TRANSFER FUNCTIONS AND OPTIMAL CONTROL

BY

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CERTIFICATION

THIS IS TO CERTIFY THAT THIS PROJECT WAS CARRIED OUT BY WAZIRI VICTOR ONOMZA, UNDER THE SUPERVISION OF DR. S . A. REJU IN THE DEPARTENT OF MATHEMATICS WITH COMPUTER SCIENCE, FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGER STATE.

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DEDICATION

Of all the living, I owe all to my beloved mother, Akwishim, by whose determination, craftmanship, faith and sweat gave me hope and the future; to my late father, Augustine A. Waziri, who gave the opportunity to grow and become; and to my wife and children, who made everything possible. To my great uncle, Bawa M.K. Yerima, who discovered my potential from childhood and has since being the guardian ever.

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I wish to use this opportunity to pay my lasting tribute to the memory of a greater teacher and uncle Mr. Pius M. Azuwah who died on the 18th August, 1997.

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May the good God bless each and everyone of the above named persons abundantly.

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ABSTRACT

The project studies expensively some application of optimal control theory; and with respect to Kalman and Ackerman equations. Pontryagin's maximum principle otherwise called the Hamiltonian form was used to obtain the Riccati Equation ; from which the optimal control of a quadratic cost function is obtained.

At the centre of the project is the Faddeev algorithm which helps to obtain the transfer functions. A computer program was developed for the algorithm to handle up to 3 dimensional matrices, indicating the number of iterations before convergence. The algorithm code was used to solve sample problems. These examples are obtained as output in the last pages of chapter 2.

CHAPTER ONE

ELEMENTS OF CONTROL THEORY

1.1 INTRODUCTION

We study in this chapter as literature review the elements of time-invariant linear system. Our concern is specifically on the representation of a control system by block diagram and analysis of state space system

DEFINITION 1.1 (CONTROL SYSTEM)

A control system maybe defined as consists of components which functions under excited signals. The signals are the input response and the output response.

1.2 BLOCK DIAGRAMS OF CONTROL SYSTEM

Systems are often represented by block diagrams. It is essential to understand and learn the rules for working with block diagrams.

Definition 1.2 (Open Loop Control System)

An open loop control system maybe defined as the simplest system whose components respond only to the input response and the output response. In this system no feedback is fed back into the system to monitor it's function.

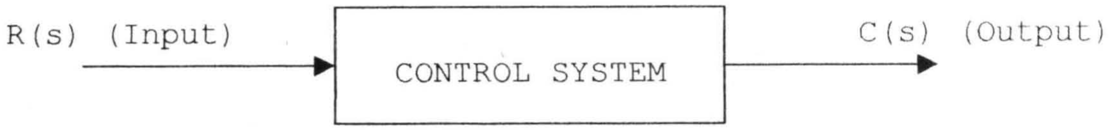


Fig 1.1 A simple control system.

R(s) is the Laplace transform of the reference (input) response and C(s) is the Laplace transform of the output response.

1.2.1 THE FEEDBACK CONTROL SYSTEM

A feedback system is representable by the block diagram below:-

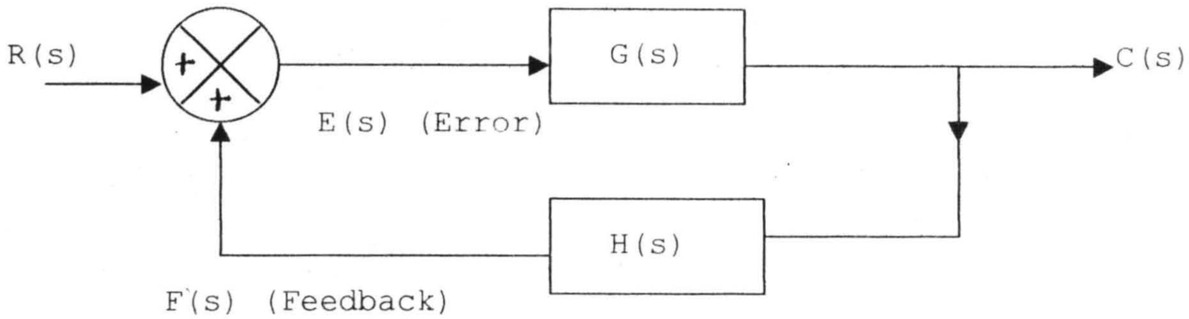


Fig. 1.2: Feedback Control System

Every feedback control system can be represented by the general block diagram shown in fig 1.2, in which G(s) is sometimes called the direct transfer function or direct transmission gain (DTG). H(s) represents all the components between the output and input summing points via the feedback path (H(s)). In many instances H(s) is merely a constant and quite frequently it is equal to unity. When H(s) = 1, the signal E(s) is the difference between the input and output

[R(s) - C(s)]•E(s) is commonly called the error. It is also called the actuating signal since a signal present at E(s) will actuate or make the system respond. The signal F(s) represents the feedback signal. When H(s) = 1,F(s) is equal to the output C(s).

Referring to figure 1.2 we can write

$$C(s) = G(s)E(s) \tag{1.2.1}$$

$$F(s) = H(s)C(s) \tag{1.2.2}$$

$$E(s) = H(s) - F(s) \tag{1.2.3}$$

Substituting (1.2.2 into (1.2.3) yields

$$E(s) = R(s) - H(s)C(s) \tag{1.2.4}$$

and now substituting (1.2.4) into (1.2.1) we get

$$\therefore \quad C(s) = G(s)[R(s) - H(s)C(s)]$$

$$= G(s)R(s) - H(s)G(s)C(s)$$

i.e.

$$C(s)[1 + H(s)G(s)] = G(s)R(s) \tag{1.2.5}$$

Then

$$C(s)/R(s) = G(s)/[1 + H(s)G(s)] \qquad (1.2.6)$$

Equation (1.2.6) which relates the output to input is called the closed loop transfer function of the system the quantity H(s) G(s) is the product of all the gains in the loop it is also the ratio of feedback signal to error signal with feedback loop

$$F(s)/E(s) = H(s)G(s) \qquad (1.2.7)$$

Equation (1.2.7) is called the open loop transfer function or simply the loop gain.

Equation (1.2.6) can be stated as

CLOSED LOOP TRANSFER = DTG/[1 + LOOP GAIN] (1.2.8)

Another equation of interest is obtained by substituting e.g (1.2.1) into (1.2.6) to get

$$E(s)/R(s) = 1/[1 + H(s)G(s)] \qquad (1.2.9)$$

called the actuating signal ratio. Equation (1.2.9) relates the error to the input.

If the input R(s) is set to zero we get

$$1 + H(s)G(s) = 0 \qquad (1.2.10)$$

called the characteristics equation of the system.

It is from this equation that information about the stability or behaviour of the system is derived.

1.3 THE DYNAMICS OF LINEAR SYSTEMS

The dynamic behaviour of many dynamic systems is quite naturally characterized by systems of first order differential equations For a general system these equations in state space notation take the form

$$x' = f(x, u, t) \tag{1.3.1}$$

and in a linear system they take the special form

$$x' = A(t)x + B(t)u \tag{1.3.2}$$

where $x = [x_1, x_2, .., x_n]$ is the system state vector and $u = [u_1, u_2, ..., u_m]$ is the input vector.

If the matrices A and B in (1.3.2) are constant matrices that is not functions of time he system dynamics is said to be "time invariant", with the state space methods the description of the system dynamic in the form of differential equations is retained throughout this project. If a subsystem is characterized by a transfer function it is often necessary to convert the transfer function to differential equations in order to proceed by state space methods.

In chapter 3 we shall develop the general formula for the solution of a vector matrix differential equation in the form of (1.3.2) in terms of a very important matrix known as the state transition matrix using the Faddeev algorithm which described how the state X(t) of the system at time t evolves into (or from) the state X(t) T time to, for time invariant system the state transition matrix is the matrix exponential function which is easily calculated

1.3.1 DERIVATION OF STATE VARIABLE MODEL

Consider the single input single output (SISO) nth-order transfer function of a system

$$H(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b)}{s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_1 s + \alpha}$$

The first order differential equations are formed from phase variable states; selected via defining the variables as

$$x_1(t) = y(t) \tag{1.3.4}$$

$$x_2(t) = x'_1(t) = y'(t) \tag{1.3.5}$$

$$x_3(t) = x'_2(t) = y''(t) \tag{1.3.6}$$

...

$$x_n(t) = x'_{n-1}(t) = y^{n-1}(t)$$

$$= \alpha_1 y(t) - \alpha_2 y'(t) - \ldots - \alpha_{n-1} y^{n-2}(t) + Ku(t) \qquad (1.3.7)$$

From which the first order d.e for the variables $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ are given by:

$$x'_1(t) = x_2(t)$$

$$x'_2(t) = x_3(t)$$

$$\ldots$$

$$x'_n(t) = - \alpha_1 x_1(t) - \alpha_2 x(t) - \ldots - \alpha_{n-1} x_{n-1} t) + Ku(t) \qquad (1.3.7)$$

$$Y(t) = (b_0 - b_n \alpha)(b_1 - \beta_v \alpha_1) \ldots (b_{n-1} - \beta_v \alpha_{v-1})X + Ku(t) \qquad (1.3.9)$$

i.e.

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \\ \ldots \\ x'_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \ldots & & \\ -\alpha_1 & -\alpha_2 & \ldots & & -\alpha_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ldots \\ K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \ldots \\ u_m \end{bmatrix} \qquad (1.3.10)$$

Equations (1.3.9) and 1.3.10) may be written in the general form as

$$x'(t) = Ax(t) + Bu(t) \qquad (1.3.11)$$

$$y(t) = Cx(t) + Du(t) \qquad (1.3.12)$$

where $u(t)$ is the single input or forcing signal $y(t)$ is the si[illegible]l output signal; A is termed the nxn-dimensional input matrix; B [illegible] he nx1 dimensional input matrix; C is the 1xn dimensional output matrix and D is the 1x1 dimensional feedforward matrix.

CHAPTER TWO

TRANSFER FUNCTION

2.1 INTRODUCTION

In analysis and design, differential equations are usually used to describe control systems. Block diagrams are devices for displaying the interrelationships of the equations pictorially. Each component is described by its transfer function. Here in this chapter, we shall study the interrelationship of these components using the faddeev algorithms.

2.2 TRANSFER FUNCTIONS

A monovariable system with input U(t) and outputy(t) is said to be linear if the relationship between u(t) and y(t) is a linear differential equation with constant coefficients (a_i and b_i);

$$an \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^y u}{dt^m} + \ldots + b_1 \frac{du}{dt} + b_0 u \quad \ldots(2.2.1)$$

Using the Laplace transforms, this equations gives the transfer functions:

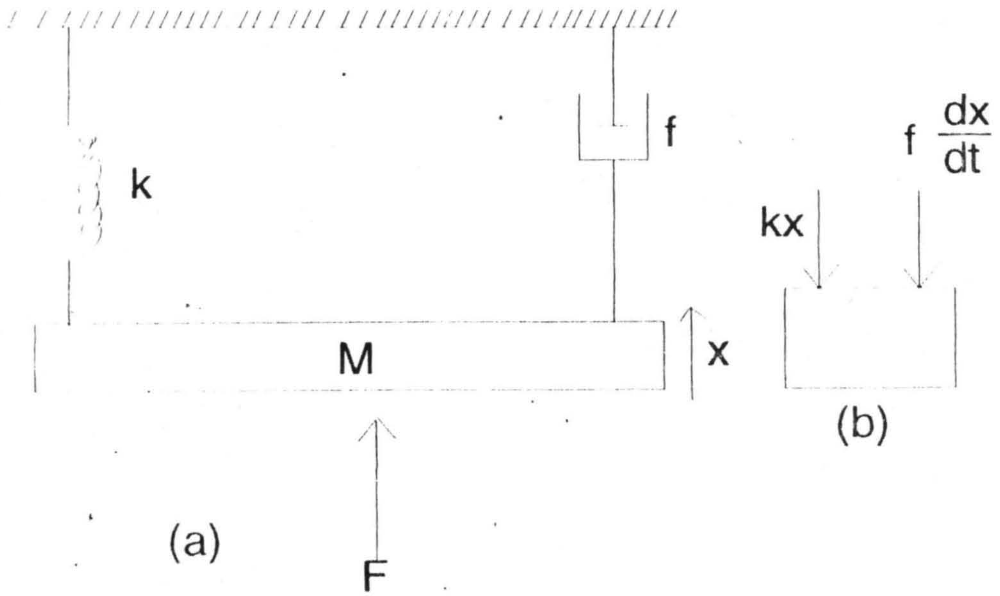
$$\frac{y(s)}{u(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0} = H(s) \qquad \ldots(2.2.2)$$

The ratio of the function defined by H(s) of the Laplace transform is called <u>transfer function</u>.

The componenets (u(t) and y(s)) are assumed at rest prior to excitation; all initial values are assumed to be zero when determining the transfer function.

EXAMPLE 2.2.1 (MECHANICAL SYSTEM).

Consider the mechanical system shown in figure 2.2.1. It is simply a mass M attached to a spring (stiffness K) and a dash pot (viscous friction coefficient f) on which the force f operates. Displacement x is positive in the direction shown.



Figures 2.2.1 (a): MASS-SPRING-DASHPOT (b):FREE-BODY DIAGRAM.

The position is taken to be at a point where the mass and spring are in static equilibrum.

By applying Newton's 2nd. law of motion to the free-body diagram the force equation can be written as

$$F - f\frac{dx}{dt} - kx = M\frac{d^2x}{dt^2} \Rightarrow F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx \quad ....(2.2.3)$$

Equation (2.2.3) shows that the dynamics of mass-spring-dashpot shown in figure (2.2.1a) is described by the second-order differential equations (2.2.3).

Taking the Laplace transform of each term of this equation, we obtain (assuming zero initial condition)

$$F(s) = Ms^2 X(S) + FS\, X(S) + K\, X(S) \quad \ldots\ldots\ldots(2.2.4)$$

Taking x(s) to be the input and F(s) as the output, the transfer function is

$$H(s) \quad \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + fs + k} \quad \ldots\ldots(2.2.5)$$

EXAMPLE 2.2.2 [ELECTRICAL SYSTEM]

The Resistor, Inductor and Capacitor are the three basic elements of electrical circuits. These circuits are analysed by the application of kirchoff's voltage and current laws.

Consider the L.R.C series circuit shown in figure (2.2.2).

The governing equations of system are

$$L\frac{di}{dt} + Ri + \frac{1}{c}\int_{\infty}^{t} idt = e \quad \ldots\ldots(2.2.6)$$

$$\frac{1}{c}\int_{\infty}^{t} idt = e_0 \quad \ldots\ldots(2.2.7)$$

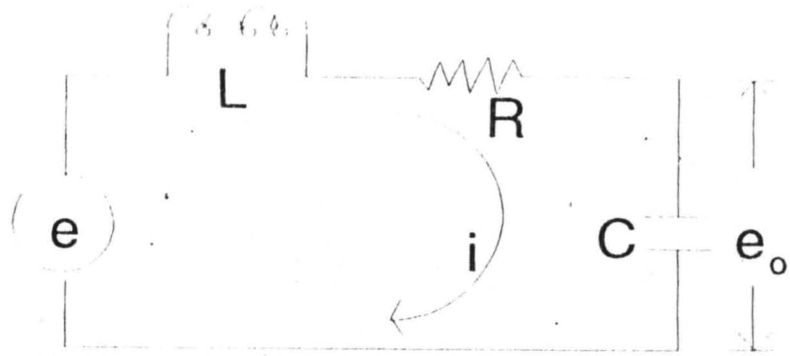


Figure (2.2.2): RLC SERIES CIRCUIT.

Taking the Laplace trasform of each term equations 2.2.6 and (2.2.7) with zero initial conditions, we have the following resulting equations:

$$SLI(s) + RI(s) + \frac{I}{SC} I(s) = E(s) \qquad \text{........(2.2.8)}$$

$$\frac{I}{SC} I(s) = E_o(s) \qquad \text{..........(2.2.9)}$$

NOW ASSUMING e is the input variable and e_o the output variable, the transfer function of the system is

$$\frac{E_o(s)}{E(s)} = \frac{I}{LCS^2 + RCS + I} \qquad \text{.....(2.2.10)}$$

From equations (2.2.1) and (2.2.2), we see that equations (2.2.5) and (2.2.9) reveal that the transfer function is an expression in S.domain, relating the output and input of the linear time-invariant system in terms of the system parameters and is independent of input.

Transfer functin of physical system is represented in Blocks diagrams. Each block describes a trasnfer function. The Block-diagram of equation (2.2.10) for example is depicted I figure (2.2.3)

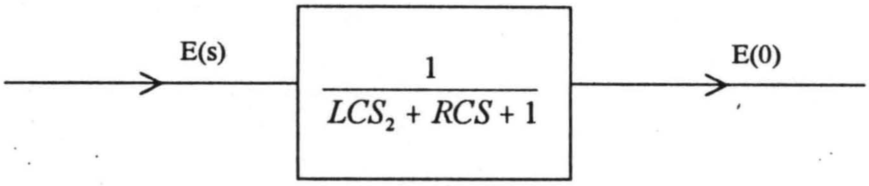


Figure (2.2.3) Block diagram for transfer function for eq. (2.2.10)

2.3 STATE PHASE -VARIABLE USING TRANSFER FUNCTIONS.

It is possible to determine the phase variable state model once the system model is known in the transfer function or in the differential equation form.

The general form of an n^{th} - order differential equation relating output y(t) and the input u(t) of a linear continuos-time system is given by equation (2.2.1) where the ai's and bi's are constants, m and n are inputs with m ≥ n and

$$Y^n \underline{\Delta} \frac{d^n y}{dt^n} \qquad \text{.... (2.3.1)}$$

The initial conditions are expressible as y(0), $y^1(0)$,.....$y^{n-1}(0)$.

Under the assumption of zero condition, the transfer function is given by equation (2.2.2).

Consider a case where the transfer function does not have zeros. Such a transfer function has the form.

$$H(s) \quad \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} \qquad \dots.(2.3.2)$$

Equation (2.3.2) has a corresponding differential equation

$$y^n + a_1 y^{n-1} + \dots\dots + a_{n-1} y + a_n y = bu \qquad \dots..(2.3.3)$$

Let the state variables as

$$\begin{aligned} x_1 &= y \\ x_2 &= y^{1} \\ x_3 &= y^{11} \\ &\vdots \\ &\vdots \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned} \qquad \dots\dots\dots\dots\dots\dots\dots\dots(2.3.4)$$

Equations (2.2.4) is reduced to a set of n-first-order differential equations given below:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ &\vdots \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x - d_{n-1} x_2 - \dots\dots - a_1 x_n + b_m \end{aligned} \qquad \dots\dots\dots\dots(2.3.5)$$

Equations (2.3.5) results in the following state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \\ +\dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots\dots & 0 \\ 0 & 0 & 1 & \dots\dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots\dots & 1 \\ a_n & a_{n-1} & a_{n-2} & a_1 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ b \end{bmatrix} u \qquad \dots..(2.3.6)$$

or

$\dot{x}^1 = Ax + Bu$ -- (2.3.7)

$y = Cx$ -- (2.3.8)

Equation (2.3.7) is called the equation of state while equation (2.3.8) is the equation of measurement.

The matrix " A" has avery special form. it has all 1's in the upper off-diagonal, its last row is comprised of negatives of the coefficients of the differential equation and all other elements are zero. This form of matrix A is known as the

BUSH-FORM OR COMPANION FORM

2.4 DERIVATION OF TRANSFER FUNCTION FROM STATE MODEL.

Consider the general state model .

$\dot{x} = Ax + Bu$ -- (2.4.1a)

$Y = Cx + Du$ -- (2.4.1b)

The transfer functions may be obtained as follows:

Taking the Laplace transform of equations (2.4.1) , we have

$Sx(s) - X_0 = Ax(s) + Bu(u)$ -- (2.4.2a)

$Y(s) - Cx(s) + Du(s)$ -- (2.4.2b)

i.e $X(s) = (SI - A)^{-1} x_0 + (SI - A)^{-1} BU(s)$ ---------------------(2.4.3)

substituting (2.4.3) in (2.44b) ,

$Y(s) = C(SI - A)^{-1} X_0 + C(SI - A)^{-1} BU + DU(s)$ -------------------- (2.4.4)

Assuming zero initial conditions, we get the system transfer as

$$H(s) = \frac{Y(s)}{u(s)} = c[SI - A]^{-1}B + D = C\frac{adj(SI - A)B}{\det(SI - A)} + D \quad \text{... (2.4.5)}$$

The quantity $(SI-A)^{-1}$ of equation (2.4.5) is called the Resolvent matrix.

EXAMPLE 2.4.1 (SINGLE - INPUT - SINGLE - OUTPUT)

Consider the linear single-input single-output (SISO) system (A,B,C^T,D) described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}u \qquad \ldots.(2.4.6)$$

$$y = |1 \;\; 0 \;\; 0|x \;\; \ldots\ldots(2.4.7)$$

$$\text{The det } (SI-A) = \det\begin{bmatrix}s & -1 & 0\\ 0 & s+1 & 1\\ 0 & 0 & s+3\end{bmatrix} \quad s(s+1)(s+3) \;\; \ldots\ldots(2.4.8)$$

Therefore the transfer function H(s) is given by

$$\frac{y(s)}{u(s)} = \frac{|1 \;\; 0 \;\; 0|\begin{bmatrix}(s+1)(s+3) & (s+3) & 1\\ 0 & (s+3) & 3\\ 0 & 0 & (s+3)\end{bmatrix}\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}}{s(x+1)(s+3)} = \frac{s+2}{s(s+1)(s+3)} \quad \ldots(2.4.9)$$

2.5 FADDEEV ALGORITHM FOR THE RESOLVENT CALCULATION

There are numerous methods for the computing of the resolvent $[(SI-A)^{-1}]$. One of such methods is the use of (2.4.5). This requires the calculations of determinants in both the numerator and denominator. Another method for calculating $(SI-A)^{-1}$ is the iterative scheme called the Faddeev ALGORITHM.

Suppose we denote (SI-A) = M then the inverse of M could be written as

$$M^{-1} = (SI-A)^{-1} = \frac{adj\ (SI-A)}{|SI-A|} \quad ....(2.5.1)$$

Letting det $(SI-A) = |SI-A| = \phi(S) = S^n + \alpha_{n-1}S^{n-1} + ..+\alpha_0$ ..(2.5.2)

and

$$adj\ (SI-A) = \Gamma_{n-1}S^{n-1} + \Gamma_{n-2}S^{n-2} + ...+ \Gamma_o$$

Where Γi are matrices and αi are constants.

Then

$$(SI-A)^{-1} = \frac{\Gamma_{n-1}S^{n-1} + \Gamma_{n-2}S^{n-2} + ...+ \Gamma_o}{S^n + \alpha_{n-1}S^{n-1} + \alpha_{n-2}S^{n-2} + ...+ \alpha_0}$$

$$= \frac{\Gamma_{n-1}S^{n-1} + \Gamma_{n-2}S^{n-2} + ...+ \Gamma_0}{\phi(s)} \quad ....(2.5.4)$$

$$\alpha_{n-1}, \alpha_{n-2}, ..., \alpha_0 \quad \Gamma_{n-1}, \Gamma_{n-2}, ... \Gamma_0$$

can be calculated in a recursive manner in the form;

$$\Gamma_{n-1} = I \quad ; \quad \alpha_{n-1} = -tr\ (A\ \Gamma_{n-1})$$

$$\Gamma_{n-2} = A\ \Gamma_{n-1} + \alpha_{n-1}\ I \quad ; \quad \alpha_{n-2} = \frac{-tr}{2}\ (A\ \Gamma_{n-2})$$

$$\vdots \qquad \qquad \vdots$$

$$\Gamma_0 = A\ \Gamma_1 + \alpha_1 I; \ \alpha_0 = \frac{-t_r}{n}\ (A\ \Gamma_0)$$

$$0 = A\,\Gamma_0 + \alpha_0 I \quad .....(2.5.5)$$

Where tr(X), the trace of x, is the sum of all the diagonal elements of the matrix X.

EXAMPLE 2.5.1 (FADDEEV ALGORITHM)

We want to compute $(SI-A)^{-1}$ which appeared in example (2.4.1) by Faddeev ALGORITHM.

SOLUTION

By applying the recursive Γ_i's and αi's, we have

$\Gamma_{3-1} = \Gamma_2 = I, \ \alpha_2 = -\ t_r(A\Gamma_2)$

$$T_r \begin{vmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} - (-0 - 3 - 1) \quad 4$$

$\Gamma_{3-2} = \Gamma_1 = A\Gamma_2 + \alpha_2 I$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha_{3-2} = \alpha_1 = \frac{T_r\,(A\Gamma_1)}{2} \quad -\frac{1}{2}\,T_r \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} \begin{vmatrix} 4 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3$$

$$\Gamma_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus from (2.5.2) and (2.5.4)

$$(SI-A)^{-1} = \frac{1}{S^3+4S^2+3S}\begin{bmatrix} s^2+4s+3 & s+3 & -1 \\ 0 & s^2+3s & -s \\ 0 & 0 & s^2+2 \end{bmatrix}$$

EXAMPLE 2.5.2 (EQUATION OF STATE TO TRANSFER FUNCTION BY FADDEEV ALGORITHM)

Consider a monovariable process described by its equations of state and of measurement as:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u \quad \ldots\ldots(2.5.6)$$

Determine the transfer function Y(s)|U(s) of this process using Faddeev algorithm.

SOLUTION

By using equaton (2.4.5)

$H(s) = Y(s)/U(s)$

$= C[SI-A]^{-1}B$ ....(2.5.7)

We can calculate the Resolvent first by the Faddeev algorithm.

Now $\Gamma_2 = I$; by definition

$\alpha_2 = -tr\ (A\Gamma_2)$

$$= t_r \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -(-4) = 4$$

$$\Gamma_1 = A\Gamma_2 + \alpha_2 I$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ -1 & -3 & 0 \end{bmatrix}$$

$$\alpha_1 = -T_r(A\Gamma_1)$$

$$= -T_r \frac{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ -1 & -3 & 0 \end{bmatrix}}{2}$$

$$= \frac{T_r}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 25 & 3 \end{bmatrix} = -\frac{1}{2}(-3-3) = -\frac{1}{2} x (-6) = 3$$

$$\Gamma_0 = A\Gamma_1 + \alpha_1 I$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ -1 & -3 & -0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\alpha_0 = -T_r(A\Gamma_0) = -T_r/3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= -t_r/3 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= -1/3(-1-1-1) = 1$$

$$Thus(SI-A)^{-1} = \frac{1}{\Phi(S)} \begin{bmatrix} S(S+4)+3 & S+4 & 1 \\ -1 & S(S+4) & S \\ -1 & -3(S+1) & S^2 \end{bmatrix}$$

where $\phi(s) = s^3 + 4s^2 + 3s + 1$

So finally, the transfer function

$$C[SI-A]^{-1}B = C \frac{\begin{bmatrix} (s+3)(s+1) & (s+4) & 1 \\ 1 & s(s+4) & s \\ -5 & -3(s-1) & s^2 \end{bmatrix}}{s^3+4s^2+4s+1} B$$

$$C(SI-A)^{-1}B = \frac{10}{(S+3)(S+1)}$$

EXAMPLE 2.5.3 (APPLICATION OF FADDEEV ALGORITHM TO TRANSFER FUNCTION)

Find, by use of the Faddeev algorithm, the transfer function of the linear system (A,B,C_T) given by

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & -3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad C^T = (-1 \quad 1 \quad 0)$$

SOLUTON

We first apply the Faddeev algorithhm, then obtain the resolvent matrix.

$$\therefore (SI\ A)^{\ 1} = \frac{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 01 & \end{bmatrix} S2 + \begin{bmatrix}4 & 0 & 1\\ 0 & 4 & 1\\ 0 & 3 & 4\end{bmatrix} S + \begin{bmatrix}7 & 3 & 2\\ 0 & 4 & 2\\ 0 & 6 & 4\end{bmatrix}}{S^3\ 6S^2\ 15s\ 14}$$

$$= -\frac{t_r}{2}\begin{bmatrix}-2 & 0 & 1\\ 0 & -2 & 1\\ 0 & -3 & -2\end{bmatrix}\begin{bmatrix}7 & 3 & 2\\ 0 & 4 & 2\\ 0 & 6 & 4\end{bmatrix} = \frac{t_r}{3}\begin{bmatrix}14 & 12 & 0\\ 0 & 14 & 0\\ 0 & 0 & -14\end{bmatrix} = -\frac{1}{3}(-42) \quad 14$$

$$\frac{\begin{bmatrix}s^2+4s+7 & -3 & s+2\\ 0 & s^2+4s+4 & s+2\\ 0 & (3s+6) & s^2+4s+4\end{bmatrix}}{s^3+6s^2+15s+14}$$

$$C^T(SA\ I)^{\ 1}B = [\ 110]\begin{bmatrix}s^2+4s+7 & -3 & s+2\\ 0 & s^2+4s+4 & s+2\\ 0 & (3s+6) & s^2+4s+4\end{bmatrix} x \begin{bmatrix}1\\ 2\\ 2\end{bmatrix} \div s^3\ 6s^2\ 15s\ 14$$

Thus the transfer function is

$$\therefore H(sO \quad \frac{s^2 - 6}{s^3+6s^2+15s+14}$$

$$Now(S\ A)^{-1} = \frac{\Gamma_{n\ 1}S^{n\ 1} + S^{n\ 2}\Gamma_{n\ 2} + \ldots + \Gamma_0}{S^n + \alpha_{n\ 1}S^{n\ 1} + \ldots + \alpha_0} \quad \ldots\ldots(2.5.8)$$

$$Now\ A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$\Gamma_2 = I$; by defnition.

$\alpha_2 = -t_r[A\Gamma_2]$

$$t_r \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad t_r \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} = -(-6) = 6$$

$\Gamma_1 = A\Gamma_2 + \alpha_2 I$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 4 \end{bmatrix}$$

$\alpha_1 = -T_r/2[A\Gamma_1]$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 4 \end{bmatrix} = -\frac{t_r}{2} \begin{bmatrix} 8 & 3 & 2 \\ 0 & 11 & 2 \\ 0 & 6 & 11 \end{bmatrix} = -\frac{1}{2}(-30) \quad 15$$

$\Gamma_0 = A\Gamma_1 + \alpha_1 I$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 2 \\ 0 & 11 & 2 \\ 0 & 6 & 11 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 2 \\ 0 & 4 & 2 \\ 0 & 6 & 4 \end{bmatrix}$$

$\alpha_0 = -t_r/3[A\Gamma_0]$

$$\frac{\left|-(s^2+4s+7)(s^2+4s+1)\right| \begin{bmatrix}1\\2\\2\end{bmatrix}}{s^3+6s^2+15s+14} = \frac{s^2-14s-7+2s^2+4s+1}{s^3+6s^2+15s+14} = \frac{s^2\ 6}{s^3+6s^2+15s+14}$$

EXAMPLE 2.5.4 (RESOLVENT AND TRANSITION MATRICES)

This example is very important for the study of the next chapter - the transition matrix.

Find the resolvents and transition for each of the following:

$$(i)\ A_1 = \begin{bmatrix}1 & 0 & 0\\1 & +2 & 0\\1 & +2 & +3\end{bmatrix} \quad (ii)\ A_2 = \begin{bmatrix}-2 & 1 & 1\\1 & -2 & 1\\1 & 1 & -1\end{bmatrix}$$

Solution:

(i)

$$A_1 = \begin{bmatrix}1 & 0 & 0\\1 & 2 & 0\\1 & 2 & 3\end{bmatrix}$$

The resolvent matrix of A_1 is:

$$(SI\ A)^{-1} = \frac{Adj[SI\ A_1]}{\det(SI\ A_1)} = \frac{\Gamma_{n-1}S^{n\ 1} + S^{n\ 2}\Gamma_{n-2} + \dots + \Gamma_0}{\phi(s)}$$

Now n = 3, thus:

Γ_2 = I; by definiton

α_2 = $-tr[A_1\Gamma_2]$

$$= -t_r \begin{bmatrix}1 & 0 & 0\\1 & 2 & 0\\1 & 2 & 3\end{bmatrix} = 1(1\ 2\ 3) = -(-4) = 4$$

$\alpha_2 = 4$

$\Gamma_1 = A_1\Gamma_2 + \alpha_2 I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$\alpha_1 = -t_r/2\,[A_1\Gamma_1]$

$$= \frac{-t_r}{2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \frac{-t_r}{2} \begin{bmatrix} 5 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 2 & 3 \end{bmatrix} = \frac{1}{2}[5-4-3] \quad 1$$

$\Gamma_0 = A_1\Gamma_0 + \alpha_1 I$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

$\alpha_0 = -t_r/3\ [A_1\Gamma_0]$

$$= \frac{\begin{bmatrix} s^2+5s+6 & 0 & 0 \\ s+3 & s^2+2s-3 & 0 \\ \underline{s+4} & 2s-2 & \\ s^2+s-2 & & \end{bmatrix}}{s^3\ 4s^2\ s\ 6}$$

$$= -\frac{t_r}{3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 2 & 2 \end{bmatrix} = \frac{t_r}{3} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \frac{-1}{3}(6\ 6\ 6) = -6$$

$$(SI\ A)^{-1} = \frac{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^2 + \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} S + \begin{bmatrix} 6 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 2 & 2 \end{bmatrix}}{s^2\ 4s^2\ s-32.7}$$

$$= \frac{\begin{bmatrix} (s+2)(s+3) & 0 & 0 \\ (s+3) & (s+3)(s-1) & 0 \\ s+4 & 2(s-1) & (s+2)(s-1) \end{bmatrix}}{(s-1)(s+2)(s+3)}$$

$$\begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ \frac{1}{(s-1)(s+2)} & \frac{1}{s+2} & 0 \\ \frac{s+4}{(s-1)(s+2)(s+3)} & \frac{2}{(s+2)(s+3)} & \frac{1}{s+3} \end{bmatrix}$$

Thus (2.5.9) is the resolvent matrix.

By taking the inverse laplace transform of (2.5.9), we obtain the transition matrix $\phi(t)$.

$$\phi(t) \begin{bmatrix} e^{t} & 0 & 0 \\ \frac{1}{3}(e^{t}-e^{-2t} & e^{-2t} & 0 \\ \frac{5}{12}e^{t}-2overe^{-2t}+\frac{1}{4}e^{-3t} & 2(e^{-2t}-e^{-3t}) & e^{-3t} \end{bmatrix}$$

Solution II:

$$(ii)\ A_2 \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\therefore [SI\ A]^{1} = \frac{S^2 \mid \Gamma_1 S \mid \Gamma_0}{S^3 \mid \alpha_2 S^2 \mid \alpha_1 S \mid \alpha_0}$$

$\Gamma_2 = I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \textit{ by definition}$$

$\alpha_2 = -tr[A\Gamma_2]$

$$tr \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = -(-6) \quad 6$$

$\Gamma_1 = A\Gamma_2 + \alpha_2 I$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$\alpha_1 \quad -tr/2[A\Gamma_1]$

$$= \frac{t_r}{2} \begin{bmatrix} .2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \frac{t_r}{2} \begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{bmatrix} = -\frac{1}{2}(-18) \quad 9$$

$\Gamma_0 = A\Gamma_1 + \alpha_1 I$

$$= \begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$\alpha_0 = -tr/3 \; A\Gamma_0$

$$\frac{t_r}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \quad \frac{t_r}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow \alpha_0 = 0$

$$\therefore [SI\ A]^{-1} = \frac{\begin{bmatrix} s^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} S + \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}}{s^3 + 6s^2 + 9s}$$

$$= \frac{\begin{bmatrix} s^3+4s+3 & s+3 & s+3 \\ s+3 & s^2+4s+3 & s+3 \\ s+3 & s+3 & s^2+4s+3 \end{bmatrix}}{s(s^2+6s+9)}$$

$$= \frac{\begin{bmatrix} (s+1)(s+3) & s+3 & s+3 \\ s+3 & (s+1)(s+3) & s+3 \\ s+3 & s+3 & (s+1)(s+3) \end{bmatrix}}{s(s+3)(s+3)}$$

$$= \begin{bmatrix} \frac{(s+1)}{s(s+3)} & \frac{1}{s(s+3)} & \frac{1}{s(s+3)} \\ \frac{1}{s(s+3)} & \frac{(s+1)}{s(s+3)} & \frac{1}{s(s+3)} \\ \frac{1}{s(s+3)} & \frac{1}{s(s+3)} & \frac{(s+1)}{s(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3s} - \frac{2}{(s+3)} & \frac{1}{3s} - \frac{1}{3(s+3)} & \frac{1}{3s} - \frac{1}{3(s+3)} \\ \frac{1}{3s} - \frac{1}{3(s+3)} & \frac{1}{3s} - \frac{2}{3(s+3)} & \frac{1}{3s} - \frac{1}{3(s+3)} \\ \frac{1}{3s} - \frac{1}{3(s+3)} & \frac{1}{3s} - \frac{1}{3(s+3)} & \frac{1}{3s} - \frac{2}{3(s+3)} \end{bmatrix} \quad \ldots\ldots\ldots(2.5.10)$$

Equation (2.5.10) is the transition matrix. Taking the inverse of Laplace transform of equation (2.5.10) gives us the transition $\Phi(t)$

$$\Phi(t) = \frac{1}{3}\begin{bmatrix} (1-2e^{-3t}) & (1-e^{-3t}) & (1-e^{-3t}) \\ (1-e^{-3t}) & (1-2e^{-3t}) & (1-e^{-3t}) \\ (1-e^{-3t}) & (1-e^{-3t}) & (1-2e^{-3t}) \end{bmatrix}$$

2.5.1. THE OUTPUT OF FADEEV ALGORITHM FOR RESOLVENT MATRIX

Given a 3 by 3 (3x 30 Matrix below:

```
      | a11  a12  a13 |
  A = | a21  a22  a23 |
      | a31  a32  a33 |
```

Solution:- the Faddeev algorithm is calculated using the following theory of resolvent matrix [SI-A]^-1.

```
Dimension For Row A1:
3
Dimension For Row A1:
3

Input For Matrix A

 0    1   .0
 3    0    2
-12  -7   -6

 α2 = -Tr(AΓ2)/1 =  6

Γ1 = AΓ2 + α2I

Solving for Γ1 ..........

 0    1    0
 3    0    2
-12  -7    6
```

The above is AΓ2 is to be added to α2 which will be keyed in below

```
Input For α2
The value of gamma1 is as shown below
 6    1    0
 3    6    2
-12  -7    0

Solving for Alpha1.......
α1 = -Tr(AΓ1)/2

 0    1    0
 3    0    2
-12  -7   -6


 6    1    0
 3    6    2
-12  -7    0
```

3	6	2
-6	-11	0
-21	-12	-14

$\alpha 1 = -Tr(A\Gamma 1)/2 = 11$

$\Gamma 0 = A\Gamma 1 + \alpha 1 I$

Solving for Γ0 ....

3	6	2
-6	-11	0
-21	-12	-14

11	0	0
0	11	0
0	0	11

The above is the result of **AΓ1 + αII** and the result of the addition is below

14	6	2
-6	0	0
-21	-12	-3

Solving for α0 ....

0	1	0
3	0	2
-12	-7	-6

14	6	2
-6	0	0
-21	-12	-3

-6	0	0
0	-6	0
0	0	-6

$\alpha 0 = -Tr(A\Gamma 0)/3 = 6$

The Number of Iteration is 5

Substituting the iterative values for the value of Γi's and αi's, we obtain the following resolvent matrix :

$$(SI - A)^{-1} = \frac{\Gamma 2 S^2 + \Gamma 1 S + \Gamma 0}{S^3 + \alpha 2 S^2 + \alpha 1 S + \alpha 0}$$

See next page for the result.

```
Therefore (SI - A)^-1 is as shown below

 1   0   0
 1   0   0
 0   0   1

 6   1   0
 3   6   2
-12 -7   0



 14   6   2
-6    0   0
-21  -12  0

S^3 + 6 S² + 11 S + 6

(SI - A)^-1 =
 1S²+ 6S+ 14
 0S²+ 1S+ 6
 0S²+ 0S+ 2

 1S²+ 3S+-6
 0S²+ 6S+ 0
 0S²+ 2S+ 0

 0S²+-12S+-21
 0S²+-7S+-12
 1S²+ 0S+-3

S^3 + 6 S² + 11 S + 6
```

OUTPUT 2

```
Solving For a2 ........ to get Trace value

Dimension For Row A1:
3
Dimension For Row A1:
3

Input For Matrix A

 0   1   0
 0  -1  -1
 0   0  -3

 a2 = -Tr(AΓ2)/1 =  4
```

$\Gamma1 = A\Gamma2 + \alpha2I$

Solving for $\Gamma1$ ....

0	1	0
0	-1	-1
0	0	-3

The above is $A\Gamma2$ is to be added to $\alpha2$ which will be keyed in below

Input For $\alpha2$

The value of gamma1 is as shown below

4	1	0
0	-3	-1
0	0	1

Solving for Alpha1.......
$\alpha1 = -Tr(A\Gamma1)/2$

0	1	0
0	-1	-1
0	0	-3

4	1	0
0	-3	-1
0	0	1

0	3	-1
0	-3	0
0	0	-3

$\alpha1 = -Tr(A\Gamma1)/2 = 3$

$\Gamma0 = A\Gamma1 + \alpha1I$

Solving for $\Gamma0$ ....

0	1	0
0	-1	-10
0	0	-3

4	1	0
0	3	-1
0	0	0

The above is the result of $A\Gamma1 + \alpha1I$ and the result of the addition is below

3	3	-1
0	0	0
0	0	0

Solving for $a0$ ....

0	1	0
0	-1	-1
0	0	-3

3	3	-1
0	0	0
0	0	0

0	0	0
0	0	0
0	0	0

$a0 = -Tr(A\Gamma 0)/3 = 0$

The Number of Iteration is 5

Substituting the iterative values for the value of Γi's and ai's, we obtain the following resolvent matrix :

$$(SI - A)^{-1} = \frac{\Gamma 2S^2 + \Gamma 1S + \Gamma 0}{S^3 + a2S^2 + a1S + a0}$$

See next page for the result.

Therefore $(SI - A)^{-1}$ is as shown below

1	0	0
1	0	0
0	0	1

4	1	0
0	3	-1
0	0	1

3	3	-1
0	0	0
0	0	0

$S^3 + 4\ S^2 + 3\ S + 0$

$(SI - A)^{-1} =$

$1S^2 + 4S + 3$
$0S^2 + 1S + 3$
$0S^2 + 0S + -1$

```
 1S² +  0S+  0
 0S² +  3S+  0
 0S² + -1S+  0

 0S² +  0S+  0
 0S² +  0S+  0
 1S² +  1S+  0

S^3  +  4  S²  +  3  S  +  0
```

OUTPUT 3

```
Solving For α2 ........ to get Trace value

Dimension For Row A:
3
Dimension For Row A:
3

Input For Matrix A

 -2   0   1
  0  -2   1
  0  -3  -2

 α2  =  -Tr(AΓ2)/1  =   6

Γ1  =  AΓ2  +  α2I

Solving for Γ1 ..........

 -2   0   1
  0  -2   1
  0  -3  -2

The above is AΓ2 is to be added to α2 which will be keyed in
below

Input For α2

The value of gamma1 is as shown below

 4   0   1
 0   4   1
 0  -3   4

Solving for Alpha1.......
α1  =  -Tr(AΓ1)/2
```

-2	0	1
0	1	1
0	-3	-2

4	0	1
0	4	1
0	-3	4

-8	-3	2
0	-11	2
0	-6	-11

$a1 = -Tr(A\Gamma 1)/2 = 15$

$\Gamma 0 = A\Gamma 1 + a1I$

Solving for $\Gamma 0$ ....

-8	-3	2
0	-11	2
0	-6	-11

15	0	0
0	15	0
0	0	15

The above is the result of $A\Gamma 1 + a1I$ and
the result of the addition is below

7	-3	2
0	4	2
0	-6	4

Solving for $a0$ ....

-2	0	1
0	-2	1
0	-3	-2

7	-3	2
0	4	2
0	-6	4

-14	0	0
0	14	0
0	0	-14

$a0 = -Tr(A\Gamma 0)/3 = 14$

The Number of Iteration is 5

Substituting the iterative values for the value of Γi's and ai's, we obtain the following resolvent matrix :

```
                 Γ2S² +  Γ1S  + Γ0
 (SI - A)^-1 =  ------------------------
                S^3  + a2S² + a1S + a0
```

See next page for the result.

Therefore (SI - A)^-1 is as shown below

```
 1    0    0
 1    0    0
 0    0    1

 4    0    1
 0    4    1
 0   -3    4

  7   -3   2
  0    4   2
  0   -6   4

S^3 + 6 S² + 15 S + 14

(SI - A)^-1 =

 1S²+ 4S+ 7
 0S²+ 0S+-3
 0S²+ 1S+ 2

 1S²+ 0S+ 0
 0S²+ 4S+ 4
 0S²+ 1S+ 2

 0S²+ 0S+ 0
 0S²+-3S+-6
 1S²+ 4S+ 4

S^3 + 6 S² + 15 S + 14
```

OUTPUT 4

Solving For a2 .... to get Trace value

Dimension For Row A1:
3
Dimension For Row A1:

3
Input For Matrix A

1	2	1
0	1	2
3	0	1

α2 = -Tr(AΓ2)/1 = -3

Γ1 = AΓ2 + α2I

Solving for Γ1 ....

1	2	1
0	1	2
3	0	1

The above is AΓ2 is to be added to α2 which will be keyed in below

Input For α2
The value of gamma1 is as shown below

-2	2	1
0	-2	2
3	0	-2

Solving for Alpha1........

α1 = -Tr(AΓ1)/2

1	2	1
0	1	2
3	0	1

-2	2	1
0	-2	2
3	0	-2

1	-2	3
6	-2	-2
-3	6	1

α1 = -Tr(AΓ1)/2 = 0

Γ0 = AΓ1 + α1I

Solving for Γ0 ....

1	-2	3
6	-2	-2
-3	6	1

0	0	0
0	0	0
0	0	0

The above is the result of $A\Gamma 1 + \alpha 1 I$ and the result of the addition is below

1	-2	3
6	-2	-2
-3	6	1

Solving for $\alpha 0$ ....

1	2	1
0	1	2
3	0	1

1	-2	3
6	-2	-2
-3	6	1

10	0	0
0	10	0
0	0	10

$\alpha 0 = -Tr(A\Gamma 0)/3 = -10$

The Number of Iteration is 5

Substituting the iterative values for the value of Γi's and αi's, we obtain the following resolvent matrix :

$$(SI - A)^{-1} = \frac{\Gamma 2 S^2 + \Gamma 1 S + \Gamma 0}{S^3 + \alpha 2 S^2 + \alpha 1 S + \alpha 0}$$

See next page for the result.

Therefore $(SI - A)^{-1}$ is as shown below

1	0	0
1	0	0
0	0	1

-2	1	2
-2	1	3
0	6	-2

```
-2     2    -2
 3    -3     0
 6    -2     1

S^3 +-3 S²  + 0 S +-10

 (SI - A)^-1 =

 1S²+-2S+ 1
 0S²+ 2S+-2
 0S²+ 1S+ 3



 1S²+ 0S+ 6
 0S²+-2S+-2
 0S²+ 2S+-2

 0S²+ 3S+-3
 0S²+ 0S+ 6
 1S²+-2S+ 1

S^3 +-3 S²  + 0 S +-10
```

2.5.2 FADEEV ALGORITHM FLOWCHART FOR THE RESOLVENT MATRIX

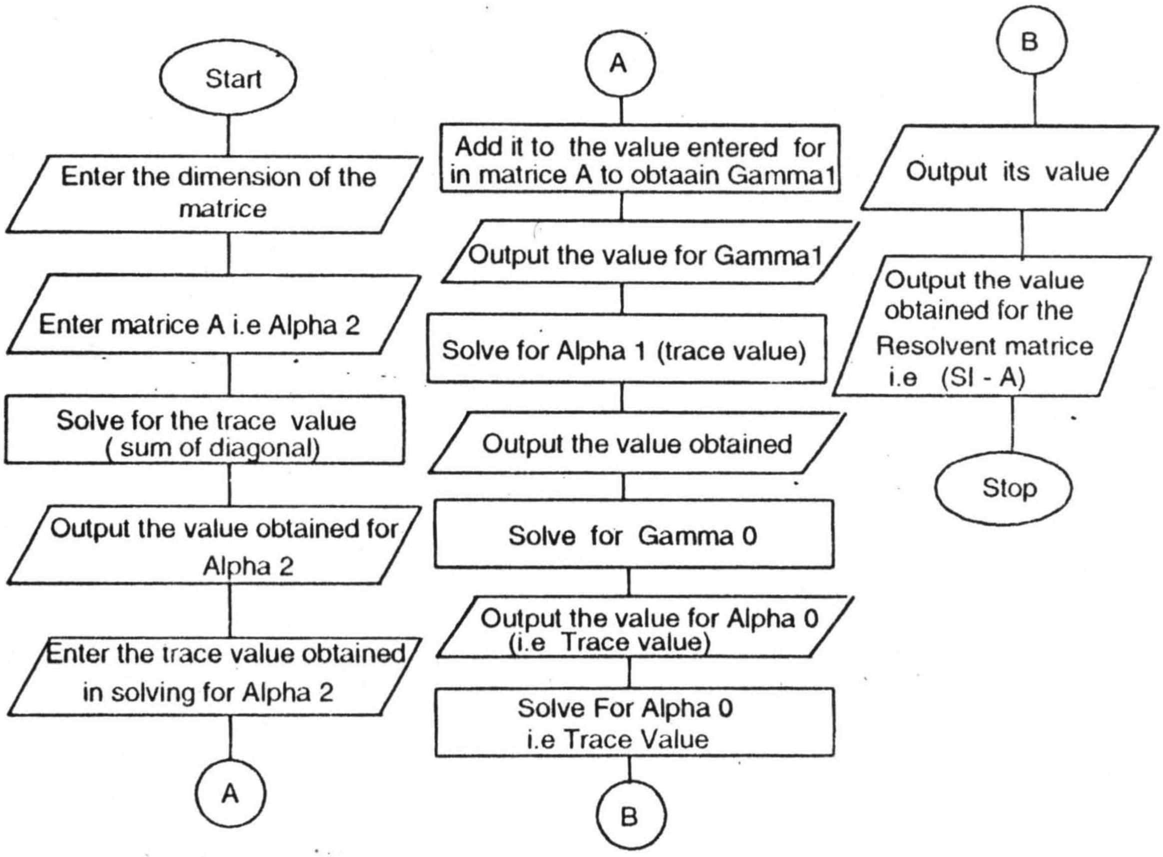


Fig. 2.2.4 Fadeev Algorithm Flowchart

CHAPTER THREE

STATE TRANSITION MATRIX AND ITS APPLICATION

3.1 INTRODUCTION

Consider a process described in the form

$$\dot{X} = AX \qquad (3.1.1)$$

$$Y = C X \qquad (3.1.2)$$

with $U(t) = 0$

Knowing that at $t = 0$, $X(t) = X(0)$, we can calculate $X(t)$ and therefore $Y(t)$ for $t > 0$, by directly integrating the equation of state (3.1.1)

$$\dot{X} = AX$$

This implies $$X(t) = e^{at} X(0) \qquad (3.1.3)$$

Definition 3.1.1 (Matrix of Transition)

The matrix e^{at}, also denoted by $\Phi(t)$, is called the matrix of transition.

Now when $U(t)$ is different from 0, the solution of the equation of state is written as :

$$X(t) \quad \Phi(t) X_0 + \int_0^t \Phi(t - \tau) BU(\tau) d\tau \qquad (3.1.4)$$

Now consider the general form of the state equation:

$$\dot{X}(t) = A X(t) + B U(t) \qquad (3.1.5)$$

The Laplace transition of (3.1.5) is given by

$$S X(s) - X_0 = AX(s) + BU(s) \qquad (3.1.6)$$

Where $X(s)$ is the Laplace transform of $X(t)$ and $U(s)$ is the Laplace transform of U(

Solving for $X(s)$ we obtain

$$X(s) = [SI - A]^{-1}X_0 + [SI - A]^{-1}B\,U(s)$$

$$X(s) = [X_0 + BU(s)]\,(SI - A)^{-1} \qquad (3.1.7)$$

The inverse of the Laplace transform of equation (3.1.7) gives the state transition equation (3.1.4); where the transition matrix is defined by

$$\Phi(t) = L^{-1}\left\{[SI - A]^{-1}\right\}, \quad \forall\, t \geq 0 \qquad (3.1.8)$$

Definition 3.1.2 (The Resolvent Matrix)

The inverse matrix $[SI - A]^{-1}$ is called the Resolvent matrix.

When the input U = 0, equation (3.1.4) reduces to

$$X(t) = e^{At}X_0 \qquad (3.1.9)$$

Definition 3.1.3 (Fundamental Matrix)

An n x n matrix function $\psi(.)$ is said to be a fundamental matrix of

$$\dot{X} = A(t)\,X \qquad (3.1.10)$$

If the n columns of ψ consists of n linearly independent solution of (3.1.10).

Example 3.1.1 (Fundamental Matrix)

Consider the dynamical equation

$$\dot{X} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad (3.1.11)$$

This actually consists of two equations

$$\dot{X}_1 = 0; \quad \dot{X}_2 = tX_1. \qquad (3.1.12)$$

The solutions of these two equations are :

Properties of Transition Matrix

We have the following very important properties of the state transition matrix.

(i) $\Phi(t, t_0) = I$ (3.1.15)

(ii) $\Phi^{-1}(t, t_0) = \Psi(t_0)\, \Psi^{-1}(t) = \Phi(t_0, t)$ (3.1.16)

(iii) $\Phi(t_2, t_0) = \Phi(t_2, t_1)\, \Phi(t, t_0)$ (3.1.17)

For any t_2, t_1, t_0 and $t \in [-\infty, \infty]$

(iv) $\Phi^{-1}(t) = \Phi(-t)$ (3.1.18)

Equation (3.1.4) can be modified by letting $t = t_0$. Solving for x_0, we obtain the following expression:

$$X_0 = \Phi^{-1}(t_0)X(t_0) - \Phi^{-1}(t_0)\int_0^{t_0} \Phi(t_0 - \tau)B(\tau)d\tau \quad (3.1.19)$$

Using (3.1.18), this equation can be written as

$$X_0 = \Phi(-t_0)X(t_0) - \Phi(-t_0)\int_0^{t}(t_0 - \tau)Bu(\tau)d\tau \quad (3.1.20)$$

Substituting equation (3.1.20) into equation (3.1.4), the following expression is obtained:-

$$X(t) = \Phi(t)\Phi(-t_0)X(t_0) - \Phi(t)\Phi(-t_0)\int_0^{t_0} \Phi(t_0 - \tau)Bu(\tau)d\tau + \int_0^{t} \Phi(t - \tau)\, Bu(\tau)d\tau \quad (3.1.21)$$

Using equation (3.1.17), equation (3.1.21) can be reduced to

$$X(t) = \Phi(t - t_0)X(t_0) + \int_0^{t} (t - \tau)\, Bu(\tau)d\tau \quad (3.1.22)$$

Equation (3.1.22) is the state equation of the system for $t \geq t_0$.

$$X_1(t) = X_1(t_0) \tag{3.1.13}$$

$$X_2(t) = 0.5t^2 X_1(t_0) + X_2(t_0) \tag{3.1.14}$$

and are linearly independent solutions

$$\psi_1 = [\,0,\ 1\,] \text{ and } \psi_2 = [\,2,\ t^2\,]$$

can easily be obtained by setting

$$X_1(t_0) = 0 \quad \text{and } X_2(t_0) = 1 \text{ and}$$

$$X_1(t_0) = 2 \quad \text{for } X_2(t_0) = t^2.$$

Hence the matrix

$$\begin{bmatrix} 0 & 2 \\ 1 & t^2 \end{bmatrix}$$

is a fundamental matrix.

Definition 3.1.4 (Transition Matrix)

Let $\psi(.)$ be any fundamental matrix of

$$\dot{X} = A(t)X(t) \tag{3.1.15}$$

and let $\Phi_1(t, t_0), \Phi_2(t, t_0), \ldots\ldots, \Phi_n(t, t_0)$ be the set of solutions of equation (3.1.15) associated with the corresponding initial conditions:

$$X_1(t_0) = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ X_2(t_0) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots\ldots X_n(t_0) = e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Combining the solutions, we define the n x n matrix

$$\Phi(t, t_0) = \left[\Phi_1(t, t_0), \Phi_2(t, t_0), \ldots\ldots, \Phi_n(t, t_0)\right]$$

which is called the transition matrix when $A = A(t)$

Theorem 3.1.1 (Fundamental Theorem)

Every fundamental matrix ψ is non-singular $\forall \in (-\infty, \infty)$

Proof:-

Let $\psi(.)$ be a solution of $\dot{X} = A(t)X$ and $\psi(t_0) \neq 0$ for some t_o, then the solution $\psi(.)$ is identically zero; that is $\psi(.) \equiv 0$. Thus $\psi = 0$ is the solution of $\dot{X} = A(t)X(t)$ with $\psi(t_0) = 0$.

Again, from the uniqueness of the solution, we conclude that $\psi(.) = 0$ is the only solution with $\psi(t_0) = 0$.

Now the proof of the theorem is by contradiction. Suppose that

$$\det \psi(t_0) = \det [\psi_1(t_0), \psi_2(t_0), \ldots, \psi_n(t_0)] = 0$$

for some t_o. Then the set of n constant column Vectors $\psi_1(t_0), \psi_2(t_0) \ldots, \psi_n(t_0)$ are linearly dependent in (R^n, R). It follows that $\exists$ real α_i for $i = 1, 2, \ldots, n$ not all zero,

$$\ni \sum_{i=1}^{n} \alpha_i \Psi_i(t_0) = 0$$

Which together with the fact that

$$\sum_{i=1}^{n} \alpha_i \Psi_i(.) \text{ is a solution of}$$

$$\dot{X} = A(t)X$$

implies

$$\sum_{i=1}^{n} \alpha_i \Psi_i(t_0) = 0$$

This contradicts the assumption that $\psi_i(.)$, for $(i = 1, 2, \ldots, n)$ are linearly independent. Hence we conclude that

$$\det \psi(t) \neq 0; \ \forall \ t \in [-\infty, \infty].$$

Example 3.1.2 (Determining State Transition Matrix)

Consider an open-loop system where the transfer function of the controlled process is given by .

$$H(s) = \frac{C(s)}{U(s)} = \frac{1}{S^2} \qquad (3.1.23)$$

Its corresponding differential equation is given by

$$\ddot{C}(t) = u(t) \qquad (3.1.24)$$

Define the state variables as

$$X_1(t) = C(t)$$

$$X_2(t) = \dot{C}(t) \qquad (3.1.25)$$

The system can be described by the following first-order differential equations:

$$\dot{X}_1(t) = X_2(t) = \dot{C}(t)$$

$$(3.1.26)$$

$$\dot{X}_2(t) = u(t)$$

Therefore the entire system can be described by the state equation

$$\dot{X}(t) = AX(t) + B\,U(t) \qquad (3.1.27)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad X(t) = \begin{bmatrix} X1(t) \\ X_2(t) \end{bmatrix},\quad \dot{X}(t) = \begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} \qquad (3.1.28)$$

Now the state transition matrix is defined by

$$\Phi(t) = L^{-1}\left\{[SI - A]^{-1}\right\} \qquad (3.1.29)$$

This can be obtained from (3.1.28). We find

$$[SI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} , \qquad (3.1.30)$$

It is known that

$$A^{-1} = \frac{adj\ A}{\det|A|} \qquad (3.1.31)$$

Therefore,

$$[SI - A]^{-1} = \frac{\begin{bmatrix} S & 1 \\ 0 & S \end{bmatrix}}{\begin{vmatrix} S & -1 \\ 0 & S \end{vmatrix}} = \frac{\begin{bmatrix} S & 1 \\ 0 & S \end{bmatrix}}{S^2} = \begin{bmatrix} \frac{1}{S} & \frac{1}{S^2} \\ 0 & \frac{1}{S} \end{bmatrix} , \qquad (3.1.32)$$

The state transition matrix obtained by equation (3.1.5) is the inverse transition of this matrix. It is given by

$$\Phi(t) = L^{-1}[SI - A]^{-1}$$

$$= \begin{bmatrix} u(t) & t \\ 0 & u(t) \end{bmatrix} .$$

Example 3.1.3 (State Vector)

With the knowledge of the state transition matrix, we can find the state variables.

Suppose the initial-state vector is given by

$$X_0 = \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad (3.1.33)$$

We can find the state vector X(t) as a function of time.

Now state vector is defined as

$$X(t) = \Phi(t) X_0 \qquad (3.1.34)$$

$$= \begin{bmatrix} u(t) & t \\ 0 & u(t) \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} u(t) & t \\ 0 & u(t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad X_1(t) = U(t) + 2t\ ;\ t > 0 \qquad (3.1.35)$$

$$X_2(t) = 2U(t) \qquad (3.1.36)$$

3.2 SOLUTION OF THE STATE TRANSITION MATRIX

The aim of this section is to show how one can obtain a complete solution for the output in the time domain of a control system by the state variable method. We will illustrate how to determine a complete solution by evaluating equation 3.1.4, the state transition equation.

Example 3.2.1 (State Transition Equation)

Consider a system described by the following differential equation:

$$\ddot{C}(t) + 2\dot{C}(t) = \dot{r}(t) + r(t) \qquad (3.2.1)$$

Determine the output C(t), given that the input r(t) is given by

$$r(t) = \text{Sin}(t) \qquad (3.2.2)$$

and the initial conditions are

$$C(0) = 1 \text{ and } \dot{C}(0) = 0 \qquad (3.2.3)$$

SOLUTION

First determine the state transition matrix, then evaluate (3.1.4) for X(t). The output C(t) is then evaluated from

$$C(t) = L\,X(t) \tag{3.2.4}$$

Now suppose the state variables are defined by

$$X_1(t) = C(t);\ X_2(t) = \dot{C}(t) \tag{3.2.4}$$

and

$$U(t) = r(t) \tag{3.2.5}$$

Then the system can be described by the following two first order differential equations:

$$\dot{X}_1(t) = X_2(t),$$
$$\dot{X}_2(t) = -2X_2(t) - X_1(t) + u(t) + \dot{u}(t) \tag{3.2.6}$$

Thus the system can be described by

$$\dot{X}(t) = AX(t) + BU(t) + \dot{U}(t) \tag{3.2.7}$$

Where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ X(t) = \begin{bmatrix} X_1(t) \\ \dot{X}_2(t) \end{bmatrix};\ \dot{X}(t) = \begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} \tag{3.2.8}$$

The transition matrix, which is defined by (3.1.4), can be obtained from (3.2.8). We find

$$[SI - A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} S & -1 \\ 1 & S+2 \end{bmatrix} \tag{3.2.9}$$

Now we know that for any matrix A, its inverse (A^{-1}) is defined as

$$A^{-1} = \frac{adj\,A}{|A|} \qquad (3.2.10)$$

Therefore

$$[SI - A] = \frac{adj[SI - A]}{|SI - A|}$$

$$= \frac{\begin{bmatrix} S+2 & 1 \\ -1 & S \end{bmatrix}}{(S+1)^2}$$

$$= \begin{bmatrix} \frac{S+2}{(S+1)^2} & \frac{1}{(S+1)^2} \\ \frac{-1}{(S+1)^2} & \frac{S}{(S+1)^2} \end{bmatrix} . \qquad (3.2.11)$$

Now by the method of partial fractions and inverse transition, (3.2.11) is given by

$$\Phi(t) = L^{-1}[SI - A]^{-1}$$

$$= \begin{bmatrix} e^{-t}(t+1) & te^{-t} \\ -te^{-t} & e^{-t}(1-t) \end{bmatrix} \qquad (3.2.12)$$

The full solution for the output can be obtained from equations (3.1.4) and (3.2.4) as follows:

$$X(t) = \Phi(t)\,X_0 + \int_0^t \Phi(t - \tau)\,U(\tau)d\tau \qquad (3.2.13)$$

$$C(t) = L\,X(t) \qquad (3.2.14)$$

Now $\Phi(t)$ is known from (3.2.12) also by inspection

$$L = [1 \;\; 0]$$

$$X_0 = \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (3.2.15)$$

For this system, the input function

$$U(\tau) + \dot{U}(\tau) = \sin\tau + \cos\tau \qquad (3.2.16)$$

Substituting all the values into the output equation:

$$C(t) = L\Phi(t)X_0 + \int_0^t L\Phi(t-\tau)BU(\tau)\,d\tau \qquad (3.2.17)$$

$$C(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} e^{-t}(t+1) & te^{-t} \\ -te^{-t} & e^{-t}(1-t)\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} +$$

$$\int_0^t \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} e^{-(t-\tau)} & (t-\tau)e^{-(t-\tau)} \\ -(t-\tau)e^{-(t-\tau)} & e^{-(t-\tau)}(1-t+\tau)\end{bmatrix} \times \begin{bmatrix}0\\1\end{bmatrix}\left[\sin\tau + \cos\tau\right]d\tau \qquad (3.2.18)$$

On simplifying, the result becomes

$$C(t) = e^{-t}(t+1) + \int_0^t \left[(t-\tau)e^{-(t-\tau)}\right](\sin\tau + \cos\tau)\,d\tau \qquad (3.2.19)$$

Integrating and simplifying, we finally obtain the output as

$$C(t) = \frac{3}{2}e^{-t} + te^{-t} + \frac{1}{2}\text{Sin } t - \frac{1}{2}\text{Cos } t \; ; \; t \geq 0 \qquad (3.2.20)$$

Example 3.2.2 (State Transition Matrix).

A very interesting ecological problem is that of rabbits and foxes in a controlled environment. If the number of rabbits were left alone, they would grow indefinitely until the food supply was exhausted. Representing the number of rabbits by $X_1(t)$, their growth rate is given by

$$\dot{X}_1(t) = A\,X_1(t) \qquad (3.2.21)$$

However, rabbit-eating foxes in the environment change this relationship to the following:

$$\dot{X}_1(t) = A\,X_1(t) - B\,X_2(t) \qquad (3.2.22)$$

Where $X_2(t)$ represents the fox population. In addition, if foxes must have rabbits to exist, their growth rate is given by

$$\dot{X}_2(t) = -C\,X_1(t) + D\,X_2(t) \qquad (3.2.23)$$

(a) Assume that A = 1, B = 2, C = 2 and D = 4. Determine the state transition matrix for this ecological model.

(b) From the state matrix, determine the response of this ecological model when $X_1(0)$ - 100 and $X_2(0) = 500$. Explain your results.

SOLUTION

(a) We have

$$\dot{X}_1(t) = A\,X_1(t) - B\,X_2(t) \qquad (3.2.24)$$

$$\dot{X}_2(t) = -C\,X_1(t) + D\,X_2(t) \qquad (3.2.25)$$

With the values of A, B, C, D given, we have the systems (3.2.24) and (3.2.25) as

$$\dot{X}_1(t) = X_1(t) - 2\,X_2(t) \qquad (3.2.26)$$

$$\dot{X}_2(t) = -2\,X_1(t) + F\,X_2(t) \qquad (3.2.27)$$

Thus we can write (3.2.26) and (3.2.24) as

$$\dot{X}(t) = A\,X(t)\ ,$$

where $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, $[SI - A] = \begin{bmatrix} S-1 & 2 \\ 2 & S-4 \end{bmatrix}$

Thus the state transition matrix is defined as

$$\Phi(t)\quad [SI - A]^{-1}\quad \frac{Adj[SI - A]}{|SI - A|}\ ,$$

$$= \frac{\begin{bmatrix} (S-4) & -2 \\ -2 & (S-1) \end{bmatrix}}{\begin{vmatrix} (S-1) & 2 \\ 2 & (S-4) \end{vmatrix}} = \frac{\begin{bmatrix} (S-4) & -2 \\ -2 & (S-1) \end{bmatrix}}{S^2 - 5S}$$

$$\Rightarrow [SI - A]^{-1} = \begin{bmatrix} \frac{S-4}{S^2 - 5S} & \frac{-2}{S^2 - 5S} \\ \frac{-2}{S^2 - 5S} & \frac{S-1}{S^2 - 5S} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{S-4}{S(S-5)} & \frac{-2}{S(S-5)} \\ \frac{-2}{S(S-5)} & \frac{S-1}{S(S-5)} \end{bmatrix} \qquad (3.2.28)$$

Applying partial fraction principle on equation (3.2.28), we have

$$[SI - A]^{-1} = \begin{bmatrix} \frac{4}{5S} + \frac{1}{5(S-5)} & \frac{2}{5S} - \frac{2}{5(S-5)} \\ \frac{2}{5S} - \frac{2}{5(S-5)} & \frac{1}{5S} + \frac{4}{5(S-5)} \end{bmatrix}$$

Taking the inverse Laplace, we have

$$\Phi(t) = \begin{bmatrix} \frac{4 + e^{5t}}{5} & \frac{2 - 2e^{5t}}{5} \\ \frac{2 - 2e^{5t}}{5} & \frac{1 + 4e^{5t}}{5} \end{bmatrix} \qquad (3.2.29)$$

Thus equation (3.2.29) is the state transition matrix.

(b) The response of this ecological model are the state variable $X_1(t)$ and $X_2(t)$

Now,

$$X_1(t) = \Phi(t)X_0 \qquad (3.2.30)$$

$$\Rightarrow \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} \qquad (3.2.31)$$

Putting equation (3.2.29) in (3.2.31)

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \frac{4+e^{5t}}{5} & \frac{2-2e^{5t}}{5} \\ \frac{2-2e^{5t}}{5} & \frac{1+4e^{5t}}{5} \end{bmatrix} \begin{bmatrix} 100 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} 80+20e^{5t} & 20-20e^{5t} \\ 40-40e^{5t} & 10+40e^{5t} \end{bmatrix}$$

Thus $X_1(t) = 80 + 20e^{5t} - 20e^{5t} + 20$

$X_1(t) = 80 + 20$

$X_1(t) = 100$

$X_2(t) = 40 - 40e^{5t} + 10 + 40e^{5t}$

$X_2(t) = 50$

Explanation:

In the initial state, $X_1(0) = 100$

and $X_2(0) = 50$

Now at $X_1(t) = 100$

and $X_2(t) = 50$

That is the state of increase of rabbits remains constant as is the state of increase of rabbit - eating foxes. The rabbits therefore continue to increase by 100 as is the increase of foxes by 50 in unit time. Should the increase of foxes surpasses the 50 constant increment, the rabbit's state of increase will diminish. With time, they shall become extinct. Also a decrease in the increase of the foxes will surge the number of rabbits such that with time, their food would be used up.

Example 3.2.3 (Transition Matrix)

Substances $X_1(t)$ and $X_2(t)$ are involved in the reaction of a chemical process. The state equations representing this reaction are as follows:

$$\dot{X}_1(t) = -4X_1(t) + 2X_2(t)$$

$$\dot{X}_2(t) = 2X_1(t) - X_2(t)$$

(a) Determine the state transition matrix of this chemical process

(b) Determine the response of this system when:

$$X_1(0) = 200{,}000 \text{ units}$$

$$X_2(0) = 10{,}000 \text{ units}$$

Solution

$$\dot{X}_1(t) = -4X_1(t) + 2X_2(t) \qquad (3.2.32)$$

$$\dot{X}_2(t) = 2X_1(t) - X_2(t) \qquad (3.2.33)$$

Equations (3.2.32) and (3.2.33) can be expressed as

$$\dot{X}_1(t) = A\,X(t)$$

where $\dot{X}(t) = \begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix}$, $X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$ and $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ (3.2.34)

Now $$[SI - A] = \begin{bmatrix} S+4 & -2 \\ -2 & S+1 \end{bmatrix} \qquad (3.2.35)$$

Also from (3.2.35)

$$[SI - A]^{-1} = \frac{\text{Adj}\begin{bmatrix} S+4 & -2 \\ -2 & S+1 \end{bmatrix}}{\begin{vmatrix} S+4 & -2 \\ -2 & S+1 \end{vmatrix}}$$

$$= \frac{\begin{bmatrix} S+1 & 2 \\ 2 & S+4 \end{bmatrix}}{\begin{vmatrix} S+4 & -2 \\ -2 & S+1 \end{vmatrix}}$$

$$= \frac{\begin{bmatrix} S+1 & 2 \\ 2 & S+4 \end{bmatrix}}{S^2 + 5S}$$

$$\therefore \; [SI - A]^{-1} = \begin{bmatrix} \dfrac{S+1}{S(S+5)} & \dfrac{2}{S(S+5)} \\ \dfrac{2}{S(S+5)} & \dfrac{S+4}{S(S+5)} \end{bmatrix} \qquad (3.2.36)$$

Expressing (3.2.36) in partial fraction, gives

$$[SI - A]^{-1} = \begin{bmatrix} \dfrac{1}{5S} + \dfrac{4}{5(S+5)} & \dfrac{2}{5S} - \dfrac{2}{5(S+5)} \\ \dfrac{2}{5S} - \dfrac{2}{5(S+5)} & \dfrac{4}{5S} - \dfrac{1}{5(S+5)} \end{bmatrix} \qquad (3.2.37)$$

Taking the inverse Laplace of (3.2.37)

$$L^{-1}[SI - A]^{-1} = \begin{bmatrix} \dfrac{1+4e^{-5t}}{5} & \dfrac{2-2e^{-5t}}{5} \\ \dfrac{2-2e^{-5t}}{5} & \dfrac{4-e^{-5t}}{5} \end{bmatrix} \qquad (3.2.38)$$

Equation (3.2.38) is the required state transition matrix.

(b) Determining the response of this equation when

$X_1(0) = 200{,}000$ units

$X_2(0) = 10{,}000$ units

The response is given by the state equation

$$X(t) = \Phi(t)X_0$$

$$\Rightarrow X(t) = \begin{bmatrix} \dfrac{1+4e^{-5t}}{5} & \dfrac{2-2e^{-5t}}{5} \\ \dfrac{2-2e^{-5t}}{5} & \dfrac{4-e^{-5t}}{5} \end{bmatrix} \begin{bmatrix} 200000 \\ 10000 \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} 40000 + 10000e^{-5t} + 4000 - 4000e^{-5t} \\ 80000 - 80000e^{-5t} + 8000 - 2000e^{-5t} \end{bmatrix}$$

$$\Rightarrow X_1(t) = 40000 + 160000e^{-5t} + 4000 - 4000e^{-5t}$$

$$= 44000 + 15600e^{-5t} \qquad (3.2.39)$$

$$X_2(t) = 88000 - 78000\,e^{-5t} \qquad (3.2.39)$$

CHAPTER FOUR

COMPUTATION OF OPTIMAL CONTROL SYSTEM OF A SINGLE INPUT SYSTEM WITH FADDEEV ALGORITHM

4.1 INTRODUCTION

In this chapter we compute the optimal control of a single-input single-output system using the Faddeev algorithms and the Kalman equation aided with the Ackerman equation.

Definition 4.1.1 (OPTIMAL CONTROL)

Control may be defined as an act of manipulation with a view to achieving or to fulfilling a desired objective. Large numbers of controls may exist to fulfil the given objective in such a case, the most desirable control in the sense of minimizing a given criterion function can be used. Such a control is said to be an optimal control.

4.2 COMPUUTATIONAL METHOD OF OPTIMAL CONTROL OF A SINGLE INPUT SINGLE OUTPUT SYSTEM THEORY

In computing the optimal control for a multi-input multi-output system (see chapter 5) a positive definite solution of a matrix P incorporated in an equation called Riccati equation must be found. For a single input single output (SISO) system the optimal control can be calculated from Kalman and Ackerman equations without the need to first obtain the P.

The Kalman equation is uniquely calculated as follows:

$$1 - F^T(-j\psi I - A^T)^{-1}(1 - F^T(j\psi I - A)^{-1}b)$$

$$= 1 + (1/r)||H(j\psi I - A)^{-1}||^2 Q \qquad (4.2.1)$$

For a single input system A, b, Q, r, are assumed to be given and the optimal control F^T may be calculated as follows

1st Step:

Note the Kalman equation (4.1.1) multiply (4.2.2) by $\Phi(s)\Phi(-s)$ to obtain

$$\det(-SI \_ A^T - bF^T)\det(SI - A - bF^T) = \Phi(s)\Phi(-s)$$

$$+ (1/r)b^T adj(SI - AT)Qadj(SI - A)b \qquad (4.2.2)$$

where

$$(SI - A)^{-1} = [adj(SI - A)]/\det(SI - A) \qquad (4.2.3)$$

The roots of the right side of equation)4.2.4) are of the form λ_1, $-\lambda_1$, $-\lambda_2$, ... $-\lambda_n$, λ_n are calculated.

2nd Step:

Find the n roots with negative real parts from the above roots of (4.1.2) Let them be denoted by λI and calculate

$$\Phi_f(s) = \prod_{i=1}^{n}(SI-\lambda_i) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_0 \qquad (4.2.4)$$

$$\text{Re } \lambda_i < 0 \ (i = 1, 2, \dots n)$$

3rd step:

Find the control laf F^T satisfying

$$\det(SI - A - bF^T) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_0 \qquad (4.2.5)$$

This can be done using Ackerman algorithm:

$$F^T = -[0, 0, \dots i][b; AB, A^2B, \dots A^{n-1}]\,\Phi_f(A) \qquad (4.2.6)$$

$$= -[0, 0, \dots i]\xi\Phi_f(A) \qquad (4.2.7)$$

EXAMPLE 4.2.1 (SINGLE-INPUT OPTIMAL CONTROL)

Find the optimal control for a linear system:

$$X' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (4.2.8)$$

which minimizes

$$J = \int_0^t XT \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} X + u2 \quad dt \qquad (4.2.9)$$

SOLUTION

The solution is obtained by the single-input method as outlined above. First we apply the Faddeev algorithm of chapter 2 to calculate

$$(SI - A)^{-1} = [\mathrm{adj}(SI - A)]/\det(SI - A)$$

$$= (\Gamma_{n-1} s^{n-1} + \Gamma_{n-2} s^{n-2} + \ldots \Gamma) \backslash \Phi(s) \qquad (4.2.10)$$

With question, we are dealing with a third order matrix. Thus n = 3.

Thus

$$\Gamma_{3-1} = \Gamma_2 = I \text{ by definition}$$

$$\Gamma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_2 = - \mathrm{Tr}[A\Gamma_2]$$

$$= - \mathrm{Tr} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = - \mathrm{Tr} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Thus

$$\alpha_2 = 0 \qquad (4.2.11)$$

$$\Gamma_1 = A\,\Gamma_2 + \alpha_2 I$$

But equation (4.2.11) gives $\alpha_2 = 0$

Therefore,

$$\Gamma_1 = A\,\Gamma_2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_1 = -Tr(A\Gamma_1)/2 = -\frac{Tr}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-Tr\ 0.5\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \qquad (4.2.12)$$

$$\Gamma_0 = A\,\Gamma_1 + \alpha_1 I$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_0 = - Tr(A\Gamma_0)/3 = 0$$

$$\Phi(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0 \qquad (4.2.13)$$

But n = 3. Then

$$\Phi(s) = - s^3$$

Therefore by a similar deduction,

$$\Phi(-s) = -1/(s^n) \qquad (4.2.14)$$

Also

$$adj(SI - A) = (\Gamma_{n-1} s^{n-1} + \Gamma_{n-2}s^{n-2} + \ldots\Gamma_0)$$

Thus

$$s^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} s^2 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} s + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} s^2 & s & 1 \\ 0 & s^2 & s \\ 0 & 0 & s^2 \end{pmatrix}$$

In a similar vein,

$$\text{adj}(-SI - A^T) = \begin{pmatrix} s^2 & 0 & 0 \\ -s & -s^2 & 0 \\ -1 & -s & -s^2 \end{pmatrix}$$

substituting the calculated values into equation (4.2.2) yields

$$\det(SI - A - bF^T)\det(-SI - A^T - bF^T) = 1 - s^4$$

$$= (1 - s^2)(1 + s^2)$$

$$s = \pm 1 \quad \text{or } s = \pm i$$

Now consider equation (4.2.4)

$$\Phi_f(s) = s + 1$$

$$\Phi_f(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (4.2.15)$$

Now applying Ackerman equation:

i.e.

$$F^T = -[0\ 0\ 1][b\ AB\ A^2B]\ \Phi_f(A)$$

But

$$[b\ AB\ A^2B] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$F^T = -(0\ 0\ 1) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$= -(0\ 0\ 1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (0\ 0\ 1) \qquad (4.2.16)$$

Hence the optimal control for the given linear system is:

$$u = F^T x = -(0\ 0\ 1) \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -x_3$$

CHAPTER FIVE

COMPUTATION OF THE OPTIMAL CONTROL OF A QUADRATIC COST FUNCTION

5.1 INTRODUCTION

This chapter studies the method for calculating the optimal control of a quadratic cost function. Pontryagin's maximum principle otherwise known as the Hamiltonian form is the instrument used to drive an equation called the Riccati equation. From the Riccati equationa symmetric positive definite matrix P is obtained. With this P known the optimal control of a quadratic cost function is obtained.

5.2 THE LINEAR QUADRATIC PROBLEM

5.2.1 THE LINEAR REGULATOR

Linearization of a linear equation around a suitable trajectory or equilibrium point is possible.

In this case we describe many system by the state-space equations.

$$X^1(t) = A(t)x(t) + B(t)u(t) \qquad (5.2.1)$$

where X is the n^{th} order state vector $\underline{U}$ is the m^{th} order control vector and A, B are respectively nxn and nxm time varying matrices.

The cost function we wish to minimize is the weighted quadratic function of state and control that is:

$$J = 0.5[X^T(tr)sx(tr) + 0.5\int_0^\infty \{X^T(t)Q(t)X(t) + U^T(t)R(t)U(t)dt \qquad ....(5.2.2)$$

Q, s are assumed to be real symmetric positive semi-definite matrices and R is a real symmetric positive definite matrix.

The assumption is that the states and controls are not bounded and that $x(t_f)$ is free. The cost function maintains the state vector near the origin.

of the state space without utilizing excessive control effort. The weighted matrices Q, R, S enable us to define the relative importance of keeping the states near the origin.

Using the Pontryagin's maximum principle as enunciated in section 5.2, we solve the above problems. Write the H function as:

$$H = \frac{1}{2}X(t)^T Q(t) X(t) + \frac{1}{2}u(t)^T R(t) u(t) + \lambda^T(t)A(t)\underline{X}(t) + \lambda^T(t)B(t)\underline{u}(t) \quad (5.2.3)$$

The optimality conditions in this case yield

$$\frac{\partial H}{\partial u} = 0, \quad u(t) = -RH^{-1}B^{1}\underline{\lambda}(t) \quad (5.2.4)$$

$$\frac{\partial H}{\partial X} = -\dot{\lambda}(t) = Q(t)X(t) + A^{1}(t)\underline{\lambda}(t) \quad (5.2.5)$$

With the terminal conditions

$$\lambda(t_1) = s(t)\, X(t_1) \quad (5.2.6)$$

substituting (5.4.4) in (5.4.1) yields

$$\dot{X}(t) = A(t)X(t) - B(t)R^{-T}(t)B^{1}(t)\lambda t \quad (5.2.7)$$

$$\text{with } X(t_0) = X_0 \quad (5.2.8)$$

Therefore our two points BVP becomes one of solving equations (5.4.5), (5.2.7) subject to I C's (5.4.6) and (5.4.8)

Suppose that the solution for the costate λ is

$$\lambda(t) = p(t)\, X(t) \quad (5.2.9)$$

Then

$$\dot{\lambda}(t) = \dot{p}(t)\, X(t) + p(t)\dot{X}(t) \quad (5.2.10)$$

Consider equations (5.4.5) and (5.4.7), equations (5.4.10) becomes

$$\dot{\lambda}(t) = -Q(t)X(t) - A^T(t)\lambda(t)$$

$$= \dot{p}(t)X(t) + p(t)\left[A(t)X(t) + B(t)R^{-1}(t)B^T(t)p(t)X(t)\right]$$

or $\left[\ \dot{p}(t) + p(t)A(t) + A^T(t)p(t) - p(t)B(t)R^{-1}(t)B^T(t)p(t) + Q(t)\ \right]X(t) = 0 \qquad (5.2.11)$

Thus we must have

$$\dot{p}(t) = -p(t)A(t) - A^T(t)p(t) + p(t)B(t)R^{-1}(t)B^T(t)p(t) - Q(t) = 0 \qquad (5.2.12)$$

with the terminal condition being given by the equation as

$$p(t_f) = S \qquad (5.2.13)$$

In equation (5.212), p is an n x n symmetric matrix having n(n+1)/2 distinct elements. This equation (5.212) is known as the matrix RICCATI Equation. The equation is integrable.

The optimal control is given by:

$$u(t) = -R^{-1}B^T(t)p(t)X(t) \qquad (5.214)$$

Thus we summarize our results:

If we wish to minimize:

$$J = \frac{1}{2}X^T(t_f)S(t_f)X(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\ X^T(t)Q(t)X(t) + u^T(t)R(t)u(t)\ \right]dt \qquad (5.2.15)$$

where Q, S are real symmetric positive semi-definite matrices whilst R is a real symmetric positive definite matrx, subject to the constraints

$$\dot{X}(t) = A(t)X(t) + B(t)u(t) \qquad (5.216)$$

$$X(t_0) = X_0 \qquad (5.2.17)$$

Then the optimal control is given by

$$u(t) = -R^{-1}B^T(t)p(t)X(t) \qquad (5.2.18)$$

where

$$\dot{p}(t) = -p(t)A(t) - A^T(t)p(t) + p(t)B(t)R^{-1}(t)B^T(t)p(t) - Q(t)$$

$$\text{with } p(t_f) = S \qquad (5.2.19)$$

and where A, B, Q, R are time invariant, $t_f \rightarrow \infty$ and if the system is controllable, then

$\underline{u} = G\underline{X}$; where G is time invariant and

$$G = -R^{-1}B^T(t)p(t) \qquad (5.2.20)$$

EXAMPLE 5.2.1 (OPTIMAL CONTROL)

Find the optimal control trajectory u(t) which minimizes

$$J = \frac{1}{2}\int_0^2 (X^2 + u^2)dt \qquad (5.2.21)$$

subject to

$$\dot{X} = 2X + 3u \qquad (5.2.22)$$

How does this control differ from the one obtained when the optimisation horizon is infinte?

SOLUTION

$$A = 2,\ B = 3,\ Q = 1 \text{ and } R = 1$$

solving using Riccati equation, we have

(a)

$$\begin{aligned}\dot{p}(t) &= -p(t)A(t) - A^T p(t) + p(t)B(t)R^{-1}(t)B^T p(t) - Q(t) \\ &= -2p(t) - 2p(t) + 9p^2 - 1\end{aligned} \qquad (5.2.23)$$

$$\therefore \int \frac{dp}{dt} = 9\int_0^2 p^2(t)\,dt - 4\int_0^2 p(t)\,dt - \int_0^2 1\,dt$$

$$= \left[3 \times p^3(t) - 2p^2(t) - t\right]_0^2$$

$$= 3 \times 8 - 2 \times 4 - 2$$

$$= 24 - 8 - 2$$

$$= 14$$

Hence p = 14.

Thus the optimal control is given by

$$\underline{u}(t) = G\underline{X}(t)$$

where $G = -R^{-1} B^T p$

$$= -1 \times 3 \times 14$$

$$= -42$$

$$\therefore u(t) = -42\, \underline{X}(t)$$

(b)

If the optimisation horizon is infinite, $t_f \to \infty$ and

$$\dot{p}(t) = 0$$

=> from equation (5.4.23) that

$$0 = -4p(t) + 9p^2(t) - 1$$

$$= 9p^2(t) - 4p(t) - 1$$

$$\text{i.e. } p^2 - \frac{4}{9}p - \frac{1}{9} = 0$$

$$\text{i.e. } p = \frac{2}{9} \pm \frac{\sqrt{13}}{9}$$

But p is positive definite by definition, thus

$$p = \frac{2}{9} + \frac{\sqrt{13}}{9}$$

Hence the optimal control law is

$$u(t) = -\frac{3(2+\sqrt{13}}{9}\underline{X}(t)$$

$$= -\frac{(2+\sqrt{13}}{3}\underline{X}(t)$$

EXAMPLE 5-2-2 (OPTIMAL CONTROL TRAJECTORY)

Find the optimal control trajectory which minimizes

$$J = \frac{1}{2}\int_0^{2}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + u^2\right)dt$$

subject to

$$\dot{X}_1 = X_2$$
$$\dot{X}_1 = u$$

SOLUTION

Consider the RICCATI matrix equation

$$\dot{p}(t) \quad -p(t)A(t) - A^T p(t) + p(t)B(t)R^{-1}(t)B^T p(t) - Q(t)$$

Here $A \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$R = 1 \Rightarrow R^{-1} = 1$

$$\therefore \frac{d}{dt}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} p_{121} & p_{22} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{12} & 0 \\ p_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_{12} \\ 0 & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ p_{12} & p_{22} \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= -\begin{bmatrix} 2p_{12} & p_{22} \\ p_{22} & 0 \end{bmatrix} + \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} p_{12}^2 - 2p_{12} + 1 & p_{12}p_{22} - p_{22} \\ p_{12}p_{22} - p_{22} & p_{22}^2 - 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \int_0^2 (p_{12}^2 - 2p_{12} - 1)dt & \int_0^2 (p_{12}p_{22} - p_{22})dt \\ \int_0^2 (p_{12}p_{22} - p_{22})dt & \int_0^2 (p_{22}^2 - 2)dt \end{bmatrix}$$

$$\Rightarrow p_{11} = \int_0^2 (p_{12}^2 - 2p_{12} + 1)dt$$

$$= 8/3 - 4 - 2$$

$$= \frac{8-18}{3} = \frac{-10}{3}$$

$$p_{22} = \int_0^2 (p_{22}^2 - 2)dt$$

$$= 8/3 - 4$$

$$= \frac{-4}{3}$$

$$\therefore p_{12} = -\frac{4}{3}\int_0^2 \left(p_{12}^2 - 1\right)dt$$

$$= -\frac{4}{3}\left[\frac{p_{12}^2(t)}{2} - t\right]_0^2$$

$$= -\frac{4}{3}\left[\frac{4}{2} - 2\right]$$

$$= 0$$

$$\therefore \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -10/3 & 0 \\ 0 & -4/3 \end{bmatrix} \qquad (5.2.24)$$

Now by (5.2.24) p is positive.

$\therefore$ The optimal control law is

$$u(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -10/3 & 0 \\ 0 & -4/3 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

$$\text{i.e.} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & -4/3 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = -\begin{bmatrix} 0 \\ -4/3 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

Hence $u_2(t) = \frac{4}{3} X_2(t)$

EXAMPLE 5.2.3 (OPTIMAL CONTROL)

Obtain the control law which minimizes the performance index

$$J = \int_0^{\infty} \left(X_1^2 + u^2\right)dt \qquad (5.2.25)$$

for the system

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Given that

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = [2]$$

We shall apply the RICCATI's matrix equation

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

We note $\dot{P} = 0$; since time is infinite

Now $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

$R = [2] \Rightarrow R^{-1} = [1/2]$

The reduced Riccati matrix equation is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & p_{12} \\ 0 & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{i.e} \begin{bmatrix} 0 & p_{11} \\ p_{11} & 2p_{12} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} p_{12}^2 & \frac{1}{2} p_{12} p_{22} \\ \frac{1}{2} p_{12} p_{22} & \frac{1}{2} p_{22}^2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (5.2.26)$$

Substituting equation (5.2.26) yields

$$\frac{1}{2}p_{12}^2 + 2 = 0$$

$$p_{11} - \frac{1}{2}p_{12}p_{22} = 0 \qquad (5.2.27)$$

$$-\frac{1}{2}p_{22}^2 + 2p_{12} = 0$$

The solution of equation (5.2.27) yields the positive definite matrix

$$p = \begin{bmatrix} 2\sqrt{2} & 2 \\ 2 & 2\sqrt{2} \end{bmatrix}$$

From equation (5.2.14), the optimal control law is given by

$$u(t) = -R^{-1} B^T p X(t)$$

$$= -[1/2]\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & 2 \\ 2 & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 2 & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

$$= -X_1(t) - \sqrt{2}\,X_2(t0$$

EXAMPLE 5.2.4 (INFINITE TIME OPTIMAL CONTROL)

Consider an infinte time regulator, this time for a system of second order.

i.e the system is described by:

$$\dot{X}_1 = -X_1 + X_2 + u_1$$
$$\dot{X}_2 = -X_2 + u_2 \qquad (5.2.28)$$

$$J = \frac{1}{2}\int_0^t \left(X_1^2 + 2X_2^2 + u_1^2 + u_2^2\right) dt$$

In this case, the Riccati equation can be written using the fact that

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix};\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that it becomes

$$\frac{d}{dt}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = -\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$+\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

since the final time $t_f \rightarrow \infty$

$$\frac{d}{dt}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \rightarrow 0$$

So that we can either solve this iteratively or by taking t_f to be sufficiently long for p_{11}, p_{12}, p_{22}, we then integrate the Riccati equation backward from

$$p_{11}(t_f) = 0;\ P_{12}(t_f) = 0,\ p_{22}(t_f) = 0$$

Suppose we take $t_f = 8$ and this gives the constant gain matrix

$$G = -\begin{bmatrix} 0.4087 & 0.1274 \\ 0.1274 & 0.7995 \end{bmatrix}$$

so that

$$u(t) = -\begin{bmatrix} 0.4087 & 0.1274 \\ 0.1274 & 0.7994 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

5·3 — **5.3 DISCRETE LINEAR QUADRATIC PROBLEM**

Many systems particularly of social economic class are most naturally represented as discrete time processes. It is therefore possible to use a discrete version of the discrete Pontryagin's maximum principle.

Consider a discrete time non-linear dynamical system with state vector $X(k)$ and control vector, $u(k)$, at the instant k. The state at the instant k+1 is related to state at instant k by

$$X(k+1) = f(X_k, u_k, k) \tag{5.3.1}$$

where f is a continuous function.

Consider the regulator problem

$$\min J = \frac{1}{2}\|X(k_f)\|^2 S(k_f) + \frac{1}{2}\sum_{k_f=0}^{k_f-1}\left\{\|X_k\|^2 Q_k + \|u_k\|^2 R_k\right\} \tag{5.3.2}$$

Where Q, S are non-negative definite whilst R is positive definite subject to the linear dynamical constants

$$X(k+1) = A(k)X(k) + B(k)u(k) \tag{5.3.3}$$

This problem is solved by forming the Hamiltonian function as

$$H = \frac{1}{2}\|X(k)\|^2 Q(k) + \frac{1}{2}\|u(k)\|^2 R(k) + \lambda^T(k+1)\left[A(k)X(k) + B(k)u(k)\right] \tag{5.3.4}$$

Applying the necessary conditions for optimality, we have:

$$\frac{\partial H}{\partial X(k)} = \lambda(k) = Q(k)X(k) + A^T(k)\lambda(k+1) \tag{5.3.5}$$

Equation (5.5.5) is not solvable unless A is invertible. But A is a state transition matrix; A^{-1} exists.

The boundary condition is

$$\lambda(k_f) = SX(k_f) \tag{5.3.6}$$

for the control

$$\frac{\partial H}{\partial u(k)} = 0 = R(k)u(k) + B^T(k)\lambda(k+1) \tag{5.3.7}$$

$$\Rightarrow u(k) = -R^{-1}(k)B^T(k)\,\lambda(k+1)$$

We can obtain the control by solving the equations

$$\underline{X}(k+1) = A(k)\,\underline{X}(k) - B(k)\,R^{-1}(k)\,B^T(k)\,\lambda(k+1) \qquad (5.3.8)$$

$$X(k_0) = X_0 \qquad (5.3.9)$$

and

$$\underline{\lambda}(k) = Q(k)\underline{X}(k) + A^T(k)\underline{\lambda}(k+1) \qquad (5.3.10)$$

with the boundary condition

$$\underline{\lambda}(k_f) = S(k_f)\underline{X}(k_f) \qquad (5.3.11)$$

As the continuous regulator treated previously, take the solution of the form:

$$\lambda(k) = p(k)\underline{X}(k) \qquad (5.3.12)$$

Considering (5.5.8) and (5.5.10) with respect to (5.5.12);

that is :-

$$X(k+1) = A(k)\,\underline{X}(k) - B(k)\,R^{-1}(k)\,B^T(k)\,p(k+1)\underline{X}(k+1) \qquad (5.3.13)$$

and

$$\Rightarrow p(k)X(k) = Q(k)\underline{X}(k) + A^T(k)p(k+1)\underline{X}(k+1) \qquad (5.3.14)$$

Solving for $\underline{X}(k+1)$ and eliminating from (5.5.13) and (5.5.14), we obtain:

$$X(k+1) = \left[I + B(k)R^{-1}(k)\,B^T(k)\,p(k+1)\right]^{-1}A(k)\,X(t)$$

Therefore considering (5.5.14),

$$p(k)\underline{X}(k) = Q(k)\underline{X}(k) + A^T(k)p(k+1)[\,I + B(k)R^{-1}(k)\,B^T(k)\,p(k+1)]^{-1}A(k)\underline{X}(t) \qquad (5.3.15)$$

Where I is the identity matrix.

This equation holds for arbitrary $\underline{X}(k)$ only if

$$p(k) = Q(k) + A^T(k)\,p(k+1)\,[\,I + B(k)R^{-1}(k)\,B^T(k)\,p(k+1)]^{-1}A(k) \qquad (5.3.16)$$

with the condition at the final stage being

$$p(k_f) = S \quad (5.3.17)$$

The optimal control requires the solution of the matrix Riccati equation (5.3.15) and (5.5.16) backwards in time from $k=k_f$ to $k=k_0$ and then

$$\underline{u}(k) = -R^{-1}(k)\, B^T(k)\, A^{-1}\, [p_k - Q]\, \underline{X}(k) \quad (5.3.18)$$

$$= G(k)\, \underline{X}(k) \quad (5.3.19)$$

where G(k) could be thought as a "Gain".

5.3.1 Infinte Stage Regulator

If A, B, Q, R are time invariant, S=0 and the system is controllable, then p(k) becomes constant as $k \to \infty$, thus G the "Gain" becomes constant.

EXAMPLE 5.3.1 (TIME INVARIANT)

Find the optimal control for the system

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

where the cost function to be minimised is

$$J = \sum_{k=0}^{\infty} \left(X_1^2(k) + X_2^2(k) + u^2 \right)$$

SOLUTION

The Riccati matrix equation is

$$P(k) = Q(k) + P(k+1)\, A^T(k)\, [I + B(k)\, R^{-1}(k)\, P(k+1)]^{-1}\, A(k)$$

with the terminal solution

$$p(k_f) = 0$$

Here $A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$; $A^1 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$; $p(k_1 - 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

$B^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \; p(k_1 - 2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 \\ 0 & 8 \end{bmatrix}$$

Now since the problem is an infinite regulatory one, we can take G to be a constant.

$$\therefore \; G = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ & \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 9 & 8 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 8 & 8 \\ 0 & 7 \end{bmatrix}$$

$$= - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 8 & 8 \\ 0 & 7 \end{bmatrix}$$

$$\Rightarrow G = - \begin{bmatrix} 16 & 16 \\ 0 & 0 \end{bmatrix} \text{ is a constant}$$

Thus the optimal trajectory is

$$u(k_1) = - \begin{bmatrix} 16 & 16 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1(k_1) \\ X_2(k_1) \end{bmatrix}$$

$$\Rightarrow u(k_1) = -16X_1(k_1) - 16X_2(k_1)$$

i.e $u(k_1) = -16X_1(k_1) + 16X_2(k_1)$

EXAMPLE 5.32 (2 POINTS BVP)

Write down the two ponit BVP for the optimal control problem of minimizing .

$$J = \frac{1}{2}\begin{bmatrix} X_1(6) \\ X_2(6) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} X_1(6) \\ X_2(6) \end{bmatrix} + \sum_{k=0}^{5}\left\{\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} \quad \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}\right\}$$

subject to the constraints

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

SOLUTION

Rewrite the question as:

minimize

$$J = \frac{1}{2}\left[X_1^2(k) + 2X_2^2(k)\right] + \sum_{k=0}^{5}\left\{\left[X_1^2(k) + 3x_2^2(k)\right] + \left[u_1^2(k) + 2u_2^2(k\right]\right\}$$

subject to

$$X_1(k+1) = 2X_1(k) + X_2(k)$$

$$X_2(k+1) = X_1(k) + X_2(k) + u_1(k) + u_2(k)$$

Then the Hamiltonian function is

$$H = X_1^2(k) + 3X_2^2(k) + u_1^2(k) + 2u_2^2(k) + \lambda_1(k+1)\left[2X_1(k) + X_2^2(k)\right] + \lambda_2(k+1)\left[X_1(k) + X_2(k) + u_1(k) + u_2(k)\right]$$

The optimality conditions are

$$\frac{\partial H}{\partial u_2} = 0 \quad => \quad 4u_2(k) + \lambda_2(k+1) = 0$$

$$=> \quad u_2(k) = -\frac{1}{4}\lambda_2(k+1)$$

Thus $\frac{\partial H}{\partial \lambda_1(k+1)} = X_1(k+1) = 2X_1(k) + X_2^2(k)$

$$\frac{\partial H}{\partial X_1} = \lambda_1(k) = 2X_1(k) + 2\lambda_1(k+1) + \lambda_2(k+1)$$

$$\frac{\partial H}{\partial X_2} = \lambda_2(k) = 6X_2(k) + \lambda_1(k+1) + \lambda_2(k+1)$$

The boudary conditions

$$\lambda_1(6) = 2X_1(6), \quad \lambda_2(6) = 12X_2(6)$$

Thus the two points BVP becomes:

$$X1(k+1) = 2X1(k) + X22(k)$$

$$X_2(k+1) = X_1(k) + X_2(k) - 1/2\lambda_1(k+1) - 1/4\,\lambda_2(k+1)$$

and

$$\lambda_1(k) = 2X_1(k) + 2\lambda_1(k+1) + \lambda_2(k+1)$$

$$\lambda_2(k) = 6X_2(k) + \lambda_1(k+1) + \lambda_2(k+1)$$

with the terminal conditions

$$\lambda_1(6) = 2X_1(6)$$

$$\lambda_2(6) = 12X_2(6)$$

EXAMPLE 533 (SCALAR DISCRETE OPTIMAL CONTROL PROBLEM)

Consider the scalar problem:

Consider the scalar problem:

$$\min J = \sum_{k=0}^{3} \frac{1}{2} X^2(k) + \frac{1}{2} u^2(k)$$

subject to $X(k+1) = X(k) + u(k)$; $X(0) = 1$

The discrete optimal control for the regulatory problem is given by

$$u(k) = -R^{-1}(k) B^{T}(k) A^{-1T}\left[p(k) - Q\right] X(k)$$

From the Riccati equation,

With the condition at the final stage being

$p(k_f) = S$

Hence $p(k_f) = p_3 = 0$, since $S = 0$

$A = 1,\ B = 1,\ Q = 1,\ R = 1$

$p(2) = 1 + 1 \times 0\,[1 + 1 \times 1 \times 1 \times 0]^{-1}\,1 = 1$

$\Rightarrow p(2) = 1$

$p(1) = 1 + 1 \times 1[1 + 1]^{-1}\,1$

$= 1 + 1/2 = 1.5$

$\therefore\ p(1) = 1.5$

$p(0) = 1 + 1.5\,[1 + 1 \times 1 \times 1.5]^{-1}\,1$

$= 1 + 1 \times 1.5\,(2.5)^{-1}$

$= 1 + 1.5/2.5 = 1.6$

$\therefore\ p(0) = 1.6$

Therefore $u(k) = -1.1\,(1.6 - 1)\,X_k$

$= -0.6\,X(k)$

$\Rightarrow \quad u(k) = -0.6\,X(k)$

5.4 CONCLUSION AND SUMMARY

The project was aimed at obtaining the solution of a state-space equation; the optimal control for a single input single output (SISO) system and the optimal control for minimizing the quadratic cost function using the multi-input multi-output (MIMO) state-space as constraints.

For the state space solution, we achieved this aim by the application of Fadeev algorithm. This Faddeev algorithm also helped in the calculation of the optimal control of a single-input-single-output system using the Kalman and Ackermann equations. The calculation of the optimal control of the quadratic cost-function of MIMO was achieved by the use of the Pontryagin's maximum principle otherwise called the Hamiltonian form. By the Hamiltonian form, we are able to find an equation called the Riccati equation from which a symmetric positive definite metrix P is calculated. This value of P is used to obtain the MIMO optimal control system:

$$U= -R^{-1}B^{T}PX \qquad (5.4.1)$$

If the system is a continuous problem, otherwise for a discrete quadratic problem, the optimal control is

$$U(K) = -R^{-1}\ (K)\ B^{T}(K)A^{-1}(P_k - Q)X_k \qquad (5.4.2)$$

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APPENDIX

```
CLS
FOR x = 40 TO 25 STEP -1
  LOCATE 10, x: COLOR 3, 0, 0: PRINT " The Application of Faddeev
Algorithm "
  LOCATE 12, x: PRINT "        To Transfer Function  "
NEXT
A$ = INPUT$(1)
CLS

OPEN "fadeev.out" FOR OUTPUT AS #1

DIM A1(20, 20), B1(20, 20), C1(20, 20), AB1(20, 20), D1(20, 20)
DIM E1(20, 20), x(20, 20), B11(20, 20), F1(20, 20), A(20, 20)

CLS
Start:

CLS
PRINT #1,
PRINT #1, "To obtain the Tranfer Function of a linear sysytem"
PRINT #1, "using the Faddeev Algorithm. Example: Given a system"
PRINT #1, "(A,B,cT) below "
 PRINT #1,
 PRINT #1, "          | 0   1   0 |        | 0 |  "
 PRINT #1, "    A = | 3   0   2 |    B = | 1 |   cT = | -1  1  0 |"
 PRINT #1, "          |-12 -7  -6 |        | 1 |  "
 PRINT #1, "Solution :- The Faddeev Algorithm is calculated using
the "
 PRINT #1, "following theory of resolvent matrix [SI-A]^-1."
 PRINT #1, "This will be demonstrated in the next page. "
 PRINT #1,
 PRINT #1, "(SI-A)^-1 = Γn-1 "
 PRINT #1, "Thus n=3 "
 PRINT #1, "Hence Γ2 = I; By definition "
 PRINT #1, "Also, α2 = -Tr(AΓ2) "

CLS
PRINT #1,
PRINT #1, "Solving For α2 ........ to get Trace value"
PRINT #1, : PRINT #1, : PRINT #1,

'***************** TO SOLVE FOR α2 **********************
NIteration = 0
GOSUB RD1
GOSUB matinputA: GOSUB diagnal:
PRINT #1, " α2 = -Tr(AΓ2)/1 = "; -1 * ((dgnal) / 1)

GOSUB iteration
```

```
'**************** TO SOLVE FOR Γ1 **********************
CLS
PRINT #1,
PRINT #1, "Γ1 = AΓ2 + α2I "
PRINT #1,
PRINT #1, "Solving for Γ1 .........."

GOSUB PROCESS

 FOR i = 1 TO irow
  FOR j = 1 TO icol
    LOCATE i + 5, j * 5
 PRINT #1, A1(i, j)
NEXT j, i
PRINT #1, "The above is AΓ2 is to be added to α2 which will be
keyed in below"

PRINT #1, : PRINT #1,
GOSUB Rd2: GOSUB MatinputB:  'To key in Alpha2 at the keyboard
GOSUB addmat    'To add the values of AΓ2 + α2
LOCATE 8, 2: PRINT #1, "The value of gamma1 is as shown below"

FOR i = 1 TO irow
  FOR j = 1 TO icol
  LOCATE 5 + i + 5, j * 7: PRINT #1, C1(i, j)

NEXT j, i

GOSUB iteration
'**************** TO SOLVE FOR α1 **********************
100 CLS
PRINT #1, : PRINT #1, : PRINT #1,
PRINT "Solving for Alpha1........"

GOSUB PROCESS
PRINT #1, "α1 = -Tr(AΓ1)/2"
 FOR i = 1 TO irow
  FOR j = 1 TO icol
   LOCATE 2 + i + 5, j * 7: PRINT #1, A1(i, j)
   LOCATE 2 + i + 5, j * 7 + 30: PRINT #1, C1(i, j)
 NEXT j, i

FOR i = 1 TO irow
FOR j = 1 TO icol
E1(i, j) = C1(i, j)
NEXT j, i

  FOR i = 1 TO irow
   FOR j = 1 TO icol2
     AB1(i, j) = 0
      FOR k = 1 TO icol
       AB1(i, j) = AB1(i, j) + A1(i, k) * E1(k, j)
```

```
        LOCATE 10 + i + 5, j * 8: PRINT #1, AB1(i, j)
  NEXT k, j, i

dgnal2 = AB1(1, 1) + AB1(2, 2) + AB1(3, 3)
PRINT #1, : PRINT #1, " α1 = -Tr(AΓ1)/2 = "; -1 * ((dgnal2) / 2)

GOSUB iteration


'***************** TO SOLVE FOR Γ0 **********************
CLS
PRINT #1, "Γ0 = AΓ1 + α1I"
PRINT #1, : PRINT #1,
PRINT #1, "Solving for Γ0 .........."

GOSUB PROCESS

 FOR i = 1 TO irow2
   FOR j = 1 TO icol2
    B11(i, j) = AB1(i, j)
    LOCATE 2 + i + 5, j * 7: PRINT #1, B11(i, j)
NEXT j, i

GOSUB Alpha1

LOCATE 15, 5: PRINT #1, "The above is the result of AΓ1 + α1I and"
LOCATE 16, 5: PRINT #1, "the result of the addition is below "

 FOR i = 1 TO irow
   FOR j = 1 TO icol
     D1(i, j) = B11(i, j) + x(i, j)
     LOCATE 12 + i + 5, j * 7: PRINT #1, D1(i, j)
 NEXT j, i

GOSUB iteration

'***************** TO SOLVE FOR α0 **********************
CLS
PRINT #1, : PRINT #1, : PRINT #1,
PRINT #1, "Solving for α0 ..........."

GOSUB PROCESS
'*********  print #1,ING OUT THE VALUE OF A   ************
 FOR i = 1 TO irow
  FOR j = 1 TO icol
    LOCATE 3 + i + 5, j * 7: PRINT #1, A1(i, j)
NEXT j, i
PRINT #1, "": PRINT #1, ""
'*********  print #1,ING OUT THE VALUE OF Γ0   ************
FOR i = 1 TO irow
  FOR j = 1 TO icol
    LOCATE 3 + i + 5, j * 7 + 30: PRINT #1, D1(i, j)
```

```
NEXT j, i

'****** MULTIPLYING THE VALUES OF A AND Γ0 {i.e -Tr(AΓ0)} ******
FOR i = 1 TO irow
FOR j = 1 TO icol
   B1(i, j) = D1(i, j)
NEXT j, i

  FOR i = 1 TO irow
   FOR j = 1 TO icol2
     F1(i, j) = 0
      FOR k = 1 TO icol
      F1(i, j) = F1(i, j) + A1(i, k) * B1(k, j)
       LOCATE 10 + i + 5, j * 8: PRINT #1, F1(i, j)
   NEXT k, j, i
PRINT #1, : PRINT #1,
dgnal3 = F1(1, 1) + F1(2, 2) + F1(3, 3)
PRINT #1, " α0 = -Tr(AΓ0)/3 = "; -1 * ((dgnal3) / 3)

GOSUB iteration

CLS
LOCATE 5, 5: PRINT #1, "The Number of Iteration is "; NIteration
PRINT #1, : PRINT #1,
PRINT #1, "Substituting the iterative values for the value of Γi's"
PRINT #1, "and αi's, we obtain the following resolvent matrix : "
PRINT #1,
PRINT #1, "                         Γ2S² +  Γ1S  + Γ0     "
PRINT #1, "  (SI - A)^-1 =   -----------------------  "
PRINT #1, "                        S^3  + α2S² + α1S + α0 "
PRINT #1, : PRINT #1, "See next page for the result. "
PRINT #1,

CLS
'FOR m = 9 TO 11
'  LOCATE m, 9: PRINT #1, CHR$(179)
'  LOCATE m, 23: PRINT #1, CHR$(179)
'    LOCATE m, 29: PRINT #1, CHR$(179)
'    LOCATE m, 43: PRINT #1, CHR$(179)
'      LOCATE m, 49: PRINT #1, CHR$(179)
'      LOCATE m, 63: PRINT #1, CHR$(179)
'  NEXT

LOCATE 7, 5: PRINT #1, "Therefore (SI - A)^-1 is as shown below "
'LOCATE 10, 24: PRINT #1, "S² +"
'LOCATE 10, 44: PRINT #1, "S +"
A(1, 1) = 1: A(1, 2) = 0: A(1, 3) = 0
A(2, 1) = 0: A(2, 1) = 1: A(2, 3) = 0
A(3, 1) = 0: A(3, 2) = 0: A(3, 3) = 1

FOR i = 1 TO irow
  FOR j = 1 TO icol
```

```
   LOCATE 5 + i + 3, j * 5 + 5: PRINT #1, A(i, j)
NEXT j, i

  FOR i = 1 TO irow
    FOR j = 1 TO icol
     LOCATE 5 + i + 3, j * 5 + 25: PRINT #1, C1(i, j)
      LOCATE 5 + i + 3, j * 5 + 45: PRINT #1, D1(i, j)
  NEXT j, i

'FOR x = 11 TO 62: LOCATE 12, x: PRINT #1, "-": NEXT
LOCATE 14, 24
PRINT #1, "S^3 +"; -1 * (dgnal); "S² +"; -1 * (dgnal2 / 2); "S +";
-1 * (dgnal3 / 3)

PRINT #1, : PRINT #1,

CLS
'FOR m = 6 TO 8
'  LOCATE m, 18: PRINT #1, CHR$(179)
'  LOCATE m, 33: PRINT #1, CHR$(179)
'    LOCATE m, 38: PRINT #1, CHR$(179)
'    LOCATE m, 50: PRINT #1, CHR$(179)
'      'LOCATE m, 49: print #1, CHR$(179)
      'LOCATE m, 63: print #1, CHR$(179)

'  NEXT

LOCATE 10, 5: PRINT #1, "(SI - A)^-1 = "
FOR i = 1 TO irow
  FOR j = 1 TO icol
    A$(i, j) = STR$(A(i, j)) + "S²"
    C1$(i, j) = STR$(C1(i, j)) + "S"
    D1$(i, j) = STR$(D1(i, j))
    D11$(i, j) = A$(i, j) + "+" + C1$(i, j) + "+" + D1$(i, j)
    LOCATE 5 + i, j * 20: PRINT #1, D11$(i, j)
NEXT j, i

'FOR x = 20 TO 70: LOCATE 10, x: PRINT #1, "-": NEXT
LOCATE 12, 30
PRINT #1, "S^3 +"; -1 * (dgnal); "S² +"; -1 * (dgnal2 / 2); "S +";
-1 * (dgnal3 / 3)

CLOSE #1

END


RD1:      'Subroutine that request for matrice dimension1

INPUT "Dimension for Row A1: ", irow
```

```
INPUT "Dimension for Column A1: ", icol
PRINT #1, "Dimension for Row A1: ": PRINT #1, irow
PRINT #1, "Dimension for Column A1: ": PRINT #1, icol

RETURN


Rd2:      'Subroutine that request for matrice dimension2

INPUT "Dimension for Row α2: ", irow2
INPUT "Dimension for Column α2: ", icol2
PRINT #1, "Dimension for Row α2: ": PRINT #1, irow2
PRINT #1, "Dimension for Column α2: ": PRINT #1, icol2

RETURN


matinputA:          'Subroutine matrice input for trace value

PRINT #1, "Input For Matrix A "
 FOR i = 1 TO irow
  FOR j = 1 TO icol
LOCATE i + 15, j * 10: INPUT A1(i, j)
 PRINT #1, A1(i, j)
 NEXT j, i
RETURN


MatinputB:          'Subroutine for matrice input for α2

PRINT #1, "Input For α2 "
 FOR i = 1 TO irow2
  FOR j = 1 TO icol2
LOCATE 5 + i + 10, j * 10: INPUT B1(i, j)
PRINT #1, B1(i, j)
NEXT j, i

RETURN


addmat:  'Subroutine that check the Rows and Columns
         'equality before matrice addition

10 CLS

IF irow <> irow2 THEN PRINT " ERROR! ": GOTO 10
IF icol <> icol2 THEN PRINT " ERROR! ": GOTO 10
GOSUB Add

RETURN

Add:      'Subroutine for matrice addition
```

```
    LOCATE i + 10, j * 10
 PRINT #1, C1(i, j)
NEXT j, i

RETURN


diagnal:    'Subroutine that picks the Trace value for α2

dgnal = A1(1, 1) + A1(2, 2) + A1(3, 3)

RETURN


matricemult:

 FOR i = 1 TO irow
  FOR j = 1 TO icol
    AB1(i, j) = 0
     FOR k = 1 TO icol2
   AB1(i, j) = A1(i, k) * C1(k, j)
   LOCATE 12 + i + 5, j * 10: PRINT #1, AB1(i, j)
 NEXT k, j, i

RETURN


PROCESS: 'Subroutine that causes little dalay

FOR x = 1 TO 150
LOCATE 4, 27: PRINT 1, "/": LOCATE 4, 27: PRINT 1, "|"
LOCATE 4, 27: PRINT 1, "\": NEXT

RETURN


iteration:

NIteration = NIteration + 1

RETURN
```