NUMERICAL SOLUTION OF BOUNDARY-VALUE PROBLEMS

BY

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JULY 1992.

DECLARATION

I hereby declare that this thesis is my own work and has not been submitted in any form for a degree, diploma or certificate at any other institution. All information got from published or unpublished work has been acknowledged in the references.

Signature.

DATE: 12/10/92

CERTIFICATION

I certify that this work was carried out by IBRAHIM S. M. of the Department of Mathematics with Computer Science, Federal University of Technology, Minna, Nigeria.

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DEDICATION

This thesis is dedicated to my parents Alhaji Ibrahim Sayiti, and Alhaja Rafiat Ibrahim with much respect for their love and endless patience.

ACKNOWLEDGEMENTS

The philosophy of life that says success comes through the dent of hardwork is incontestable and indisputable but it can not be achieved by working in isolation. Bearing this in mind, I am indebted to various people that contributed to make my dream come true.

Firstly, I utmostly thank Allah for all He has done for me. My love, respect and endless thanks go to my parents who toiled relentlessly to meet my needs financially and morally. My profound gratitude goes to my able supervisor, Dr. K. R. Adeboye, whose constructive criticisms and suggestions on the original manuscript have contributed immensely to successful completion of this project. Many thanks to my H.O.D., Mr. T. Bankefa and other members of staff of my Department.

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ABSTRACT

This thesis discussed the Numerical Solution of Boundary value problems for Ordinary differential equations occuring in many branches of mathematics and engineering.

The notion and nature of Bounding-value problems constitute the introductory chapter. The finite-difference method, Galerkin and Collocation methods of solving boundary value problems are discussed under the Numerical methods, chapter three contains a program written in Pascal language for solving a boundary value problem using a finite difference scheme. Various Errors arising in the methods discussed are considered in the last chapter with comparison of results.

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CHAPTER ONE

INTRODUCTION

1.1 NOTION OF BOUNDARY-VALUE PROBLEM

Boundary value problem for ordinary differential equation is a differential equation which is associated with the conditions specified at the and points of an interval and its solution over the interval is to be determined such that the end conditions are satisfied.

The name boundary value problem derives from the fact that the points $x = a$ and $x = b$ at which conditions on the dependent variable are specified usually coincide with some physical boundaries in the problem. For example, in the elementary theory of the strength of material, a simply supported beam of length L is a flexible beam supported at each end in such a way that the points of support are on the same horizontal level as shown in (FIG 1). It is shown that if the beam is homogeneous with mass (M) per unit length, the moment of inertia of its cross-sectional is I, and Young's Modulus for the material is E, then provided the deflection Y is small, it satisfies the differential equation.

$$d^2Y/dX^2 = M/2EI\ (x^2 - Lx) \qquad (1.0)$$

The determination of the deflection Y is a two point boundary value problem (b.v.p) for the equation (1.0) because y must satisfy the two boundary conditions

$y(0) = 0$ and $y(L) = 0$ (1.1)

These conditions require the end points to experience no deflection, as the beam is rigidly supported at its ends.

The boundary value problems occur in many branches of mathematics and Engineering. In practice, in most cases we fail to find the exact solution of these problems. This happens mainly not because we do not know the way in which the exact solution is found but usually owing to the fact that the desired solution is not expressible in elementary or other function usually known to us. So that recourse has to be made to Numerical approach. Therefore numerical methods assume ever greater importance, especially in connection with the increasing role of mathematical methods in various fields of Science and Technology and owing to the advent of highly efficient electronic computers.

By numerical methods, we mean the methods of solving problems which are reduced to arithmetic and certain logical operations on numbers, that is to the operations usually performed by computers.

1.2 NATURE OF BOUNDARY VALUE PROBLEM

We use the example of a second-order differential equation

$$F(x,y,y',y'') = 0 \qquad (1.2)$$

to discuss the solution of the boundary-value problem for ordinary differential equations.

The simplest two-point boundary value problem for (1.2) is posed as follows: We have to find the function $y= y(x)$ which satisf[illegible] equation (1.2) within the interval [a,b] and the boun[illegible] [illegible]itions

$\alpha_0 y(a) + \alpha_1 y'(a) = A$

$\beta_0 y(b) + \beta_1 y'(b) = B$ (1.3)

Let us consider some kinds of two-point boundary value problem (1.2)

Assume for instance that we are given a second-order differential equation

$y'' = f(x,y,y')$ (1.4)

With the boundary condition $y(a) = A$, $y(b) = B$ $(a < b)$ ie the values of the required function $y = y(x)$ at the boundary points $x = a$ and $x = b$ are known. Then in terms of geometry, the solution of equation (1.4) is an integral curve $y = y(x)$ which passes through the given points $M(a,A)$ and $N(b,B)$ as shown in FIG 2

Assume that for equation (1.4) we are given the values of the derivatves of the required function at boundary points, ie $y'(a) = A_1$ $y'(b) = B_1$. Then in terms of geometry, the solution of equation (1.4) means that we have to find an integral curve $y = y(x)$ of this equation which would cut through the straight lines $x = a$ and $x = b$ at the angles $\alpha = \arctan A$ and $\beta = \arctan B_1$ respectively, as shown in FIG 3.

Assume, finally that for equation (1.4), we know the value of the required function $y(a) = A$ at one boundary point and the value of the derivative of this function $y'(b) = B_1$ at other the point. A boundary - value problem of this kind is known as the third (mixed) boundary - value problem.

In terms of geometry, the solution of equation (1.4) means that we have to find an integral curve $y = y(x)$ of this equation which would pass through the point $M(a,A)$ and cut the straight line $x = b$ at angle $\beta = \arctan B_1$ as shown in FIG 4.

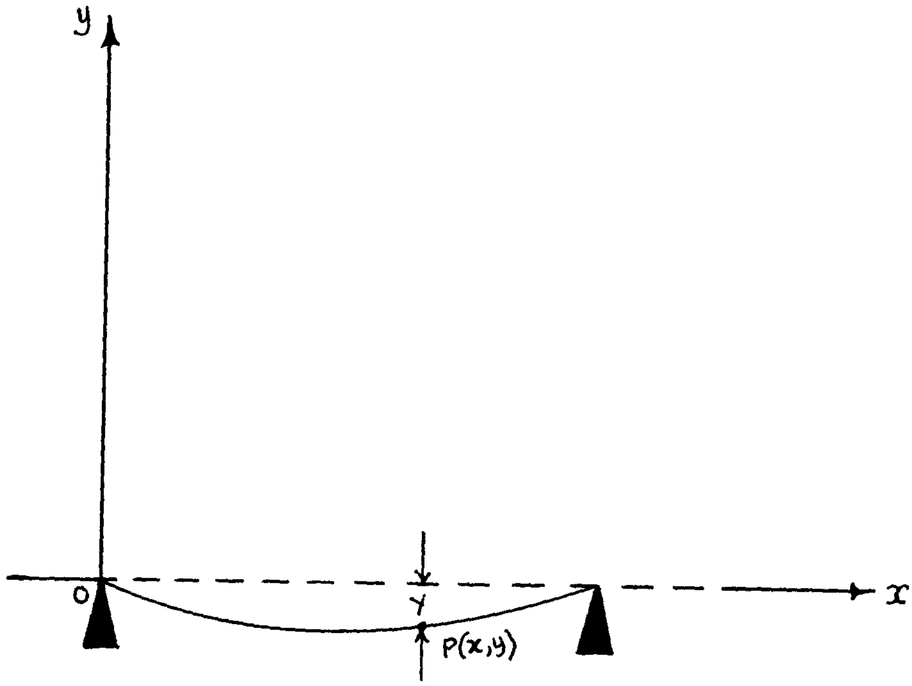

FIG 1

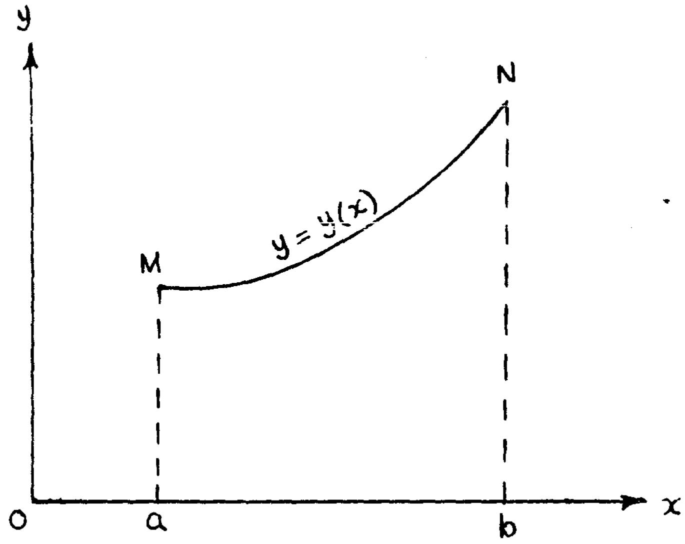

FIG 2

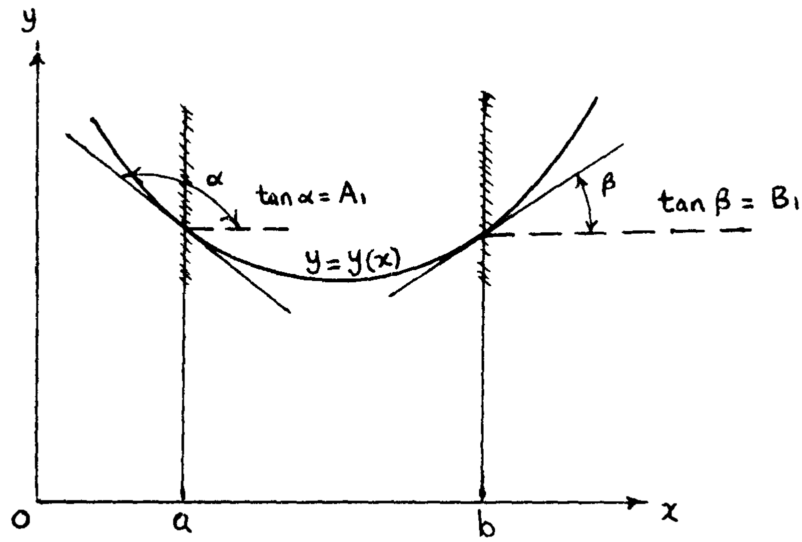

FIG 3

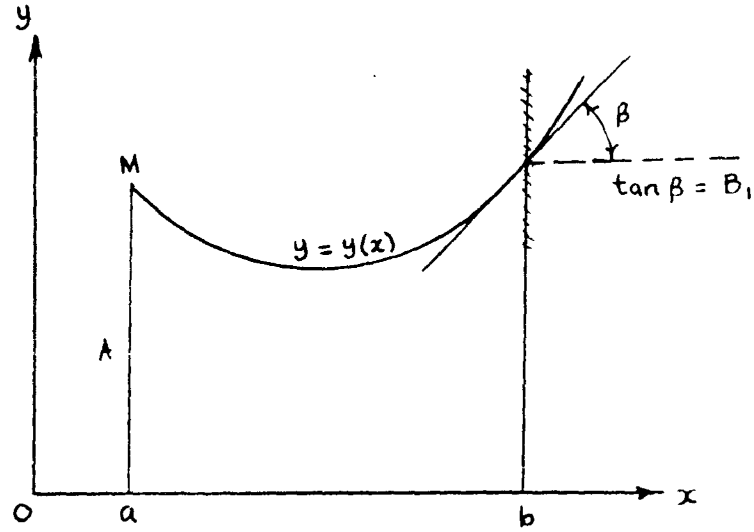

FIG 4

CHAPTER TWO

NUMERICAL METHODS OF SOLVING BOUNDARY-VALUE PROBLEMS

2.1. THE FINITE DIFFERENCE METHOD

2.1.1. BASIC CONCEPT.

We assume that we have a linear differential equation of order greater than one, with conditions specified at the end (boundary) points of an interval [a,b]. We divide the interval [a,b] into N equal parts of width N. We set $x_o = a$, $x_N = b$ and $x_i = x_o + ih$ $(i = 1, 2, ..., N-1)$.
Altogether forming a system of equally spaced points with spacing

$$h = \frac{b - a}{N}$$

We define $x_i = x_o + ih$ $(i = 1, 2, ..., N-1)$ as the interior mesh points. The corresponding values of y at these mesh points are denoted by $y_i = y(x_o + ih)$ $(i = 0, 1..., N)$.
We shall sometimes have to deal with points outside the interval [a,b]. These will be called the exterior mesh points, those to the left of x_o being denoted by $x_{-1} = x_o - h$, $x_{-2} = x_o - 2h$, etc, and those to the right of x_N being denoted by $x_{N+1} = x_{N+1} + h$, $x_{N+2}+ = x_{N+2h}$ etc.
The corresponding values of y at the exterior mesh points are denoted in obvious way as Y_{-1}, Y_{-2}, Y_{N+1}, Y_{N+2} etc.
To solve a boundary-value problem by the method of finite

differences, every derivative appearing in the equation, as well as in the boundary conditions, is replaced by an appropriate difference approximations. Central difference approximations are usually preferred because they lead to greater accuracy.

2.1.2. FINITE-DIFFERENCE APPROXIMATIONS.

Analytically if y = f(x), the first derivation of y is defined by:

$$dy/dx = \lim \quad y/x = \frac{f(x+ \quad x) - f(x)}{x}$$

As shown in FIG 5 higher-order derivation are similarly defined.

In the method of finite differences x does not approach zero but is given a finite value h. The derivations of y at x = x_i are approximated by formulae which use values of y at points spaced at distance h apart. With reference to FIG 6, the first derivative can be represented in one of the three ways.

1. Using forward differences.

$$\left(\frac{dy}{dx}\right)_i \approx \frac{y_{i+1} - y_i}{h}$$

2. Using central differences,

$$\left(\frac{dy}{dx}\right)_i \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

3. Using backward differences,

$$\left(\frac{dy}{dx}\right) \approx \frac{y_i - y_{i-1}}{h}$$

The second derivative of y can also be represented in one of the three ways.

1. Using forward differences, by

$$(d^2y/dx^2)_i \approx \frac{y_i - 2y_{i+1} + y_{i+2}}{h^2}$$

2. Using central differences, by

$$(d^2y/dx^2)_i \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

3. Using backward difference, by

$$(d^2y/dx^2)_i \approx \frac{y_{i-2} - 2y_{i-1} + y_i}{h^2}$$

The third derivatives are approximated by.

1. Using forward differences,

$$(d^3y/dx^3)_i \approx \frac{-y_i + 3y_{i+1} - 3y_{i+2} + y_{i+3}}{h^3}$$

2. Using central differences,

$$(d^3y/dx^3)_i \approx \frac{-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}}{2h^3}$$

3. Using backward differences,

$$(d^3y/dx^3)_i \approx \frac{-y_{i-3} + 3y_{i-2} - 3y_{i-1} + y_i}{h^3}.$$

Higher-order derivations can be similarly defined. In each case of finite-difference representation is an $O(h^2)$ approximations to the res[p]tive derivative.

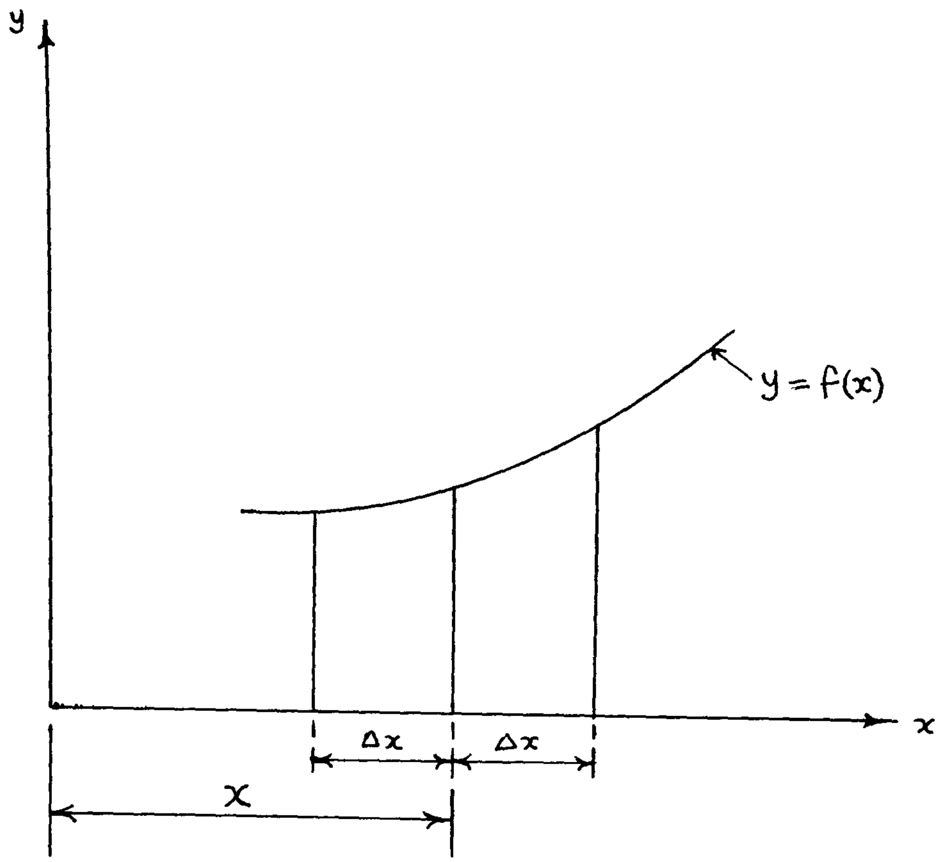

FIG 5 : INFINITESIMAL DIFFERENCES

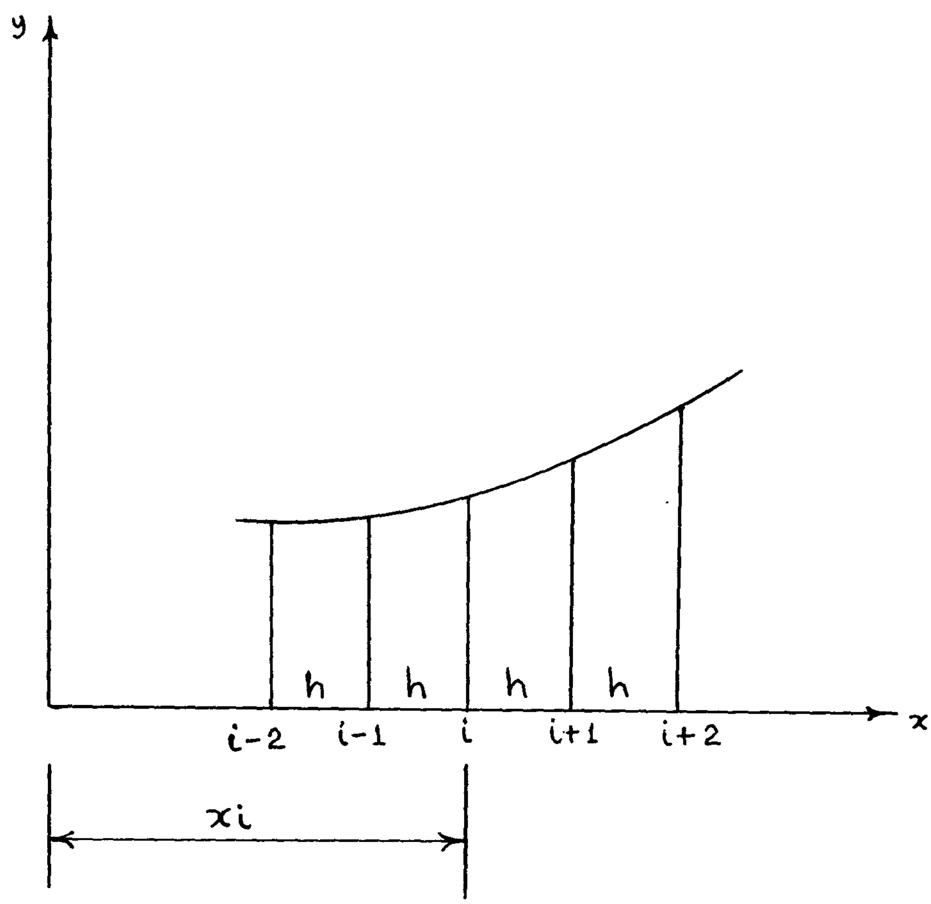

FIG 6 : FINITE DIFFERENCES

2.1.3. **STATEMENT OF PROBLEMS**

For illustration,we consider the linear second-order.differential equation.

$$y''(x) + f(x)\ y' + g(x)y = q(x) \qquad (2.0)$$

Under the boundary conditions.

$$Y(x_0) = \alpha \qquad (2.1.)$$

$$Y(x_n) = \beta \qquad (2.2).$$

The finite-difference approximation to equation (2.0) is

$$\frac{y_{i-} - 2y_i + y_{i+1}}{h^2} + \frac{f(x_n)(y_{i+1} - y_{i-1})}{2h^2} + g(x_i)y_i = q(x_i)$$

Setting $f(x_i) = f_i$, $g(x_i) = g_i$ and $q(x_i = q_i$

$(\ i = 1,2,\ldots,\ n - 1)$

We have

$$\frac{2y_{i-1} - 4y_i + 2y_{i+1} + hf_i\ y_{i+1} - hf_iy_{i-1}}{2h^2} + g_i\ y_i = q_i$$

Multiplying through by h^2 we have

$$\frac{2y_{i-1} - 4y_i + 2y_{i+1} + hf_i\ y_{i+1} - hf_i\ y_{i-1} + h^2\ g_i\ y_i}{2} = h^2q_i$$

Collecting liked terms, we have

$$(1 - h/2f_i)y_{i-1} + (-2 + h^2g_i)y_i + (1+h/2f_i)y_{i+1} = h^2q_i$$

$$(i = 1,2,\ldots N - 1). \qquad (2.3)$$

Since y_o and y_n are specified by the conditions (2.1) and (2.2), Hence (2.3) is a linear system of N - 1 equations in N unknowns.

Writing out equation (2.3) and replacing y_o by α and y_n by β, the system takes the form.

$$(-2 + h^2g_1)y_1 + (1+ h/2f_1)y_2 = h^2q_1 - (1-h/2f_1)\alpha.$$

$$(1 - h/2f_2)y_1 + (-2 + h^2g_2)y_2 + (1+h/2f_2)y_3 = h^2q_2.$$

$$(1 - h/2f_3)y_2 +(-2+h^2g_3)y_3 + (1+h/2f_3)y_4 = h^2q_3 \qquad (2.4)$$

$$(1-h/2f_{N-2})y_{N-3} + (-2+h^2g_{N-2})y_{N-2} + (1+h/2f_{n-2})y_{N-1} = h^2q_{n-2}.$$

$$(1-h/2f_{N-1})y_{N-2} + (-2+h^2g_{N-1})y_{N-1} = h^2q_{N-1} - (1+h/2f_{N-1})\beta.$$

The coefficients in (2.4) can, of course be computed, since f(x), g(x) an q(x) are known functions. This linear system may be written in the

$$Ay = b \qquad (2.5)$$

Where A is the N-1 Tridiagonal matrix of coefficients $y = [y_1, y_2, \ldots, y_{N-1}]_T$, representing the vector of unknowns, b representing the vector of known quantities. The matrix A has a special from.

$$A = \begin{bmatrix} d_1 & c_1 & & & & & 0 \\ a_2 & d_2 & c_2 & & & & \\ & & a_3 & d_3 & c_3 & & \\ - - - & & & & & & \\ & & & & a_{N-1} & d_{N-1} & c_{N-1} \\ 0 & & & & & & \end{bmatrix}$$

The system Ay = b can be solved easily by Gauss elimination method.

Furthermore if the boundary condition (2.1) is of the form ie at the point $x = x_0$

$$y'(x_o) + \tau y(x_o) = 0 \qquad (2.6)$$

So that derivatives are involved, we must make an approximation to (2.6) using Central differences ie we replace (2.6) by.

$$\frac{y(x_0 + h) - y(x_o - h)}{2h} + \tau y(x_o) = 0.$$

Which on rearrangement yields

$$y_1 - y_{-1} + 2h\tau y_o = 0 \qquad (2.7)$$

Since we have introduced an exterior point y_{-1} we must now consider y_o as well as $y_1, y_2, \ldots\ldots y_{N-1}$ as unknowns. We now have N unknowns so we must have N equations for the solution of the linear system. We can obtain additional equation by taking i = 0 in (2.3). If we then eliminate y_{-1} using (2.7) ie $y_{-1} = y_1 + 2h\tau y_o$

We will have for the first two equations

$$[2h\tau(1 - h/2f_o) + (-2 + h^2 g_o)]\, y_o + 2y_1 = h^2 q_o \qquad i = 0.$$

$$(1 - h/2f_1)y_o + (-2 + h^2 g_1)y_1 + (1+h/2f_1)y_2 = h^2 q_1 \qquad i = 1.$$

The remaining equations will be the same as those appearing in (2.4). The system is still tridiagonal but now of order N. It can also be solved using the same Gauss elimination method.

Example 1.

Using the method of finite-difference, let us find the solution of the boundary-value problem.

$$xy'' + x^2 y' = 1 \qquad (2.8)$$

$$y(1) = 0, \; y(1.4) = \tfrac{1}{2}\ln^2(1.4) = 0.0566$$

Solution.

We firstly replace equation (2.8) by corresponding difference approximations.

$$x_i \frac{(y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{x_i^2 (y_{i+1} - y_{i-1})}{2h} = 1$$

$= 2x_i\ y_{i+1} - 4x_i\ y_i + 2x_i y_{i-1} + hx_i^2\ y_{i+1} - hx_i^2\ y_{i-1}$

$= 2h^2$

Collecting the like terms we've

$(2x_i - hx_i^2)y_{i-1} - (4x_i)y_i + (2x_i + hx_i^2)y_{i+1}$

$= 2h^2$

We divide the interval [1,1.4] into parts with a step h = 0.1, then we get five nodal points with abscissa $x_i = x_o + ih$ hence $x_0 = 1$, x_1, = 1.1, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$,

Writing equation (2.10) for each of the interior points x_i (i = 1,2,3,) w get.

For i = 1

$(2x_1 - hx_1^2)y_{1-1} - (4x_1)y_1 + (2x_1 + hx_1^2)y_{1+1} = 2h^2$

$= (2.08y_o - 4.40y_1 + 2.32y_2 = 0.02$

For i = 2

$(2x_2 - hx_2^2)y_{2-1} - (4x_2)y_2 + (2x_2 + hx_2^2)y_{2+1} = 2h^2.$

$= 2.26y_1 - 4.80y_2 + 2.54\ y_3 = 0.02$

for i = 3.

$(2x_3 - hx_3^2)y_{3-1} - (4x_3)y_3 + (2x_3 + hx_3^2)y_{3+1} = 2h^2.$

$= 2.43y_2 - 5.20y_3 + 277y_4 = 0.02.$

Altogether we have the system.

$$2.08y_o - 4.40y_1 + 2.32y_2 = 0.02$$

$$2.26y_1 - 4.80y_2 + 2.54y_3 = 0.02$$

$$2.43y_2 - 5.20y_3 + 2.77y_4 = 0.02$$

From the boundary conditions $y_o = 0$ and $y_4 = 0.0566$ we have

$$-4.40y_1 + 2.32y_2 = 0.02$$

$$2.26y_1 - 4.80y_2 + 2.54y_3 = 0.02$$

$$2.43\ y_2 - 5.20\ y_3 = -\ 0.137$$

Solving this system by Gauss elimination method we

$$\begin{vmatrix} -4.40 & 2.32 & 0 \\ 2.26 & -4.80 & 2.54 \\ 0 & 2.43 & -5.20 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 0.02 \\ 0.02 \\ 0.02 \end{vmatrix}$$

$$\left| \begin{array}{ccc|c} -4.40 & 2.32 & 0 & 0.02 \\ 2.26 & -4.80 & 2.45 & 0.02 \\ 0 & 2.43 & -5.\ 20 & -0.137 \end{array} \right|$$

Multiplying the first row by 0.5136363 and adding to the second row to eliminate y_1 from the second to the third equation we get.

$$\left| \begin{array}{ccc|c} -4.40 & 2.32 & 0 & 0.02 \\ 0 & -3.61 & 2.54 & -0.03 \\ 0 & 2.43 & -5.20 & -\ 0.137 \end{array} \right|$$

Finally multiplying the second row by 0.6731301 and adding to the third row to eliminate y_2 from the third equation we get

$$\left|\begin{array}{ccc|c} -4.40 & 2.32 & 0 & 0.02 \\ 0. & -3.61 & 2.54 & 0.03 \\ 0. & 0 & -3.49 & -0.117 \end{array}\right|$$

Using back substitution, we get

$y_3 = 0.034$

$y_2 = 0.016$

$y_1 = 0.004$

Example 2

Let us solve by finite - difference method the boundary value problem

$$\frac{d^2y}{dx^2} + y = 0 \quad y(0) = 0,\ y(1) = 1 \qquad (2.1.1)$$

Solution

$$y'' + y = 0 \quad y(0) = 0,\ y(1) = 1 \qquad (2.1.2)$$

Replacing (2.1.2) by its equivalent finite difference approximations we have

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i = 0 \qquad (2.1.3$$

we now divide the interval [0,1] into parts with a Step $h = 0.25$. Then we get four nodal points with abscissa $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.50$, $x_3 = 0.75$, $x_4 = 1$.

Simplifying equation (2.1.3) we have

$$y_{i-1} - 2y_i + y_{i+1} + h^2 y_i = 0$$

$$= y_{i-1} + (-2+h^2)y_i + y_{i+1} = 0 \qquad (2.1.4)$$

$(i = 1,2,3,)$

Writing out the equation (2.1.4) for each of the mesh points $(i = 1,2,3,)$ we have

$$y_0 - 1.9375y_1 + y_2 = 0$$

$$y_1 - 1.9375y_2 + y_3 = 0$$

$$y_2 - 1.9395y_3 + y_4 = 0$$

From the boundary conditions $y_0 = 0$ and $y_4 = 1$ we get the system

$$- 1.9375y_1 + y_2 = 0$$

$$y_1 - 1.9375y_2 + y_3 = 0$$

$$y_2 - 1.9375y_3 = -1$$

Solving the resulting system by Gauss elimination

$$\begin{bmatrix} -1.9375 & 1 & 0 \\ 1 & -1.9375 & \\ 0 & 1 & -1.9375 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1.9375 & 1 & 0 & 0 \\ 1. & -1.9375 & 1 & 0 \\ 0. & 1 & -1.9375 & -1 \end{array}\right]$$

Multiplying the first row by 0.516129 and Subtracting the Second row from it, to eliminate y_1 from the second equations we get

$$\left[\begin{array}{ccc|c} -1.9375 & 1 & 0 & 0 \\ 0 & -1.4214 & -1 & 0 \\ 0 & 0 & -1.9375 & -1 \end{array}\right]$$

Again multiplying the second row by 0.7035317 and subtracting the third row from it we get

$$\left[\begin{array}{ccc|c} -1.9375 & 1 & 0 & 0 \\ 0 & 1.4214 & -1 & 0 \\ 0 & 0 & 1.2340 & 1 \end{array}\right]$$

Using back substitution, we get

$$y_3 = 0.8104$$

$$y_2 = 0.5701$$

$$y_1 = 0.2942$$

Example 3

Using finite difference method, let us find the boundary - value problem

$$y'' - xy' + 2y = x + 1 \qquad (2.1.5)$$

$$y(0.9) - 0.5y'(0.9) = 2 \qquad (2.1.6)$$

$$y(1.2) = 1$$

Solution

We divided the interval [0.9, 1.2] into parts with a step h = 0.1 to get the nodal points with abscissa $x_0 = 0.9$, $x_1 = 1.0$, $x_2 = 1.1$, $x_3 = 1.2$

Using the boundary conditions (2.1.6) we set up a finite difference equations at the end points to replace the derivative we have,

$$y_0 - \frac{0.5(y_1 - y_{-1}) = 2}{2h} \qquad (2.1.7)$$

$$= 0.2y_o - 0.5y_1 + 0.5y_{-1} = 0.4 \qquad (2.1.8)$$

from (2.1.8) we get

$$y_{-1} = 0.8 - 0.4y_0 + y_1 \qquad (2.1.9)$$

At x = 0.9 the approximation to (2.1.5) is

$$\frac{y_1 - 2y_0 + y_{-1}}{(0.1)^2} - \frac{0.9(y_1 - y_{-1})}{0.2} + 2y_o = 1.90 \qquad (2.2.0)$$

$$= 0.2(y_1 - 2\,y_o + y_{-1}) - 0.009(y_1 - y_{-1}) + 0.004y_o$$

$$= 0.0038$$

Simplifying and collecting the like terms we get

$$0.191y_1 - 0.396y_o + 0.209y_{-1} = 0.0038 \qquad (2.2.1)$$

Substituting for y_{-1} in (2.2.1) using (2.1.9) we obtain

$$0.4y_1 - 0.4796y_0 = -0.1634$$

$$= 1.20y_0 - y_1 = 0.409 \qquad (2.2.2)$$

The finite difference approximation to (2.1.5) is

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{x_i y_{i+1} - y_{i-1}}{2h} + 2y_i = x_i + 1 \quad (i = 1,2)$$

$= 2y_{i+1} - 4y_i + 2y_{i-1} - hx_i y_{i+1} + hx_i\, y_{i-1} + 4h^2 y_i$

$= 2h^2(x_i + 1)$

Collecting the like terms and taking into account that $h = 0.1$ we have

$(2+hx_i)y_{i-1} - 4(1-h^2)y_i + (2-hx_i)\, y_{i+1} = 2h^2(x_i + 1)$

$$= (2 + 0.1x_i)y_{i-1} - 4(1-0.01)y_i + (2-0.1x_i)y_{i+1} = 0.02(x_i+1) \tag{2.2.3}$$

Writing (2.2.3) for each of the mesh points $(i = 1,2)$

$2.1\, y_0 - 3.96y_1 + 1.9y_2 = 0.04$

$2.11y_1 - 3.96\, y_2 + 1.89\, y_3 = 0.042$

This problem with the boundary conditions reduce to the solution of the system of equations

$1.20y_0 - y_i = 0.409$

$2.1\, y_0 - 3.96y_1 + 1.9y_2 = 0.04$

$2.11\, y_1 - 3.96\, y_2 + 1.89\, y_3 = 0.042$

$\Longrightarrow 2.11\, y_1 - 3.92y_2 = -1.848$

From $y_3 = 1$

Solving this system by Gauss elimination method

$$\left[\begin{array}{ccc|c} 1.20 & -1 & 0 & 0.409 \\ 2.1 & -3.96 & 1.9 & 0.04 \\ 0 & 2.11 & -3.96 & -1.848 \end{array}\right]$$

Multiplying the first row by 2.1/1.20 and subtracting the second row from it we have

$$\begin{bmatrix} 1.20 & -1 & 0 & | & 0.409 \\ 0 & 2.21 & -1.9 & | & 0.676 \\ 0 & 2.11 & -3.96 & | & -1.848 \end{bmatrix}$$

Multiplying the second row by 2.11 and subtracting the third row from it we get

$$\begin{bmatrix} 1.20 & -1 & 0 & | & 0.409 \\ 0 & 2.21 & -1.9 & | & 0.676 \\ 0 & 0 & 2.15 & | & 2.493 \end{bmatrix}$$

Using back Substitution we get

$y_2 = 1.160$

$y_1 = 1.303$

$y_0 = 1.426$

2.2 THE PASSAGE METHOD

We observed that the accuracy of the finite difference method can be considerably increase if the mesh length chosen for the interval c the Boundary - value problem becomes smaller. However, the system of lin equations obtained becomes large and the solution of this system of equa- tions becomes rather cumbersome. We introduced a simple method develope specially for solving such kind of system called **The passage method**

2.2.1 Basic Concept

Let us consider a linear boundary - value problem

$$Y'' + P(x)y' + q(x)y = f(x) \qquad (2.20)$$

$$\alpha_0 y(a) + \alpha_1 y'(a) = A$$

$$\beta_0 y(b) + \beta_1 y'(a) = B \qquad (2.2.1)$$

Replacing (2.20) and 2.21) by their Central finite - difference relations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{P_i(y_{i+1} - y_{i-1})}{2h} + q_i y_i = f_i$$

Simplifying and collecting the like terms we get

$$= (2 + hP_i)y_{i+1} + (2q_i h^2 - 4)y_i + (2-hP_i)y_{i-1} = 2h^2 f_i$$

Dividing through by $(2+hP_i)$

$$= y_{i+1} + \frac{(2q_i h^2-4)y_i}{2+hP_i} \quad \frac{(2-hP_i y)_{i-1}}{2+hP_i}$$

$$= \frac{2h^2 f_i}{2 + hP_i} = \qquad (2.2.2)$$

$(i = 1,2\ldots n-1)$

The boundary conditions are given by

$$\alpha_0 y_o \; \alpha_1 \frac{y_1 - y_0}{h} = A$$

$$\beta_n y_n + \beta_1 \frac{y_{n+1} - y_{n-1}}{2h} = B \qquad (2.2.3)$$

We write first n-1 equations of system (2.2.2) in the form

$$y_{i+1} + m_i y_i + k_i y_{i-1} = 2h^2/2+hP_i = \phi_i$$

where $m_i = (2q_i\ h^2 - 4)/(2+hP_i$

$$k_i = (2-hP_i)/(2+hP_i) \qquad (2.2.4)$$

We then reduce these equations to the recurrence form

$$y_i = C_i(d_i - y_{i+1}) \quad (i = 1,2,\ \ldots\ldots,\ n-1) \quad (2.2.5)$$

where the coefficients C_i, d_i are computed by the following for $i = 1$

$$C_1 = \frac{\alpha_1 - \alpha_0 h}{m_1(\alpha_1 - \alpha_0 h) + k_1 \alpha_1},$$

$$d_1 = \frac{2f_1 h^2 \; K_1}{2 + hP_1} + \frac{Ah}{\alpha_1 - \alpha_0 h} = \phi_1 + \frac{K_1 Ah}{\alpha_1 - \alpha_0 h}$$

(2.2.6)

for $i = 2, \ldots .n,$

The recurrence formula below can be used

$$C_i = 1/m_i - k_i \, C_{i-1},$$

$$d_i = 2f_i h^2/2+hP_i - k_i C_{i-1} d_{i-1} = \phi_i - k_i C_{i-1} - d_{i-1} \quad (2.2.7)$$

These computations are carried out in the following succession

Direct procedure: We find m_i, k_i by formulae (2.2.4)

i.e $M_i = 2q_i h^2 - 4/2 - hp_i,$

$k_i = 2 - hP_i/2 + hP_i$

We compute C_1, d_1 from (2.2.6) and then using recurrence formulae (2.7), we successively find C_i, d_i ($i = 2, \ldots, n$).

Reverse procedure: We write down equation (2.2.5) for $i = n$, $i = n-1$ and the last equation of (2.2.3)

$$y_n = C_n \, (d_n - y_{n+1}) \qquad (2.2.8)$$

$$y_{n+1} = C_n \, (d_{n-1} \, y_n$$

$$\beta_o y_n + \beta_1 \frac{y_{n+1} - y_{n-1}}{2h.} = B$$

Solving this system with respect to y_n we have

$$y_n = \frac{2Bh - \beta_1(d_n - C_{n-1} d_{n-1})}{2\beta_o h + \beta_1(C_{n-1} - 1/C_n)} \qquad (2.2.9)$$

Using the knowing number c_n, d_n, c_{n-1}, we find y_n

The value of y_i ($i = n-1, \ldots 1$) are obtained from the reoccurrence formula (2.2.5), Y_o being computed from the equation

$$\alpha_o y_o + \alpha_1 \frac{y_1 - y_o}{h} = A$$

$$y_o = \frac{\alpha_1 y_1 - Ah}{\alpha_1 - \alpha_o h}$$

2.2.1 Statement of problem

Example 4.

Using the passage method let us find the approximate solution of the boundary -value problem

$y'' - 4xy' - 2y = -8x$ (2.30)

Satisfying the boundary conditions

$y(0) - y'(0) = 0 \quad 2y(1) - y'(1) = 1$ with $h = 0.1$ (2.31)

Solution:

We first replace equation (2.30) and (2.3.1) by the Corresponding finite - difference equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4x_i \frac{y_{i+1} - y_{i-1}}{2h} - 2y_i = -8x_i$$

$= 2y_{i+1} - 4y_i + 2y_{i-1} - 4hx_i y_{i+1} + 4hx_i y_{i-1}$

$- 4h^2 = -16h^2 x_i$

Collecting the like terms

$(2-4hx_i)y_{i+1} - (4+4h^2)y_i + (2+4hx_i)y_{i-1} =$

$- 16h^2 x_i$

Dividing through by 2 we get

$(1 - 2x_i h)\, y_{i+1} - (2+2h^2)y_i + (1+2x_i h)y_{i-}$

$$y_{i+1} \quad \frac{2 - 2h^2}{1-2x_i h} y_i + \frac{1+2x_i h}{1-2x_i h} y_{i-1} = \frac{-8h^2 x_i}{1-2x_i}$$

Thus we have

$m_i = (-2+2h^2)/(1-2x_i h)$, $K_i = (1+2x_i h)/(1-2x_i h)$,

$\phi = (-8h^2 x_i)/(1-2x_i)$ $(i = 0,1,2,...10)$

From (2.3.1) we have

$\alpha_o = 1$, $\alpha_1 = -1$, $\beta_o = 2$, $\beta_1 = -1$, $A = 0$, $B = 1$

Direct procedure:- We enter the number $x_i = 0.1_i$, in the Table 1 below and Compute the value of m_i, k_i, ϕ_i $(i = 1,2, ...10)$

Then using formulae (2.26) we compute for Example C_i

$C_1 = (\alpha_1 - \alpha_o)/(m_i (\alpha_1 - \alpha_o)+K_1 \alpha_1) =$

$(-1.1)/(2.06(1.1) - 1.041) = (-1.1)/(1.226) = - 0.897$

$d_1 = (2+f_1^2)/(2+hp_1) + (k_1 Ah)/(\alpha_1 - \alpha_0 h) =$

$\phi_1 + (k_1 Ah)/\alpha_1 - \alpha_0 h) = -0.008 + 0 = - 0.008$

We write the obtained values in Table 1 and proceed with successive computation of C_i and d_i by formulae (2.2.7)

Thus for $i = 2$ we get for example

$C_2 = 1/(m_2 - k_2 c_1) = 1/(-2.104 +(1.083) (0.897)) = 0.883.$

$d_2 = \phi_2 - k_2 c_1 d_1 = - 0.017 - 1.083 * 0.897 0.008 = -0.025$

Computation for $i = 3,4, ...10$ are carried out analogoushy. All the value are entered in the columns for the direct procedure of Table 1.

Reverse procedure

We determine y_{10} by the formula (2.2.9) i.e

$$y_i = \frac{2Bh - \beta_1 \quad (d_n - C_{n-1} \quad d_{n-1})}{2\beta_0 + \beta_1 \; (c_{n-1} - 1/c_n)}$$

$$y_{10} = \frac{0.2-0.636 - (0.726*0.492)}{0.4 + 0.726 - 1/0.096} = 0.793/0.311 = 2.54$$

We write down the obtained value in the last row of table 1, we then successively find y_i ($i = 9,8...1$) using recursive formula.

$y_i = C_i(d_i - y_i+1)$ for example.

$y_9 = c_9 \; (d_9 - y_{10} = 0.726 \; (-0.49 - 2.54) = 2.20$

$y_8 = c_8 \; (d_7 - y_9) - 0.754 \; (-0.37 - 2.20) = 1.94$

$y_7 = c_7 \; (d_7 - y_8) = - 0.871 \; (-0.27 - 1.94) = 1.73$

$y_6 = c_6 \; (d_6 - y_7) = -0.806 \; (-0.20 - 1.73) = 1.56$

$y_5 = c_5 \; (d_5 - y_6 = - 0.829 \; (0.13 - 1.56) = 1.40$

Computation for y_i ($i=4,3,....1$) are carried out in the same way.

Finally by formula $y_0 = (\alpha_1 \; y_1 - Ah)/(\alpha_1 - \alpha_0 h)$ we get

$y_0 = (-0.93)/(-1.1) = 0.85$

i	X	M	K	E	Direct procedure C	d	Reverse procedure y
0	0.0						0.85
1	0.1	- 2.061	1.041	- 0.008	- 0.897	- 0.008	0.93
2	0.2	- 2.104	1.083	- 0.017	- 0.883	- 0.025	1.03
3	0.3	- 2.149	1.128	- 0.026	- 0.867	- 0.051	1.14
4	0.4	- 2.196	1.174	- 0.035	- 0.849	- 0.087	1.27
5	0.5	- 2.244	1.222	- 0.044	- 0.829	- 0.134	1.40
6	0.5	- 2.295	1.273	- 0.055	- 0.806	- 0.196	1.56
7	0.7	- 2.349	1.326	- 0.065	- 0.781	- 0.274	1.73
8	0.8	- 2.405	1.381	- 0.076	- 0.754	- 0.372	1.94
9	0.9	- 2.463	1.439	- 0.088	- 0.726	- 0.492	2.20
10	1.0	- 2.525	1.500	- 0.100	- 0.696	- 0.636	2.54

2.3. GALERKIN'S METHOD

The previously considered method enables us to approximate the solution of a boundary - value problem in tabular form. We now treat an analytical method which make it possible to find the approximate solution of linear boundary value problem in the form of an analytical expression named Galerkin's method.

2.3.1 Basic Concept

Suppose we have a linear boundary - value problem

$y'' + P(x)\ y' + q(x)y = f(x)$ (2.3.2)

$\alpha_0\ y(a) + \alpha_1 y'(a) = A$

$\beta y(0) + \beta_1 y'(b) = B$ (2.3.3)

Where P(x), q(x), f(x) are known quantities continuous on the interval [
of the function α_0, α_1, β_0, β_1, A, B are given constants and $|\alpha_0| + |\alpha_1| =$
and $|\beta_0| + |\beta_1| = 0$

Let us now introduce the following notation.

$L[y] = y'' + P(x)y' + q(x)y$ (2.3.4)

$\Gamma_a[y] = \alpha_0 y(a) + \alpha_1 y'(a) = A$

$\Gamma_b[y] = \beta_0 y(b) + \beta_1 y'(b) = B$ (2.3.5)

Let us in an interval [a,b] there be given system of basis functions.

$U_0(x), U_1(x), \ldots, U_n(x)$ (2.3.6)

Satisfying any of the following conditions.1

1. System (2.3.6) is orthogonal i.e.

$U_i(x)U_j dx = 0$ for $i = j$

$U_i^2(x)dx = 0$ (.2.3.7)

2. System (2.3.6) is a complete one, i.e there is no other non-zero function which is orthogonal to all the functions $U_i(x)$ $(i=0,1,2\ldots,n)$

3. The finite system of basis function $\{U_i(x)\}$ $(i=0,1,\ldots,n)$ is chosen so that the function $U_0(x)$ satisfies the non-homogeneous boundary condition.

$\Gamma_a[U_0] = A, \quad \Gamma_b[U_0] = B$ (2.3.8)

and the function $U_i(i)$ $(i=1,2,..,n)$ satisfy the homogenecus boundary conditions.

$\Gamma_a[U_i] = \Gamma_b[U_i] = 0 \quad (i = 1,2, \ldots n)$ (2.3.9)

We shall look for the solution of the boundary - value problem (2.3.2) & (2.3.3) in the form

$y(x) = U_0(x) + \Sigma_{i=1} C_i U_i(x)$ (2.40)

It follows from the conditions (2.38) and (2.3.9) that this function satisfies the boundary condition (2.3.3) above.

Now let us consider the expression called the residual

$R(x, c_1, c_2 \ldots . C_n) = L[U_0 + \Sigma_{i=1} c_i L(U_i) - f(x)$ (2.4.1)

We choose the coefficients C_i so as to obtain the least value of the integral of the squared residual.

$$\int R^2(x, c_1 c_2, \ldots, C_n)dx \quad (2.4.2)$$

It is proved that this is achieved only if the residual $R(x, c_1, c_2, \ldots, c_n)$ is orthogonal to all the basis functions

U_i $i = (1,2,..n)$

Let us now write down the condition of orthogonality.

$$\int U_i(x) R(x, c_1 c_2 \ldots c_n)dx = 0 \quad (k=1,2,\ldots,n)$$

or in full

$$\Sigma_{i=1} c_i \int U_i(x) L[U_i]dx$$

$$= \int U_i (x) \{f(x) - L[U_0]\}dx.$$

Thus, we obtained a system of linear algebraic equation with respect to C_i $(i = 1,2,\ldots..n)$

We note that in choosing the basis functions the condition of orthoganality (1) is not obligatory if the coefficients are chosen proceeding form minimality condition of the integral (2.4.2)

For instance, taking for the basis functions, a complete system of orthogonal functions on the interval [a,b], we may cnoose as the basis functions the linear combinations of functions from the system. It is only necessary and sufficient that the chosen function be linearly independent on the interval [a,b].

2.3.2 Statement of Problems

Example 5

Let us for example approximate the solution of the following equation using Gelerkin's method

$y'' - y' \quad y(0) = 0, \; y(1) = 1 \qquad (2.4.3)$

Solution

As the system of the basis functions $U_i(x)$ (i =,0,1,2) Let us choose the following polynomial functions

$U_1 = x(x-1)$ and $U_2 = x^2(1,x)$ where $U_0 = 0$

These functions are linear independent on the interval [0,1] also U_1 and U_2 satisfy zero boundary condition. We look for an approximate solution of the problem in the form

$y = C_1U_1 + C_2U_2$

$\Longrightarrow y = C_1\,(x^2 - x) + C_2\,(x^2 - x^3$

$= C_1x^2 - C_1x + C_2\,x^2 - C_2x^3$

$y' = 2c_1 - c_1 + 2c_2x - 3c_2\,x^2$

$y'' = 2c_1 + 2c_2 - 6c_2x$

$y'' - y' - 1 = 0 = R(x,c,c_2)$

$\Longrightarrow R\,(x,c_1,c_2) = 2c_1 + 2c_2 - 6c_2x - 2c_1x + C_1 -$
$2c_2x + 3c_2x^2 - 1$

Taking into consideration the orthogonality of the function R with respect to the functions $U_1(x)$ and U_2 we have

$$\int_0^1 U_1\ R\ (x,\ c_1,\ c_2) = \ (x^2-x)\ (2c_1 + 2c_2 - 6c_2x$$

$$- 2c_1x\ c_1 - 2c_2x+3c_2x^2-1)\ dx = 0$$

$$= \int_0^1 [2c_1x^2+2c_2x^2-6c_2x^3 - 2c_1$$

$$+ c_1x^2-2c_2x^3 + 3c_2x^4 - x^2$$

$$- 2c_1x-2c_2x+6c_2x^2+2c_1x^2-c_1x+c_2x^2$$

$$- 3c_2x^3+x]dx=0$$

$$= 2/3c_1X^3+ 2/3c_2X^3-6/4c_2X^4-2/4+c_1X^4$$

$$+c_1/3X^3 - 2/4c_2x^4 + 3/5c_2x^5 - x^3/3$$

$$- 2/2c_2{}^2+6/3c_2x^3 +2/3c_1x^3-c_1/2x^2$$

$$+2/3c_2x^3-3/4c_2x^4+\tfrac{1}{2}x^2$$

$$= 2/3c_1 + 2/3c_2-3/2\ c_2 - 1/2c_1 +c_1/3 - 1/2c_2+3/5c_2$$

$$-1/3 - c_1/1 - c_2/1 + 2c_2/1 + 2c_1/3 - c_1/2 + 2c_2/3$$

$$- 3c_2/4 + \tfrac{1}{2}$$

$$= -1/3c_1 + 11/60\ c_2 - 1/6$$

Also

$$\int_0^1 U_2\ R(x,c_1,\ c_2\ )\ \int_0^1 (x^2-x^3)$$

$$[2c_1 +2c_2-6c_2x-2c_1x+c_1-2c_2x+3c_2x^2-1]dx = 0$$

$$= \int_0^1 [2c_1\ x^2 + 2c_2\ x^2 - 6c_2\ x^3 -$$

$$2c_1\ x^3 + c_1\ x^2 - 2c_2\ x^3 +$$

$$3c_2\ x^4 - x^2 - 2c_1\ x^3 - 2c_2x^3 + 6c_2\ x^4$$

$$+ 2c_1\ x^4-c_1x^3 + 2c_2\ x^4 - 3c_2x^5 +x^3]dx =0$$

$$2/3c_1x^3 + 2/3c_2x^3 - 6/4c_2\ x^4 -2/4c_1x^4+1/3c_1$$

$$x^3 - 2/4c_2x^4 + 3/5c_2\ x^5 - x^3/3 - 2/4c_1x^4 -$$

$$2/4c_2x^4 + 6/5c_2x^5 + 2/5c_1x^5\ c_1/4x^4$$

$$+ 2/5c_2x^5 - 3/6c_2x^6 + x^4/4\ \Big|_0^1 = 0$$

$= 2/3c_1 + 2/3c_2 - 6/4c_2 - 2/4c_1 + c_1/3 - 2c_2/4 +$

$3/5c_2 - 1/3 - 2c_1/4 - 2c_2/4 + 6/5c_2 + 2c_1/5$

$- c_1/4 + 2c_2/5 - 3/5C_2 + \frac{1}{4} = 0$

$= 3/20\ C_1 - 21/15c_2 = 1/12$

We now have a system of linear algebraic equations for deter mining the coefficients C_1 and C_2

$-1/3\ c_1 + 11/60\ C_2 = -1/6$

$3/20\ c_1 - 2/15c_1 = 1/12$

Solving this system we get

$C_1 = 75/183, \quad C_2 = -330/2013$

From $y = C_1\ U_1 + C_2\ U = C_1\ (x^2-x) + c_2(x^2-x^3)$

Substituting the values of C_1 and C_2 we get

$y = 75/183\ (x^2 - x) - 330/2013\ (x^2 - x^3)$

For comparison purpose Table (1) gives the values of the obtained approximate solution and the exact solution

$1/(1 - e) + 1/(e-1)e^x - x$

Table (1)

Approximate and Exact Solution of Example 5

x	0.25	0.50	0.75
y	0.085	- 0.123	-0.099
y_t	-0.085	- 0.122	- 0.099

Example 6

Using Gakerkin's method, let us approximate the solution of the equation

$y'' + y + x = 0 \qquad (2.4.4)$

Satisfying the boundary conditions

$y(0) = y(1) = 0 \qquad (2.4.5)$

Solutions

Choosing the following functions as the system of basis functions U_i $(i = 1,2,)$

$$U_i(x) = x(1-x), \quad U_2(x) = x^2(1,-x)$$

They are linearly independent and satisfy the Homogeneous boundary condition. We look for an approximate solution of the problem in the form

$$y = C_1 U_1 + C_2 U_2 = C_1 x (1-x) + C_2x^2 (1-x)$$

$$y = C_1x - C_1x^2 + C_2x^2 - C_2 x^3$$

$$y' = C_1 - 2C_1 x + 2C_2 x^2 - 3c_2 x^2 \quad \text{and}$$

$$y'' = -2c_1 + 2c_2 - 6c_2x$$

$$y'' + y + x = - 2c_1+2c_2 -6c_2x + c_1x - c_1x^2 - c_2x^3 + x = 0$$

$$R(x, c_1, c_2) = -2c_1+2c_2 - 6c_2x +C_1x - c_1x^2 - c_2 x^3 +x$$

$\int (wx?$

$$\int_0^1 U_1 (x) R(x,c_1 c_2)dx = 0$$

$$= \int_0^1 (-2c_1 + 2c_2 - 6c_2x + C_1x - c_1x^2 - c_2x^3 + x)dx = 0$$

$$\int_0^1 [-2c_1x + 2c_2x - 6c_2 x^2 + c_1x^3$$

$$+ c_2 x^3 - c_2x^4 + x^2 + 2c_1x^2 - 2c_2x^2$$

$$+ 6c_2x^3 - C_1x^3 + C_1x^4 -C_2x^4 +C_2x^5$$

$$- x^3]dx = 0$$

$$= - C_1x^2 +c_2x^2 - 2c_2x^3 + 1/3C_1x^3 -$$

$$\tfrac{1}{4}C_1x^4 + \tfrac{1}{4}c_2x^4 - 1/5c_2x^5 +1/3x^3 + 2/3c_1x^3 - 2/3c_2 x^3 + 6/4c_2x^4$$

$$- C_1/4x^4 + 1/5c_1x^5 - 1/5c_2/3x^5 + c_2/6 x^6 - x^4/4 \Big|_0^1$$

For Comparison Purpose Table (3) gives the value of the obtained approximate solution and the exact solution

$- c_1 + c_2 - 2c_2 + \frac{c_1}{3} - \frac{c_1}{4} + \frac{c_2}{4} - \frac{c_2}{5} + \frac{1}{3} - \frac{2}{3} + \frac{2}{4} - \frac{6}{4}$

$- c_1/4 + c_1/5 - c_2/5 + c_2/6 - \frac{1}{4}$

$= 3/10c_1 + 3/20c_2 = 1/12$

Also $\int_0^1 U_2(x)\ R(x, c_1, c_2)dx = 0$

$= \int_0^1 (x^2 - x^3)(-2c_1 + 2c_2 - 6c_2x + c_1x - c_1x^2 + c_2x^2 - c_2x^3 +$

$+ x)dx = 0$

$= \int_0^1 [-2c_1x^2 + 2c_2x^2 - 6c_2x^2 - c_1x^3 - c_1x^4 + c_2z^4 - c_2x^5 + x^3$

$+ 2c_1x^3 - 2c_2x^3 + 6c_2x^4 - c_1x^4 + c_1x^5 + c_2x^6 - x^4]dx = 0$

$= -2/3c_1x^3 + 2/3c_2x^3 - 6/4c_2x^4 + \frac{1}{4}c_1x^4 - 1/5c_1x^5 + 1/5c_2x^5$

$- 1/6c_2x^6 + \frac{1}{4}x^4 + 2/4c_1x^4 - 2/4c_2x^4 + 6/5c_2x^5 - 1/5c_1x^5$

$+ 1/6c_1x^6 - 1/6c_2x^6 + 1/7c_2x^7 + 1/5x^5 \Big|_0^1$

$= -2/3c_1 + 2/3c_2 - 6/4c_2 + \frac{1}{4}c_1 - 1/5c_1 + 1/5c_2 - 1/6c_2 + \frac{1}{4}$

$+ 2/4c_1 - 2/4c_2 + 6/5c_2 - 1/5c_1 + 1/6c_1 - 1/6c_2$

$+ 1/7c_2 + 1/5$

$= 3/20c_1 + 13/105c_2 = 1/20$

We have system of linear algebraic equation from which we can determine the coefficient c_1 and c_2. Solving thie system we get

$c_1 = 71/369$ and $c_2 = -71/41$

From $y = c_1(x-x^2) + c_2(x^2-x^3)$

We have

$y = 71/369(x-x^2) + 7/41(x^2-x^3)$

Table (3)

xi	0.25	0.50	0.75
yi	0.044	0.069	0.060
y	0.044	0.070	0.060

2.4. THE COLLOCATION METHOD

2.4.1 Basic Concept

Let us assume that we have a second - order linear boundary - value problem

$Ly = y'' + P(x)y' + q(x)y = f(x)$ 2.4.6)

$\alpha_0\ y(a) - \alpha_1\ y(a) = A$

$\beta_0\ y(a) - \beta_1\ y'(b) = B \quad |\ \alpha_0\ |_* + |\beta_0| \neq 0$ (2.4.7)

We look for the solution of the b.v.p (2.4.6) & (2.4.7) in the form

$y(x) = U_0(x) + \sum_{i=1}^{n} C_i\ U_i(x)$ (2.4.8)

where $U_i(x)$ $(i = 0,1,\ldots,n)$ are linearly independent functions satisfying the conditions.

$\Gamma_a[U_0] = A,\ \Gamma_b[u_0] = B$ (2.4.9)

$\Gamma_a\ [u_i] = \Gamma_b[u_i] = 0 \quad (i = 1,2,\ldots n)$ (2.5.0)

Let us require that the residual

$R(x,c_1,\ c_2,\ldots,c_n) = L(y) - f(x) + \sum_{i=1}^{n} C_i L[u_i]$ (2.5.1)

Vanish for a certain system of points $x_1,\ x_2\ \ldots x_n$ of the interval [a,b] called the collocation points (the number of such points must equal the number of the coefficients C_1 $C_2\ \ldots\ C_n$ in the expression (2.4.8). Then for determining C_1, $C_2\ \ldots C_n$ we get the following system of equations.

$R(x_1,C_1,\ \ldots,C_n) = 0$

$R(x_2, c_1, \ldots, c_n) = c$

....

$R(x_n, c_1\ c_2, \ldots\ldots c_n) = 0$

Solving the obtained linear system of equations, the solution c_1 $(i = 1,2,\ldots n)$ is substituted into (2.4.8) to obtained the desired approximate solution.

The basis functions $U_i\ (x)$ are usually chosen so as to have one or more of the following properties

(i) The $U_i(x)$ are continuously differentiable on $[a,b]$

(ii) The $U_i(x)$ are orthogonal over the interval $[a,b]$, i.e

$$\int_a^b U_i(x)\ U_i(x)dx = 0 \text{ for } i = j$$

(iii) The $U_i(x)$ are "simple" functions such as polynomials or trigonometric functions

(iv) The $U_i(x)$ satisfy the homogeneous boundary conditions

2.4.? ement of problems

Example 7

Using collocation method let us approximate the solution of the boundary - value problem

$y''+x^2y - x = 0$ (2.5.2)

with the boundary condition

$y(-1) = y(1) = 0$ (2.5.3)

Solution

Let us choose for the basis functions the polynomials

$U_0(x) = 0$, $U_1(x) = 1 - x^2$, $U_2(x) = x^2(1-x^2)$ which all satisfy the boundary conditions.

We now seek for the solution of the problem in the form

$y = c_1(1-x^2) + c_2 x^2(1-x^2)$

$= c_1 - c_1x^2 + c_2x^2 - c_2x^4$

$y' = -2c_1x + 2c_2x - 4c_2x^3$

$y'' = -2c_1 - 2c_2 - 12c_2x^2$

The residual $R(x_1, c_1, c_2) = -2c_1+2c_2-12c_2x^2$

$+ x^2[c_1 - c_1x^2 + c_2x^2 - c_2x^4]- x = 0$

$= -2c_1 +2c_2 - 12c_2x^2 + c_1x^2-c_1x^4 +c_2x^4$

$- c_2x^6 - x = 0$

$= (-x^4 +x^2-2) c_1 + (-x^6+x^4 -12x^2+2) - x = 0$

Taking $x_0 = 0$ and $x_1 = ½$ as the collection points we've

$- 2c_1 + 2c_2 = 0$

$- 5/16c_1 + 61/64c_2 = ½$

Solving this linear system, we get

$c_1 = c_2 = 32/41$

Hence we have approximate solution

$y = 32/41 (1-x^2) = 32/41 (x^2 -x^4)$

Example 8

Let us find the approximate solution of the differential equation

$y'' - y' = 1$ (2.5.3)

$y(0) = 0,\ y(1) = 1$ (2.5.4)

Using collocation method.

Solution

Choosing $U_o = 0$, $U_1 = x(1-x)$ and $U_2 = x^2(1-x)$

as the basis functions, we look for solution in the form

$y = c_1 U_1 + c_2 u_2(x)$

$y = c_1(x-x^2)+c_2(x^2-x^3)$

$y = c_1x-c_1x^2+c_2x^2 - c_2 x^3$

$y' = c_1 - 2c_1x + 2c_2x -3c_2x^2$

$y'' = -2c_1 +2c_2 - 6c_2x$

$R(x,c_1,c_2) = y'' - y' - 1 = 0$

$R(x,c_1 c_2) = -2c_1 +2c_2 - 6c_2x -$

$(c_1 -2c_1 x+2c_2x - 3c_2x^2) - 1 = 0$

$= - 2c_1 +2c_2 - 6c_2x - c_1 + 2c_1x -$

$2c_2x + 3c_2x^2 - 1 = 0$

$= -3c_1 + 2c_2 - 8c_2x+2c_1 x+3c_2x^2 - 1 = 0$

$(2x -3)c_1 + (3x^2 - 8x+2)c_2 = 1$

Taking $x_0 = \frac{1}{4}$ and $x_1 = 3/4$ as the collocation points we obtained the following system

$- 10/4\ c_1 + 3/16\ c_2 = 1$

$- 6/4c_1 - 37/16c_2 = 1$

Solving this linear system, we get

$c_1 = -40/97$ and $c_2 = - 16/97$

Hence we have an approximate solution

$$y \approx -40/97\ (x - x^2) - 16/97(x^2 - x^3)$$

CHAPTER THREE

3.0 COMPUTER ANALYSIS

Given the task of developing a program to solve a problem like boundary problem, we begin with a very clear idea about how the program is to be written. Putting into consideration the Methodology for the development of the program and the language in which the program is to be written. Development of program involves one or more stages that, by careful integration and control, will bring order and direction to the flow of the program. We have stages like Problem Analysis, Program Implementation and Program Debugging & Testing.

3.1 Problem Analysis

Before we can hope to develop a program to solve any problem, we must understand exactly what the problem is. In this particular case, our problem is to develop a program in a Pascal Language to solve a boundary-value problem using a finite-difference scheme that has been discussed earlier in this project. At this stage, a concise statement of the problem to be solved and the constraints that exist for its solution is made. A functional specification is produced.

3.2 Program Design

The central task of this phase is to take the agreed Functional Specification and derive from it a design that will satisfy it. At this stage we design how the data are to be stored and manipulated by the computer. A flowchart containing necessary steps to be taken in writing the program is drawn. We adopted a modular design for solving this problem, in which case the program is divided into subprograms (procedures) that can be tested separately. A main program invokes these modules as they are needed.

3.3 **Program Implementation**

The implementation phase of this program is concerned with translating the design specifications into source code (Pascal). The primary goal of this phase is to write source code and internal documentation so that conformance of the code to the functional specification of the program can be easily verified, and so that debugging, testing and modification are eased. This goal is achieved by making the source code as clear and straightforward as possible. Source code clarity is enhanced by structured coding technique with Pascal language.

3.4 **Program Debugging and Testing**

Program debugging deals with correcting known errors in the program. One become aware of errors in the program in three ways. First, and explicit diagnostic provide us with the exact location and nature of the error. Second, a diagnostic may occur but the exact location of error is unclear. For example, the computer may indicate that an error has occurred in a line representing a long equation including user defined functions. This error could be due to syntax error in the equation or it could be due to syntax in the statements defining the functions. Third, no diagnostic occur but the program does not operate properly. after debugging, the program is finally tested in modules.

CONCLUSION AND RECOMMENDATION

CONCLUSION

The importance of Numerical approach to boundary-value problems arising in the area of engineering and physical sciences has been emphasized, the concepts of Numerical methods like Finite-differences, passage, Galerkins and Collacation have been discussed.

It has been found out that Numerical methods of solving boundary-value problems reduce the problems to arithmetic and certain logical operations on numbers, that is to the operations usually performed by computers.

The solution obtained by any of the numerical methods discussed is usually approximate, that is it has some error. The following are some of the sources of error in an approximate solution:

(i) The error of the Initial data (Input parameters)

(ii) The error of the method of solution

(iii) Round-off error in arithmetic and other type of operations on the numbers involved.

Accuracy attainable with finite-difference method clearly depend upon the fineness of the mesh and upon the order of the finite-difference approximation. As the mesh is refined, the number of equations to be solved increases and become more difficult. The use of higher-order approximations will yield greater accuracy for the same mesh size but results in considerable complication.

In Galerkin and Collocation methods one can obtain a sequence of approximations by increasing the number N of basis functions. An estimate of the accuracy can then be obtained by comparing these approximate solution at a fixed set of points on the Interval [a,b].

RECOMMENDATION

Since Numerical methods of solving boundary-value problems reduce the problems to arithmetic and certain logical operations on numbers which are usually performed by computers, also considering the speed and ease at which computer can evaluate these operations to give the solution to these problems, more effort should be geared towards the study of Numerical approach to boundary-value problems. In particular I recommend further study on other numerical methods, like variational, least square and shooting methods for solving boundary-value problems.

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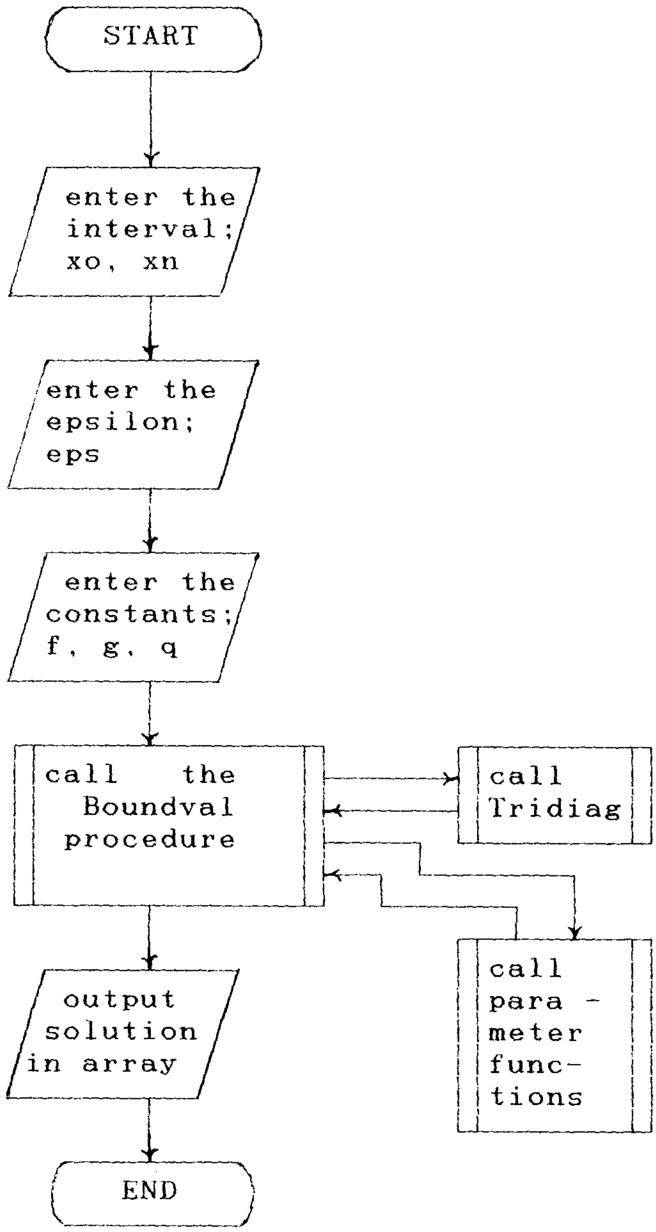
APPENDIX A

FLOW CHART



APPENDIX B

```
Program Numsol;
uses crt;

const npts = 50;

type vect1 = array[1..50] of real;

Var    xo,xn,h,x:real;
       i,f,g,q:integer;
       mesh : integer;
       f1,g1,q1:real;
       eps:real;
       y_out:vect1;

Function fnc1(x:real):real;
 begin
  for i := 1 to mesh do
   x := xo+(i*h);
   fnc1 := f*(xo+(i*h))
 end;

Function fnc2(x:real):real;
 begin
  for i := 1 to mesh do
   x := xo+(i*h);
   fnc2 := g*(xo+(i*h))
 end;
Function fnc3(x:real):real;
 begin
  for i := 1 to mesh do
   x := xo+(i*h);
   fnc3 := q*(xo+(i*h))
 end;

Procedure Assign(f,g,q:real);
  begin
{  f1:=fnc1;
  g1:=fnc2;
  q1:=fnc3;   }
  end;

Procedure tridiag(n: integer; a, b, c: vect1; var d: vect1;
                  eps: real);
{
**********************************************************************
*
*      Solution of simultaneous linear equations with  tridiagonal
*      coefficient matrix. N is the number of linear equations. a,
*      b, and c are the arrays of the subdiagonal, diagonal, and
*      superdiagonal elements, respectively. d is the constant
*      vector on input and the solution vector on output. The matrix
*      is singular or ill-conditioned if a diagonal element becomes
*      < eps .
*
```

```
***********************************************************************
}
type
  flag = (continue, singular);

Var
  k: integer;
  stasus: flag;

begin
  if (b[1] < eps ) then stasus := singular
    else stasus := continue;

  {perform LU factorization}

  for k := 2 to n do
    while (stasus = continue) do
      begin
        a[k] := a[k]/b[k-1];
        b[k] := b[k] - a[k]*c[k-1];

        {check that diagonal element doesn't become too small}

        if (b[k] < eps) then stasus := singular;
        d[k] := d[k] - a[k]*d[k-1]
     end;

  case stasus of
    continue: {perform backward and forward substitution}
      begin
        d[n] := d[n]/b[n];
        for k := n-1 downto 1 do
          d[k] := (d[k] - c[k]*d[k+1])/b[k]
        end;

     singular:  {Error section}
       writeln('***Error***Matrix is singular or ',
         'ill-conditioned')
   end {of case}
end;


Procedure Boundval(npts:integer;xo,xn:real;var y:vect1;eps:real;f1:re
{
***********************************************************************
*          Solution of the second-order boundary value problem
*
*                        y" + f(x)y' + g(x)y = q(x)
*
*        using the finite difference method with npts mesh points.
*        x0 = x(1) and x(N) = x(npts).  y(1) and y(0) are known; this
*        procedure determines the solution at the interior mesh points.
*        The data type vector is defined as array [1..npts] of
*        real in the calling program. f, g, and q are user-supplied
*        functions. eps is the smallest tolerable diagonal element for
```

```
*        the tridiagonal matrix to be considered nonsingular.
*
*
*********************************************************************
}
var
  a, b, c, d: vect1;
  temp, h : real;
  i, mesh: integer;

begin
      {calculate step size}

      h := (xn - xo) / (npts - 1 );
      mesh := npts - 2;

     {Evaluate tridiagonal matrix and constant vector}
     for i := 1 to mesh do
       begin
         temp := fnc1(x) * h/2.0;
         a[i] := 1 - temp;          {Subdiagonal elements}
         b[i] := -2 + fnc2(x)*sqr(h); {Diagonal elements}
         c[i] := 1 + temp;          {superdiagonal elements}
         d[i] := fnc3(x)*sqr(h)        {Constant vector}
       end;

         {Add end conditions}

         d[i] := d[i] - a[i]* y[i];
         d[mesh] := d[mesh] - c[mesh] * y[npts];

         {Solve equations and store solution in y}

         tridiag(mesh, a, b, c, d, eps);
         for i := 1 to mesh do
          y[1 + i] := d[i]
      end;


  begin
   clrscr;
   write('Enter the first point');
   readln(xo);
   write('Enter the last point');
   readln(xn);
   write('Enter the value of f ');
   readln(f);
   write('Enter the value of g ');
   readln(g);
   write('Enter the value of q ');
   readln(q);
   write('Enter the value of eps');
   readln(eps);
   boundval(npts,xo,xn,y_out,eps,f,g,q);
   writeln('running');
```

```
 for i := 1 to mesh do
   writeln(y_out[1+i]);
 readln;
end.
```