A11: A Fourth-Order Four-Stage Trigonometrically-Fitted Improved Runge-Kutta Method for Oscillatory Initial Value Problems

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Abstract

The present work pertains to the derivation, analysis and application of a fourth-order four-stage trigonometrically-fitted Improved Runge-Kutta (TFIRK4-4) method for solving the initial value problem (IVP) $y'(x) = f(x, y(x))$. The method is known to integrate exactly the initial value problem whose solution is a linear combination of the functions $\sin(\omega x)$ and $\cos(\omega x)$, or equivalently $e^{i\omega x}$ and $e^{-i\omega x}$, where $\omega > 0$, being the principal frequency of the problem, is used to enhanced the accuracy of the method. The numerical results show the efficacy of the new method in comparison with other existing methods.

Keywords: improved Runge-Kutta (IRK) method, initial value problem, oscillating solution, trigonometric fitting.

1. Introduction

A method of choice among researchers in the numerical solution of the first order IVP

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0 \qquad x \in [x_0, X] \tag{1}$$

is the Runge-Kutta method. The general $s-$stage Runge-Kutta method is described by the relation

$$\left.\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i \\ k_i &= f\left(x_n + c_i h, y_n + h\sum_{j=1}^{s} a_{i,j} k_j\right), i = 1, \dots, s \end{aligned}\right\} \tag{2}$$

where the $k_i{'}s$ are multiple estimates of the slope $f(x,y)$ within the subinterval $[x_n, x_{n+1}]$. The number of evaluations of $f(x,y)$ per step, denoted by the integer s, is known as the number of stages of the method and measures its complexity. Thus, at least for the lower-order methods, the number of stages usually represents the order of the method. Most attempts at improving the order of Runge-Kutta methods have been done by increasing the number of Taylor series terms used and thus the number of function evaluations, which in effect reduces the efficiency of the method.

2. Literature Review

To address this deficiency, a great many authors have made attempts to improve the efficiency of Runge-Kutta methods by reducing the number of function evaluations required. In this regard, the works of Goeken and Johnson (2000), Xinyuan (2003), Phohomsiri and Udwadia (2004) and Udwadia and Farahani (2008) readily come to mind. The Improved Runge-Kutta (IRK) methods is a class of two-step methods that require lower number of function evaluations, i.e., stages, compared to the classical Runge-Kutta method; therefore, they are computationally more efficient at achieving the same order of accuracy. More so, the IRK can be used for autonomous as well as non-autonomous systems. The IRK method with $s - stages$ for solving (1) has the form:

$$y_{n+1} = y_n + h\left(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})\right) \tag{3}$$

for $1 \leq n \leq N - 1,$ where

$$\left.\begin{aligned} k_1 = f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1}) \\ k_i = f\left(x_n + c_i h, y_n + h\sum_{i=1}^{s} a_{i,j}k_j\right), 2 \le i \le s \\ k_{-i} = f\left(x_{n-1} + c_i h, y_{n-1} + h\sum_{i=1}^{s} a_{i,j}k_{-j}\right), 2 \le i \le s \end{aligned}\right\} \tag{4}$$

for $c_2 \dots c_s \epsilon [0, 1]$ and f depend on both $x\ and\ y$ while $k_i\ and\ k_{-i}$ depend on the value of $k_j\ and\ k_{-j}$ for $j = 1, \dots\dots, i - 1$. Here s is the number of function evaluations performed at each integration step and increases with the order of local accuracy of the IRK method. A one step method must provide the approximate solution of y_1 at the first step since the method is not self-starting and must ensure that the difference, $y_1 - y(x_1)$, is of order p or higher (Rabiei and Ismail, 2012).

Trigonometric fitting is a method to approximate a function f by series of trigonometric functions. The approximation g of f can be written as

$$g(x) = \sum_{i=1}^{m} \rho_i \cos\left(\omega_i x + \varphi_i\right)$$

where m is the number of terms we want to approximate f with, ρ_i is the amplitude, ω_i is the frequency and φ_i is the amplitude of the ith cosine function. In the past and recent years, the numerical solution of initial value problem (1) whose solution shows pronounced oscillatory behaviour has attracted the interest of many researchers. For instance, Paternoster (1998) developed Runge-Kutta Nyström methods for ODEs with periodic solutions based on trigonometric polynomials. Kalogiratou and Simos (2002) constructed trigonometrically and exponentially fitted Runge-Kutta Nyström methods for the numerical solution of the Schrödinger equation and related problems which is eighth algebraic order. Sakas and Simos (2005) developed a fifth algebraic order trigonometrically fitted modified Runge-Kutta Zonneveld method for the numerical solution of orbital problems. Yang and Wu (2008) constructed trigonometrically fitted adapted Ruge-Kutta Nyström methods for perturbed oscillators. Recently, Demba *et al*. (2016) constructed an explicit trigonometrically fitted Runge-Kutta Nyström method using Simos technique. In this present work, we analyse the construction of fourth-order four-stage trigonometrically fitted (TFIRK4-4) methods based on two-step explicit fourth-order four-stage Improved Runge-Kutta (IRK4-4) method proposed by Rabiei *et al*. (2013). Ismail *et al.* (2018) developed algebraic order conditions for two-point block hybrid method up to order five using the approach of B-series. A fifth order two-point block explicit hybrid method for solving special second order ordinary differential equations (ODEs) was derived based on the order conditions. The existing explicit hybrid method of order five is employed as the method at the first point. Subsequently, the method is trigonometrically fitted in order to be suitable for solving highly oscillatory problems arising from special second order ODEs. The trigonometrically-fitted block method is then validated with a set of oscillatory problems over a very large interval.

3. Methodology

A more compact form of the general IRK method (3) and (4) is

$$y_{n+1} = y_n + hb_1 f(x_n, y_n) - hb_{-1} f(x_{n-1}, y_{n-1}) + h\sum_{i=2}^{s} b_i\big(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_{-i})\big) \quad (5)$$

where,

$$Y_i = y_n + h\sum_{j=1}^{i-1} a_{i,j}\, f\big(x_n + c_j h, Y_j\big) \quad (6)$$

$$Y_{-i} = y_{n-1} + h\sum_{j=1}^{i-1} a_{i,j}\, f\big(x_{n-1} + c_j h, Y_{-j}\big) \quad (7)$$

with y_{n+1} and y_n being an approximation to $y(x_{n+1})$ and $y(x_n)$ respectively.

Trigonometrically- fitted techniques are derived to approximate exactly the initial value problems whose solution are linear combination of the functions $\{e^{\omega x}, e^{-\omega x}\}$, where ω can be real or complex number (Berghe *et al.*, 2000).

If a function $y(x)$ is integrated exactly by TFIRK method for all problems whose solution is $y(x)$ then,

$$y_n = y(x_n) = e^{i\omega x_n} \quad (8)$$

$$y_{n-1} = y(x_{n-1}) = y(x_n - h) = e^{i\omega(x_n - h)} \quad (9)$$

$$y'_n = i\omega e^{i\omega x_n} = f(x_n, y_n) \quad (10)$$

$$y'_{n-1} = i\omega e^{i\omega(x_n - h)} = f(x_{n-1}, y_{n-1}) \quad (11)$$

$$Y_i = e^{i\omega(x_n + c_i h)} \quad (12)$$

$$Y_{-i} = e^{i\omega(x_{n-1} + c_i h)} \quad (13)$$

Consequently, we obtain the recursive relations

$$cos(c_i z) = 1 - z\sum_{j=1}^{i-1} a_{ij}\sin(c_j z), \quad i = 2,3,\dots,s \tag{14}$$

$$sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\, cos(c_j z), \qquad i = 2,3,\dots,s \tag{15}$$

$$cos(z) = 1 - zb_{-1}\, sin(z) - z\sum_{i=2}^{s} b_i\, sin(c_i z) + z\sum_{i=2}^{s} b_i\, sin\big(z(c_i - 1)\big) \tag{16}$$

$$sin(z) = zb_1 - zb_{-1}\, cos(z) + z\sum_{i=2}^{s} b_i\, cos(c_i z) - z\sum_{i=2}^{s} b_i\, cos\big(z(c_i - 1)\big) \tag{17}$$

The relations (14), (15), (16) and (17) are relations of order conditions of the trigonometrically-fitted method. These relations replace the equations of order conditions of two-step Improved Runge-Kutta method, which can be solved to give the coefficients of a particular method based on existing coefficients.

To derive a trigonometrically-fitted method with order $p = 4$ and stage $s = 4$, consider the order conditions up to order four from IRK methods (Rabiei *et al*., 2013).

$$\left.\begin{array}{rl}
\text{First order:} & b_1 - b_{-1} = 1 \\
\text{Second order:} & b_{-1} + \displaystyle\sum_{i=2}^{s} b_i = \frac{1}{2} \\
\text{Third order:} & \displaystyle\sum_{i=2}^{s} b_i c_i = \frac{5}{12} \\
 & \displaystyle\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3} \\
\text{Fourth oder:} & \displaystyle\sum_{i=2,j=1}^{s} b_i a_{i,j} c_j = \frac{1}{6} \\
 & \displaystyle\sum_{i=2}^{s} b_i c_i^3 = \frac{31}{120}
\end{array}\right\} \tag{18}$$

And the classical fourth order four stage (IRK4-4) has the butcher tableau as

Table 1. Coefficients of IRK4 − 4 methods

	0				
	$\frac{1}{5}$	$\frac{1}{5}$			
	$\frac{3}{5}$	0	$\frac{3}{5}$		
	$\frac{4}{5}$	$\frac{2}{15}$	$\frac{4}{25}$	$\frac{38}{75}$	
$\frac{122825}{161448}$	$\frac{19}{288}$	$\frac{307}{288}$	$\frac{-25}{144}$	$\frac{25}{144}$	$\frac{125}{288}$

To derive the fourth order four stage Trigonometrically-fitted IRK method, we substitute $s = 4, c_1 = 0$ in the recursive relations $(14) - (15)$
for $i = 2$

$$\cos(c_2 z) - 1 = 0 \tag{19}$$
$$\sin(c_2 z) - z a_{2,1} = 0 \tag{20}$$

for $i = 3$

$$\cos(c_3 z) - 1 + z a_{3,2} \sin(c_2 z) = 0 \tag{21}$$
$$\sin(c_3 z) - z[a_{3,1} + a_{3,2} \cos(c_2 z)] = 0 \tag{22}$$

for $i = 4$

$$\cos(c_4 z) - 1 + z[a_{4,2} \sin(c_2 z) + a_{4,3} \sin(c_3 z)] = 0 \tag{23}$$
$$\sin(c_4 z) - z[a_{4,1} + a_{4,2} \cos(c_2 z) + a_{4,3} \cos(c_3 z)] = 0 \tag{24}$$

Now, substituting $s = 4$, $c_1 = 0$ in equations $(16) - (17)$

$$\cos(z) - 1 + z b_{-1} \sin(z) + z[b_2 \sin(c_2 z) + b_3 \sin(c_3 z) + b_4 \sin(c_4 z)] - z[b_2 \sin((c_2 - 1)z) + b_3 \sin((c_3 - 1)z) + b_4 \sin((c_4 - 1)z)] = 0 \tag{25}$$
$$\sin(z) - z b_1 + z b_{-1} \cos(z) - z[b_2 \cos(c_2 z) + b_3 \cos c_3 z + b_4 \cos c_4 z] + z[b_2 \cos((c_2 - 1)z) + b_3 \cos((c_3 - 1)z) + b_4 \cos((c_4 - 1)z)] = 0 \tag{26}$$

Equations $(23) - (26)$ are now the equations of order conditions for fourth order four stage trigonometrically-fitted method that replaces order conditions (18) of the original method. To obtain the coefficients of the new method, we solve the system of two equations
(25) and (26) together with additional equations from the order condition (18) namely,

$$b_1 - b_{-1} = 1 \tag{27}$$

$$b_{-1} + b_2 + b_3 + b_4 = \frac{1}{2} \tag{28}$$

$$b_2c_2 + b_3c_3 + b_4c_4 = \frac{5}{12} \tag{29}$$

These sum up to five equations in eight unknowns $(b_{-1}, b_1, b_2, b_3, b_4, c_2, c_3$ and $c_4)$. The equations are solved in terms of three free parameters $(c_2 = \frac{1}{5}, c_3 = \frac{3}{5}, c_4 = \frac{4}{5})$ whose values are obtained from Table 1. Equations(27), (28) and (29) are chosen to augment the updated (25) and (26) so that $(b_{-1}, b_1, b_2, b_3$ *and* $b_4)$ are not taken as free parameters. Solving equations (25), (26), (27), (28) and (29) the following values are obtained for b_{-1}, b_1, b_2, b_3, and b_4

$$\left.\begin{aligned} b_{-1} &= \frac{1}{6}\frac{M_1}{M_2} \\ b_1 &= \frac{1}{6}\frac{\mathrm{M}_3}{\mathrm{M}_4} \\ b_2 &= -\frac{1}{12}\frac{\mathrm{M}_5}{\mathrm{M}_6} \\ b_3 &= \frac{1}{6}\frac{M_7}{M_8} \\ b_4 &= -\frac{1}{12}\frac{M_9}{M_{10}} \end{aligned}\right\} \tag{30}$$

where,

$M_1 =$ [-6cos$(\frac{1}{5}z) - 18\cos\left(\frac{3}{5}z\right) + 6\cos\left(\frac{4}{5}z\right) + 18\cos\left(\frac{2}{5}z\right) + 18z\sin\left(\frac{1}{5}z\right) + 18z\sin\left(\frac{4}{5}z\right) - 18z\sin\left(\frac{3}{5}z\right) - 18z\sin\left(\frac{2}{5}z\right) + 6\cos\left(\frac{1}{5}z\right)\cos(z) - 6\cos\left(\frac{4}{5}z\right)\cos(z) - 18\sin(z)\sin\left(\frac{1}{5}z\right) - 18\sin(z)\sin\left(\frac{4}{5}z\right) + 7z\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right) + 7z\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right) - 7\cos\left(\frac{1}{5}z\right)z\sin\left(\frac{4}{5}z\right) - 7\cos\left(\frac{1}{5}z\right)z\sin\left(\frac{1}{5}z\right) + 9\sin\left(\frac{1}{5}z\right)\cos\left(\frac{3}{5}z\right)z - 9\sin\left(\frac{1}{5}z\right)\cos\left(\frac{2}{5}z\right)z + 10\sin\left(\frac{3}{5}z\right)\cos\left(\frac{1}{5}z\right) - 10\sin\left(\frac{3}{5}z\right)\cos\left(\frac{4}{5}z\right) + 9\sin\left(\frac{4}{5}z\right)\cos\left(\frac{3}{5}z\right)z - 9\sin\left(\frac{4}{5}z\right)\cos\left(\frac{2}{5}z\right)z + 10\sin\left(\frac{2}{5}z\right)\cos\left(\frac{1}{5}z\right)z - 10\sin\left(\frac{2}{5}z\right)\cos\left(\frac{4}{5}z\right)z + 18\sin\left(\frac{3}{5}z\right)\sin(z) + 18\cos(z)\cos\left(\frac{3}{5}z\right) + 18\cos(z)\cos(\frac{2}{5}z)$]

$M_2 = z$[-3sin$(\frac{1}{5}z) + 3\sin\left(\frac{3}{5}z\right) - 3\sin\left(\frac{4}{5}z\right) + 3\sin\left(\frac{2}{5}z\right) + 6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{1}{5}z\right) +$
$6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{4}{5}z\right) - \cos\left(\frac{1}{5}z\right)\sin(z) - 6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right) - 6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right) +$
$\cos\left(\frac{4}{5}z\right)\sin(z) + 3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{1}{5}z\right) + 3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{4}{5}z\right) - 3\cos\left(\frac{3}{5}z\right)\sin(z) -$
$3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{1}{5}z\right) - 3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{4}{5}z\right) + 3\cos\left(\frac{2}{5}z\right)\sin(z)\,z + 3\cos(z)\sin\left(\frac{1}{3}z\right) +$
$3\cos(z)\sin\left(\frac{4}{5}z\right) - 3\cos(z)\sin\left(\frac{3}{5}z\right) - 3\cos(z)\sin\left(\frac{2}{5}z\right) - 5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{3}{5}z\right) -$
$5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{2}{5}z\right)z + 5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{3}{5}z\right) + 5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{3}{5}z\right) + 5\cos(\frac{4}{5}z)\sin(\frac{2}{5}z)$]

$M_3 =$ [-6cos$(\frac{1}{5}z) - 18\cos\left(\frac{3}{4}z\right) + 6\cos\left(\frac{4}{5}z\right) + 18\cos\left(\frac{2}{5}z\right) + 6\cos\left(\frac{1}{5}z\right)\cos(z) -$
$6\cos\left(\frac{4}{5}z\right)\cos(z) - 18\sin(z)\sin\left(\frac{4}{5}z\right) - 29\text{zcos}\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right) - 29z\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right) +$
$6z\cos\left(\frac{4}{5}z\right)\sin(z) + 18\cos(z)\,z\sin\left(\frac{4}{5}z\right) + 18\cos(z)\,z\sin\left(\frac{1}{5}z\right) + 29\cos\left(\frac{1}{5}z\right)z\sin\left(\frac{4}{5}z\right) -$
$6\cos\left(\frac{1}{5}z\right)z\sin(z) + 29\cos\left(\frac{1}{5}z\right)z\sin\left(\frac{1}{5}z\right) - 18\cos(z)\sin\left(\frac{3}{5}z\right)z - 18\cos(z)\sin\left(\frac{2}{5}z\right)z -$
$18\sin(z)\cos\left(\frac{3}{5}z\right)z + 18\sin(z)\cos\left(\frac{2}{5}z\right)z + 27\sin\left(\frac{1}{5}z\right)\cos\left(\frac{2}{5}z\right)z -$
$20\sin\left(\frac{3}{5}z\right)\cos\left(\frac{1}{5}z\right)z + 20\sin\left(\frac{3}{5}z\right)\cos\left(\frac{4}{5}z\right)z + 27\sin\left(\frac{4}{5}z\right)\cos\left(\frac{3}{5}z\right)z -$
$27\sin\left(\frac{1}{5}z\right)\cos\left(\frac{2}{5}z\right)z - 20\sin\left(\frac{2}{5}z\right)\cos\left(\frac{1}{5}z\right)z + 20\sin\left(\frac{2}{5}z\right)\cos\left(\frac{4}{5}z\right)z +$
$18\sin\left(\frac{3}{5}z\right)\sin(z) + 18\sin\left(\frac{2}{5}z\right)\sin(z) + 18\cos(z)\cos\left(\frac{3}{5}z\right)z - 18\cos(z)\cos(\frac{2}{5}z)$]

$M_4 = z$[-3sin$(\frac{1}{5}z) + 3\sin\left(\frac{3}{5}z\right) - 3\sin\left(\frac{4}{5}z\right) + 3\sin\left(\frac{2}{5}z\right) + 6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{1}{5}z\right) +$
$6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{4}{5}z\right) - \cos\left(\frac{1}{5}z\right)\sin(z) - 6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right) - 6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right) +$
$\cos\left(\frac{4}{5}z\right)\sin(z) + 3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{1}{5}z\right) + 3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{4}{5}z\right) - 3\cos(z)\sin\left(\frac{2}{5}z\right) -$
$5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{3}{5}z\right) - 5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{2}{5}z\right) + 5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{3}{5}z\right) + 5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{2}{5}z\right)$]

$M_5 =$ [12-36cos$(\frac{1}{5}z) - 48\cos\left(\frac{4}{5}z\right) + 36\cos\left(\frac{4}{5}z\right) + 48\cos\left(\frac{2}{5}z\right) + 12z\sin(z) +$
$29z\sin\left(\frac{1}{5}z\right) + 29z\sin\left(\frac{4}{5}z\right) - 47z\sin\left(\frac{3}{5}z\right) - 47z\sin\left(\frac{2}{5}z\right) + 12\cos(z^2) +$
$36\cos\left(\frac{1}{5}z\right)\cos(z) - 36\cos\left(\frac{4}{5}z\right)\cos(z) - 36\sin(z)\sin\left(\frac{1}{5}z\right) - 36\sin(z)\sin\left(\frac{4}{5}z\right) +$
$7z\cos\left(\frac{4}{5}z\right)\sin(z) + 7\cos(z)\sin\left(\frac{4}{5}z\right) + 7\cos(z)\sin\left(\frac{1}{5}z\right) - 7\cos\left(\frac{1}{5}z\right)\text{zsin}(z) -$

$12\sin(z^2)-\cos(z)\sin\left(\frac{3}{5}z\right)-\cos(z)\sin\left(\frac{2}{5}z\right)z-\sin(z)\cos\left(\frac{3}{5}z\right)+\sin(z)\cos\left(\frac{2}{5}z\right)+$
$25\sin\left(\frac{1}{5}z\right)\cos\left(\frac{3}{5}z\right)z-25\sin\left(\frac{1}{5}z\right)\cos\left(\frac{2}{5}z\right)z+25\sin\left(\frac{3}{5}z\right)\cos\left(\frac{1}{5}z\right)z-$
$25\sin\left(\frac{3}{5}z\right)\cos\left(\frac{4}{5}z\right)z+25\sin\left(\frac{4}{5}z\right)\cos\left(\frac{3}{5}z\right)z-25\sin\left(\frac{4}{5}z\right)\cos\left(\frac{2}{5}z\right)z+$
$25\sin\left(\frac{2}{5}z\right)\cos\left(\frac{1}{5}z\right)z-25\sin\left(\frac{2}{5}z\right)\cos\left(\frac{4}{5}z\right)z+48\sin\left(\frac{3}{5}z\right)\sin(z)+48\sin\left(\frac{2}{5}z\right)\sin(z)+$
$48\cos(z)\cos\left(\frac{3}{5}z\right)-48\cos(z)\cos\left(\frac{2}{5}z\right)-24\cos(z)]$

$M_6=z[-3\sin\left(\frac{1}{5}z\right)+3\sin\left(\frac{3}{5}z\right)-3\sin\left(\frac{4}{5}z\right)+3\sin\left(\frac{2}{5}z\right)+6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{1}{5}z\right)+$
$6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{4}{5}z\right)-\cos\left(\frac{1}{5}z\right)\sin(z)-6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right)-6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right)+$
$\cos\left(\frac{4}{5}z\right)\sin(z)+3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{1}{5}z\right)+3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{4}{5}z\right)-3\cos\left(\frac{3}{5}z\right)\sin(z)-$
$3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{1}{5}z\right)-3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{4}{5}z\right)+3\cos\left(\frac{2}{5}z\right)\sin(z)+3\cos(z)\sin\left(\frac{1}{5}z\right)+$
$3\cos(z)\sin\left(\frac{4}{5}z\right)-3\cos(z)\sin\left(\frac{3}{5}z\right)-3\cos(z)\sin\left(\frac{2}{5}z\right)-5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{3}{5}z\right)-$
$5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{2}{5}z\right)+5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{3}{5}z\right)+5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{2}{5}z\right)]$

$M_7=[18-36\cos(z)+18z\sin(z)-27z\sin\left(\frac{1}{5}z\right)-27z\sin\left(\frac{4}{5}z\right)-30\cos\left(\frac{1}{5}z\right)+$
$30\cos\left(\frac{4}{5}z\right)+18\cos(z^2)+30\cos\left(\frac{1}{5}z\right)\cos(z)-30\cos\left(\frac{4}{5}z\right)\cos(z)+18\sin(z)\sin\left(\frac{1}{5}z\right)+$
$18\sin(z)\sin\left(\frac{4}{5}z\right)-18\sin(z^2)-25z\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right)-25z\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right)+$
$10z\cos\left(\frac{4}{5}z\right)\sin(z)-9\cos(z)z\sin\left(\frac{4}{5}z\right)+9\cos(z)z\sin\left(\frac{1}{5}z\right)+25\cos\left(\frac{1}{5}z\right)z\sin\left(\frac{4}{5}z\right)-$
$10\cos\left(\frac{1}{5}z\right)z\sin(z)+25\cos\left(\frac{1}{5}z\right)z\sin\left(\frac{1}{5}z\right)]$

$M_8=z[-3\sin\left(\frac{1}{5}z\right)+3\sin\left(\frac{3}{5}z\right)-3\sin\left(\frac{4}{5}z\right)+3\sin\left(\frac{2}{5}z\right)+6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{1}{5}z\right)+$
$6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{4}{5}z\right)-\cos\left(\frac{1}{5}z\right)\sin(z)-6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right)-6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right)+$
$\cos\left(\frac{4}{5}z\right)\sin(z)+3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{1}{5}z\right)+3\cos\left(\frac{3}{5}z\right)\sin(\frac{4}{5}z)-3\cos\left(\frac{3}{5}z\right)\sin(z)-$
$-3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{1}{5}z\right)-3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{4}{5}z\right)+3\cos\left(\frac{2}{5}z\right)\sin(z)+3\cos(z)\sin\left(\frac{1}{5}z\right)+$
$3\cos(z)\sin\left(\frac{4}{5}z\right)-3\cos(z)\sin\left(\frac{3}{5}z\right)-3\cos(z)\sin\left(\frac{1}{5}z\right)-5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{3}{5}z\right)-$
$5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{2}{5}z\right)+5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{3}{5}z\right)+5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{2}{5}z\right)]$

$M_9 = [24 - 36\cos\left(\frac{1}{5}z\right) + 12\cos\left(\frac{3}{5}z\right) - 36\cos\left(\frac{4}{5}z\right) - 12\cos\left(\frac{2}{5}z\right) + 24z\sin(z) -$
$29z\sin\left(\frac{1}{5}z\right) - 7z\sin\left(\frac{3}{5}z\right) - 7z\sin\left(\frac{2}{5}z\right) + 24\cos(z^2) + 36\cos\left(\frac{1}{5}z\right)\cos(z) -$
$36\cos\left(\frac{4}{5}z\right)\cos(z) + 36\sin(z)\sin\left(\frac{1}{5}z\right) + 36\sin(z)\sin\left(\frac{4}{5}z\right) + 7z\cos\left(\frac{4}{5}z\right)\sin(z) -$
$7\cos(z)\,z\sin\left(\frac{4}{5}z\right) - 7\cos(z)\,z\sin\left(\frac{1}{5}z\right) - 7\cos\left(\frac{1}{5}z\right)z\sin(z) - 24\sin(z^2) +$
$19\cos(z)\sin\left(\frac{3}{5}z\right) + 19\cos(z)\sin\left(\frac{2}{5}z\right) + 19\sin(z)\cos\left(\frac{3}{5}z\right)z + 19\sin(z)\cos\left(\frac{2}{5}z\right)z -$
$25\sin\left(\frac{1}{5}z\right)\cos\left(\frac{3}{5}z\right)z + 25\sin\left(\frac{1}{5}z\right)\cos\left(\frac{2}{5}z\right)z + 25\sin\left(\frac{3}{5}z\right)\cos\left(\frac{3}{5}z\right)z -$
$-25\sin\left(\frac{3}{5}z\right)\cos\left(\frac{4}{5}z\right)z - 25\sin\left(\frac{4}{5}z\right)\cos\left(\frac{3}{5}z\right) + 25\sin\left(\frac{4}{5}z\right)\cos\left(\frac{2}{5}z\right)z +$
$25\sin\left(\frac{2}{5}z\right)\cos\left(\frac{1}{5}z\right)z - 25\sin\left(\frac{2}{5}z\right)\cos\left(\frac{4}{5}z\right)z - 12\sin\left(\frac{3}{5}z\right)\sin(z) - 12\sin\left(\frac{2}{5}z\right)\sin(z) -$
$12\cos(z)\cos\left(\frac{3}{5}z\right) + 12\cos(z)\cos\left(\frac{2}{5}z\right) - 48\cos(z)]$

$M_{10} = z[-3\sin\left(\frac{1}{5}z\right) + 3\sin\left(\frac{3}{5}z\right) - 3\sin\left(\frac{4}{5}z\right) + 3\sin\left(\frac{2}{5}z\right) + 6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{1}{5}z\right) +$
$6\cos\left(\frac{1}{5}z\right)\sin\left(\frac{4}{5}z\right) - \cos\left(\frac{1}{5}z\right)\sin(z) - 6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{1}{5}z\right) - 6\cos\left(\frac{4}{5}z\right)\sin\left(\frac{4}{5}z\right) +$
$\cos\left(\frac{4}{5}z\right)\sin(z) + 3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{1}{5}z\right) + 3\cos\left(\frac{3}{5}z\right)\sin\left(\frac{4}{5}z\right) - 3\cos\left(\frac{3}{5}z\right)\sin(z) -$
$-3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{1}{5}z\right) - 3\cos\left(\frac{2}{5}z\right)\sin\left(\frac{4}{5}z\right) + 3\cos\left(\frac{2}{5}z\right)\sin(z) + 3\cos(z)\sin\left(\frac{1}{5}z\right) +$
$3\cos(z)\sin\left(\frac{4}{5}z\right) - 3\cos(z)\sin\left(\frac{3}{5}z\right) - 3\cos(z)\sin\left(\frac{1}{5}z\right) - 5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{3}{5}z\right) -$
$5\cos\left(\frac{1}{5}z\right)\sin\left(\frac{2}{5}z\right) + 5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{3}{5}z\right) + 5\cos\left(\frac{4}{5}z\right)\sin\left(\frac{2}{5}z\right)]$

In order to recover the original method $IRK4-4$ as z approaches zero, the Taylor expansions of the coefficients b_{-1}, b_1, b_2, b_3 and b_4 in (30) are obtained as follow

$$\left.\begin{aligned} b_{-1} &= \frac{19}{288} + o(z^2) \\ b_1 &= \frac{307}{288} + o(z^2) \\ b_2 &= \frac{-25}{144} + o(z^2) \\ b_3 &= \frac{25}{144} + o(z^2) \\ b_4 &= \frac{125}{288} + o(z^2) \end{aligned}\right\} \quad (31)$$

It is clear from (31) that, as z approache zero, the classical method $IRK4-4$ presented in Table 1 is recovered exactly.

To verify the order of the method as earlier claimed, we substitute the coefficients of the method into order conditions (18) up to order four and take the Taylor expansion of each to obtain

$$\left.\begin{aligned} &\text{Order 1: } b_1 - b_{-1} = 1 + o(z^2) \\ &\text{Order 2: } b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} + o(z^2) \\ &\text{Order 3: } \sum_{i=2}^{s} b_i c_i = \frac{5}{12} + o(z^2) \\ &\text{Order 4: } \quad \sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3} + o(z^2) \\ &\qquad \sum_{i=2,j=1}^{s} b_i a_{i,j} c_j = \frac{1}{6} + o(z^2) \end{aligned}\right\} \quad (32)$$

From (32), as z tends to zero, the order conditions of the Improved Runge-Kutta method up to order four are recovered; this signifies that the coefficients of the Trigonometrically-fitted fourth order four stage method satisfies the IRK order four conditions.

4. Results and Discussion

In order to evaluate the effectiveness of the derived method $TFIRK4-4$, we seek the approximate solutions on the partition $[x_0, x_N]$ where errors are calculated at the endpoints $|y_n - y(x_n)|$ with y_n, $y(x_n)$ as approximate and exact solutions at the point x_n respectively. The

$TFIRK4-4$ method is implemented in fixed step. The solutions to the sample problems are obtained using Maple 2019 software package and the result presented in Tables $2-5$.

Problem 1:
$y'(x) = -2\cos(8x) - 8\sin(8x)\,,\, y(0) = 1\,,\ \omega = 8$
Exact solution: $y(x) = -\frac{1}{4}\sin(8x) + \cos(8x)$
Source: Senu *et al*. (2009).

Problem 2 (Inhomogeneous problem):
$y'(x) = \cos(x) - \sin(x) + 1,\, y(0) = 1,\, \omega = 1$
Exact solution: $y(x) = \sin(x) + \cos(x) + x$
Source: Al-khasawneh *et al*. (2007).

Table 2 Results of Problem 1 on $[0,1]$, $h = 0.05$, $\omega = 8$

x	Exact	$TFIRK4-4$	Error	$IRK4-4$	Error
0.05	0.8237064080	0.7521265630	0.0715798450+000	0.7521265630	0.0715798450+000
0.10	0.5173676860	0.5173676866	1.8117440000E-24	0.5173676275	5.9125125585E-08
0.15	0.1293479830	0.1293479830	1.0394260000E-24	0.1293478387	1.4425101549E-07
0.20	-0.2790929230	-0.2790929231	2.1950190000E-24	-0.2790931650	2.4193816346E-07
0.25	-0.6434711930	-0.6434711933	7.3809490000E-24	-0.6434715300	3.3676391676E-07
0.30	-0.9062595100	-0.9262595107	1.3699615000E-23	-0.9062599244	4.1375737399E-07
0.35	-1.0259693780	-1.0259693782	2.0153440000E-23	-1.0259698390	4.6076296117E-07
0.40	-0.9837012390	-0.9837012399	2.5723512000E-23	-0.9837017103	0.7035952965E-07
0.45	-0.7861283050	-0.7861283055	2.9530430000E-23	-0.7861287465	4.4103199228E-07
0.50	-0.4644429970	-0.4644429970	3.0973170000E-23	-0.4644433374	3.7741052236E-07
0.55	-0.06994323510	-0.0694323515	2.9823955000E-23	-0.0694326410	2.8953955107E-07
0.60	0.3365401350	0.3365401356	2.6264220000E-23	0.3365399443	1.9129197267E-07
0.6	0.6893803350	0.689380335	2.0855969000E-	0.689380237	9.8178919528E-

5		2	23	1	08
0.70	0.9333825380	0.9333825380	1.4453049000E-23	0.9333825131	2.4900895363E-08
0.75	1.0300241610	1.0300241612	8.0663300000E-24	1.0300241782	1.6973111048E-08
0.80	0.9640476170	0.9640476175	2.7041520000E-24	0.9640476384	2.0832114819E-08
0.85	0.7458691520	0.7458691526	7.8692900000E-25	0.7458691386	1.3933135895E-08
0.90	0.4099343480	0.4099343486	1.8557450000E-24	0.4099342667	8.1833972424E-08
0.95	0.0092799246	0.0092799246	3.3355300000E-25	0.0092797524	1.7215034568E-07
1.00	-0.3928395950	-0.3928395955	3.5393280000E-24	-0.3928398661	2.7062328621E-07

Table 3 Results of Problem 2 on $[0,1]$, $h = 0.05$, $\omega = 1$

x	Exact	$TFIRK4-4$	Error	$IRK4-4$	Error
0.05	1.0987294300	1.0974385550	0.0012908750+000	1.0974385550	0.0012908750+000
0.10	1.1948375820	1.1948375819	6.4294365045E-18	1.1948375819	4.3436993214E-14
0.15	1.2882092100	1.2882092104	1.2631979148E-17	1.2882092104	8.4420634217E-13
0.20	1.3787359090	1.3787359086	1.8601264304E-17	1.3787359086	1.2284848525E-13
0.25	1.4663163810	1.4663163810	2.4331511364E-17	1.4663163810	1.5862449670E-13
0.30	1.5508566960	1.5508566958	2.9817537194E-17	1.5508566958	1.9165924717E-13
0.35	1.6322705200	1.6322705203	3.5054769086E-17	1.6322705203	2.2187016699E-13
0.40	1.7104793360	1.7104793363	4.0039256184E-17	1.7104793363	2.4918174459E-13

0.45	1.7854126360	1.7854126365	4.4767679364E-17	1.7854126365	2.7352571525E-13
0.50	1.8570081000	1.8570081004	4.9237359531E-17	1.8570081004	1.8570081005E-13
0.55	1.9252117510	1.9252117510	5.3446264308E-17	1.9252117510	1.9252117510E-13
0.60	1.9899780880	1.9899780883	5.7393013125E-17	1.9899780883	3.2818149408E-13
0.65	2.0512702040	2.0512702043	6.1076880663E-17	2.0512702043	2.0512702043E-13
0.70	2.1090598750	2.1090598745	6.4497798670E-17	2.1090598745	3.4886940676E-13
0.75	2.1633276290	2.1633276289	6.7656356130E-17	2.1633276289	3.5439913268E-13
0.80	2.2140628000	2.2140628002	7.0553797794E-17	2.2140628002	3.5669826299E-13
0.85	2.2612635510	2.2612635510	7.3192021063E-17	2.2612635510	3.5576105106E-13
0.90	2.3049368780	2.3049368779	7.5573571252E-17	2.3049368779	2.3049368779E-13
0.95	2.3450985940	2.3450985943	2.7701635225E-17	2.3450985943	3.4419505395E-13
1.00	2.3817732910	2.3817732907	7.9580033426E-17	2.3817732907	3.3359517774E-13

Table 4 Results of Problem 1 on $[0, 100]$, $h = 0.05$, $\omega = 8$

h	$TFIRK4-4$	$IRK4-4$	$NFEs$
$\frac{1}{20}$	1.5972881000E-23	4.7165770500E-07	8000
$\frac{1}{40}$	2.8342763090E-21	7.5075822572E-09	16000
$\frac{1}{80}$	1.5259107062E-18	1.1821122475E-10	32000
$\frac{1}{160}$	7.5887651341E-17	1.8534369893E-12	64000
$\frac{1}{320}$	9.1214384286E-14	2.9007149252E-12	128000
$\frac{1}{640}$	3.9372296697E-11	4.5359484770E-16	256000

Table 5 Results of Problem 2 on $[0, 100]$, $h = 0.05$, $\omega = 1$

h	$TFIRK4-4$	$IRK4-4$	$NFEs$
$\frac{1}{20}$	1.5669465801E-14	2.2329866638E-12	8000
$\frac{1}{40}$	7.1609149749E-12	3.4715053785E-14	16000
$\frac{1}{80}$	6.8495955422E-10	5.4104357702E-16	32000
$\frac{1}{160}$	4.4036834142E-08	8.4429855162E-18	64000
$\frac{1}{320}$	2.9626952912E-05	1.3183696630E-19	128000
$\frac{1}{640}$	3.8431900551E-03	2.0593030000E-21	256000

Table 2 shows the result of dissimilarity of performance between $TFIRK4-4$ and $IRK4-4$ over the interval $[0, 1]$ when applied to Problem 1. It was revealed that TFIRK4-4 performs better than the classical $IRK4-4$ by displaying lesser errors. In addition, Table 3 reveals the results of the comparison between $TFIRK4-4$ and $IRK4-4$ when applied to Problem 2, which shows $TFIRK4-4$ display more accurate performance than the non-fitted $IRK4-4$ method. In Table 4, the results of maximum errors obtained from the interval $[0, 100]$

when $TFIRK4-4$ and $IRK4-4$ are compared in application to Problem 1. In Table 5, it is clear that the performance of $TFIRK4-4$ over $IRK4-4$ reduces as z approaches zero. Lastly, in Table 5, the $TFIRK4-4$ started off better than $IRK4-4$ with reduced level of computational accuracy as the step size diminishes.

5. Conclusion

A fourth-order four-stage trigonometrically-fitted Improved Runge-Kutta method has been derived. The method is applied to solve first order initial value problems with oscillatory behaviours. The results obtained revealed that the trigonometrically-fitted technique derived perform better than the classical method having the same number of function evaluations per step. The $TFIRK4-4$ method requires only four function evaluations at each integration step and in general requires $\left(4\cdot\left(\frac{T}{h}\right)\right)$ NFEs on the entire interval of integration. The effectiveness (in terms of accuracy) and efficiency (in terms of execution time) of the method as displayed by the numerical results, demonstrates the superiority of the method over other existing methods in literature.

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