

COMPUTER APPLICATION TO ITERATIVE  
METHODS FOR THE SOLUTION OF NON-LINEAR  
ALGEBRAIC EQUATIONS.

BY

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CERTIFICATION

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## DEDICATION

This work is dedicated to God Almighty whose love remains as solid as a rock and to my precious parents. Thank you Mum for praying for me.

## ACKNOWLEDGEMENT

Words are not enough to express my heart-felt gratitude and appreciation to all those who contributed one way or the other to the successful completion of this project. I will like to thank the Almighty God Jehova, for his consistent love and guardians. Without his miraculous help I would not have even started the PGD program not to talk of writing this project. All glory and honour be to him, amen. A heart felt thank you also goes to my hard-working supervisor and Head of Department - Professor K. R. Adeboye. His expert advice guided me especially in the development of the iterative method of Chapter Four. A big thanks also goes to the lecturers and staffs of the Maths/Computer science department, Federal University of Technology. They have all one way or the other contributed to this project.

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May the good and merciful God bless and reward you all for

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## LIST OF SPECIAL SYMBOLS

$x$	Variable
$[a,b]$	Domain of permissible values of $x$
$a$	Minimum value of $x$ in $[a,b]$
$b$	Maximum value of $x$ in $[a,b]$
$P_n(x)$	Algebraic Function of $n$ th order
$P_n(x)=0$	Algebraic equation of $n$ th order
$P_n^1(x)$	First derivative of $P_n(x)$
$P_n^{11}(x)$	Second derivative of $P_n(x)$
$\epsilon(x)=x$	Equivalent equation of the
OR	
$g(x)=x$	Equation $P_n(x)=0$
$\in$	Contained in
$e$	error between the exact solution $x_t$ and the approximate solution $\xi$
$x_t$	exact solution of $P_n(x)=0$
$\xi$	approximate solution of $P_n(x)=0$
$c_k$	Midpoint of the interval $(a_x, b_k)$
$M$	Maximum value of $P_n^1(x)$
$m$	Minimum value of $P_n^1(x)$
$\lambda$	a constant
$\exists$	there exists
$\forall$	for all

# CHAPTER ONE

## INTRODUCTION

### 1.1 Why Iterative Method?

In practice, most problems are reduced to mathematical equations in order to obtain their solutions. Some of these equations include: differential, algebraic, integral, transcendental, etc. equations. In this project, however, we are going to concentrate on non-linear algebraic equations in one variable. This type of equation is generally represented as follows:

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n \dots \dots \dots (1.1)$$

where  $a_i$ ,  $i = 0, 1, 2, \dots, n$  are constant coefficients and  $a_0 > 0$

The main aim of converting problems to mathematical models, as earlier mentioned, is to obtain solutions which are then used to answer questions or solve the problems which the equations represent. Although the solutions of some equations can be obtained by exact methods, most equations, especially those with higher degrees or orders, cannot be solved by these methods. Also, in real life problems, most data obtained from experiments, measurements, etc. are approximate or rounded off decimal numbers. Hence, the coefficients of some equations are approximate numbers e.g.:

$$2.1222x^4 - 2.15x^3 + 2x^2 + 7x = F_4(x)$$



The issue then of finding the exact roots or solution of such equations may be quite impossible.

In such situations where the exact roots of the mathematical models are either too complicated or quite impossible to obtain, the approximate solutions are calculated and used. The approximate solution is not the exact solution but differs slightly from it and can be used to substitute for it. Iterative methods are used to obtain the approximate solutions of mathematical equations. The iterative method used depends on the type of equation involved. Normally, iterative methods involved the execution of several computations which are sometimes quite cumbersome to carry out manually. The invention of the computer has been of immense help to iterative methods because it facilitates the processes involved in the methods.

As we go on in this project, we shall see the various iterative methods for obtaining the roots of non-linear algebraic equations and also how the computer can be applied to those methods.

## 1.2 General Properties of Algebraic Equation

We re-write the  $n$ th-degree algebraic equation in 1.1 above:

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

Where  $n$  is the highest degree of the unknown  $x_1$  and  $a_0, \dots, a_n$  are real coefficients.  $x_1$  is a root of the equation if  $P_n(x_1) = 0$ . Also if  $x_1$  is a root, then  $P_n(x)$  is divisible by  $(x - x_1)$

Theorem 1.1 (The fundamental theorem of algebra) Every polynomial with any numerical coefficients whose degree is not lower than unity has at least one root which may be complex in the general case.

Corollary 1.1: Every polynomial  $P_n(x)$  of degree  $n(n > 1)$  with any numerical coefficient has exactly  $n$  roots, real or complex.

The function  $P_n(x)$  can be re-written in the form

$$x = \varphi(x) \text{ i.e. } \varphi(x) - x = 0$$

Hence  $\varphi(x) - x = P_n(x)$  and  $x$  is the root of the equation  $P_n(x)$

The function  $\varphi(x)$  has a unique solution in a given interval  $(x_0, x_{otr})$  if it satisfies the conditions in theorem 1.2.

Theorem 1.2: Let on the interval  $[x_0, x_{otr}]$  the function  $\varphi(x)$  satisfy a lipchits condition with constant  $a$  and

$$0 < a < 1$$

$$0 \leq \varphi(x_0) - x_0 \leq (1 - a)$$

Then the equation has the unique solution

$$x = \lim_{k \rightarrow \infty} x_k$$

And  $a = \text{Max} |\varphi'(x)| (> 0)$  if  $\varphi(x)$  is  
[a, b]

continuously differentiable on the internal  $[x_0, x_{otr}]$ . The lipschitz condition

is fulfilled if  $|\varphi(x_1) - \varphi(x_2)| \leq a|x_1 - x_2|$

where  $x_1, x_2 \in [x_0, x_{otr}]$

### Types of Error

During the solution of majority of practical problems with a certain degree of conventionality, the problem can be represented as two successive stages: (1) Mathematical description of the problem and (2) the solution of the formulated mathematical model.

Each of the stages introduces one form of error or the other which contributes to the final error of approximation in final results. In the first stage, errors are introduced when there is lack of correspondence between the mathematical equation and the problem under study, and also when the parameters for the mathematical equation are inexact since they are obtained from an experiment which gives only approximate results. Errors due to this are non-removable and independent of the second stage while the processes of solution continues.

In the second stage, the errors are introduced during rounding off and other type operations on the numbers involved, and also during the execution of the processes of the iterative method employed. Errors involved in the second stage may be avoidable and also dependent on the processes of the first stage.

The error between the exact solution and approximate solution is called the error of approximation. Absolute error  $\Delta_a$  is given as

$$\Delta_a \geq |A - a|$$

Where  $A$  is the exact solution and  $a$  is the approximate solution. Generally  $\Delta_a$  is the upper bound of the deviation of the exact number  $A$  from its approximate.

$$\text{i.e. } a - \Delta_a \leq A \leq a + \Delta_a$$

Errors of iterative method shall be discussed in chapter 3.

## CHAPTER TWO

### Methods For Separating Roots

#### Introduction

Before calculating the approximate roots of an algebraic equation, we must (a) determine the number of roots that the equation has, (b) separate the roots and (c) compute the values of the approximate roots with specified degree of accuracy. In theorem 1.1 in chapter 1, we have seen that the function  $P_n(x)$  with degree  $n \geq 1$  has  $n$  number of roots which are either real or complex. Descartes' rule of sign can be used to determine the number of positive or negative real roots of an algebraic equation.

Descartes' rule of sign: If an equation  $P_n(x) = 0$  is incomplete, the number of positive real roots of the algebraic equation  $P_n(x) = 0$  with real coefficients either is equal to the number of sign changes in the sequence of the coefficients of the equation  $P_n(x) = 0$  or is less than the number of sign changes by an even integer (the coefficients equal to zero are not considered). The number of negative roots of the equation is equal to the number of sign changes in the sequence of the coefficients of  $P_n(-x)$  or is smaller by an even integer. On the other hand, if the equation  $P_n(x)$  is complete, then the number of its positive real roots is equal to the number of variations of sign in the sequence or is smaller by an even integer and the number

of negative roots is equal to the number of constancies of sign or is smaller by an even integer. By the number of constancies of sign, we mean the number of time the sign remains constant, that is the number of time the sign (+ or -) did not vary from one term to the next. The following examples illustrates Descartes' rules of signs.

Example 2.1: Find the number of positive and negative real roots of the equation.

$$x^4 - 18x^3 + 6x^2 + 10x - 5 = 0 \dots\dots\dots (2.1)$$

Equation 2.1 is complete in the sense that no term is missing. The equation has four roots (at least one of which is real). The sequence of the coefficients is +---. There are three sign changes and this means that there are either three or one positive roots or there are none. The number of sign constancies is 1 and constancies is 1 and consequently, the equation has one negative root.

Example 2.2: Find the number of positive and negative roots of the equation.

$$x^6 - 8x^4 - x^3 + x^2 - 1 = 0 \dots\dots\dots (2.2)$$

The equation (2.2) is incomplete (with zero coefficients for  $x^5$  and  $x$ ) and it has six roots according to theorem 1.1. The sequence of sign is +- - + -. There are three sign changes and consequently there are either three positive roots or one root. Also

$$P_n(-x) = x^6 - 8x^4 + x^3 + x^2 - 1 = 0 \dots\dots\dots (2.3)$$

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In equation 2.3, the sequence of signs is + - + + - . There are also three signs changes here and so there are either three negative roots or one.

Having determined the number of positive and negative real roots of the equation  $P_n(x)$ , we then separate the roots of the equation. The root  $x_i$  of the equation  $P_n(x) = 0$  is considered to be separated on the interval  $[a, b]$  if the equation  $P_n(x) = 0$  has no other root on this interval. In other words, to separate the root means to divide the whole domain of permissible value into intervals in each of which there is one root of the equation  $P_n(x)$ . Before going over to the methods of separating roots the following definitions would be useful to know.

Definition 2.1: A function  $P_n(x)$  which is single valued in a domain  $[a, b]$  is said to be differentiable within the domain if  $\forall x \in [a, b]$ ,  $P_n'(x)$  exists and it is unique.

Definition 2.2: A function  $P_n(x)$  is continuous with a domain  $[a, b]$  if  $\forall x \in [a, b]$ .

Definition 2.3: The function  $P_n(x)$  is said to be analytic (or regular) in a domain  $[a, b]$  if the function is one valued and differentiable  $\forall x \in [a, b]$ .

Definition 2.4: If the function  $P_n(x)$  is defined analytically, then the domain of existence (domain of definition) of the function is the set of all the real values of the argument  $x$ , for which the analytical expression defining the function does not lose the numerical sense and

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assumes only real values.

Definition 2.5: The function  $P_n(x)$  is said to be monotonous in a given interval  $[a, b]$  if it satisfies the condition  $F(x_2) \geq F(x_1)$  or the condition  $F(x_2) \leq F(x_1)$  for any  $x_2 > x_1 \in [a, b]$ .

There is also Sturm's theorem which enable us to be more precise in determining the number of roots of an algebraic equation. Without loosing generality, we assume that the roots of the equation  $P_n(x) = 0$  are all simple roots. Also let us assume that all the roots are in the interval  $[a, b]$  where  $a < b$ . We find the first derivative  $P_n^1(x)$  and divide  $P_n(x)$  by it. We take the remainder of the division  $P_n(x)$  by  $P_n^1(x)$  with the opposite sign and denote it by  $S_1(x)$ . Next, we divide  $P_n^1(x)$  by  $S_1(x)$ , take the remainder obtained with the opposite sign and denote it by  $S_2(x)$ . Again, we divide  $S_1(x)$  by  $S_2(x)$  and taking the remainder with the opposite sign we denote it by  $S_3(x)$ . This dividing process continues until we get a remainder which is a constant quality. We then take that quantity also with the opposite sign. The result is a sequence of functions:

$$P_n(x), P_n^1(x), S_1(x), S_2(x), S_3(x), \dots, S_{n-1}(x), S_n = \text{Constant}.$$

This system is known as Sturm's system. Next, we substitute  $a$ , first and then  $b$  for  $x$  in this sequence and count the number of sign changes in both cases. WE designate the numbers of sign changes in both cases with  $W(a)$  and  $W(b)$  respectively.





$$\begin{array}{r}
 40x-9 \quad \overline{) \quad 120x^2-160} \\
 \underline{120x^2-27x} \phantom{0} \\
 (27x-160) \times 40 \\
 \underline{1080x-6400} \\
 1080x-234 \\
 \underline{-1970}
 \end{array}$$

$$S_2 = -(-4070) = 4070$$

Table 2.1

X	$P_n(x)$	$P_n^1(x)$	$S_1(x)$	$S_2(x)$	$W(x)$
$-\infty$	-	+	-	+	3
0	+	-	-	+	2
$+\infty$	+	+	+	+	0

The number of real roots is given by  $W(-\infty) - W(+\infty) = 3 - 0 = 3$ . Hence there are three real roots. The number of negative roots is given by  $W(-\infty) - W(0) = 3 - 2 = 1$ . The number of positive real roots is given by  $W(0) - W(+\infty) = 2 - 0 = 2$

## 2.2 Graphical Method

One of the methods for separating roots is the graphical method. This method can be applied in two ways.

1<sup>st</sup> Technique: This method involves the construction of the graph of the function  $y = P_n(x)$ . The points of intersection of the graph and the x-axis yields the values of the root. From the graphs we can easily locate two numbers a and b which include one root between them. For example, we can separate the roots of the equation

$$x^3 - 3x - 1 = 0 \dots\dots\dots (2.4)$$

Example 2.4

Let us first determine the maximum and minimum points (if they both exist) of equation 2.4. This will help in determining how the graph must look like.

$$p^I_3(x) = 3x^2 - 3 = 0$$

$$x = 1, \text{ therefore } x = \pm 1$$

$$p^{II}_3(x) = 6x = 0$$

$$\text{at } -1, p^{II}_3(-1) < 0,$$

$$\text{at } 1, p^{II}_3(1) > 0$$

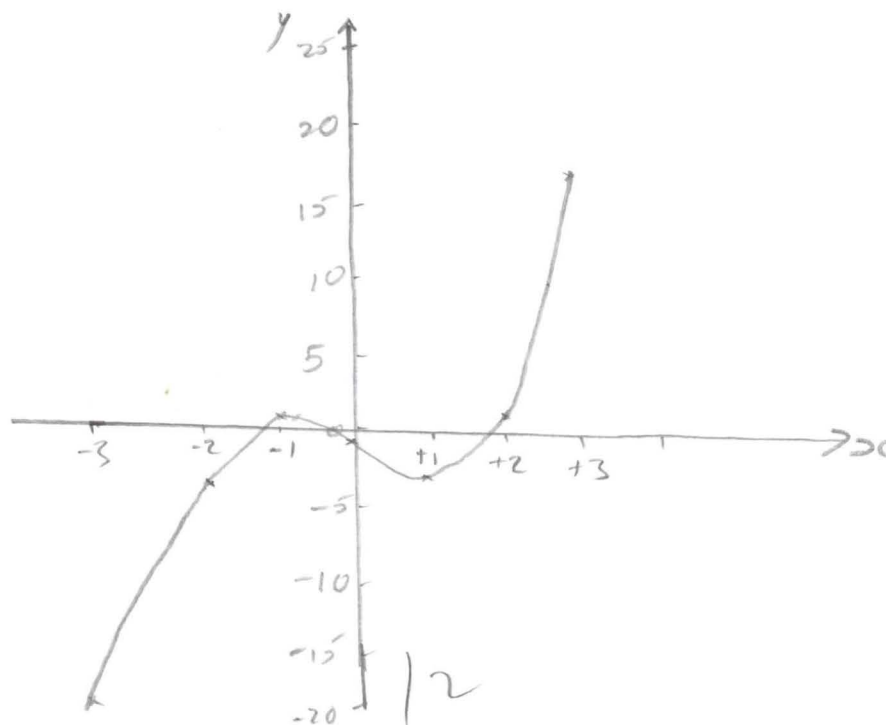
Hence we have a minimum point at  $x = 1$

Hence we have a maximum point at  $x = -1$

Table 2.2

x	-3	-2	-1	0	1	2	3
y	-19	-3	1	-1	-3	1	17

Fig. 2.1



From figure 2.1 we can see that the graph of  $y = x^3 - 3x - 1$  cuts the  $x$ -axis at the points  $a$ ,  $b$  and  $c$ . These are the roots of the equation in (2.4). They lie in the domains  $[-2, -1]$ ,  $[-1, 0]$  and  $[1, 2]$  respectively.

2<sup>nd</sup> Techniques: In this techniques, all the terms of the equation  $P_n(x)$ . is re-written in the form  $F(x) = g(x)$ . The graphs at  $F(x)$  and  $g(x)$  are then plotted and the abscissas of the points of inter-section of the two graphs are the roots of  $P_n(x)$ . Finally, from the graph plotted, the domains containing these points are determined. We shall apply this technique to the equation 2.4

Example 2.5

$$y = x^3 - 3x - 1 = 0$$

Re-writing the equation in the form  $F(x) = g(x)$ , we have

$$x^3 = 3x + 1$$

$$g(x) = x^3, \quad F(x) = 3x + 1$$

Table 2.3

$$g(x) = x^3$$

$x$	-3	-2	-1	0	1	2	3
$g(x)$	-27	-8	-1	0	1	8	27

Table 2.4

$$f(x) = 3x+1$$

$x$	-3	-2	-1	0	1	2	3
$f(x)$	-8	-5	-2	1	4	7	10

From Fig. 2.2, we can see that  $g(x)$  and  $f(x)$  intersect at the points  $a$ ,  $b$  and  $c$  which are contained in the intervals  $[-2, -1]$ ,  $[-1, 0]$ ,  $[1, 2]$ .

Figure 2.1 shows the graph  $P_n(x)$  cuts the graph at three points and hence it has three roots. If, however the curve touches the abscissa, Fig. (2.3), then the equation has a repeated root at that point. It means that the values of  $P_n(x)$  and  $P_n^{ll}(x)$  at that point is zero. If  $P_n(x)$  has three roots (where  $n = 3$ ) and it's curve cuts the  $x$ -axis at just one point, then the equation  $P_n(x)$  has a real root which is being repeated three times, fig. 2.4: In other words, the root has a multiplicity of three or a multiple root. In this case, the values of  $P_n(x)$ ,  $P_n^l(x)$  and  $P_n^{ll}(x)$  at that point are all equal to zero.

Fig. 2.3

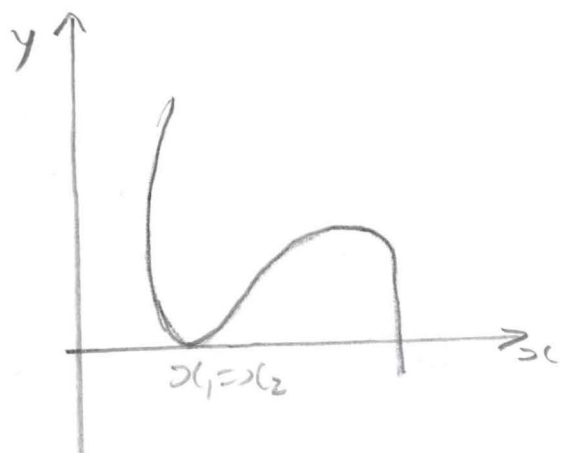
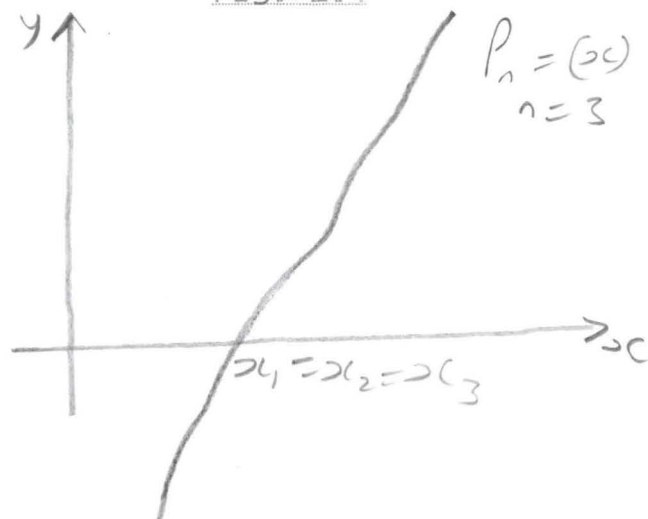


Fig. 2.4



The graphical method is not very precise but gives a roughly determined interval of separation of the roots. However, it is very helpful in cases of repeated roots.

## Analytical Method

Using some properties of functions studied in the course of mathematical analysis, we can separate the roots of the equation  $P_n(x) = 0$ .

Theorem 2.2 (Intermediate value theorem):

If a function  $P_n(x)$  is continuous on the interval  $[a, b]$  and assumes values of unlike signs at the end-point of this interval, when at least one root of the equation  $P_n(x) = 0$  lies within this interval.

Theorem 2.3 :- If a function  $P_n(x)$  is continuous and monotonic on the interval  $[a, b]$  and assumes values of unlike signs at the end-points of this interval, there is a root of the equation  $P_n(x) = 0$  within the interval and the root is unique.

Theorem 2.4:- If the function  $P_n(x)$  is continuous on the interval  $[a, b]$  and assumes values of unlike signs at the end-point of this interval and derivative  $P_n'(x)$  retains sign within the interval then there is a root of the equation  $P_n(x) = 0$  within the interval and the root is unique.

The following sequence of operations is used to separate the roots using analytical method:

- (a) Find the first derivative  $P'_n(x)$  of the function  $P_n(x)$ .
- (b) Compile a table of signs of the function  $P_n(x)$  where  $x$  is the critical value (or root) of the derivative  $P'_n(x)$  and/or values close to it and the boundary values (obtained from the domain of permissible values of the unknown,  $x$ ).
- (c) Determine the end-points of the intervals in which the functions  $P_n(x)$  assumes opposite signs. These intervals contains only one root within each interior.

Example 2.6

We shall apply, the analytical method to equation (2.4) to illustrate how the method works.

$$P_n(x) = x^3 - 3x - 1$$

$$P'_n(x) = 3x^2 - 3 = 0$$

The roots of  $P'_n$  are  $x = 1$  and  $x = -1$ .

We then compile a table of signs of the function  $P_n(x)$  at and around these points.

Table 2.5

$x$	$-\infty$	$-1$	$0$	$1$	$+\infty$
signs of $P_n(x)$	-	+	-	-	+

From the signs of  $P_n(x)$  in table 2.5 we can see that the roots according to theorem 2.1 lies in the interval  $[-\infty, -1]$ ,  $[-1, 0]$  and  $[1, +\infty]$ . We now have to obtain the end-

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points  $a$  and  $b$  to replace  $-\infty$  and  $+\infty$  in the first and last intervals such that the signs of the values of  $P_n(a)$  and  $P_n(b)$  corresponds to the signs of  $P_n(-\infty)$  and  $P_n(+\infty)$  respectively. Now, we have that  $P_n(-2) = -3 < 0$  and  $P_n(2) = > 0$ .

$x$	-2	-1	0	1	2
signs of $P_n(x)$	-	+	-	-	+

Hence we choose  $-2$  and  $2$  to replace the end-points  $-\infty$  and  $+\infty$  respectively. Hence, the roots lie in the intervals  $[-2, -1]$ ,  $[-1, 0]$  and  $[1, 2]$



# CHAPTER THREE

## Iterative Methods and Their Applications

### 3.1 Bisection Method

Let us consider an algebraic equation  $P_n(X) = 0$  which is continuous and separated on an interval  $[a, b]$ . We have to find an approximate root  $\xi \in [a, b]$  with an accuracy  $\epsilon$ . Since the equation is separated on the interval  $[a, b]$  then there exists an exact solution  $X_1 \in [a, b]$  and the end point  $a, b$  have opposite signs or the function  $P_n(X)$ . That is  $P_n(a) \cdot P_n(b) < 0$ .

The error of the approximate solution  $\xi$  must not exceed the length of the interval  $b - a$ . That is  $b - a \leq \epsilon$ . If however,  $b - a > \epsilon$  then the requirement for accuracy of calculation has not been attained. To attain the requirement i.e.  $b - a \leq \epsilon$ , we must choose new values of the end-points  $a = a_0$  and  $b = b_0$  so that  $P_n(a_0) \cdot P_n(b_0) < 0$ . We continue this process of replacing the end-points  $a, b$  until we obtain the interval  $[a_n, b_n]$  such that  $P_n(a_n) \cdot P_n(b_n) < 0$  and  $b_n - a_n \leq \epsilon$ .

In order to achieve the aforesaid replacements of the end-points  $a, b$ , we have to obtain the mid-point  $c_0$  of the interval  $[a_0, b_0]$  where  $a = a_0$  and  $b = b_0$ . Where

$$C = a_k + b_k / 2 \quad k = 0, 1, 2 \dots \dots \dots (3.1)$$

✓

Having obtained  $C_0$ , we note that the sign of  $P_n(C_0)$  must coincide either with the sign of  $P_n(a_0)$  or with the sign of  $P_n(b_0)$ . At the end-points of the interval  $[a_0, c_0]$  or the interval  $[c_0, b_0]$ , the function  $P_n(x)$  has the same signs, but has opposite signs at the end-points of the other interval. We chose the interval in which  $P_n(x)$  has opposite signs at the end-points and reject the other interval since the root  $\xi$  of the equation  $P_n(x) = 0$  is contained in the chosen interval, according to the theorem.

We denote the retained interval by  $[a_1, b_1]$

where:

$$\begin{aligned}
 a_1 &= \begin{cases} C_0, & \text{sign } F(a_0) = \text{sign } F(C_0) \\ b_0, & \text{sign } F(a_0) \neq \text{sign } F(C_0) \end{cases} \\
 b_1 &= \begin{cases} C_0, & \text{sign } F(b_0) = \text{sign } F(C_0) \\ b_0, & \text{sign } F(b_0) \neq \text{sign } F(C_0) \end{cases}
 \end{aligned}$$

In other words,

$$\begin{aligned}
 a_{k+1} &= \begin{cases} C_k, & \text{sign } F(a_k) = \text{sign } F(C_k) \\ a_k, & \text{sign } F(a_k) \neq \text{sign } F(C_k) \end{cases} \\
 b_{k+1} &= \begin{cases} C_k, & \text{sign } F(b_k) = \text{sign } F(C_k) \\ b_k, & \text{sign } F(b_k) \neq \text{sign } F(C_k) \end{cases}
 \end{aligned}$$

If however,  $F(C_k) = 0, (0 \leq k \leq m)$  then  $C_k$  is the exact root of the equation  $P_n(x) = 0$ . The graph  $P_n(x)$  below explains the bisection method geometrically.

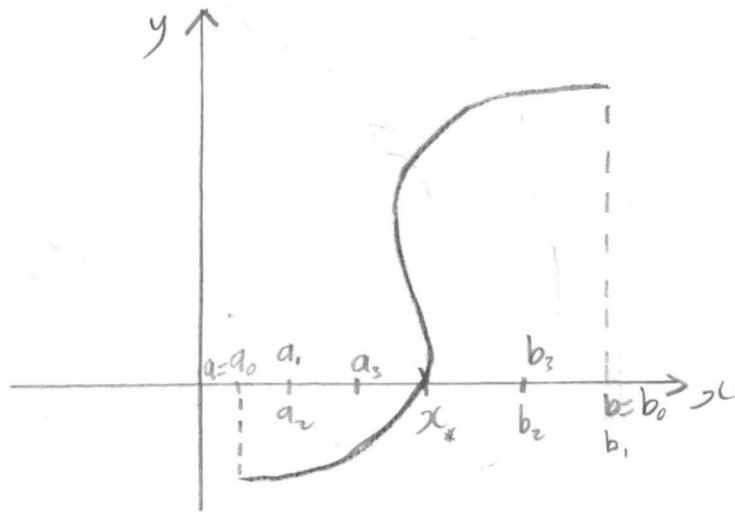


Fig. 3.1

When we finally obtain an interval  $[a_n, b_n]$  such that  $b_n - a_n \leq \epsilon$ , then numbers  $a_n$  and  $b_n$  are the approximate roots of the equation  $P_n(X)$  and we may choose  $\xi$  such that  $\xi = a_n$  or  $\xi = b_n$ .

Error of Approximation: The error of approximation  $\epsilon$  is given by:

$$b - a \leq \epsilon$$

At the  $k$ th interval  $[a_k, b_k]$ , we obtain the length of the interval by the formula:

$$b_k - a_k = (b - a) / 2^k$$

where  $k$  indicates how many divisions have been performed. If  $\xi = (a_n + b_n) / 2$  is taken to be the approximate root, then the error  $\epsilon$  does not exceed  $(b - a) / 2^{n+1}$ .

$$\text{Hence } |x - \xi| = b_n - a_n \leq (b - a) / 2^{n+1} \leq \epsilon$$

Example 3.1

We shall now apply the bisection method to equation 3.2 below:

$$P_3(x) = x^3 + 3x^2 - 3x = 0 \dots\dots\dots (3.2)$$

where  $\epsilon = 10^{-3}$  (0.001)

First, we shall separate the roots of the equation 3.2 using the analytical method.

$$P_3^1(x) = 3x^2 + 6x$$

$$3x^2 + 6x = 0$$

$$3x(x+2) = 0 \quad , \quad x_1 = 0 \text{ and } x_2 = -2$$

We shall now compile a table of signs of the function  $x^3 - 3x^2 - 3$  for some values of  $x$  to determine the places of sign changes and hence the intervals of the roots.

Table 3.1

x	$-\infty$	-2	-1	0	1	$+\infty$
signs of	-	+	-	-	+	+
$P_3(x)$						

There are three sign changes. The first root lies in the interval  $[-\infty, -2]$ .

$P_3(-3) = -3$ . Since  $P_3(-3)$  has the same sign with  $P_3(-\infty)$ , we replace  $-\infty$  with -3. Also  $P_3(1) = +1$  and since  $P_3(1)$  has the same positive sign with  $P_3(+\infty)$ , we replace  $+\infty$  with 1.

Table 3.2

x	-3	-2	-1	0	1
signs of	-	+	-	+	+
$P_3(x)$					

This means that the roots of the equation  $x^3+3x^2-3=0$  lie in the intervals  $[-3, -2]$   $[-2, -1]$ ,  $[0, 1]$ .

Let us obtain an approximate value of the root  $\xi$ , with an accuracy equal to  $\epsilon = 10^{-3}(0.001)$ . Let the root  $\xi$  be the smallest root lying in the interval  $[-3, -2]$ .

We have that

$$P_3(-3) = -3 \quad \text{and} \quad P_3(-2) = 1$$

$$P_3(-3) \cdot P_3(-2) = -3 < 0.$$

Hence, let  $a_0 = -3$  and  $b_0 = -2$ .

Using equation 3.1, we obtain  $c_0$

$$C_0 = \frac{-3+(-2)}{2} = -5/2 = -2.500$$

$$P_3(-2.500) = 0.125$$

Since  $P_3(c_0)$  and  $P_3(-2)$  have the same sign.

We now have the new interval  $[-3, -2.500]$

Hence,  $a_1 = a_0 = -3$  and  $b_1 = -2.500$

$$C_1 = \frac{-3+(-2.500)}{2} = -2.750$$

$$P_3(c_1) = -1.111.$$

Hence  $a_2 = C_1 = -2.750$  and  $b_2 = b_1 = -2.500$

Table 3.3 gives the rest of the computation

Table 3.3

K	$a_k$	$b_k$	$x_k = \frac{a_k+b_k}{2}$	$x_k^3$	$3x_k^2$	$P_3(x_k)$
0	-3	-2	-2.500	-15.625	18.750	0.125
1	-3	-2.500	-2.750	-20.800	22.689	-1.111
2	-2.750	-2.500	-2.625	-17.990	20.670	-0.320
3	-2.625	-2.500	-2.563	-16.840	19.701	0.139
4	-2.563	-2.500	-2.532	-16.320	19.233	0.003
5	-2.563	-2.532	-2.548	-16.540	19.479	-0.071

6	-2.548	-2.532	-2.540	-16.390	19.356	-0.034
7	-2.540	-2.532	-2.536	-16.310	19.293	-0.014
8	-2.536	-2.532	-2.534	-16.270	19.263	-0.007
9	-2.534	-2.532	-2.533	-16.250	19.248	-0.002
10	-2.533	-2.532				

In the table above, the signs - and + at the upper indices of  $a_k$  and  $b_k$  means that  $P_3(a_k) < 0$  and  $P_3(b_k) > 0$ . The root of the equation  $\xi = -2.532$ . Note that the above procedure could also be applied to obtain the roots in the intervals  $[-2, -1]$  and  $[0, 1]$ .

### 3.2 Chords Method

This method is also known as the "false position method" or the method of linear interpolation. The idea of this method is that on a sufficiently small interval  $[a, b]$ , the arc of the curve  $y = P_n(x)$  is replaced by the chord and the  $x$ -axis is taken as the approximate value of the root. Where the function  $P_n(x)$  is continuous with first and second derivations in the interval  $[a, b]$  and  $P_n(a) \cdot P_n(b) > 0$ . The method of chords comprises of two cases. Case 1: In this case, the first and the second derivatives of  $P_n(x)$  are of the same sign. That is,  $P_n^I(x) \cdot P_n^{II}(x) > 0$  in both cases. The graph of the equation  $P_n(X) = y$  passes through the points  $A_0 (a, P_n(a))$  and  $B(b, P_n(b))$ . The required root of the equation is the abscissa of the point of intersection of the graph and the  $X$ -axis. We do not know this point and so we choose  $x_1$ , which is the point of intersection of the chord  $A_0 B$  and the  $X$ -axis, as an

approximation to the required root  $x$ . To obtain the value of  $x_1$ , we consider the equation of the chord  $A_0B$ . This is given as:

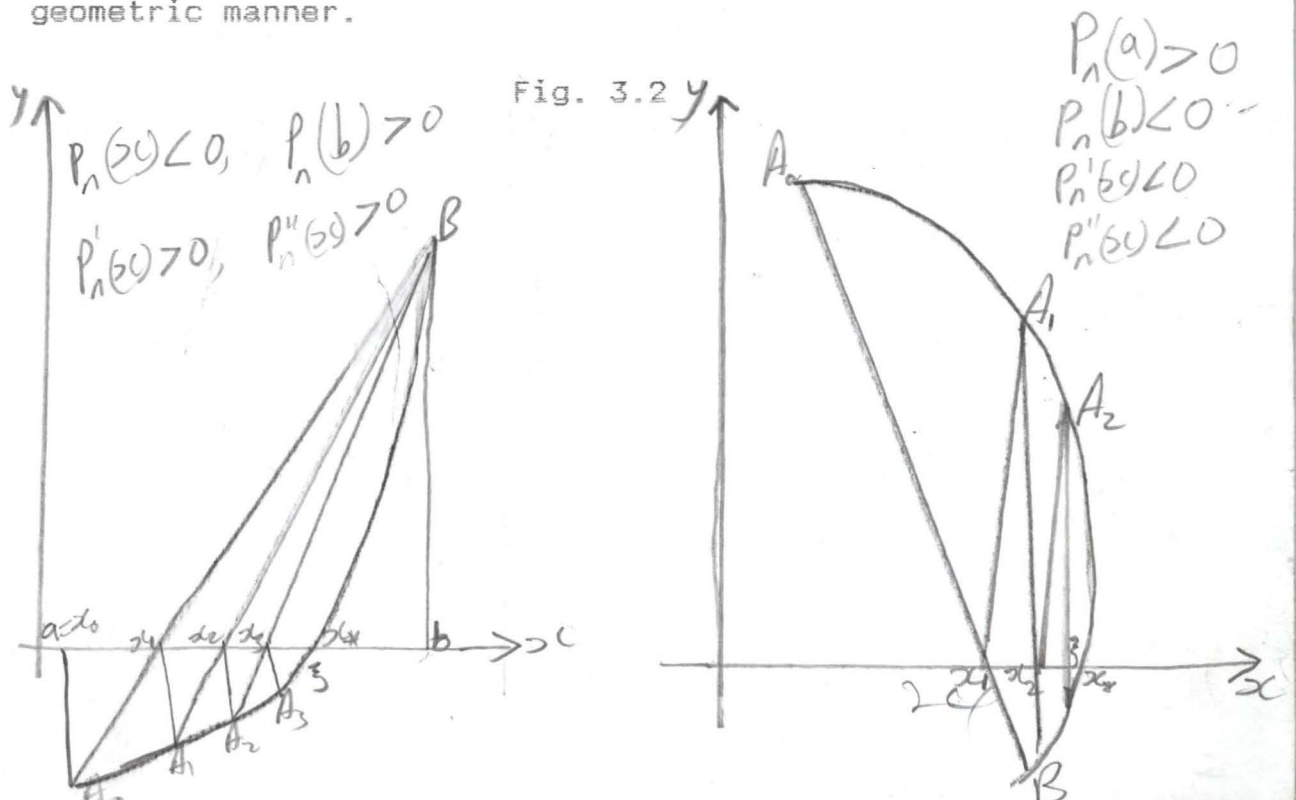
$$\frac{y - P_n(a)}{P_n(b) - P_n(a)} = \frac{x_1 - a}{b - a} \quad \dots\dots\dots (3.3)$$

However, since we are interested in obtaining the value of  $x_1$ , the root of equation 3.3, at which  $y = 0$ , then equation 3.3 becomes:

$$\frac{-P_n(a)}{P_n(b) - P_n(a)} = \frac{x_1 - a}{b - a}$$

$$x_1 = a - \frac{P_n(a)(b-a)}{P_n(b) - P_n(a)} \quad \dots\dots\dots (3.4)$$

The root is now within the interval  $[x_1, b]$ . However, we can refine this interval until it suits our purpose. The graphs below illustrates the idea of this method in a geometric manner.



In Fig. 3.2(a), we connect point  $A_1(x, P_n(x))$  with point  $B(B, P_n(b))$  and find  $x_2$ .  $x_2$  is the point of intersection of the chord  $A_1B$  and the  $x$ -axis.

$$x_2 = x_1 - \frac{P_n(x_1)(b-x_1)}{P_n(b)-P_n(x_1)}$$

Continuing this process, we have:

$$x_{k+1} = x_k - \frac{P_n(x_k)(b-x_k)}{P_n(b)-P_n(x_k)} \quad \dots \dots \dots (3.5)$$

Case 2:- In this case, we have that  $P_n'(x) \cdot P_n''(x) < 0$  and  $P_n(a) \cdot P_n(b) < 0$ , of course.

Let  $P_n'(x) < 0$  and  $P_n''(x) > 0$ . The graph of the equation  $P_n(x) = y$  is given below:

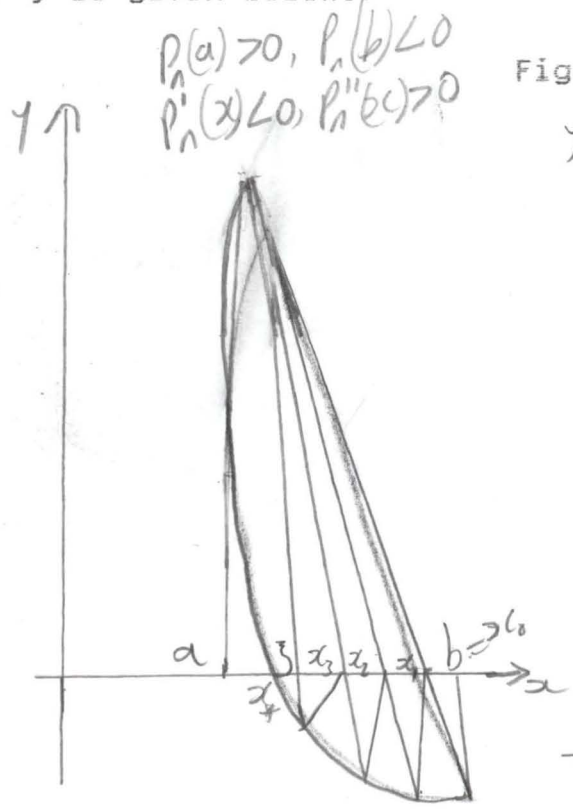
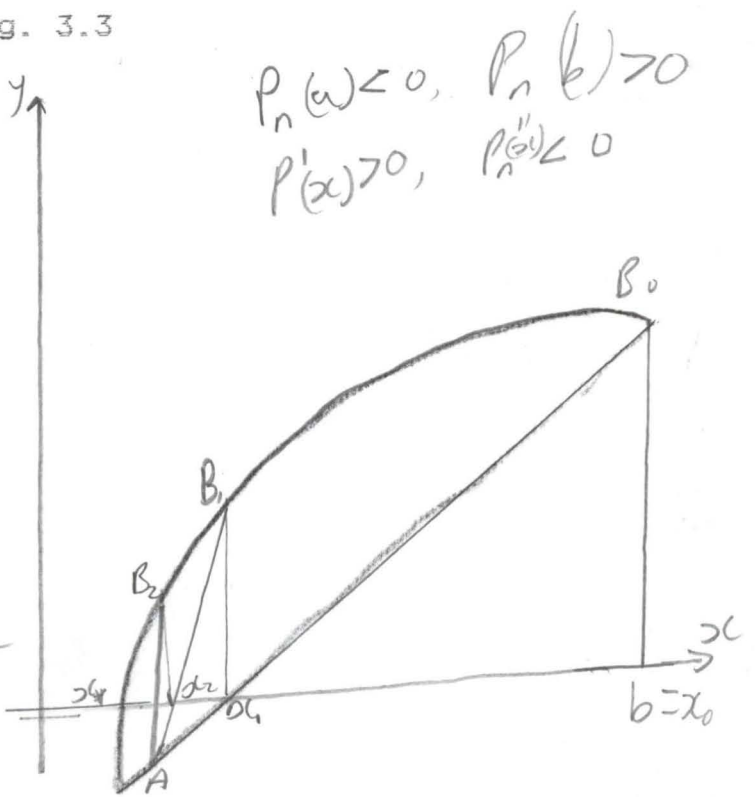


Fig. 3.3





We consider Fig. 3.3(a) where  $P_n(a) > 0$ , and  $P_n(b) < 0$  and  $P_n(x) < 0$ ,  $P_n''(x) > 0$ . We connect the points  $A(a, P_n(a))$  and  $B_0(b, P_n(b))$  and write the equation of the chord which passes through the points  $A$  and  $B_0$ .

$$\frac{y - P_n(b)}{P_n(b) - P_n(a)} = \frac{x - b}{b - a} \quad \dots\dots\dots (3.6)$$

Equation (3.6) is similar to equation (3.3). The only difference is that while  $b$  is a stationary point in equation (3.3),  $a$  is a stationary point in equation (3.6).

Like in case 1,  $x_1$  is the desired solution to the equation (3.6) and also an approximation to the exact root  $x$ .  $x_1$  is the intersection of the chord  $AB_0$  and the  $x$ -axis.

At  $x = x_1$ ,  $y = 0$ .

Hence equation (3.6) becomes

$$x_1 = b - \frac{P_n(b)(b-a)}{P_n(b) - P_n(a)}$$

Continuing the process we have

$$x_{k+1} = x_k - \frac{P_n(x_k)(x_k - a)}{P_n(x_k) - P_n(a)} \quad \dots\dots\dots (3.7)$$

The root  $\xi$  is now  $x_{k+1}$  which is in the interval  $(a, x_k)$ . We can determine the stationary end-point and have the formula to apply by applying the following rule:

The stationary end-point of an interval is the end-point for which the sign of the function coincides with the sign of the second derivative i.e. if  $P_n(b) \cdot P_n''(x) > 0$ , then the stationary

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point is  $b$  and hence we make use of equation (3.5)

However, if  $P_n(x) \times P_n'(x) > 0$ , then the stationary point is  $a$  and so we make use of equation (3.7).

Error of Estimation: If we have that  $M \leq 2m$  where  $M = \text{Max}/P_n'(x)/$  and  $m = \text{Min } |P'(x)|$ , then we can use the formula below to determine the error of the root lying in the interval  $[a, b]$ .

$$|x_k - \xi| < |x_k - x_{k-1}| \dots \dots \dots (3.8)$$

Where  $\xi$  is the approximate of the root  $x$ .  $x_k$  and  $x_{k-1}$  are the approximations at the  $k$ th and  $(k-1)$ th stages respectively.

### Example 3.2

We shall once again consider equation 3.2 in example 1. Again let  $\epsilon = 10^{-3}$  (i.e. 0.001) and let us consider the root in the interval  $[-3, -2]$ .

Before we start, we must verify that the condition  $m \leq 2M$  is fulfilled in the interval  $[-3, -2]$ .

$$|P_3'(x)| = |3x^2 + 6x|$$

$$M = \text{Max}_{[-3, -2]} |P_3'(x)| = |27 - 18| = 9$$

$$m = \text{Min}_{[-3, -2]} |P_3'(x)| = |12 - 12| = 0$$

Hence  $m > 2M$ . So the condition is not fulfilled. We now reduce the interval by taking the midpoint. Midpoint of

$[-3, -2]$  is  $-2.5$ .  $P_n(-2.5) = 0.125$ .

So we replace  $-2$  with  $-2.5$  since the function of both points

are positive. Hence we consider the new interval  
 $[-3, -2.5]$ .

$$M = \text{Max} \{ P_n^1(x) \} = 9, \\ [-3, -2.5]$$

$$m = \text{Min} \{ P_n^1(x) \} = 3.75 \\ [-3, -2]$$

Again  $M > 2m$ . So we find the midpoint of the interval  $[-3, -2.5]$ . The midpoint is  $x = -2.75$  and  $F(-2.75) < 0$ .

Hence, we replace  $-3$  with  $-2.75$  and so we again have a new interval  $[-2.75, -2.5]$ .

$$M = \text{Max} \{ P_n^1(x) \} = 6.189, \\ [-2.75, -2.5]$$

So  $M < 2m$  is fulfilled. Therefore, to estimate the error of the root lying on the interval  $[-2.75, -2.5]$ , we may apply equation 3.8.

The next thing is now to determine the formula that must be used for calculation by determining the sign of the second derivation.

$$P_n^{II}(x) = 6x + 6. < 0 \text{ for all } x$$

$$P_n(-2.75) < 0 \text{ and } P_n(-2.5) > 0, P_n(-2.75). P_n^{II}(x) > 0$$

Hence  $x = -2.75$  is the stationary point i.e.  $a$  is the stationary point. Hence we employ equation 3.7

$$x_{k+1} = x_k - \frac{P_n(x_k)(x_k - a)}{P_n(x_k) - P_n(a)}$$

Where  $a = -2.75$  and  $P_n(a) = -1.111$

Table 3.4 below gives the values of  $x_k$

where  $x_0 = -2.5$ .

Table 3.4

K	$x_k$	$x_k^3$	$3x_k^2$	$P_n(x)$	$x_k - a$	$\frac{-P_n(x_k)(x_k - a)}{P_n'(x_k) - P_n'(a)}$
0	-2.5	-15.625	18.75	0.125	0.25	-0.025
1	-2.525	-16.098	19.1268	0.0288	0.225	-0.006
2	-2.531	-16.213	19.2180	0.0050	0.219	-0.0009
3	-2.5319					

$x_2 = -2.531$  and  $x_3 = -2.5319$  ( $x_3 - x_2$ ) = 0.0001 < 0.001 and so

rounding off to the thousands place, we get  $\xi = 2.532$ .

### 3.3 Newton's Method

The idea behind Newton's method, geometrically, is the replacement of the arc of the curve  $P_n(x) = y$  with tangent to the curve. Like in the method of chords, the Newton's method involves two cases.

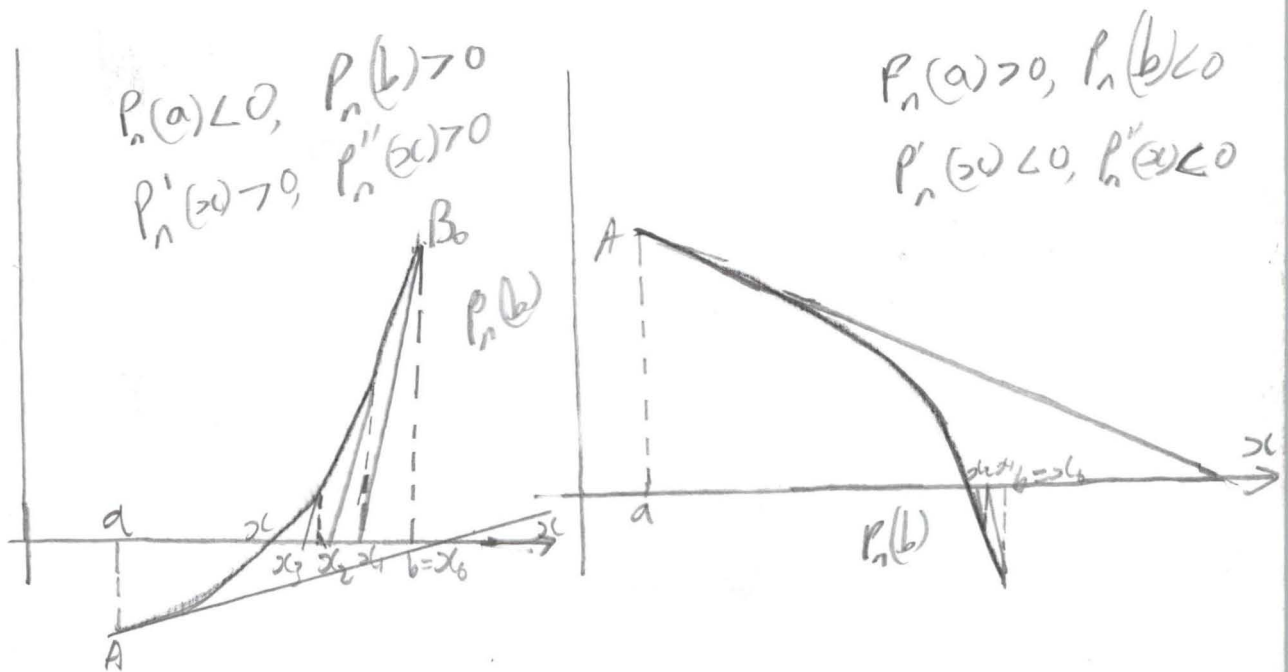
Case 1: In this case the signs of the first and second derivatives of the function  $P_n(x)$  are alike. That is assuming that  $P_n(x) = 0$  has been separated on the interval  $[a, b]$  and  $P_n'(x)$  and  $P_n''(x)$  are continuous and retain constant signs throughout the interval  $[a, b]$ .

So, considering cases where  $P_n'(x), P_n''(x) > 0$  we have  $P_n(a) < 0, P_n(b) > 0, P_n'(x) > 0, P_n''(x) > 0$  or  $P_n(a) > 0, P_n(b) < 0, P_n'(x) < 0, P_n''(x) < 0$ .

The graphs in fig. 3.4 show the tangents to these curves

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Fig. 3.4



In Fig. 3.4(a) we draw a tangent to the curve  $y = P_n(x)$  at the point  $B_0(b, F(b))$  and find the abscissa of the point of intersection of the tangent and the x-axis. The equation of the tangent at the point  $B_0(b, F(b))$  is given by:-

$$y - P_n(b) = P_n'(b) (x-b) \dots\dots\dots (3.9)$$

$x = x_1$  and  $y = 0$  at  $x_1$ . Hence (3.9)

becomes:

$$x_1 - b = - \frac{P_n(b)}{P_n'(b)}$$

The root is now on the interval  $[a, x_1]$  Fig. (3.4). Applying the Newton's method again, we draw a tangent to the curve at the point  $B_1(x_1, F(x_1))$

We obtain

$$x_2 = x_1 - \frac{P_n(x_1)}{P_n'(x_1)}$$

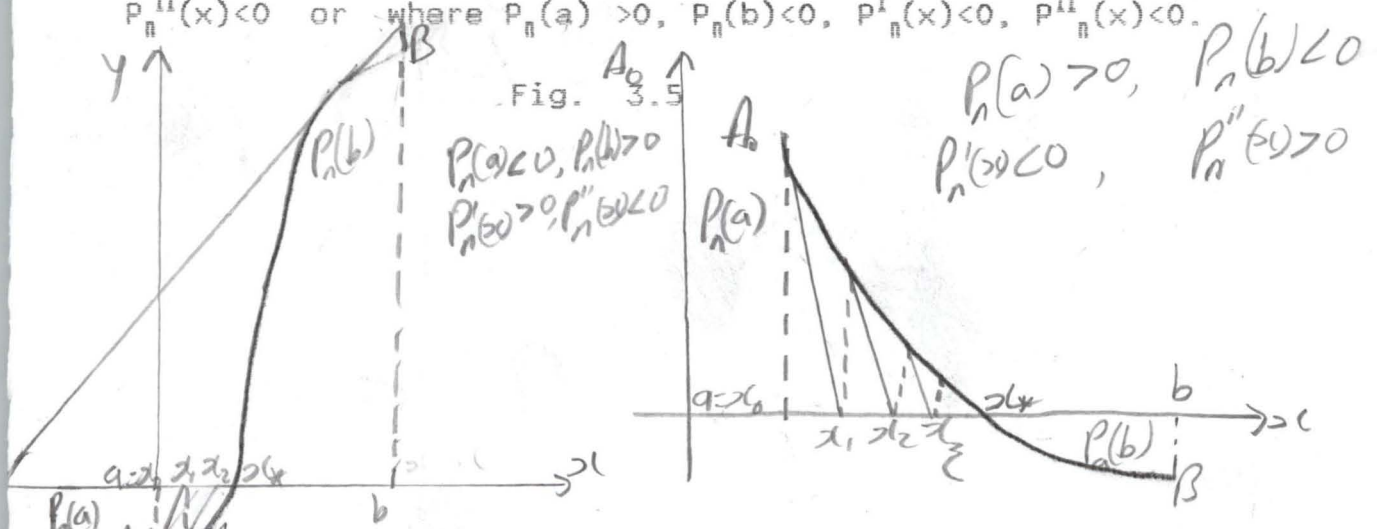
Hence, we have generally:

$$x_{k+1} = x_k - \frac{P_n(x_k)}{P_n'(x_k)} \quad \dots \dots \dots (3.10)$$

From fig. 3.4, we can see that all the values of  $x_k$  are approximate values of the exact root  $x_k$ , by excess.

Case 2: In this case,  $P_n'(x) > 0$ ,  $P_n''(x) < 0$ .

We are talking of cases where  $P_n(a) < 0$ ,  $P_n(b) > 0$ ,  $P_n'(x) > 0$ ,  $P_n''(x) < 0$  or where  $P_n(a) > 0$ ,  $P_n(b) < 0$ ,  $P_n'(x) < 0$ ,  $P_n''(x) < 0$ .



In fig. 3.5 we can see that if we draw a tangent to the curve  $P_n(x)=y$  at the point B it will cut the x-axis at the point which does not belong inside the interval  $[a, b]$ . As a result of this, we draw a tangent at  $A_0 (a, P_n(b))$ .

The equation for this tangent is:

$$y - P_n(a) = P_n^1(a) (x-a) \text{ ----- (3.11)}$$

At the point  $x = x_1$ , we see from fig. 3.5 that  $y = 0$  at the first tangent. Hence equation (3.11) becomes:

$$\begin{aligned} - P_n(a) &= P_n^1(a) (x_1 - a) \\ \therefore x_1 &= a - \frac{P_n(a)}{P_n^1(a)} \text{ ..... (3.12)} \end{aligned}$$

The root of the equation is now in the interval  $[x_1, b]$ . Continuing this process of replacing  $x_1$ ,  $r = 1, 2, \dots, k-1$  in each of the tangents to the curve (fig. 3.5), where  $x_1$  in each case have a value of  $y = 0$ , we obtain the general equation

$$x_{k+1} = x_k - \frac{P_n(x_k)}{P_n^1(x_k)} \text{ ..... (3.13)}$$

In this case, the successive approximate values  $x_k$  obtained are less than the exact value  $x_t$ . That is the approximate value  $\xi$  approximates  $x_t$  by defect.

Like in the method of chords, the formula used depends on the stationary point. In this method, the stationary point in the interval  $[a, b]$  is the endpoint at which the sign of the function does not coincide with the sign of the second derivative. That is to say that the endpoint whose sign coincides with the sign of the second derivative is the varying endpoint. For example, if  $P_n(b) \cdot P_n^1(x) > 0$ , then  $b$  is the varying endpoint and the initial point is  $b_0$ .

Alternatively, if  $P_n(a) \cdot P_n^l(x) > 0$ , then the initial point is  $a_0$ .

Error of approximation:- Let us assume that

$$\xi = x_{k+1}. \text{ Then we have that:}$$

$$|x_k - \xi| \leq |x_{k+1} - x_k|$$

but in (3.13) we have

$$|x_{k+1} - x_k| = \frac{P_n(x_k)}{P_n^l(x_k)}$$

Hence,

$$|x_{k+1} - x_k| = \left| \frac{P_n(x_k)}{P_n^l(x_k)} \right|$$

$$\text{Therefore } |x_k - \xi| \leq \frac{P_n(x_k)}{P_n^l(x_k)}$$

Generally, we use the formula

$$|x_k - \xi| \leq \left| \frac{P_n(x_k)}{m} \right| \dots \dots \dots (3.14)$$

Where  $m = \text{Min } |P_n^l(x)|$   
 $[a, b]$

The formula in (3.14) can be used in the chords method. Also like in the chords method;  $M \leq 2m$  must be fulfilled, in the interval  $[a, b]$ .

Finally, if the derivative  $P_n^l(x)$  varies slightly on the interval  $[a, b]$  or if it is computer in a complicated way, we may use the formula in 3.15 to simplify calculations.



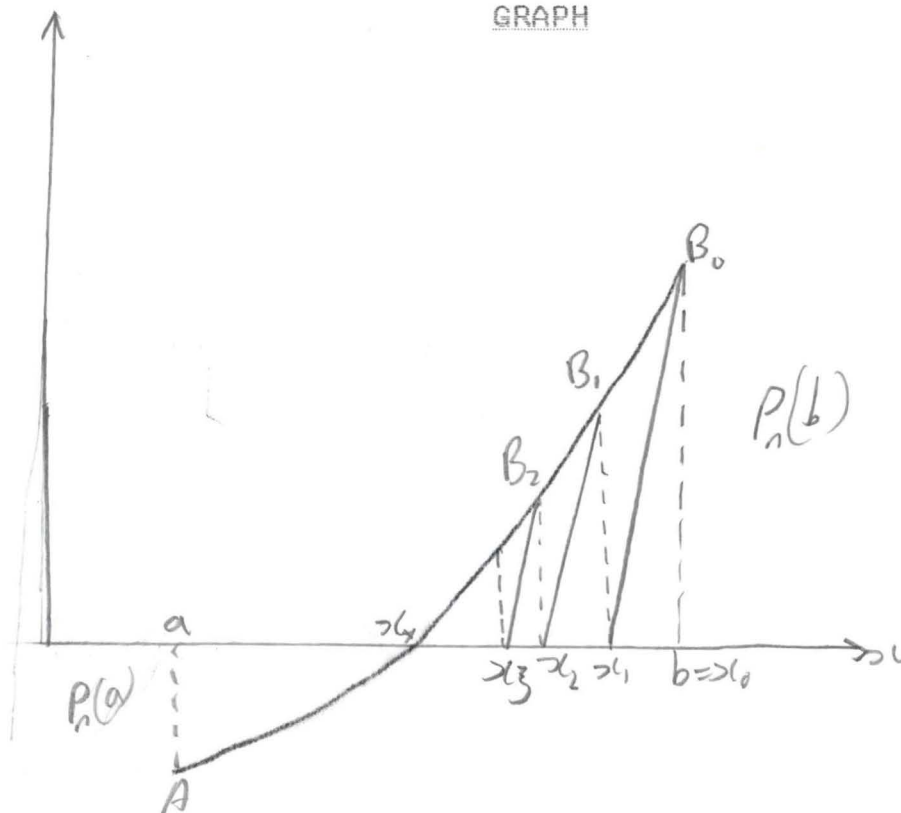
$$x_{k+1} = x_k - \frac{P_n'(x_k)}{P_n'(x_0)} \dots \dots \dots (3.15)$$

$$k = 1, 2$$

Formula 3.15 is called the Newton's simplified formula. That is, it is sufficient to calculate the value of the derivative at the origin only once. Geometrically, this means that the tangent at the point  $B_0(x_0, P_n(x_0))$

we replaced by straight lines which are parallel to the tangent drawn to the curve  $y = P_n(x)$  at the point  $B_0(x_0, P_n(x))$ .

GRAPH



### Example 3.3

We shall again solve equation 3.2 using Newton's method. Already, we have seen in example 3.2 that the root of the equation lies in the interval

$$[-2.75, -2.5]$$

$$P_n(-2.75) < 0$$

$$P''_{\epsilon_n}(x) = 6x + 6 < 0 \quad \forall x \in [-2.75 - 2.5]$$

$$\text{Hence } P_n(-2.75), P''_n(x) > 0$$

So -2.75 is the vary point and -2.5 is the stationary point.

$$P''_n(-2.75) = 6.188.$$

For the sake of convenience and easy computation, we shall employ Newton's simplified formula in equation 3.15.

Table 3.5 below gives the values of  $x_k$ .

Table 3.5

K	$x_k$	$x_k^3$ (1)	$3x_k^2$ (2)	$P_n(k)$ $\frac{(1)+(2)-6.188}{3}$	$P(k)$ 6.188
0	-2.75	-20.797	22.688	-1.109	-0.179
1	-2.71	-16.994	19.830	-0.160	-0.027
2	-2.514	-16.465	19.416	-0.049	-0.008
3.	-2.536	-16.310	19.294	-0.016	-0.003
4	-2.533	-16.252	19.248	-0.004	-0.006
5.	-2.5324				

The approximate value  $\xi$  of  $x_5$  is -2.532 approximating to three decimal place. We see from table 3.5 that  $|x_5 - x_4| = |-2.5524 + 2.533| = 0.0006 < 0.001$ .

### 3.4 Method of Successive Approximation

This method is simply referred to as the iterative methods in some books. It is one of the most important methods for solving algebraic equations. The operations carried out at each stage are of the same kind and so computation is made easy.

Let  $P_n(x)$  be a function which is continuous on an interval  $[a, b]$  and there exists a point  $x_f \in [a, b]$  at which  $P_n(x)$  vanishes. That is,  $P_n(x_f) = 0$ . In order to apply the method of successive approximation to the problem of finding the approximate value  $\xi$  of the exact solution,  $x_f$ , we must first replace the equation  $P_n(x)=0$  with an equivalent equation  $\varphi(x) = x$  ----- (3.16)

The equation in (3.16) must have the same root with  $P_n(x)=0$ . The root of the equation  $P_n(x)=0$  is  $x_f$ .  $P_n(x_f) = 0$ . Hence the root of equation 3.16 is also  $x_f$

$$\varphi(x_f) = x_f \text{ ----- (3.17)}$$

There are numerous ways of arriving at the equation in (3.16). However, when we finally do arrive at it, the next thing to do is to choose the initial approximation  $x_0 \in [a, b]$ . We substitute  $x_0$  on the left hand side of equation (3.16) to obtain  $x_1$

$$\varphi(x_0) = x_1 \text{ ----- (3.18)}$$

Continuing this process of substitution, we obtain  $x_2, x_3, \dots$ . Generally, equation (3.18) is of the form:

$$\varphi(x_k) = x_{k+1} \text{-----} \quad (3.19)$$

If there exist a limit  $\lim x_n$ , then the limit is the exact solution  $x_*$  of equation (3.16)

$$\lim x_k = x_* \text{.....} \quad (3.20)$$

where  $n$  is sufficiently large and (3.19) is convergent.

We have already mentioned that there are numerous ways of arriving at the equivalent equation  $\varphi(x)$ . It is important to note that the equation  $\varphi(x)=x$  which we choose to replace  $P_n(x)=0$  must be the appropriate one. It must be convergent. Before we discuss the general technique for constructing the function  $\varphi(x)$ , let us introduce a theorem which defines the conditions for convergence of the function  $\varphi(x)$ .

Theorem 3.1: Let us assume that the following conditions are fulfilled:

- (1) The function  $\varphi(x)$ . is defined and differentiable on the interval  $[a, b]$ .
- (2) All the values of  $\varphi(x)$ .  $\in [a, b]$  for  $x \in [a, b]$ .
- (3) There is a number  $\alpha < 1$  such that

$$|\varphi'(x)| \leq \alpha < 1 \quad \forall x \in [a, b]$$

(in theorem 1.2 we have already seen that

$$= \text{Max } |\varphi'(x)| \\ [a, b]$$

Then, the equation (3.19) converges irrespective of the choice of the initial approximation  $x_0 \in [a, b]$  and the  $\lim x_k = x_*$  is the unique and simple root the

equation (3.19) on the interval  $[a, b]$ .

In order to generate the function  $\varphi(x)$ , we will consider first the case where  $P_n^1(x) > 0$ . Let  $P_n(x)$  have a unique solution  $x_f \in [a, b]$  and let  $P_n^1(x)$  have a unique solution  $x_f \in [a, b]$  and let  $P_n^1(x)$  exist  $\forall x \in [a, b]$  and retain sign so that  $m_1 \leq P_n^1(x) \leq$

$M_1$  where  $m_1 = \text{Min}/P_n^1(x)$  and  $M_1 = \text{Max}/P_n(x)/$ . We will replace equation  $[a, b]$

$$P_n(x) = 0$$

with an equivalent equation  $x = x - \lambda P_n(x) \dots \dots \dots (3.21)$ .

In order to fulfil condition (3) of theorem 3.1 above, we must choose the right value for the constant  $\lambda$ .

$$\varphi(x) = x - \lambda P_n(x) \dots \dots \dots (3.22)$$

$$\varphi^1(x) = x - \lambda P_n^1(x) \dots \dots \dots (3.23)$$

In condition (3) of theorem 3.1, we have that

$$\varphi^1(x) < 1. \text{ Hence } |1 - \lambda P_n^1(x)| < 1 \dots \dots \dots (3.24)$$

Solving inequality (3.24), we have:

$$-1 < 1 - \lambda P_n^1(x) < 1 \dots \dots \dots (3.25)$$

From the right hand side of 3.25, we have

$$1 - \lambda P_n^1(x) < 1$$

$$0 < \lambda P_n^1(x)$$

$$0 < \lambda \text{ or } \lambda > 0$$

From the left hand side of 3.25 we have

$$-1 < 1 - \lambda P_n^1(x)$$

$$\lambda P_n^1(x) < 2$$

$$\lambda < 2/P_n^1(x)$$

Hence  $0 < \lambda < 2/P_n^1(x)$ .

Usually, we assume  $1/M_1$  to be  $\lambda$ . Where  $M_1 = \max_{[a, b]} |P_n^1(x)|$

Hence, equation (3.22) becomes:

$$\varphi(x) = x - \frac{P_n(x)}{M_1} \quad (3.26)$$

However, if  $P_n^1(x) < 0$ , then equation 3.24 becomes

$$|1 + \lambda P_n^1(x)| < 1 \quad (3.27)$$

Inequality (3.26) becomes

$$-1 < 1 + \lambda P_n^1(x) < 1 \quad (3.28)$$

The right hand side of (3.28) gives:

$$1 + \lambda P_n^1(x) < 1$$

$$\lambda P_n^1(x) < 0$$

$$\lambda < 0$$

The left-hand side of (3.28) gives:

$$-1 < 1 + \lambda P_n^1(x)$$

$$-2 < \lambda P_n^1(x)$$

$$-2/P_n^1(x) < \lambda$$

Hence we have

$$-2/P_n^1(x) < \lambda < 0$$

Let us assume  $\frac{1}{M_1} = \lambda$ . Then equation 3.26 becomes

$$\varphi(x) = \frac{P_n(x)}{x + \frac{1}{M_1}} \quad (3.29)$$

Error of Estimation: Let us start by considering the difference between the exact  $x_s$  and the approximate value  $\xi = x_k$

$$|x_s - x_k| = |\varphi(x_s) - \varphi(x_{k-1})| \quad (3.30)$$

where  $\varphi(x_s) = x_s$  and  $\varphi(x_{k-1}) = x_k$  in equations 3.17 and 3.19 respectively.

$$|\varphi(x_s) - \varphi(x_{k-1})| = |x_s - x_k| \leq \alpha |x_s - x_{k-1}| \\ = \alpha |x_s - x_k + x_k - x_{k-1}| \leq \alpha |x_s - x_k| + \alpha |x_k - x_{k-1}| \quad (3.31)$$

From equation 3.31 we have

$$|x_s - x_k| \leq \alpha |x_k - x_s| + \alpha |x_k - x_{k-1}| \\ |x_s - x_k| - \alpha |x_s - x_k| \leq \alpha |x_k - x_{k-1}| \\ |x_s - x_k| (1 - \alpha) \leq \alpha |x_k - x_{k-1}| \\ \therefore |x_s - x_k| \leq \frac{\alpha}{1 - \alpha} |x_k - x_{k-1}| \quad (3.32)$$

Also

$$|x_s - x_k| \leq \frac{\alpha}{1 - \alpha} |x_k - x_{k-1}| \leq \frac{\alpha^k}{1 - \alpha} |x_1 - x_0| \quad (3.33)$$

Hence for  $|x_s - x_k|$  not to be longer than  $\epsilon$  it is sufficient that

$$\frac{\alpha^k}{1 - \alpha} |x_1 - x_0| \leq \epsilon \quad (3.34)$$

Now, from (3.34) :-

$$\alpha^k \leq \frac{\epsilon (1 - \alpha)}{|x_1 - x_0|}$$

Taking the log of both sides, we have

$$K \log \alpha \leq \text{Log} \frac{\epsilon (1-\alpha)}{|x_1 - x_0|}$$

$\alpha < 1$ , hence  $\log \alpha < 0$

$$\therefore K(\epsilon) \geq \frac{\text{Log} \frac{\epsilon (1-\alpha)}{|x_1 - x_0|}}{\text{Log} \alpha} \dots \dots \dots (3.35)$$

Relation (3.35) makes if possible, after the first iteration, to find the maximum number of Iterations, K, necessary to calculate the approximate root with the special accuracy  $\epsilon$ .

Where  $x_k = \xi$ .

If  $\alpha \leq \frac{1}{2}$ , then equation (3.32) becomes

$$|x_2 - x_k| \leq \frac{\frac{1}{2}}{1 - \frac{1}{2}} |x_k - x_{k-1}|$$

$$= |x_2 - x_k| \leq |x_k - x_{k-1}| \dots \dots \dots (3.36)$$

That is if  $\alpha \leq 0.5$ , equation (3.36) may be used to estimate error where

$$\alpha = \max_{[a, b]} \phi^1(x)$$

and from (3.23)  $\phi^1(x) = 1 - \lambda P_n^1(x)$ . Therefore  $\alpha = 1 - \lambda P_n^1(x)$ . We assume

$$\lambda = M_1 \text{ and } |P_n^1(x)| = m_1.$$

Hence,

$$\alpha = 1 - \frac{m_1}{M_1} \dots \dots \dots (3.37)$$

Where  $M_1 = \text{Min}_{[a, b]} |P_n^1(x)|$   $M_1 = \text{Max}_{[a, b]} |P_n^1(x)|$

Example 3.4

We shall again solve equation (3.2) using the method of successive approximation.. In equation (3.2) the equation

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$P_3(x) = x^3 + 3x^2 - 3x = 0$  has a root lying on the interval  $[-2.75, -2.5]$  with an accuracy of 0.001.

$$P_3^1(x) = 3x^2 + 6x$$

$$M_1 = \max_{[-2.75, -2.5]} |P_3^1(x)| = |P_3^1(-2.75)| = 6.189$$

$$m_1 = \min |P_3^1(x)| = |P_3^1(-2.5)| = 3.75$$

$$(M_1 \leq 2m_1)$$

$$\alpha = 1 - \frac{m_1}{M_1} = 1 - \frac{3.75}{6.189} = 0.39 < 0.5$$

Hence, we may make use of relation (3.36) to estimate the error of approximation. In other words for  $|x_k - x|$  not to be larger than  $\epsilon$ , it is necessary and sufficient that  $|x_k - x_{k-1}| < \epsilon$

$$\text{Now, } P_3^1(x) > 0 \quad \forall x \in [-2.75, -2.5]$$

Hence equation 3.26 gives the equivalent equation.

$$\varphi(x) = x - \frac{P_3(x)}{M_1}$$

$$\varphi(x) = x - \frac{x_k^3 + 3x_k^2 - 3}{6} \quad \text{Where } M_1 \approx 6$$

Hence according to (3.19)

$$x_{k+1} = x_k - \frac{x_k^3 + 3x_k^2 - 3}{6} \quad \dots \dots \dots (3.38)$$

Equation (3.38) gives the values of  $x_k$  for  $k = 1, 2, 3 \dots$ .  
Where  $x_0 = -2.75$ .

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Table (3.6) below gives the computations.

Table 3.6

K	$x_k$	$x_k^3$	$3x_k^2$	$-\frac{(x_k^3)+3x_k^2}{3}$
0	-2.75	-20.797	22.688	0.185
1	-2.565	-16.876	19.738	0.023
2	-2.542	-16.426	19.385	0.007
3	-2.533	-16.252	19.249	0.0005
4	-2.5325			

From table 3.6, we have that  $x_5$

$$= -2.5325 \text{ and } x_4 = 2.533. \quad |x_5 - x_4|$$

$$= |-2.5325 + 2.533| = 0.0005 < \epsilon = 0/001$$

Hence  $\xi = -2.533$  approximating to three decimal places.

$$|x_5 - \xi| < \epsilon.$$

LB

# CHAPTER FOUR

## Development of an iterative Method

### 4.1 Introduction

In section 3.4, we did mention that the equivalent equation ( $\phi(x) = x$ ) given in equation 3.16 can be derived in numerous ways. Based on this statement of fact, we shall derive an iterative method which must, of course, satisfy all the conditions given in theorem 3.1. Later in this chapter, we shall apply the iterative method to equation 3.2 and other non-linear algebraic equations to see how it works.

In a branch of Mathematics called functional analysis, a strictly convex functional,  $g$ , is defined as follows:

Definition 4.1:- Let  $K$  be a convex subset of a normed space,  $X$ . Then a functional  $g$  defined on  $K$  is called a strictly convex functional if

$$g(ax+(1-a)y) \leq ag(x)+(1-a)g(y) \dots \dots \dots (4.1)$$

$$x, y, \in K.$$

Now, let  $g(x) = x$ . If  $x = x_{n+1}$  and

$y = x_n$  Then from equation (3.19)

$$g(x_n) = x_{n+1}$$

If the exact solution  $x^* = x_{n+1}$

$$\text{then, } g(x_{n+1}) = x_{n+1}$$

$$\text{Now, } y = x_n$$

$$\text{Then } g(y) = x$$

$$g(x) = x$$

Therefore,  $g(y) = g(x)$ . Substituting these in equation (1.4), we have

$$g(x) = ax + (1-a)g(x) \dots \dots \dots (4.2)$$

#### 4.2 Development of the Method

We shall re-write equation 4.2 as thus

$$g(x) = x\lambda + (1-\lambda)g(x) \dots \dots \dots (4.3)$$

where  $\lambda$  is a constant, to be determined.

Now,  $g(x)$  and  $f(x) = 0$  where

$P_n(x)$  is the initial algebraic equation

$$\dots \quad x - g(x) = 0$$

$$\dots \quad x - g(x) = P_n(x)$$

$$\dots \quad x - P(x) = g(x) \dots \dots \dots (4.4)$$

Substituting equation (4.4) into (4.3) we have:-

$$x = x\lambda + (1-\lambda)(x - P_n(x))$$

$$x = x\lambda + (x - P_n(x)) - x\lambda + \lambda P_n(x)$$

$$x = x\lambda + x - P_n(x) - x\lambda + \lambda P_n(x)$$

$$x = x - P_n(x) + \lambda P_n(x) \dots \dots \dots (4.5)$$

Equation (4.5) is the equivalent equation we need. As we have already mentioned, this equation must satisfy all the conditions given in theorem 3.1.

$$g^1(x) < 1 \quad (\text{Condition 3, theorem 3.1})$$

$$g^1(x) = 1 - P_n^1(x) + \lambda P_n^1(x) \dots \dots \dots (4.6)$$

$$- 1 < 1 - P_n^1(x) + \lambda P_n^1(x) < 1 \quad \text{is a necessary condition}$$

for convergency

of equation (4.5)

$$- 1 < 1 - P_n^1(x) + \lambda P_n^1(x) < 1 \dots \dots \dots (4.7)$$

$$- 1 < 1 - P_n^1(x) + \lambda P_n^1(x) \quad \quad \quad 1 - P_n^1(x) + \lambda P_n^1(x) < 1$$

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$$- \frac{2 + P_n^1(x)}{P_n^1(x)} < \lambda < \frac{P_n^1(x)}{P_n^1(x)} = 1$$

$$\therefore \lambda < 1$$

Hence, (4.8) 
$$\frac{P_n^1(x) - 2}{P_n^1(x)} < \lambda < 1$$
 .....

The inequality in 4.8 gives the interval within which  $\lambda$  exists. Let us

assume then that 
$$\lambda = \frac{P_n^1(x) - 1}{P_n^1(x)} \dots\dots\dots (4.9)$$

Since 
$$\frac{P_n^1(x) - 2}{P_n^1(x)} < \frac{P_n^1(x) - 1}{P_n^1(x)} < 1$$

Where 
$$P_n^1(x) = \text{Max}_{[a, b]} \{ P_n^1(x) \} = M$$

Hence, equation (4.5) becomes

$$g(x) = x - P_n(x) + \frac{M-1}{M} P_n(x) \dots\dots\dots (4.10)$$

However, if  $P_n^1(x) < 0 \quad \forall x \in [a, b]$

then equation (4.6) becomes

$$g^1(x) = 1 + P_n^1(x) - \lambda P_n^1(x) \dots\dots\dots (4.11)$$

Then,

$$-1 < 1 + P_n^1(x) - \lambda P_n^1(x) < 1$$

i.e 
$$\lambda < \frac{2 + P_n^1(x)}{P_n^1(x)} \quad \text{and} \quad \frac{P_n^1(x)}{P_n^1(x)} = K \lambda$$

i.e 
$$1 < \lambda < \frac{2 + P_n^1(x)}{P_n^1(x)} \dots\dots\dots (4.12)$$

$$\text{Let } \lambda = \frac{1+P_n^1(x)}{P_n(x)}$$

$$\text{where } P_n^1(x) = M = \text{Max}_{[a, b]} \{ P_n^1(x) \}$$

Equation (4.5) becomes

$$g(x) = x - P_n(x) + \frac{1+M}{M} P_n(x) \dots\dots\dots (4.13)$$

So, given an interval  $[a, b]$  which contains one solution of equation  $P_n(X) = 0$  and such that the condition  $M < 2m$  is satisfied

$$\text{where } M = \text{Max}_{[a, b]} \{ P_n^1(x) \} \quad \text{and} \quad m = \text{Min}_{[a, b]} \{ P_n^1(x) \}$$

then equation 4.10 and 4.13 can be used as substitutes or equivalent equations for  $P_n(X) = 0$  in order to obtain the solution in  $[a, b]$ .

$$\text{i.e } g(x_{n-1}) = x_{n-1} - P_n(x_{n-1}) + \frac{M-1}{M} P_n(x_{n-1}) \dots\dots\dots (4.14)$$

where  $\forall x, \in [a, b], g(x) \in [a, b], g'(x) < 1,$  and  $P_n^1(x) > 0$

$$g(x_{n-1}) + x_{n-1} - P_n(x_{n-1}) + \frac{1+M}{M} P_n(x_{n-1}) \dots\dots\dots (4.15)$$

where  $\forall x \in [a, b], g(x) \in [a, b], g'(x) < 1$  and  $P_n^1(x_{n-1}) < 0.$

### 4.3 Applications

Let us now apply equations 4.14 and 4.15 to three different algebraic equations. In each equation, we shall obtain at least two solutions of the equations. We shall be considering third, fourth and fifth order algebraic equations.

### Application 1      Third order Equation

The third order equation we shall consider is equation 3.2. Already this equation has been separated in tables 3.1 and 3.2. The roots of the equation lie in the intervals  $[-3, -2]$ ,  $[-2, -1]$  and  $[0, 1]$ , respectively.

In chapter 3, the root of the equation lying in the interval  $[-3, -2]$  was obtained using the four iterative methods discussed. In this section, however, we shall obtain the root lying in this interval as well as the largest root lying in the interval  $[0, 1]$ , using our iterative method.

$$P_3(x) = x^3 + 3x^2 - 3 = 0$$
$$x \in [-3, -2] \quad \text{and} \quad \epsilon = 0.001$$
$$P_3^1(x) = 3x^2 + 6x$$
$$P_3^1(-3) = 9 \qquad P_3^1(-2) = 0$$
$$\text{Hence } M > 2m$$

So we reduce the interval finally to  $(-2.75, -2.5)$  (see section 3.1)

$$P_3^1(-2.75) = 6.19 \qquad P_3^1(-2.5) = 3.75$$
$$M < 2m$$

where  $M = 6.19$  and  $m = 3.75$ .

So we have

$$f(x) = x^3 + 3x^2 - 3$$
$$x \in [-2.75, -2.5]$$

Since  $P_3^1(x) > 0 \quad \forall x \in [-2.75, -2.5]$   
then we shall employ equation 4.14

$$g(x_{n-1}) = x_{n-1} - \frac{P_3(x_{n-1})}{P_3'(x_{n-1})} + \frac{M-1}{M} P_3(x_{n-1})$$

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$$\frac{M-1}{M} = \frac{5.19}{6.19} = 0.8384$$

Hence, equation 4.14 becomes

$$g(x_{n-1}) = x_{n-1} - P_3(x_{n-1}) + 0.8384 P_3(x_{n-1}) \dots\dots\dots (4.16)$$

$$g^1(x) = 1 - P_3^1(x_n) + 0.8384 P_3^1(x_n)$$

$$g^1(-2.75) = 1 - 6.19 + 5.19 = 0 < 1$$

$$g^1(-2.5) = 1 - 3.75 + 3.15 = 0.39 < 1$$

$$(1) \quad P_3(-2.75) = -1.1094$$

$$\begin{aligned} g(-2.75) &= -2.75 + 1.1094 - (0.8384)(1.1094) \\ &= -2.5707 \end{aligned}$$

$$(2) \quad P_3(-2.5707) = -0.1630$$

$$\begin{aligned} g(-2.5707) &= -2.5707 + 0.1630 - 0.1367 \\ &= -2.5444 \end{aligned}$$

$$(3) \quad P_3(-2.5444) = -0.0505$$

$$\begin{aligned} g(-2.5444) &= -2.5444 + 0.0505 - 0.0423 \\ &= -2.5362 \end{aligned}$$

$$(4) \quad P_3(-2.5362) = -0.0167$$

$$\begin{aligned} g(-2.5362) &= -2.5362 + 0.0167 - 0.0140 \\ &= -2.5335 \end{aligned}$$

$$(5) \quad P_3(-2.5335) = -0.0057$$

$$\begin{aligned} g(-2.5335) &= -2.5335 + 0.0057 - 0.0048 \\ &= -2.5326 \end{aligned}$$

$$(6) \quad P_3(-2.5326) = 0.0021$$

$$\begin{aligned} g(-2.5326) &= -2.5326 + 0.0021 - 0.0018 \\ &= -2.5323 \end{aligned}$$

$$(7) \quad P_3(-2.5323) = 0.0009$$

$$g(-2.5323) = -2.5323 + 0.0009 - 0.0008$$



$$= -2.5322$$

$$(8) \quad P_3(-2.5322) = -0.00046$$

$$g(-2.5322) = -2.5322 + 0.00046 - 0.00039 \\ = -2.53213$$

$$|-2.53213 + 2.5322| = 0.00007 < 0.001$$

$$\text{Hence, } \xi = -2.532$$

$$(8) \quad P_3(x) = x^3 + x^2 - 3$$

$$x \in [0, 1] \quad \epsilon = 0.001$$

$$P_3^1(x) = 3x^2 + 6x$$

$$P_3^1(1) = 9 \quad P_3^1(0) = 0$$

$$M = 9 \quad m = 0$$

$M > 2m$  So, we need to reduce the interval  $[0, 1]$ .

$$\text{Now, } P_3(0.5) = (-2.125). \text{ Also } P_3(0) = -3$$

So, 0.5 replaces 0 in the interval  $[0, 1]$

We have  $[0.5, 1]$

$$P_3^1(0.5) = 3.75 = M \quad P_3^1(1) = 9$$

$$M = 9, \quad m = 3.75$$

$$M > 2m$$

$$\frac{1+0.5}{2} = \frac{1.5}{2} = 0.75$$

$$P_3^1(0.75) = -0.8906, \text{ so } 0.75$$

replaces 0.5 in the interval  $[0.5, 1]$

We have,  $[0.75, 1]$

$$P_3^1(0.75) = 6.1875 \quad P_3^1(1) = 9$$

$$m = 6.1875 \quad M = 9$$

$$M < 2m.$$

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So the interval we need is  $[0.75, 1]$

Hence, we have

$$P_3(x) = x^3 + 3x^2 - 3$$

$$x \in [0.75, 1] \quad \epsilon = 0.001$$

Since  $P_3'(x) > 0 \quad \forall x \in [0.75, 1]$ ,

we shall make use of equation (4.14)

$$g(x_n) = x_{n-1} - \frac{P_3(x_{n-1}) + M-1 P_3(x_{n-1})}{M}$$

$$\text{where} \quad \frac{M-1}{M} = \frac{8}{9} = 0.89$$

so,

$$g(x_{n-1}) = x_{n-1} - P_3(x_{n-1}) + 0.89 P_3(x_{n-1})$$

$$g^1(x) = 1 - P_3(x) + 0.89 P_3(x)$$

$$g^1(0.75) = 1 - 6.1875 + 5.5069$$

$$= 0.3194 < 1$$

$$g^1(1) = 1 - 9 + 8.01 = 0.01 < 1$$

$$(1) \quad P_3(0.75) = -0.8906$$

$$g(0.75) = 0.75 + 0.8906 - 0.7926$$

$$+ 0.848$$

$$(2) \quad P_3(0.848) = -0.2329$$

$$g(0.848) = 0.848 + 0.2328 - 0.2073$$

$$= 0.8736$$

$$(3) \quad P_3(0.8736) = -0.0433$$

$$g(0.8736) = 0.8736 + 0.0433 - 0.0385$$

$$= 0.8784$$

$$(4) \quad P_3(0.8784) = -0.0074$$

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$$g(0.8784) = 0.8784 + 0.0074 - 0.0066$$

$$= 0.8792$$

$$(5) \quad P_3(0.8792) = -0.0014$$

$$g(0.8792) = 0.8792 + 0.0014 - 0.0012$$

$$= 0.8794$$

$$(6) \quad P_3(0.8794) = 0.00008$$

$$g(0.8794) = 0.8794 + 0.00008 - 0.00007$$

$$= 0.87941$$

$$\text{Now, } |0.8794| - 0.5794 | = 0.00001$$

$$0.1 < 0.0001$$

$$\text{Hence } \xi = 0.8794$$

#### Application 2    Fourth order equation

$$P_4(x) = x^4 + 5x - 3 \quad \epsilon = 0.001$$

Let us use the analytical method discussed in chapter two to separate the roots of the fourth order equation above.

Table 4.1

$-\infty$	$-3$	$-2$	$-1$	$0$	$1$	$2$	$3$	$+\infty$
$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$

Hence, from table 4.1 we can see that the roots lie in the intervals  $[-2, -1]$  and  $[0.1]$ . We shall obtain the two roots in these intervals

$$P_4(x) = x^4 + 5x - 3$$

$$x \in [-2, -1] \quad \epsilon = 0.01$$

$$P_4^1(x) = 4x^3 + 5$$

$$|P_4^1(-2)| = |-27| = 27 = M$$

$$|P_4^1(-1)| = |1| = 1 = m$$

$$M > 2m$$

So we reduce  $[-2, -1]$  finally to  $[-2, -1.75]$  by using the procedure we used in application 1 above.

$$\text{Where } |P_4^1(-2)| = |-27| = 27 = M$$

$$|P_4^1(-1.75)| = |-16.44| = 16.44 = m$$

$$\text{Hence } M < 2m$$

$$\text{Since } P_4^1(x) < 0 \quad \forall x \in [-2, -1.75]$$

We shall employ equation 4.15

$$g(x) = x - P_4(x) + \frac{1+M}{M} P_4(x)$$

$$\frac{1+M}{M} = \frac{28}{27} = 1.04$$

$$\therefore g(x_{n-1}) = x_{n-1} - P_4(x_{n-1}) + 1.04 P_4(x_{n-1})$$

$$g^1(x) = 1 - P_4^1(x) + 1.04 P_4^1(x)$$

$$g^1(-2) = 1 + 27 - 28.08 = 0.08 < 1$$

$$g^1(-1.75) = 1 + 16.44 - 17.1 = 0.34 < 1$$

$$(1) \quad P_4(-2) = 3$$

$$g(0.11) = -2 - 3 + 3.12 = 1.88$$

$$(2) \quad P_4(-1.88) = 0.092$$

$$g(0.11) = -1.88 - 0.092 + 0.0957$$

$$= -1.8763$$

$$(3) \quad P_4(-1.8763) = 0.0124$$

$$g(0.11) = -1.8763 - 0.0124 + 0.0129$$

$$= -1.8758$$

$$\text{Now, } |-1.8763 + 1.8758| = 0.0005 < 0.001$$

Hence,  $\xi = -1.876$

$$(B) \quad P_4(x) = x^4 + 5x - 3$$

$$x \in [0, 1] \quad \epsilon = 0.001$$

$$P_4^1(x) = 4x^3 + 5$$

$$P_4^1(0) = 5 \quad P_4^1(1) = 9$$

$$M = 9 \quad \text{and} \quad M = 5$$

$M < 2m$  Hence we may use the interval  $[0, 1]$ .

Now, since  $P_4^1(x) > 0 \quad \forall x \in [0, 1]$ .

We shall use equation 4.14

$$\text{i.e. } g(x_n) = x_{n-1} - P_4(x_{n-1}) + \frac{M-1}{M} P_4(x_{n-1})$$

$$M = \max_{[0,1]} |P_4^1(x)| = 9$$

$$\frac{M-1}{M} = \frac{8}{9} = 0.89$$

$$\therefore g(x_{n-1}) = x_{n-1} - P_4(x_{n-1}) + 0.89 P_4(x_{n-1})$$

$$g^1(x) = 1 - P_4^1(x) + 0.89 P_4^1(x)$$

$$g^1(0) = 1 - 5 + 4.45 = 0.45 < 1$$

$$g^1(1) = 1 - 9 + 8.01 = 0.01 < 1$$

$$(1) \quad P_4(0) = -3$$

$$g(0) = 0 + 3 - 2.67 = 0.33$$

$$(2) \quad P_4(0.33) = -1.3381$$

$$g(0.11) = 0.33 + 1.3381 - 1.1909 \\ = 0.4772$$

$$(3) \quad P_4(0.4772) = -0.5621$$

$$g(0.4772) = 0.4772 + 0.5621 - 0.5003$$

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$$\begin{aligned}
 &= 0.539 \\
 (4) \quad P_4(0.539) &= -0.2206 \\
 g(0.539) &= 0.539 + 0.2206 - 0.19630066 \\
 &= 0.5633 \\
 (5) \quad P_4(0.5633) &= -0.0828 \\
 g(0.5633) &= 0.5633 + 0.0828 - 0.0737 \\
 &= 0.5724 \\
 (6) \quad P_4(0.5724) &= -0.0307 \\
 g(0.5724) &= 0.5724 + 0.0307 - 0.00273 \\
 &= 0.5758 \\
 (7) \quad P_4(0.5758) &= -0.111 \\
 g(0.5758) &= 0.5758 + 0.0111 - 0.0099 \\
 &= 0.577 \\
 (8) \quad P_4(0.577) &= -0.0042 \\
 g(0.577) &= 0.577 + 0.0042 - 0.0037 \\
 &= 0.5775 \\
 |0.5775 - 0.577| &= 0.0005 < 0.001 \\
 \text{Hence } \xi &= 0.577
 \end{aligned}$$

Application 3

Fifth order equation

$$\begin{aligned}
 F(x) &= x^5 - 5x + 2 = 0 \\
 &= 0.001
 \end{aligned}$$

Let us separate the roots using the analytical method.

Table 4.2

$-\infty$	-2	-1	0	1	2	$+\infty$
-	-	+	+	-	+	+

The roots of the equation lie in the intervals  $[-2, -1]$ ,  $[0, 1]$  and  $[1, 2]$

respectively. Let us find in smallest root in the interval  $[-2, -1]$  and the largest root in the interval  $[1, 2]$ .

$$(A) \quad P_5(x) = x^5 - 5x + 2 = 0$$

$$[-2, -1] \quad \epsilon = 0.001$$

$$P_5^1(x) = 5x^4 - 5$$

$$|P_5^1(-2)| = 75 \quad |P_5^1(-1)| = 0$$

$$M = 75 \quad m = 0$$

$$M < 2m$$

We reduce  $[-2, -1]$  to  $[-2, -1.75]$

$$\text{where } |P_5^1(-2)| = 75 \quad \text{and} \quad |P_5^1(-1.75)| = 41.89$$

$$M = 75 \quad m = 41.89$$

$$M < 2m$$

Hence, we have

$$P_5(x) = x^5 - 5x + 2 = 0$$

$$x \in [-2, -1.75] \quad \epsilon = 0.001$$

Since  $P_5^1(x) > 0 \quad \forall x \in [-2, -1.75]$  then we will make use of equation 4.14

$$\text{i.e.} \quad g(x_{n-1}) = x_{n-1} - \frac{P_5(x_{n-1})}{M} + \frac{M-1}{M} P_5(x_{n-1})$$

$$\frac{M-1}{M} = 0.99$$

$$\text{Then, } g(x_{n-1}) = x_{n-1} - P_5(x_{n-1}) + 0.99 P_5(x_{n-1})$$

$$g^1(x) = 1 - P_5^1(x) + 0.99 P_5^1(x)$$

$$g^1(-2) = 1 - 75 + 74.25 = 0.25 < 1$$

$$g'(-1.75) = 1 - 41.89 + 41.47 = 0.58 < 1$$

(1)  $P_5(-2) = -20$   
 $g(-2) = -2 + 20 - 19.8 = -1.8$

(2)  $P_5(-1.8) = -7.8957$   
 $g(-1.8) = -1.8 + 7.8957 - 7.8167$   
 $= -1.721$

(3)  $P_5(-1.721) = -4.4925$   
 $g(-1.721) = -1.721 + 4.4925 - 4.4476$   
 $= -1.6761$

(4)  $P_5(-1.6761) = -2.8477$   
 $g(-1.6761) = -1.6761 + 2.8477 - 2.8192$   
 $= -1.6476$

(5)  $P_5(-1.6476) = -1.9031$   
 $g(-1.6476) = -1.6476 + 1.9031 - 1.8841$   
 $= -1.6286$

(6)  $P_5(-1.6286) = -1.3140$   
 $g(-1.6286) = -1.6286 + 1.3140 - 1.3009$   
 $= -1.6155$

(7)  $P_5(-1.6155) = -0.9261$   
 $g(-1.6155) = -1.6155 + 0.9261 - 0.9168$   
 $= -1.6062$

(8)  $P_5(-1.6062) = -0.6595$   
 $g(-1.6062) = -1.6062 + 0.6595 - 0.6529$   
 $= -1.5996$

(9)  $P_5(-1.5996) = 0.4747$   
 $g(-1.5996) = -1.5996 + 0.4747 - 0.4700$   
 $= -1.5949$



$$(10) \quad P_5(-1.5949) = -0.3452$$

$$g(-1.5949) = -1.5949 + 0.3452 - 0.3417 \\ = -1.5914$$

$$(11) \quad P_5(-1.5914) = -0.2500$$

$$g(-1.5914) = -1.5914 + 0.25 - 0.2475 \\ = -1.5889$$

$$(12) \quad P_5(-1.5889) = -0.1825$$

$$g(-1.5889) = -1.5889 + 0.1825 - 0.1807 \\ = -1.5871$$

$$(13) \quad P_5(-1.5871) = -0.1343$$

$$g(-1.5871) = -1.5871 + 0.1343 + 0.1330 \\ = -1.5848$$

$$(14) \quad P_5(-1.5858) = -0.0996$$

$$g(-1.5858) = -1.5858 + 0.0996 - 0.0986 \\ = -1.5848$$

$$(15) \quad P_5(-1.5848) = -0.0731$$

$$g(-1.5848) = -1.5848 + 0.0731 - 0.0724 \\ = -1.5851$$

$$\text{Now, } |-1.585| + 1.5848 = 0.0003 < 0.001$$

$$\text{Hence, } \xi = 1.585.$$

$$(B) \quad P_5(X) = X^5 - 5X + 2 = 0$$

$$x \in [-2, -1] \quad \epsilon = 0.001$$

$$P_5^1(x) = 5x^4 - 5$$

$$|P_5^1(-1)| = 0 \quad |P_5^1(2)| = 75$$

$$m = 0$$

$$M = 75$$

$$M > 2m$$

We reduce the interval  $[1, 2]$  to  $[1.25, 1.375]$  using the process described

in section 3.1

$$\begin{aligned}
 P_5^1(1.25) &= 7.2070 & P_5^1(1.375) &= 12.8723 \\
 m &= 7.2070 & M &= 12.8723 \\
 M &< 2m
 \end{aligned}$$

So,  $P_5(x) = x^5 - 5x + 2 = 0$   
 $x \in [1.25, 1.375] \quad \epsilon = 0.001$

Since  $P_5^1(x) > 0 \quad \forall x \in [1.25, 1.375]$  then we will make use of equation 4.14

i.e  $g(x_{n-1}) = x_{n-1} - \frac{P_5(x_{n-1})}{M} + \frac{M-1}{M} P_5(x_{n-1})$

Where the constant  $\frac{M-1}{M} = \frac{11.8723}{12.8723} = 0.9223$

$\therefore g(x_{n-1}) = x_{n-1} - P_5(x_{n-1}) + 0.92 P_5(x_{n-1})$

$g^1(x) = 1 - P_5^1(x) + 0.92 P_5^1(x)$

$g^1(1.25) = 1 - 7.2070 + 6.6304 = 0.4234 < 1$

$g^1(1.375) = 1 - 12.8723 + 11.8425 = 0.0298 < 1$

(1)  $P_5(1.25) = -1.1982$

$g(1.25) = 1.25 + 1.1982 - 1.1023 = 1.3459$

(2)  $P_5(1.3459) = -0.3131$

$g(1.3459) = 1.3459 + 0.3131 - 0.2881 = 1.3709$

(3)  $P_5(1.3709) = -0.0125$

$$g(1.3709) = 1.3709 + 0.0125 - 0.0115$$

$$= 1.3719$$

$$(4) P_5(1.3719) = 0.00023$$

$$g(1.3719) = 1.3719 - 0.00023 + 0.0021$$

$$= -1.37188$$

$$\text{Now, } |1.37188 - 1.3719| = 0.00002$$

$$0.00002 < 0.00$$

$$\text{Hence, } \xi = 1.372$$

#### 4.4 Error of estimation:-

Since  $g(x_k) = x_k$  and  $g(x_{k-1}) = x_{k-1}$ , the iterative method, uses the same principles as in the estimation of error in the method of successive approximation.

The inequality relation in 3.36 is used to estimate the error where the constant  $\alpha \leq 0.5$  (see equation 3.30 to 3.36)

$$|x_k - x_{k-1}| \leq |x_{k-1} - x_{k-2}| < \epsilon$$

Where  $\alpha = \text{Max } |g'(x)|$ .

$$\text{i.e } |g'(x)| = 1 - P_n^1(x) + R P_n^1(x)$$

$$\text{Where } R = \frac{M-1}{M} \text{ in (4.14) and } R = \frac{M+1}{M} \text{ in (4.15)}$$

$$= 1 - M + RM \text{ where } M = \text{Max } |P_n^1(x)|$$

$$[a,b]$$

# CHAPTER FIVE

## COMPUTER APPLICATION AND CONCLUSION

### 5.1 Computer Application

The development of computers have been of immense benefit to the science world in general and Mathematics in particular. Time spent in manual works of all types is been saved and hence enhancing quality and quantity of production.

In chapters three and four, we observed that the iterative methods discussed involve time consuming and mistake-prone computations. For higher order equations like the fifth, sixth, etc, orders, we are required to carry out mere computations. This is where the computer application comes in. The computer is designed to execute these cumbersome computations in minutes and thereby saving time. Also, the computer is more accurate and enables the user to have time to solve other problems. In fact, the immense benefits of the computer cannot be over emphasised.

If the number of computations does not exceed a thousand, then the desk piano key computer can be used. If however the computations are more than a thousand then, one needs a high speed computer so as to facilitate the solution being found in the shortest possible time. However, in order to use the computer to carry out any mathematical computation, the processes in the computation will have to be written in high-level languages like Basic, Fortran, Pascal, etc. The set of instructions, written in any high-level language, which is

designed to solve a particular problem and obtain result is called a program.

After writing the program, in any desired language, it is then coded into the computer for execution. Coding involves the transcription of instructions to the form of language acceptable by the computer. This language is called the machine language and it exists in binary numbers. The source program written in high level language is translated into a machine language object program with the help of another program called a compiler.

In this chapter, we shall construct program in basic language to carry out the computations in the examples given in chapter four. The results of the program shall also be displayed. Let us now see the different stages of a program development before going to the programs in section 5.3.

## **5.2 Stages of Program Development**

Program Planning:- This is the first stage in a program development. At this stage, the aim of the program must be understood. That is the problem which the program would solve must be defined and the input data must be identified.

(b) Program design:- This is the listing and ordering of the successive steps and instructions in a high level language, required to make the computer achieve the desired results. This, of course is the most important stage of a

5

program development. The instructions, pseudocodes must be clearly and properly written.

• Coding:- Once the steps of the program has been outlined and ordered, the next stage is to transfer the program to the form understandable by the computer. As we have earlier seen in section 5.1, this language is called the machine language.

(d) Debugging: The coding process involves the help of the compiler, which translates the program from high-level language to machine language. Normally, the compiler may detect one or more errors (bugs). The detecting and removing of these errors is called debugging. There are two kinds of error: syntax and logic error. Syntax error involves incorrect punctuation, incorrect word sequence, undeclared identifiers or misuse of terms. These errors are also called coded errors and are easily found the language compiler points them out to the programmer. Logic error is the error that makes the computer to fail to print the expected result even when there is no more syntax error. Logic error is associated with the problem of the program logic and the way to remove it is by checking the planning and design of the program.

(e) Testing:- This refers to the process of running the program and evaluating the program result in order to determine if any error exists. The testing is done by running the program using or imputing values with known results as to be sure the expected result is got

(f) Implementation:- Once the program has been tested and found to be working, it may then be applied to the problem it

was designed for.

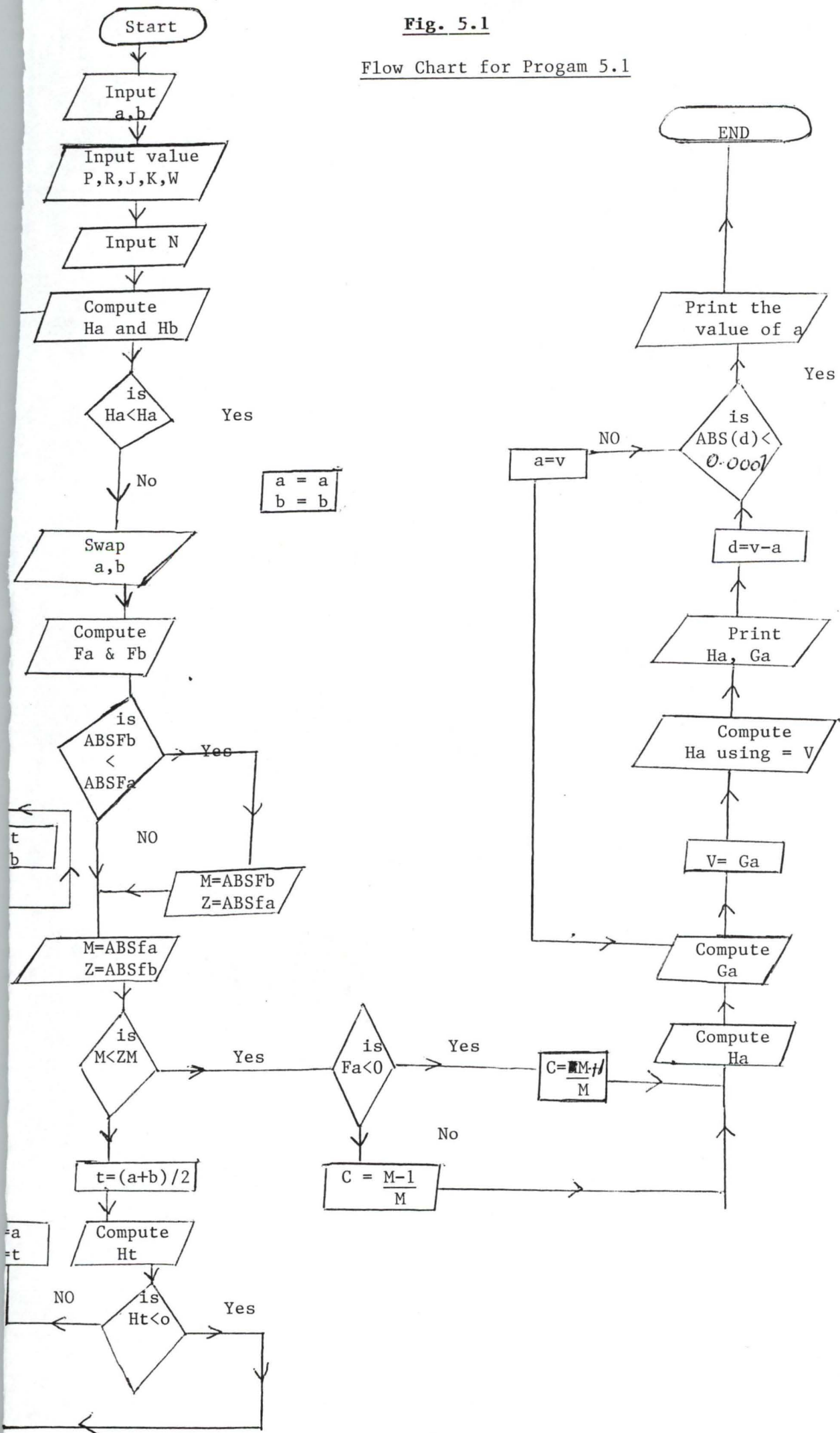
(g) Documentation:- This is the description of the proper form for users and to enhance maintainability. It describes the working of the program and how the expected problem could be solved. Documentation enables the users to understand the program better and also gives room for modifying the program whenever necessary.

### 5.3 PROGRAM

In order to see the plan of a program at a glance and to facilitate the construction of the program itself, it is advisable to draw up a flow chart of the program. A flow chart is a design which shows the objectives and processes of the program. Flow chart is made up of arrows, circles, boxes and rhombuses. All these have special and important functions in a flow chart. Before we go to the flow chart in fig. 5.1, it will be most helpful to look at the different parts of a flow chart and also their various functions.

Fig. 5.1

Flow Chart for Program 5.1





SYMBOLS	FUNCTION
	Page connector
	Start and end symbol
	Process Symbol
	Decision Symbol
	Input and output symbol

### PROGRAM 5.1

```

10  CLS
15  REM SOLUTION OF AN ALGEBRAIC EQUATION USING
16  REM THE DERIVED ITERATIVE METHOD IN CHAPTER 4
20  INPUT "ENTER THE LIMITS OF THE EQUATION as (a,b)"; a,b
22  INPUT "ENTER THE COEFFICIENTS P,R,J,K, AND THE CONSTANT
    W" P, R, J, K, W
23  "Input Enter The Index N"; N
28  Hb=(P*(b^N)) + R*(b^(N-1))+(J*(b^(N-2)))+(K*(b^(N-3)))+w
30  Ha=(P*(a^N))+(R*(a^(N-1)))+(J*(b^(N-2)))+(k*(b^(N-3)))+W
32  If (Ha<1+b) THEN 35
33  SWAP a,b
34  GOTO 40
35  a=a
36  b=b

```

by

```

40 Fa=(P*N*(a^(N-1)))+(R*(N-1)*(a^(N-2)))+(J*(N-2)*(a^(N-
3)))+(K*(N-3)*(a^(N-4)))
50 Fb=(P*N*(B^(n-1)))+(R*(N-1)+(b^(N-2)))+(J*(N-2)*(a^(N-
3)))+(K*(N-E)*(a^(N-4)))
60 IF (ABS(Fb) > ABS (Fa)) THE 100
70 M=ABS(Fa)
80 Z=ABS9Fb)
90 GOTO 120
100 M=ABS(Fb)
110 Z=ABS(Fa)
120 IF (M<(2*2)) then 210
130 t = (a+b)/2
135 Ht = (P*(t^N))+R*(t^(N-1))+J*(t^(N-2))+K*(t^(N-3))+W
140 IF(Ht<0)THEN 180
150 a=a
160 b+t
170 GOTO 30
180 a=t
190 b=b
200 GOTO 30
210 IF(Fa<0) THEN240
220 C=(M-1)/M
230 GOTO 250
240 C=(m+1)/M
245 Ha=P*(a^N)+(R*(a^(N-1))+t*(a^(N-2)))+(K*(a(N-3))+W
250 Ga=a-Ha+C* Ha
260 V=Ga

```

```

270 Ha=(P*(V^N))+(R*(V^(N-1)))+(J*(V^(N-2)))+(K*(V(N-3))+W
274 PRINT Ha, Ga
278 d = v-a
280 IF (absd) <.0001)then 310
285 a=v 310"PRINT THE APPROXIMATE VALUE a =", a
300 GOTO 250
320 END

```

The program in 5.1 is a general program for the algebraic equations in chapter 4. It was tested with the third order equation in application 1 of Chapter 4 and it ran giving the values in fig. 5.2. The program was ran in a Abasic environment. This environment would not work with small letter m in lines 80 and 110. Each time m was inputted in lines 80 and 110, the big letter M in lines 70 and 100 authomatically changes to m too. Hence m was replaced with Z in lines 80 and 110.

Lines 28 and 30 tive the values of the algebraic equation  $P_n(x)$  where  $x=b$  and  $x=a$  respectively. Also, lines 40 and 50 give the values of  $P_n(x)$  where  $a=b$  and  $a$  respectively. The program would run for 4th and 5th order equation if the number of terms in lines 28 and 30 are increase to suit the equation.

The program would also run using Turbo basic environemnt, however, Qbasic is recommended since it spaces out the inputs automatically and supplies explanations for each error messages ecountered during debugging process.

Fig 5.2

Values of program 5.1

Enter the limits of the equation as:- (a,b)? -3,-2

Enter the coefficients P, R, J, K, W ? 1,3,0,0,-3

Enter the index of the equation (N)? 3

(Ha)	(Ga)	
-0.1630028	-2.570707	
(Ha)	(Ga)	
-5.030598E-02		-2.544363
(Ha)	(Ga)	
-1.682884E-02		-2.536233
(Ha)	(Ga)	
-5.765715E-03		-2.533513
(Ha)	(Ga)	
-1.991515E-03		-2.532581
(Ha)	(Ga)	
-6.895912E-04		-2.532259
(Ha)	(Ga)	
-2.394437E-04		-2.532148
(Ha)	(Ga)	
-8.331636E-05		-2.532109

The approximate value is= -2.532148

Press any key to continue

## 5.4

## Conclusion

The methods discussed in chapters three and four converge, however the method of chords converges faster than the rest. The iterative method developed in chapter four converges slower than the bisection which is the slowest of all the methods.

However, the main advantage of the iterative method in chapter four is that the operations carried out at each stage are very easy and are of the same kind and hence this makes it considerably easier to set up programs for a computer.

A good knowledge of computer programming language is required if one must write programs for any of the iterative methods. Also, the programs must be tested to ensure that they run. This is a major problem. Making a program run takes time, patience and a good knowledge of the chosen programming language. Otherwise, one may write programs that will never run. In developing an iterative method like we did in chapter four, patience and persistence is also needed. In the iterative method of chapter 4, it was discovered that if the condition  $M > 2m$  is not fulfilled where  $M = \max |P_n^1(x)|$  and  $m = \min |P_n^1(x)|$ , the method will not be applicable to some intervals of an algebraic equation even when it is applicable to the other intervals of the same equations.

Finally, in the solution of algebraic equations, the method of chords is good for quick convergency however, the method of successive iteration is recommended since it is very easy to set up a program for the computer using this method

and besides it converges faster than the other methods except  
the method of chords.

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