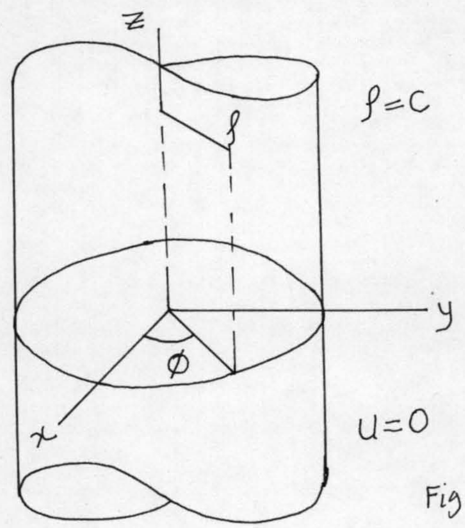


Fig. 3.2.

Let the internal surface ρ=c of an infinitely long circular cylinder (see fig. 3.2), or a cylinder of finite altitude with insulated base, be kept at temperature zero. The initial temperature distribution is a function f(ρ) of distance from the axis, only.

The heat equation in cylinderical coordinates is given as:.ls1

(1) $\delta u = k(\frac{\delta^2 u}{\delta \rho^2} + 1/\rho \frac{\delta u}{\delta \rho})$ $(0<\rho<C,\ t>0)$

Also, when t>0, the function U is to be continuous throughout the cylinder, in particular at the axis $\rho=0$.

The particular solution of the equation (1) above can be obtained by the method of variable separation as follows:

The general solution of equation (1) is

(2) $U(\rho,t) = R(\rho).T(t)$

The partial differential of equation (2) gives

$\delta u/\delta t = \dot{T}R$, $\delta u/\delta r = R'T$, $\delta^2 u/\delta r^2 = R''T$.

Putting this partial derivatives in equation (1) we get;

$\dot{T}R = K(TR'' \_ 1/\rho TR')$

(3) $\dot{T}R/R = TR'' + 1/\rho R'/R$

Dividing (3) through by TR, we get

(4) $\dot{T}/KT = R''/R + 1/\rho R'/R$

For a purely function of T to be equal to a purely function of R each of their values must be equal to a constant.

That is,

$\dot{T}/KT = R''/R + 1/\rho R'/R = - \lambda^2$

This implies that $T/KT = -æ^2$

or $\dot{T}/T = -kæ^2$

$dT/(dt/T) = -kæ^2$

Therefore $\S dT/T = -\S kæ^2 dt$

$\ln T = -kæ^2 t$

(5) $T = e^{kæ_2 t}$

where k, æ, t are all constant

Also, $1/R(R'' + R'/\rho) = -æ^2$

or $R'' + R'/\rho = -Ræ^2$ or

(6) $\rho R''(\rho) + R'(\rho) + æ^2 \rho R(\rho) = 0$

The differential equation (6) in R is Bessel's equation with the parameter æ, in which n=0.

Equation (6) above can be written in terms of the variable x, where we let $x=æ^2\rho$ so as to make eqaution (6) æ - free. Then, equation (6) becomes

$xy''(x) + xy'(x) + x^2y(x) = 0$ or

(7) $x^2y''(x) + xy'(x) + x^2y(x) = 0.$

The solution to above equation can be obtained using the Frobenins method by assuming a trial solution of the form

$$y = X^p \Sigma a_j X^j = \Sigma a_j X^{p+j}$$

By differentiating and substituting into (7) we get

$$y' = \Sigma(p+j)a_j X^{p+j-1}$$

$$y'' = \Sigma(p+j)(p+j-1)a_j X^{p+j-2}$$

That is from equation (7)

$\Sigma[(p+j-1)(p+j) + (p+j) + X^2]a_j X^{p+j} = 0$ or

$\Sigma[(p+j)^2]a_j X^j + \Sigma a_k X^{k+2} = 0$

The last sum can be written as

$\Sigma\, a_{j-2}X^j$

Hence the equation becomes

(8) $$p^2 a_o + (p+1)(p+1)a_1 X + \Sigma[(p+j)(p+j)a_j + a_{j-2}]X^j = 0$$

Since equation (8) is an identity, for the coefficient of the powers of X to vanish, we must impose a condition. That is , p=o or p=-1. If $a_o=0$ and we have that $(p+j)(p+j)a_j + a_{j-2} = 0$ (j= 2,3,...). By making p=j, we can obtain the recurrence relation, that is,

$$2j(2j)a_j = -\, a_{j-2} \quad \text{OR}$$

(9) $$a_j = \frac{-1}{2^2 j^2}\, a_{j-2}$$

$$a_{2j} = \frac{-1}{2^2 . 2^2 j^2}\, a_{2j-2}$$

and $$a_{2j-2} = \frac{-1}{4^2(j-1)^2} a_{2j-4}$$

So

$$U_{2j} = \frac{(-1)^2}{4^2.4^2j^2(j-1)^2} U_{2j-4}$$

Continuing this process through j-2, further steps we get;

$$a_{2j} = \frac{(-1)^j}{j^{j2!}\ldots(j+1)} \frac{a_o}{2^{2j}}$$

Let $a_o = \frac{1}{2^n n!}$

(10) - Therefore, $a_{2j} = \frac{(-1)^j}{j^{2!}2^{n+2j}}$ (j=0,1,2,...)

$$J_n(x) = \Sigma \frac{(-1)^j}{j^{2!}} (x/2)^{n+2j}$$

Where $J_n(x)$ is the solution to the Bessel equation.

$$J_n(x) = \frac{x^n}{2^n n!} [1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{3^2 2^6} + ,,,]$$

Since our n=0 we get

(11) $$J_o(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 2^4} - \frac{x^6}{3^2 2^6} + \ldots$$

Therefore, the general solution to our physical problem is

$$U(\rho,t) = T(t) \cdot R(\rho)$$

(12) $$= e^{-kx2t}J_o(x)$$

And a particular solution can be obtained if initial values for constants k, λ and t and the variable x are given. In this case for example, if $\lambda = 0$, then the solution to equation (12) becomes

$$U(\rho,t) = J_o(x)$$

3.2 SOLUTION TO DIFFERENTIAL EQUATIONS

Bessel function is one of the important functions used in determining the solution to differential equations of the second order.

Let us take a look at how we can use the Bessel function to find the solution to some of the differential equations.

Example 1:

Solve the differential equation

(1) $$x^2y''(x) + xy'(x) + (x^2 - \tfrac{1}{4})y(x) = 0.$$

The solution to above equation can be obtained using the Frobenins method by assuming a trial solution stated earlier in the previous section.

$$y = x^p \Sigma a_j x^j = \Sigma a_j x^{p+j}$$

$$y' = \Sigma (p+j) a_j x^{p+j-1)}$$

$$y'' = \Sigma (p+j-1)(p+j) a_j x^{p+j-2}$$

By putting this derivative in (1), we get

$$\Sigma [(p+j)(p+j-1) + (p+j) + (x^2-\tfrac{1}{4})] a_j x^{p+j} = 0$$

or $$\Sigma [(p+j)^2 - \tfrac{1}{4}] a_j x^j + \Sigma a_k x^{k+2} = 0$$

The last term can be written as

$$\Sigma a_{j-2} x^j;$$

Hence the equation becomes

(2) $$(p-\tfrac{1}{2})(p+\tfrac{1}{2}) a_o + (p+\tfrac{1}{2})(p+3/2) a_1 x +$$

$$\Sigma[(p-\tfrac{1}{2}+j)(p+\tfrac{1}{2}+j) a_j + a_{j-2}] x^j = 0$$

For the coefficient of the power of x to vanish, we musst impose a condition. That is, $p=\tfrac{1}{2}$ or $p=-\tfrac{1}{2}$

If $a_1 = 0$, then

$$(p-\tfrac{1}{2}+j)(p+\tfrac{1}{2}+j) a_j + a_{j-2} = 0$$

Let $p=\tfrac{1}{2}$

$$j(j+1) = - a_{j-2}$$

Therefore $$a_j = \frac{-1}{j(j+1)} a_{j-2}$$

$$a_{2j} = \frac{-1}{2j(2j+1)} a_{2j-2}$$

$$a_{2j-2} = \frac{-1}{(2j-2)(2j-1)} a_{2j-4}$$

$$a_{2j-4} = \frac{-1}{(2j-4)(2j-3)} a_{2j-6}$$

Therefore $a_{2j-2} = \frac{(-1)^2}{(2j-4)(2j-3)(2j-2)(2j-1)} a_{2j-6}$

So $a_{2j} = \frac{(-1)^3}{(2j+1)(2j-1)(2j-2)(2j-3)(2j-4)} a_{2j-2}$

Continuing this process further by j-2 steps, we will have that

$$a_{2j} = \frac{(-1)_j}{(2j+1)!^{2j}} a_o \quad (j=0,1,2,3...)$$

Let $a_o = 1/2^n n!$

Therefore, $J_n(x) = \Sigma \frac{(-1)^j}{(2j+1)!} (x/2)^{n+2j}$

$$= a_o x^n [1 - \frac{x^2}{3!2^2} + \frac{x^4}{5!2^4} - \frac{x^6}{7!^6} + \quad ...]$$

$$= a_o x^{\frac{1}{2}} [1 - \frac{x^2}{3!2^2} + \frac{x^4}{5!2^4} - \frac{x^6}{7!2^6} + ...]$$

Example 2

Solve the equation

(1) $y''(x) + (d-1)y'(x)/X + (\lambda - m^2/X^2)Y(x) = 0$

The parameter d≥1 is the dimension, the parameter æ is the eigen value, and the parameter m is the angular frequency. For solutions in cylindrical coordinates, we need d=2, while for problems in spherical coordinates, we shall need d=3.

Solution:

We use the Frobenins method to look for the solution in power series. Let

(2) $y = X^r \Sigma a_n X^n = \Sigma a_n x^{n+r}$

Where $(r, a_o, a_1, ...)$ are to be determined. The power series will converge for all x and this can be shown by differenting equation (2), and substituting inti (1) and then rewriting the result asa a single power series.

Thus,

(2.1) $y' = \Sigma(n+r)a_n X^{n+r-1}$

(2.2) $y'' = \Sigma(n+r-1)(n+r)a_n x^{n+r-2}$

From equation (1) we have

$$\Sigma[(n+r)(n+r-1)+(d-1)(n+r)-m^2]a_n x^{n+r-2} = 0$$

$$= \quad \Sigma[(n+r)(n+r+d-2)-m^2]a_n x^{n+r-2}$$

In order for this to be equal to the series λy, the two series must agree, term by term. The series for λy begins with the power x^r the above series begins with x^{r-2}. Therefore we must have

(2.3) $[r(r+d-2) - m^2 a_o = 0 \quad n = 0$

(2.4) $[(1+r)(r+d-1) - m^2]a_1 = 0 \quad n = 1$

$[(n+r)(n+r+d-2) - m^2]a_n + xa_{n-2} = 0 \quad n=2,3,\ldots$

We obtain a non zero solution by taking $a_o <> 0$; $U_1 = 0$.

$$r = 1 - \tfrac{1}{2}d + [m^2 + (\tfrac{1}{2}d-1)^2]^{\frac{1}{2}}$$

The exponent r, which is positive, is the largest root of the Indicial equation $r(r+d-2) - m^2 = 0$, from (2.5). To determine Un $n \geq 2$, we use the Indicial equation to write.

$(n+r)(n+r+d-2)-m^2 = n^2+n(r+d-2) + nr +r(r+d-2)-m^2$.

Thus (2.5) becomes

$n(n+2r+d-2)a_n + \lambda a_{n-2} = 0 \quad n=2,3,\ldots$

$a_1 = 0$ implies that $a_3 = 0$ and $a_5 = 0$ while

$$a_2 = \frac{-\lambda}{2(d+2r)} a_o$$

$$a_4 = \frac{-\lambda}{2(d+2r)} \frac{-\lambda}{4(d+2r+2)} a_o$$

$$a_{2n} = \frac{(-\lambda)^n}{2(d+2r)4(d+2r+2)\ldots 2n(d+2r+2_{n-2}} a_o$$

Hence we have obtained the sought-after function

$$y(x) = a_o x^r[1 + \Sigma \frac{(-\lambda)nX^{2n}}{2(d+2r)4(d+2r+2)\ldots 2n(d+2r+2_{n-2}}$$

<u>Example 3</u>

Solve the differential equation:-

(1) $x^2y''(x) + y'(x) + (x^2-16)y(x) = 0.$

<u>Solution</u>

Comparing equation (1) with the general Bessel equation, we have that, n=4. The solution can be obtained in the form of a power series multiplied by x^p, where p is some constant. Our aim is to determine p and the coefficients a_j so that the function

(2) $y = x^p\Sigma a_j x^j = \Sigma a_j X^{p+j}$ satisfies equation (1). By differentiating the function (2) and substituting its derivatives into (1), we get

$$y' = \Sigma(p+j)a_jX^{p+j-1}$$

$$y'' = \Sigma(p+j-1)(p+j)a_jX^{p+j-2}.$$

Combining equation (1) and (2), we obtain

$$\Sigma[(p+j)(p+j-1) + (p+j) - (x^2-16)]a_jx^{p+j} = o$$

$$\Sigma[(p+j)^2 - 16]a_j x^j + \Sigma a_k X^{p+j} = 0$$

We can rewrite the last sum as

$$\Sigma a_{j-2} x^j$$

thus the equation becomes

$$(p-4)(p+4)a_o + (p-4+1)(p+4+1)a_1 x +$$

$$\Sigma[(p-4+j)(p+4+j)a_j + a_{j-2} x^j = 0$$

Since the solution (2) is an identity in x, for the coefficient of each power of x to vanish, then, p=4 or p=-2. So that the constant term vanishes and if $a_1 = 0$ and $(p-4+j)(p+4+j)a_j = -a_{j-2}$ (j=2,3,...)

By making the choice p=4; and we have the recurrence relation

(3) $$a_j = \frac{-1}{2^2(8+j} a_{2j-2} \qquad (j=2,3,\ldots)$$

It should be noted that 8+j <> 0 in (3). The choice p=-4 fails to give a recurrence relation. Since $a_1 = 0$, relation (4) requires that $a_3 = 0$ and that $a_5 = 0$... etc. thus

(4) $a_{2j-1} = 0$.

For the remaining coefficients the recurrence relation is

$$a_{2j} = \frac{-1}{2^2 j(4+j)} a_{2j-2} \qquad (j=1,2,\ldots)$$

$$a_{2j-2} = \frac{-1}{2^2(j-1)(j+3)} a_{2j-4}$$

$$a_{2j-4} = \frac{-1}{2^2(j-2)(j+2)} a_{2j-6}$$

Therefore $a_{2j-6} = -1 \dfrac{a_{2j-8}}{2^2(j-3)(j+1)}$

This implies that

$$a_{2j} = (-1)^4 \frac{}{4^4(j+4)(j+3)(j+2)(j+1)(j)(j-1)\ldots} a_{2j-8}$$

Continuing this process through j-2 further steps, we find that

$$a_{2j} = (-1) \frac{a_o}{(4+j)! \quad 2^{2j}}$$

Thus a_o is a common factor in all terms of the series. To simplify the series, we select this value for a_o

$$a_o = \frac{1}{2^4 4!}$$

Then our formular for the non-vanishing coefficients becomes

(5) $$a_{2j} = \frac{(-1)^j \qquad 1}{(4+j)!\ldots 2^{4+2j}} \qquad (j=0,1,2,3\ldots)$$

Therefore our Bessel's equation can be written in view of formulas (4) and (5).

That is,

$$y + J_n(x)$$

Whose J_n is Bessel's function of the first order of n, or index n, definced by the equation

(6) $$J_n(x) = \Sigma \frac{(-1)^j}{(4+j)!} (\tfrac{1}{2}x)^{4+2j}$$

$$= a_o x^4 [\tfrac{1}{4} - \frac{x^2}{5!2^2} + \frac{x^4}{6!2^4} - \frac{x^6}{7!2^3} + \ldots]$$

CHAPTER FOUR

4.0 ANALYTIC FUNCTION OF $J_n(x)$.

The Bessel function, $J_n(x)$, which has n degree is said to be analytic for any value of x. This can be obtained using the formula.

$$J_n(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 2^4} - \frac{x^6}{3^2 2^6} + \ldots \quad x=1,2,\ldots,$$

Which was the solution to our application problem discussed in the previous chapter.

A first method to test for this analytic function, $J_n(x)$ any value of x is by writing a computer program which will accomplish this task. The solution to the equation can not be written in an infinite form as far as the available compiler can allow. For this constraint, the program has assembled to handle a solution up to a maximum of order 5 (i.e $x <= 5$)

4.1 COMPUTER PROGRAMMER TO TEST FOR THE ANALYTIC FUNCTION OF $J_n(x)$

When a programmer sits down to write a program to solve a particular task, he or she takes some things into consideration. One of the things he/she does is to write instructions in the program.

This however, is only one activity in the programming problem. Infact there are other five main activities that must be completed.

These activities include:

(i) Definition of the problem (i.e problem analysis)

(ii) Program design (i.e algorithm).

(iii) Program coding

(iv) Program testing

(v) Program documentation.

Let us now briefly describe each of these activities in the subsequent sections.

4.2 **PROBLEM ANALYSIS**

Naturally, the first step in the programming process is to understand and define the problem to be solved. To understand the problem, it means one would have to determine its requirements and how they can be met. It is essential that the programmer must know precisely the task that will be performed by his/her program. This usually means that the programmer will have to know what output the program will produce and what computation it must perform. In addition, the programmer must determine what resources are available to meet the requirements. This includes determining the available input, Specifying the input, computation, and output requirements for the problem definition.

4.3 **ALGORITHM**

Now that the programmer has gained an understanding of the problem, the programmer can begin to design a program to solve it. The programmer makes sure that the sequence of steps that are necessary to solve the problem are carefully planned. This sequence can take the form of a flowchart or may use some other techniques. It is therefore, necessary that before a program can be written, the programmer must develop the algorithm to solve the problem.

For our own application problem, see appendix 1 for the algorithm testing for the analycity of Bessel function using the equation.

4.4 **PROGRAM CODING.**

The program can now be written after the algorithm to solve it has be written. Writing of this program is what is called "coding". The programmer uses any computer language at his disposal to write the program, so far he or she understands the language, an understanding of the program to be solved, and the algorithm determined previously. With that background the programmer codes the program to solve the problem by writing the necessary computer language instructions. See Appendix 2 for the program in FORTRAN for the solution to our application problem.

4.5 **PROGRAM TESTING.**

After the program has been coded the next thing is to test the program by running it on the computer with some input data. Then, the output of the program produced is compared with the expected output; any discrepancy indicates an error which must be corrected. Testing the program in this manner will not necessarily find all the errors but it will usually points out any serious problem. The actual proces of determining correctness of the program involves much more than just testing the program on the computer. A program is correct because it makes sense logically. The programmer makes sure of this as he or she plans and codes the program.

4.6 **PROGRAM DOCUMENTATION.**

Included in the program documentation are program listing, descriptions of the input and output data and a flowchart. Documentation enables other programmers to understand how the program functions. Often it is necessary to return to the program after a period of time and to make corrections or changes. With adequate documentations, it is easier to understand a program's operation.

4.7 **MANUAL CALCULATION**

The test for the analytic function of our application problem can be done mainly by using the equation.

$$J_o(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{576} + \ldots +, \quad x=1,2,3,\ldots$$

For x = 2, we get $J_o = 0.1388$

For x = 3, we get $J_o = -1.2500$

For x = 4, we get $J_o = -6.1111$

For x = 5, we get $J_o = -22.6111$

For x = 6, we get $J_o = -59.7500$

4.8 **COMPARISON OF RESULTS**

The test for analycity of Bessel function, $J_o(x)$ are obtained using two methods viz: Computer and Manual computations. The two results are compared in order to see if at all they agree with each other. Using the Bessel function, $J_o(x)$ with our variable x being <=5, we get the following results.

By manual computations

The value are:

$J_o(2) = 0.1388$

$J_o(3) = -1.2500$

$J_o(4) = -6.1111$

$J_o(5) = -22.6111$

$J_o(6) = -59.7500$

By computer evolution

The values are:

$J_o(2) = 0.1388$

$J_o(3) = -1.2500$

$J_o(4) = -6.1111$

$J_o(5) = -22.6111$

$J_o(6) = -59.7500$

Therefore we can see that the two methods agree with each other. This means that any of the two methods can be used to determined the analycity, but the latter (i.e using the computer is faster.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATION

In this chapter, we try to summarize briefly what I have done so far for the benefit of understanding.

5.1 CONCLUSION

Bessel function as a topic cannot in any way be discussed without the knowledge of the differential equations. Like we had already mentioned in chapter one, differential equation are equation that relate function f to its derivatives. And we have three types of differential equations. The first one is that in which dy/dx is a function of x alone or of y alone. The second consists of ordinary differential equations of the first order of the form dy/dx = f(x,y), and the third is made up of partial differential equations of the first order. And it is finding solution to the partial differential equation using the methods of variable separales that gives rise to the use of the Bessel's equation. For instance, in boundary value problems that involves the Laplacian ^2U expressed in cylindrical coordinates, the process of separating variable often produce and equation of the form.ls1

$$\rho^2 \frac{d^2y}{d\rho^2} + \rho \frac{dy}{d\rho} + (k^2 \rho^2 - v^2)y = 0, \text{ which}$$

is a Bessel equation with the function of the cyclindrial coordinate, ρ. And its solution can be obtained using method of variable separable.

In the second chapter, we take a look at the various properties of the Bessel function and tried to explained detailly their relevance and where applicable.

In chapter three, we discussed the two important area where Bessel functions are used. That is, using them to solve physical problems and finding solution to the differential equations.

In chapter four I tried to test for the analyticity of the solution to our application problem to see if actually my solution can be analytic for any particular value of my variable x. This I tried to accomplish by writing a computer program to do it and using the result to compare with values I got by manual computation.

5.2 <u>RECOMMENDATION.</u>

It will amount to over exaggeration for me to say that I have recorded all the information that is required on Bessel functions. For this reason, I will not say that this piece of work has exhaustively treated all the information about Bessel function. Infact there is a lot to say about this very important and special type of differential equations. I therefore recommend that more work should be done on the properties of Bessel functions. Also, the Bessel function has a lot of applications. To say that I have exhausted all these applications is like passing a camel through the eye of a needle. I recommend that more of the properties not mention here be fetched out.

Finally, since Bessel function is a significant aspect of the ordinary differential equations, all information related to it should be brought together and published for the benefit of this generation and the future generations.

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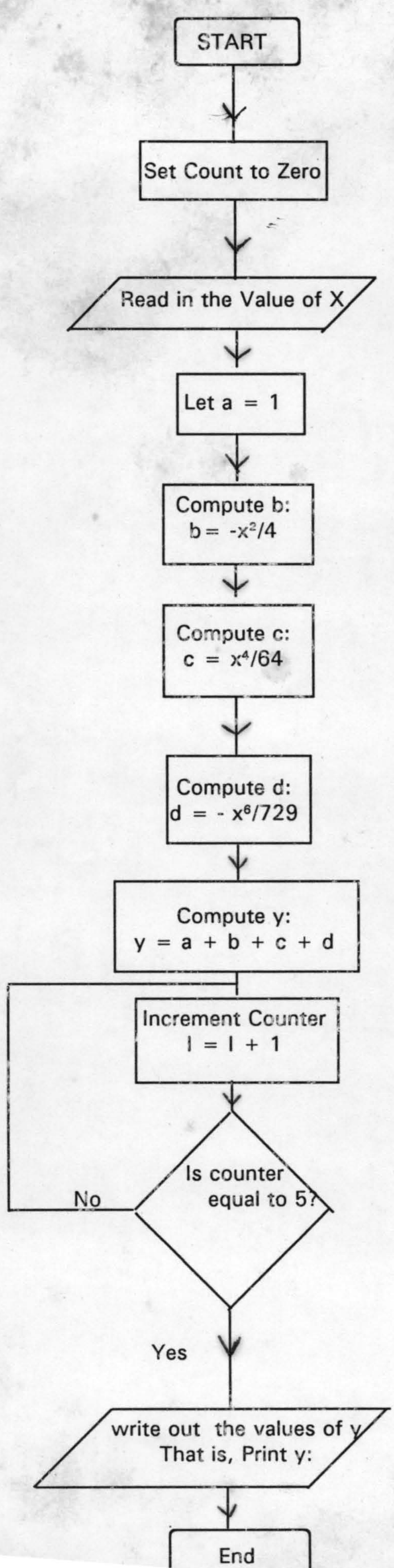
START
Set Count to Zero
Read in the Value of X
Let a = 1
Compute b:
b = -x²/4
Compute c:
c = x⁴/64
Compute d:
d = - x⁶/729
Compute y:
y = a + b + c + d
Increment Counter
I = I + 1
Is counter equal to 5?
No
Yes
write out the values of y
That is, Print y:
End

```
C      ***** This program tests for the convergent of a variable X,
C      ***** in the equation
C      ****** Y(x)=1-x^2/4 + x^4/64 - x^6/729 + ...
C      ****** which is the solution to the application discussed in the
C      ****** in the previouse chapter.

       dimension is(8), y(8)

       write (*,*)'This program tests for the analytic function of the'
c      write (*.*)'Bessel Equation:J(x)=1-x**2/4 + x**4/64 - x**6/576 '
       write (*,*)'Which is the solution to the application discussed in'
       write (*,*)'the previous chapter'
       write (*,*)

       open(2,file='adamu1', status='old')
       open(3,file='adamu2', status='new')

       Do 16 k= 1,5
       read(2,10)is(k)
  10   format(1x,i2)

       ia= 1
       b= -(is(k)**2)/4
       c= (is(k)**4)/64
       d= -(is(k)**6)/576

       y(k)= ia+b+c+d
       write (*,*)y(k)

c     write(3,15)y(k)
  15   format(1x,f9.1)
  16   continue
       stop
       end
```