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DYNAMIC ANALYSIS OF A TRANSVERSE DISPLACEMENT OF ROD WITH HARMONIC LOAD

BY

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CERTIFICATION

This thesis titled "DYNAMIC ANALYSIS OF A TRANSVERSE DISPLACEMENT OF ROD WITH HARMONIC LOAD" by Adamu Alhaji Mohammed meets the regulations governing the award of the degree of Masters of Technology in Mathematics, Federal University of Technology Minna and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This project is dedicated to Almighty Allah. It is also dedicated to my father, Alhaji Abubakar Ahmed and my beloved mother, Malama Aishetu Abubakar for their fearless support and encouragement in my quest for knowledge.

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ABSTRACT

Only system that can be modeled as one-dimensional continuous mechanical or electrical media, in particular continuum models of rods, which undergo uniaxial deformation due to longitudinal forces, will be considered in this dissertation. Systems that by their continuous nature are said to possess an infinite number of degrees of freedom. One-dimensional continuous dynamics models lead to partial differential equations (PDE) of motion. Hamilton's principle was used to derive the equations of motion with the significant benefit of providing the natural boundary conditions. We define the terms in the approximate variational indicator and then use the calculus of variations to obtain the equations of motion as well as the natural boundary conditions. We also discover that the amplitude of vibration of the rod rises when the cross sectional area of the rod and the speed at which the wave propagate increases.

CHAPTER ONE

1.1 INTRODUCTION

One dimensional continuous mechanical models lead to partial differential equations (PDE) of motion. In particular, partial differential equation is an equation involving one or more partial derivatives of an unknown function of two or more independent variables.

Rod is thin, straight piece of metal. Rod is a structure, which undergo uniaxial deformation due to longitudinal forces.

The basic requirements for formulating the equation of motion for mechanical system are:-

(i) Geometric requirements on the motions

(ii) Dynamic requirements on the forces, and

(iii) Constitutive requirements for all the system elements and fields.

In formulating the equations of motion for a specific model, we begin by defining the geometric constraints on the system which is a requirement that restricts or imposes the system's spatial motion.

Requirement 2 satisfies the Hamilton's principle. Hamilton's principle states that the variation of the kinetic energy and potential energy plus the variation of the work done by the nonconservative force considered during any time interval t to t must equal zero. This approach enables us to obtain all boundary condition with ease.

Requirement 3 will be satisfied by introducing the constitutive information characterizing the system's elements and field. This is accomplished in terms of work like and energy like expressions, enunciated in accordance with the constitutive relations, which are rendered in terms of the geometrically admissible states.

A continuous system-containing rod considered is of two-degree-of-freedom systems. An admissible variation of a generalized coordinate is a hypothetical contemporaneous change from one geometrically allowable state to a neighbouring geometrically allowable state. The number of independent admissible variations in a complete set of admissible variations is the number of degree of freedom.

Generally rods are being excited by externally applied forces, the distributed longitudinal force and the prescribed force. Rod models are useful in analyzing longitudinal vibration and wave propagation in ship propulsion systems, such as those in ocean liners and aircraft carriers.

1.2 AIMS AND OBJECTIVES

The aim of this study is to carryout investigation on the longitudinal vibration of a rod, which undergoes uniaxial deformation due to longitudinal forces.

The objectives of this study include the following:-

(i) To derive the equation of motion and the boundary condition defining the longitudinal vibration of the rod, this is formulated using the variational indicator of the Hamiltonian principle.

(ii) To obtain the analytical solution to the derived partial differential equation for system models containing no generalized forces and system containing a longitudinal forces distribution.

(iii) To provide graphical summaries of the system responses

1.3 SCOPE AND LIMITATION

The governing equation of motion was formulated using variational indicator of the Hamiltonian principle. The analytical results for the initial boundary value problem for system containing a longitudinal force distribution f(x,t) was obtained only when f(x,t) is separable in to spatial and temporal functions and the temporal function is harmonic. The boundary condition assumed is the simple-supported ends condition.

1.4 FEATURE OF THE DISSERTATION

Chapter 1 gives general overview of the problem. In chapter 2 relevant scientific literatures are reviewed together with brief explanation on Hamiltonian principle as well as review of partial differential equations. Chapter 3 gives the analytical methods of solutions, where the governing equation of motion was derived using the variational indicator of the Hamiltonian principle. In chapter 4 the problem statement was formulated and the initial-boundary value problem of the Rod was solved using the method of separation of variable. Chapter 5 gives the numerical simulation as well as the discussion of simulated results with the graphical representation of the results. Finally conclusion and recommendation for future study were also stated.

CHAPTER TWO

2.1 LITERATURE REVIEW

Continuous systems are systems that are generally referred to as system with infinite number of degree of freedom. One dimensional continuous dynamics models lead to partial differential equation (PDE) of motion. Various researchers over the years have immensely contributed in the field of dynamic especially continuous systems containing rod.

The works of the following researchers are relevant in this dissertation.

Sergey B and Olga N. (2002) considered the features of longitudinal compression waves propagating in a finite-cross-section homogeneous elastic rod. They provided a new method for obtaining exact analytical solutions for vibrant elastic systems. They also proved that the velocity of accompanying transversal wave's propagation is equal to the velocity of longitudinal wave, while the velocity of accompanying longitudinal wave is equal to the velocity of transversal wave when equal velocities of the main longitudinal and transversal waves are involved. In addition, they proved that in case of equal coefficients of longitudinal and transversal stiffness, the direction of vibrations of an elastic lumped system will always coincide with the direction of the external force action.

Wang and Varadan (2002) Presented the results of longitudinal wave propagation in piezoelectric coupled rod structures. They based their deduction of non-dispersive or dispersive characteristics of the structures on a classical rod model and the Mindlin–Herrmann rod model. Using classical model, they introduced correction factors for piezoelectric effects to provide remedy for the discontinuity of the normal stress at the interface of the host rod and the piezoelectric layer. The model they used is more accurate in predicting the dispersive characteristics of the structures. They obtained the results of the long- and short-wavelength limits as by-products. The results of phase velocity by the two models are agreeable at low wave number by adjusting the correction factor defined in the classical model.

Fairhurst (1961) describe what happened when two rods impact. Across the plane of contact two conditions must be fulfilled during impact: (1) the contact forces in the standard penetration test (SPT) hammer and the rod must be equal; and (2) the absolute spatial velocities of the striking end of the hammer and the struck end of the rod must be equal at all times when the two surfaces are in contact.

Kupka and Kupková (2001). Presented some results of a theoretical investigation on the free flexural vibration of a slender prismatic rod containing the materials for which the elastic moduli in tension and compression may differ. For simplicity they considered rod that possessed a longitudinal plane of symmetry, and the initial condition adjustment and support are also symmetric with respect to this plane. The equation of motion they derived is nonlinear, no matter how low the amplitude of flexural vibration is. The nonlinearity is based on the fact that the flexural rigidity of the rod depends on the sign of a local curvature of the bent rod treated as an elastic line. They also investigated the particular solution that corresponds to the fundamental mode of flexural vibration. They observed that in the mode, the rod performs periodic, though not simple harmonic, vibration. The frequency of this vibration is derived as a function of the material and geometric properties of the rod

Lafortune (2003) worked on systems appearing in elastic rod theory. The dynamical equations of elastic rods widely in his research work are due to Kirchhoff. These equations are applicable to the case when local deformations relative to an undistorted configuration remain small, although rotations may be large. Almost all the analyses done on the Kirchhoff equations are shown to support time-dependent solutions.

In a series of papers, Goriely et al (1997,1999,2001) obtained the amplitude equations describing the dynamics of a rod beyond its first writhing instability.

Shim (2002) described an eigen value conforming model for a vibration rod. He named the model as a spectral conforming discrete model, which estimates the n lowest eigen values of the continuous system with uniform accuracy. The essential ingredient he used in building up such a model is the inverse eigen value problem of reconstructing a chain of mass-spring system with prescribed spectral data. On his future research in this work, he extends the method to include tapered elements that can better capture the geometry of a non-uniform rod. Broadening the method over two and three-dimensional elements appears to be a challenging problem.

Stoneley (1924) considered a more general problem of the wave propagation at the separation surface of two solid media. He showed that in media there must propagate as waves similar to Rayleigh waves, and their amplitudes must reach the maximum at the separation surface. He also studied the generalized type of love wave, which propagates along the interior stratum limited from both sides by the thick layers of material distinguishing by its elastic properties.

Karavashkin and Karavashkina (2002) proved that in case of equal coefficients of longitudinal and transversal stiffness, the direction of vibrations of an elastic lumped system will always coincide with the direction of the external force action. Also longitudinal deformation will be accompanied by transversal deformation. And the transversal deformation will be accompanied by that longitudinal deformation.

2.2 HAMILTON'S PRINCIPLE

Newton's laws of motion are often considered to be fundamental postulates for describing the motion of particles in a gravitational field, at least from our daily viewpoint. In a more general picture this is not so. Not only are they just a result of the general theory of relativity, they can also be derived from a more general principle, namely Hamilton's principle. Newton's laws of motion are just examples of equations that can be deduced from Hamilton's principle.

Hamilton's principle is an ``integral principle", which means that it considers the entire motion of a system between time t_1 and t_2. What is meant by this needs to be specified somewhat. The instantaneous configuration of the system is described by the values of n generalized coordinates $q_1,\ldots,q_n$, and corresponds to a particular point in a Cartesian hyperspace where the *q*-s form the *n* coordinate axes. This *n*-dimensional space is known as the configuration space. As the time evolves, the system point moves in this configuration space, tracing out a curve. This curve describes the path of motion of the system. The configuration space can be very different from the physical three-dimensional space, where only three coordinates are needed to describe a position at any give time. For example, a system that is being described both by the spatial coordinates and the velocities would have a six-dimensional configuration space at any given point in time.

Hamilton's principle is a version of the integral principle which considers the motion of a mechanical system described by a scalar potential that may be a function of the coordinates, velocities and time. The integral, often also reffered to as the action,

$$V.I' = \int_{t_1}^{t_2} \left[\delta(T^* - V) + \sum_{j=1}^{n} E_j \delta\xi_j - \sum_{i=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j \right] dt$$

$$= \int_{t_1}^{t_2} \left[\delta(T^* - V) + \sum_{j=1}^{n} E_j \delta\xi_j \right] dt - \sum_{i=1}^{n} \left[\frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j \right]_{t_1}^{t_2} \qquad (2.1)$$

Where V.I is the variational indicator, T and V being the Kinetic and Potential energy, respectively. E_j (j=1,2,...) is the generalized forces and ξ_j (j=1,2,...) being the generalized coordinates and the n associated admissible variations $\delta\xi_j$.

If we adopted the convention that all admissible variations vanish at $t_1 = t$ and

$t = t_2$. In this case, the last term in equation (2.1) vanishes, which result in to

$$V.I = \int_{t_1}^{t_2}\left[\delta(T^* - V) + \sum_{j=1}^{n} E_j \delta\xi_j\right]dt \qquad (2.2)$$

Equation (2.2) is Hamilton's principle which, in words, we express as the variational indicator for mechanical systems.

2.3 PARTIAL DIFFERENTIAL EQUATION

Partial differential equations occur in various physical and engineering problems when the functions involved depend on two or more independent variables.

An equation involving one or more partial derivatives of an unknown function of two or more independent variables is called a partial differential equation. The order of the highest derivative is called the order of the equation. We consider partial differential equation of orders one and two only since these are the ones which are mostly used in this dissertation.

The most general second order linear partial differential equation in two independent variables is

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{dx\partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g \qquad 2.3$$

where a, b, c, d, e, f, and g, are functions of x and y. if g = 0, the equation 2.3 is said to be homogeneous.

The general linear partial differential equation of order higher than one in two independent variables is

$$A_0\frac{\partial^n Z}{\partial x^n}+A_1\frac{\partial^n Z}{\partial x^{n-1}\partial y}+\cdots+A_n\frac{\partial^n Z}{\partial y^n}+B_0\frac{\partial^{n-1}Z}{\partial x^{n-1}}+\cdots+M\frac{\partial Z}{\partial x}+N\frac{\partial Z}{\partial y}+PZ=f(x,y)$$

Because of the nature of the solution of equation 2.3 the equation is often classified into three categories, namely;

Elliptic if $b^2 - 4ac < 0$

Parabolic if $b^2 - 4ac = 0$

Hyperbolic if $b^2 - 4ac > 0$

Some important linear partial differential equations of the second order are

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad (two-\dim ensional\ Laplace's\ equation)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \quad (two-\dim ensional\ Poisson\ equation)$$

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \quad (one-\dim ensional\ heat\ equation)$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad (one-\dim ensional\ wave\ equation),$$

Where c is a constant, t is the time variable and (x, y) are Cartesian coordinates.

A boundary – value problem involving a partial differential equation seeks all solutions of a partial differential equation which satisfy conditions called boundary conditions

There are many methods by which boundary value problem involving linear partial differential equations can be solved. The following are among the most important

I. General solution

II. Separation of variables

III. Laplace transform methods

IV. Complex variables methods

The method of separation of variables is assumed that a solution can be expressed as a product of unknown functions each of which depends on only one of the independent variables. The success of the method hinges on being able to write the resulting equation so that one side depends only on one variable while the other side depends on the remaining variables so that each side must be a constant. By repetition of this, the unknown functions can then, be determined. Superposition of these solutions can then be used to find the actual solution

CHAPTER THREE

3.1 INTRODUCTION

The purpose of this chapter is to derive the equation of motion and the boundary conditions for the system containing a rod undergoing longitudinal motion. The derivation of the equation of motion for the system with the boundary condition is presented in section 3.2. This is followed by solving the governing equation of motion for system models containing no (nonconservative) generalized forces in section 3.3. Finally, the solution to the governing equation of motion for system containing a longitudinal forces distributed is illustrated in section 3.4.

3.2 DERIVATION OF THE GOVERNING EQUATION OF MOTION

Considering the system containing a rod undergoing longitudinal motion as sketched below.

ρ = density

A = cross-sectional area

E = elasticity

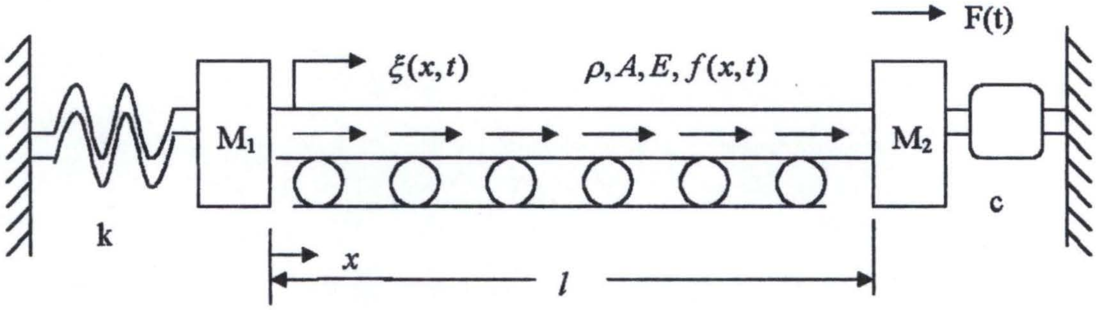
L = equilibrium length

M_1 = mass attached to the left-hand end of the rod (x=0)

K = an elastic element having a spring constant

M_2 = mass attached to the right-hand end of the rod (x=l)

c = dissipative element characterized by a linear dashpot constant.



To derive the equation of motion and the boundary condition for the system, we consider the Hamilton's principle for mechanical systems

$$V.I = \int_{t_1}^{t_2} [\delta(T^* - V) + \sum_{j=1}^{n} E_j \delta\xi_j] dt \qquad 3.1$$

Where

$$T^* = \frac{1}{2} M_1 \left[\frac{\partial \xi(0,t)}{\partial t} \right]^2 + \int_0^l \frac{1}{2} \rho A \left[\frac{\partial \xi(x,t)}{\partial t} \right]^2 dx + \frac{1}{2} M_2 \left[\frac{\partial \xi(l,t)}{\partial t} \right] \qquad 3.2$$

is the kinetic coenergy function consists of contributions due to the masses M_1 and M_2 and the mass ρA (per unit length) of the rod.

$$V = \frac{1}{2} K [\xi(0,t)]^2 + \int_0^l \frac{1}{2} EA \left[\frac{\partial \xi(x,t)}{\partial t} \right]^2 dx \qquad 3.3$$

is the potential energy function consists of contributions due to the spring K and the strain energy of the elastic rod.

$$\sum_{j=1}^{n} E_j \delta\xi_j = f(x,t)\delta\xi(x,t) + F(t)\delta\xi(l,t) - c\frac{\partial \xi(l,t)}{\partial t}\delta\xi(l,t) \qquad 3.4$$

is the no conservative force contributions.

ξ (x,t) which is the generalized coordinate for the rod is the longitudinal displacement of the section whose equilibrium position is x. Because of the selection of a continuous model, we observe that the generalized coordinate is not only a function of time but is also a function of space. Thus we resolve that

$$\xi_j : \xi(x,t) \quad \text{and} \quad \delta\xi_j : \delta\xi(x,t) \qquad 3.5$$

Note that throughout the calculation δ behave mathematically as d.

Substitution of equation 3.2 through 3.5 in to equation 3.1 gives

$$V.I = \int_{t_1}^{t_2} \Bigg\langle \delta\Bigg\{ \frac{1}{2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2 + \int_0^l \frac{1}{2}\rho A\left[\frac{\partial\xi(x,t)}{\partial t}\right]^2 dx + \frac{1}{2}M_2\left[\frac{\partial\xi(l,t)}{\partial t}\right]^2$$

$$-\frac{1}{2}K[\xi(0,t)]^2 - \int_0^l \frac{1}{2}EA\left[\frac{\partial\xi(x,t)}{\partial x}\right]^2 dx \Bigg\} + f(x,t)\delta\xi(x,t)$$

$$+\left[F(t) - c\frac{\partial\xi(l,t)}{\partial t}\right]\delta\xi(l,t) \Bigg\rangle dt \qquad 3.6$$

$$= \int_{t_1}^{t_2} \delta\left\{\frac{1}{2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2\right\}dt + \int_{t_1}^{t_2} \delta\left\{\int_0^l \frac{1}{2}\rho A\left[\frac{\partial\xi(x,t)}{\partial t}\right]^2 dx\right\}dt$$

$$+\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}M_2\left[\frac{\partial\xi(l,t)}{\partial t}\right]^2\right\}dt + \int_{t_1}^{t_2} \delta\left\{-\frac{1}{2}K[\xi(0,t)]^2\right\}dt$$

$$+\int_{t_1}^{t_2} \delta\left\{-\int_0^l \frac{1}{2}EA\left[\frac{\partial\xi(x,t)}{\partial x}\right]^2 dx\right\}dt$$

$$+\int_{t_1}^{t_2} \left\{f(x,t)\delta\xi(x,t) + \left[F(t) - c\frac{\partial\xi(l,t)}{\partial t}\right]\delta\xi(l,t)\right\}dt \qquad 3.7$$

We now evaluate equation 3.7 term by term with the notation that

$$\delta\left(\frac{\partial\xi}{\partial t}\right)=\frac{\partial(\delta\xi)}{\partial t} \qquad 3.8$$

and

$$\delta\left(\frac{\partial\xi}{\partial x}\right)=\frac{\partial(\delta\xi)}{\partial x} \qquad 3.9$$

$$\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2\right\}dt$$

$$=\int_{t_1}^{t_2}\frac{1}{2}M_1\delta\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2 dt$$

$$=\int_{t_1}^{t_2}\frac{1}{2}M_1 2\left[\frac{\partial\xi(0,t)}{\partial t}\right]\ \delta\left[\frac{\partial\xi(0,t)}{\partial t}\right]dt$$

Applying equation 3.8, we have

$$\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2\right\}dt$$

$$=\int_{t_1}^{t_2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]\left[\frac{\partial\delta\xi(0,t)}{\partial t}\right]dt \qquad 3.10$$

Solving 3.10 by integration by parts yields

$$\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2\right\}dt$$

$$=M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]\delta\xi(0,t)\Bigg|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left\{M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]\right\}\delta\xi(0,t)dt \qquad 3.11$$

In accordance with Hamilton's principle that the variations vanish at t_1 and t_2, the first term on the right-hand side of equation 3.11 vanish.

$$\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}M_1\left[\frac{\partial\xi(0,t)}{\partial t}\right]^2\right\}dt = -\int_{t_1}^{t_2}M_1\left[\frac{\partial^2\xi(0,t)}{\partial t^2}\right]\delta\xi(0,t)dt \qquad 3.12$$

Also $\int_{t_1}^{t_2}\delta\left\{\int_0^l\frac{1}{2}\rho A\left[\frac{\partial\xi(x,t)}{\partial t}\right]^2 dx\right\}dt$

$$= \int_0^l dx\int_{t_1}^{t_2}\frac{1}{2}\rho A2\left[\frac{\partial\xi(x,t)}{\partial t}\right]\delta\left[\frac{\partial\xi(x,t)}{\partial t}\right]dt$$

$$= \int_0^l dx\int_{t_1}^{t_2}\rho A\left[\frac{\partial\xi(x,t)}{\partial t}\right]\left[\frac{\partial\delta\xi(x,t)}{\partial t}\right]dt$$

$$= \int_0^l dx\left\{\rho A\frac{\partial\xi(x,t)}{\partial t}\delta\xi(x,t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left[\rho A\frac{\partial\xi(x,t)}{\partial t}\right]\delta\xi(x,t)dt\right\} \qquad 3.13$$

In accordance with Hamilton's principle that the variations vanish at t_1 and t_2, the first term on the right-hand side of equation 3.13 vanishes.

$$\int_{t_1}^{t_2}\delta\left\{\int_0^l\frac{1}{2}\rho A\left[\frac{\partial\xi(x,t)}{\partial t}\right]^2 dx\right\}dt = -\int_{t_1}^{t_2}dt\int_0^l\rho A\left[\frac{\partial^2\xi(x,t)}{\partial t^2}\right]\delta\xi(x,t)dx \qquad 3.14$$

Also by considering the evaluation of the first integral in 3.7 we observed that

$$\int_{t_1}^{t_2}\delta\left\{\frac{1}{2}M_2\left[\frac{\partial\xi(l,t)}{\partial t}\right]^2\right\}dt = -\int_{t_1}^{t_2}M_2\left[\frac{\partial^2\xi(l,t)}{\partial t^2}\right]\delta\xi(l,t)dt \qquad 3.15$$

More so

$$\int_{t_1}^{t_2}\delta\left\{-\frac{1}{2}K[\xi(0,t)]^2\right\}dt = \int_{t_1}^{t_2}-\frac{1}{2}K.2[\xi(0,t)]\delta\xi(0,t)dt$$

$$= -\int_{t_1}^{t_2}K\xi(0,t)\delta\xi(0,t)dt \qquad 3.16$$

Also

$$\int_{t_1}^{t_2}\delta\left\{-\int_0^l \frac{1}{2}EA\left[\frac{\partial\xi(x,t)}{\partial x}\right]^2 dx\right\}dt = \int_{t_1}^{t_2}dt\int_0^l -\frac{1}{2}EA.2\left[\frac{\partial\xi(x,t)}{\partial x}\right]\delta\left[\frac{\partial\xi(x,t)}{\partial x}\right]dx$$

$$= \int_{t_1}^{t_2}dt\int_0^l -EA\left[\frac{\partial\xi(x,t)}{\partial x}\right]\left[\frac{\partial\delta\xi(x,t)}{\partial x}\right]dx$$

$$= \int_{t_1}^{t_2}dt\left\{-EA\frac{\partial\xi(x,t)}{\partial x}\delta\xi(x,t)\;\Bigg|\; + \int_0^l \frac{\partial}{\partial x}\left[EA\frac{\partial\xi(x,t)}{\partial x}\right]\delta\xi(x,t)dx\right\}$$

$$= \int_{t_1}^{t_2}dt\left\{-EA\frac{\partial\xi(l,t)}{\partial x}\delta\xi(l,t) + EA\frac{\partial\xi(0,l)}{\partial x}\delta\xi(0,t) + \int_0^l \frac{\partial}{\partial x}\left[EA\frac{\partial\xi(x,t)}{\partial x}\right]\delta\xi(x,t)dx\right\} \qquad 3.17$$

We now substitute equations (3.12), (3.14), (3.13), (3.16) and (3.17) in equation (3.7) which gives

$$\text{V.I} = -\int_{t_1}^{t_2}M_1\left[\frac{\partial^2\xi(0,t)}{\partial t^2}\right]\delta\xi(0,t)dt - \int_{t_1}^{t_2}dt\int_0^l \rho A\left[\frac{\partial^2\xi(x,t)}{\partial t^2}\right]\delta\xi(x,t)dx$$

$$-\int_{t_1}^{t_2}M_2\left[\frac{\partial^2\xi(l,t)}{\partial t^2}\right]\delta\xi(l,t)dt - \int_{t_1}^{t_2}K\xi(0,t)\delta\xi(0,t)dt$$

$$+\int_{t_1}^{t_2}dt\left\{-EA\frac{\partial\xi(l,t)}{\partial x}\delta\xi(l,t) + EA\frac{\partial\xi(0,t)}{\partial x}\delta\xi(0,t) + \int_0^l \frac{\partial}{\partial x}\left[EA\frac{\partial\xi(x,t)}{\partial x}\right]\delta\xi(x,t)dx\right\}$$

$$+\int_{t_1}^{t_2}\left\{f(x,t)\delta\xi(x,t) + \left[F(t) - c\frac{\partial\xi(l,t)}{\partial t}\right]\delta\xi(l,t)\right\}dt \qquad 3.18$$

On rearranging we have,

$$\text{V.I} = \int_{t_1}^{t_2}\left\langle\int_0^l\left\{-\rho A\left[\frac{\partial^2\xi(x,t)}{\partial t^2}\right] + \frac{\partial}{\partial x}\left[EA\frac{\partial\xi(x,t)}{\partial x}\right] + f(x,t)\right\}\delta\xi(x,t)dx\right.$$

$$+\left\{-M_1\left[\frac{\partial^2\xi(0,t)}{\partial t^2}\right] - K\xi(0,t) + EA\frac{\partial\xi(0,t)}{\partial x}\right\}\delta\xi(0,t)$$

$$\left\{-M_2\left[\frac{\partial^2\xi(l,t)}{\partial t^2}\right]+F(t)-c\frac{\partial\xi(l,t)}{\partial t}-EA\frac{\partial\xi(l,t)}{\partial x}\right\}\delta\xi(l,t)\ \Big\rangle dt$$

Hamilton's principle requires that the variation indicator in equation (3.19) vanishes for arbitrary admissible variations of the generalized coordinates. Thus, the necessary conditions that equation (3.19) vanishes for arbitrary values of $\delta\xi(x,t)$ in (0, l) for arbitrary values of $\delta\xi(0,t)$ at x = 0, and for arbitrary values of $\delta\xi(l,t)$ at x = l is that each term in (3.19) vanishes independently.

Thus;

$$-\rho A\frac{\partial^2\xi}{\partial t^2}+\frac{\partial}{\partial x}\left(EA\frac{\partial\xi}{\partial x}\right)+f(x,t)=0 \qquad 0<x<l$$

$$\Rightarrow \rho A\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial x}\left(EA\frac{\partial\xi}{\partial x}\right)+f(x,t) \qquad 0<x<l \tag{3.20}$$

Equation (3.20) is the governing partial differential equation for the longitudinal motion of the rod. More also;

$$-\mathrm{M}_1\frac{\partial^2\xi}{\partial t^2}-\mathrm{K}\xi+EA\frac{\partial\xi}{\partial x}=0 \qquad x=0$$

$$\mathrm{M}_1\frac{\partial^2\xi}{\partial t^2}+\mathrm{K}\xi=EA\frac{\partial\xi}{\partial x}=0 \qquad x=0$$

3.21

with initial conditions $\xi(x,0)$ and $\frac{\partial\xi(x,0)}{\partial t}$, $0<x<l$, which are the longitudinal displacement and longitudinal velocity throughout the rod at time t = 0.

and

$$-\mathrm{M}_2\frac{\partial^2\xi}{\partial t^2}+F(t)-c\frac{\partial\xi}{\partial t}-EA\frac{\partial\xi}{\partial x}=0 \qquad x=l$$

$$M_2\frac{\partial^2\xi}{\partial t^2}+c\frac{\partial\xi}{\partial t}+EA\frac{\partial\xi}{\partial x}=F(t) \qquad x=l \qquad 3.22$$

Equations (3.21) and (3.22) are called the natural boundary conditions, where f(x,t) and F(t) are the external forces. We shall seek solutions to the governing partial differential equations in equation (3.20).

3.3 SOLUTION TO THE GOVERNING EQUATION OF MOTION FOR SYSTEM MODELS CONTAINING NO (NONCONSERVATIVE) GENERALIZED FORCES. (FREE MOTION)

Considering the PDE in equation (3.20), since the system contain no generalized forces, we seek the solution to the generic equation of motion (3.20) with f(x,t) = 0; namely

$$\rho A\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial x}(EA\frac{\partial\xi}{\partial x})$$

$$\Rightarrow \qquad p_1\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial x}(p_2\frac{\partial\xi}{\partial x}) \qquad , \qquad 3.23$$

where the parameters P_1 = –A and P_2 = EA are assumed to be constant throughout the extent of the one-dimensional continua.

Equation (3.23) can be written as

$$p_1\frac{\partial^2\xi}{\partial t^2}=p_2\frac{\partial^2\xi}{\partial x^2}$$

$$\frac{\partial^2\xi}{\partial t^2}=\frac{p_2}{p_1}\frac{\partial^2\xi}{\partial x^2}$$

$$\frac{\partial^2\xi}{\partial t^2}=c_q^2\frac{\partial^2\xi}{\partial x^2} \qquad , \qquad 3.24$$

where $$c_q^2 = \frac{p_2}{p_1}$$

We shall use the method of separation variable to solve equation (3.24).

Let assume the solution to be

$$\xi(x,t) = X(x)T(t) \quad , \qquad 3.25$$

where X (x) a function of x only, T (t) is is a function of t only.

Substitution of equation (3.25) in to equation (3.24) gives

$$X(x)\frac{d^2T}{dt^2} = c_q^2 T(t)\frac{d^2X}{dx^2} \qquad 3.26$$

Dividing both side of (3.26) by X(x)T(t), we have

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = \frac{c_q^2}{X(x)}\frac{d^2X}{dx^2} \qquad 3.27$$

The left hand side and the right-hand side of equation (3.27) are functions of t and x only respectively. Hence, in order for equation (3.27) to hold for all values of t and for all values of x, $0 < x < l$, both side of equation (3.27) must be equal to a constant say α.

Therefore equation (3.27) becomes

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = \alpha$$

$$\frac{d^2T}{dt^2} - \alpha T(t) = 0 \qquad 3.28$$

and

$$\frac{c_q^2}{X(x)}\frac{d^2X}{dx^2}=\alpha$$

$$\frac{d^2X}{dx^2}-\frac{\alpha}{c_q^2}X(x)=0 \qquad 3.29$$

Equation (3.28) and (3.29) gives two ordinary differential equation, which we now solve.

Considering equation (3.28), the characteristic equation is given as

$$r^2=\alpha \Rightarrow r=\pm\sqrt{\alpha} \qquad 3.30$$

In which the general solution of (3.28) is given as

$$T(t)=a'e^{r_1t}+b'e^{r_2t} \qquad 3.31$$

Substituting (3.30) in (3.31) gives

$$T(t)=a'e^{\sqrt{\alpha}t}+b'e^{-\sqrt{\alpha}t}$$

where a' and b' are two unknown coefficients.

$$let\alpha=-\omega^2$$

Then $T(t)=a'e^{it}+b'e^{-it}$

Where $i\equiv\sqrt{-1}$

Now by making the substitution of $e^{\pm i\omega t}=\cos\omega t\pm i\sin\omega t$, we have

$$T(t)=a'\cos\omega t+a'i\sin\omega t+b'\cos\omega t-b'i\sin\omega t$$

$$=(a'+b')\cos\omega t+(a'-b')i\sin\omega t$$

$$=a\cos\omega t+bi\sin\omega t \qquad \omega\neq 0 \qquad 3.32$$

The case when α in equation (3.28) is zero, we have

$$\frac{d^2T}{dt^2} = 0$$

which gives the general solution to be

$$T(t) = a + bt \qquad \omega = 0 \qquad 3.33$$

Note that a and b in equation (3.32) and a and b in equation (3.33) are not the same.

The function T(t) is known as time function.

Considering equation (3.29), the characteristic equation is given as

$$m^2 = \frac{\alpha}{c_q^2} = -\frac{\omega^2}{c_q^2} \quad \text{since} \quad \alpha = -\omega^2$$

$$\therefore m = \pm\frac{\omega}{c_q}i$$

which gives the general solution of (3.29) to be

$$X(x) = c\cos\frac{\omega}{c_q}x + d\sin\frac{\omega}{c_q}x \qquad \alpha \neq 0 \qquad 3.34$$

The case where α is zero, we have

$$\frac{d^2X}{dx^2} = 0$$

$$X(x) = c + dx \qquad \omega = 0 \qquad 3.35$$

where c and d in equation (3.34) and c and d in equation (3.35) are unknown coefficients, which are not equal.

The function X(x) is known as Eigen functions.

Substituting equation (3.32) and (3.33) in equation (3.25) and substituting equation (3.34) and (3.35) in equation (3.25) give (3.36) and (3.37) respectively.

That is

$$\xi(x,t) = (a\cos\omega t + b\sin\omega t)(c\cos\frac{\omega}{c_q}x + d\sin\frac{\omega}{c_q}x) \ , \qquad \omega \neq 0 \qquad 3.36$$

and

$$\xi(x,t) = (a+bt)(c+dx) \, , \qquad \omega = 0 \qquad 3.37$$

The boundary condition will be discussed letter.

3.4 SOLUTION TO THE GOVERNING EQUATION OF MOTION FOR SYSTEM CONTAINING A LONGITUDINAL FORCE DISTRIBUTION f(x,t)

Here we consider only when f(x,t) is separable in to spatial and temporal functions and the temporal function is harmonic.

Considering PDE (3.20)

$$\rho A\frac{\partial^2\xi}{\partial t^2} = EA\frac{\partial^2\xi}{\partial x^2} + f(x,t)$$

Let assume the harmonic force distribution function to be

$$f(x,t) = f_o \sin\Omega t$$

where f_o is the constant amplitude of the force, and Ω is excitation frequency.

Therefore equation (3.20) can be written as

$$\rho A\frac{\partial^2\xi}{\partial t^2} = EA\frac{\partial^2\xi}{\partial x^2} + f_o \sin\Omega t \qquad 3.38$$

Since the excitation is harmonic, the forced response will also be harmonic and at the same frequency Ω. This is a property of linear systems. We assume the solution of equation (3.38) to be

$$\xi(x,t) = X(x)\sin\Omega t \qquad 3.39$$

where X(x) is the unknown spatial distribution of the generalized coordinate

$\xi(x,t)$.

Substituting equation (3.39) in equation (3.38) gives

$$\rho A \frac{\partial^2}{\partial t^2}\{X(x)\sin \Omega t\} = EA\frac{\partial^2}{\partial x^2}\{X(x)\sin \Omega t\} + f_0 \sin \Omega t$$

which gives

$$\rho A X(x)\frac{\partial^2}{\partial t^2}\sin \Omega t = EA \sin \Omega t \frac{\partial^2}{\partial x^2}X(x) + f_0 \sin \Omega t$$

$$-\rho A \Omega^2 X(x)\sin \Omega t = EA\frac{d^2X}{dx^2}\sin \Omega t + f_0 \sin \Omega t$$

$$\left(\rho A \Omega^2 X(x) + EA\frac{d^2X}{dx^2}\right)\sin \Omega t = -f_0 \sin \Omega t \qquad 3.40$$

Since equation (3.40) holds for all time t, both side of (3.40) may be divided by

$\sin \Omega t$,

yielding

$$EA\frac{d^2X}{dx^2} + \rho A \Omega^2 X(x) = -f_0$$

$$\frac{d^2X}{dx^2} + \frac{\rho}{E}\Omega^2 X(x) = -\frac{f_0}{EA}$$

$$\frac{d^2X}{dx^2} + \frac{\Omega^2}{c_q^2}X = -\frac{f_0}{EA} \qquad 3.41$$

where $c_q^2 = \dfrac{E}{\rho}$

To solve equation (3.41), we find the homogeneous solution and a particular solution.

The characteristic equation of (3.41) is

$$m^2 + \frac{\Omega^2}{c_q^2} = 0 \Rightarrow m = \pm \frac{\Omega}{c_q} i$$

the homogeneous solution is given as

$$X_h(x) = c_1 e^{i\frac{\Omega}{c_q}x} + c_2 e^{-i\frac{\Omega}{c_q}x}$$

$$c_1 \cos\frac{\Omega}{c_q}x + c_2 \sin\frac{\Omega}{c_q}x \qquad 3.42$$

where c_1 and c_2 are unknown coefficients.

We now assume the particular integral to be

$$X_p(x) = c_3, X_p^1(x) = 0, X_p^{11}(x) = 0 \qquad 3.43$$

where c_3 is another unknown coefficient.

Substitute equation (3.43) in (3.41) we have

$$c_3 \frac{\Omega^2}{c_q^2} = -\frac{f_0}{EA}$$

$$\Rightarrow \qquad c_3 = -\frac{f_0 c_q^2}{EA\Omega^2} = -\frac{f_0 \, EA/\rho A}{EA\Omega^2}$$

$$X_p(x) = c_3 = -\frac{f_0}{\rho A\Omega^2} \qquad 3.44$$

Summation of equations (3.42) and (3.44) gives the solution to equation (3.41) as

$$X(x) = c_1 \cos\frac{\Omega}{c_q}x + c_2 \sin\frac{\Omega}{c_q}x - \frac{f_0}{\rho A\Omega^2} \qquad 3.45$$

Substitution of equation (3.45) in to equation (3.39) gives the solution to equation (3.38) as

$$\xi(x,t) = \left(c_1 \cos\frac{\Omega}{c_q}x + c_2 \sin\frac{\Omega}{c_q}x - \frac{f_0}{\rho A\Omega^2} \right) \sin \Omega t \qquad 3.46$$

CHAPTER FOUR

4.1 INTRODUCTION

The purpose of this chapter is to find general solution to a particular initial-boundary problem. Firstly the boundary conditions for (longitudinal) vibration of rod for various end condition (simple boundary condition) is presented in section 4.2. This followed by stating the particular initial-boundary-value problem in section 4.3. Finally, the general solution to the governing partial differential equations of motion (the dynamic response of the rod) subject to an initial displacement and initial velocity is illustrated in section 4.4.

4.2 SIMPLE BOUNDARY CONDITION

There are four such cases:

(1) Clamped at the both ends

$\xi(0,t) = \xi(l,t) = 0$ for all time t that require $X(0) = X(l) = 0$

(2) Clamped at left and free at right

$\xi(0,t) = \left.\frac{\partial \xi(x,t)}{\partial x}\right|_{x=l} = 0$ for all time t that require $X(0) = X^1(l) = 0$

(3) Free at left and clamped at right

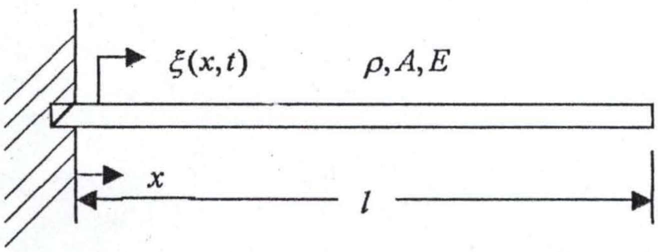
$\left.\frac{\partial \xi(x,t)}{\partial x}\right|_{x=0} = \xi(l,t) = 0$ for all time t that require $X^1(0) = X(l) = 0$

(4) Free at both ends

$\left.\frac{\partial \xi(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial \xi(x,t)}{\partial x}\right|_{x=l} = 0$ for all time t that require $X^1(0) = X^1(l) = 0$

4.3 PROBLEM STATEMENT

Consider a uniform rod of density ρ, cross-sectional area A, modulus of elasticity E and equilibrium length l. The left hand end of the rod is clamped and the right-hand end is free. Find the dynamic response of the rod when it is subjected to an initial displacement $\xi(x,0) = A_0 \cos\left(\frac{\pi x}{2l}\right)$ and zero initial velocity throughout its length.



4.4 SOLUTIONS

Equation (3.36) gives the response of the rod.

$$\xi(x,t) = (a\cos\omega t + b\sin\omega t)(c\cos\frac{\omega}{c_q}x + d\sin\frac{\omega}{c_q}x) \;,\; \omega \neq 0 \;,\; c_q = \sqrt{E/\rho} \qquad 4.1$$

This can still be written as

$$\xi(x,t) = X(x)T(t)$$

where

$$X(x) = c\cos\frac{\omega}{c_q}x + d\sin\frac{\omega}{c_q}x$$

is called the eigen function and

$$T(t) = a\cos\omega t + b\sin\omega t$$

is called the time response.

Since the rod is clamped-free, we have the condition

X(0) = 0

Which implies c = 0

$$X(x) = d\sin\frac{\omega}{c_q}x$$

and $X'(l) = 0$

which implies

$$X'(x) = \frac{d\omega}{c_q}\cos\frac{\omega}{c_q}x$$

$$X'(l) = \frac{d\omega}{c_q}\cos\frac{\omega}{c_q}l = 0,$$ but

$$\cos\frac{\omega}{c_q}l = 0$$

$$\frac{\omega}{c_q}l = (n - \tfrac{1}{2})\pi$$

$$\Rightarrow \quad \omega_n = (n - \tfrac{1}{2})\frac{\pi}{l}c_q, \qquad n = 1,2,3,\ldots$$

If we denote the frequency ω for each value n = 1,2,3,... by ω_n

i.e. $\omega_n = (n - \tfrac{1}{2})\frac{\pi}{l}c_q$

The eigen function $X_n(x)$ corresponding to the n[th] natural frequency ω_n is given by

$$X_n(x) = d_n \sin\frac{\omega_n}{c_q}x, \quad n = 1,2,3,\ldots$$

equation (4.1) becomes

$$\xi(x,t)=\sum_{n=1}^{\infty}(a_n\cos\omega_n t+b_n\sin\omega_n t)d_n\sin\frac{\omega_n}{c_q}x$$

Since a_n, b_n and d_n are constants, we can write the above equation as

$$\xi(x,t)=\sum_{n=1}^{\infty}(a_n\cos\omega_n t+b_n\sin\omega_n t)\sin\frac{\omega_n}{c_q}x \qquad 4.2$$

In other to obtain the coefficients a_n and b_n, we apply the initial conditions

$$\xi(x,0)=A_0\cos\left(\frac{\pi x}{2l}\right)$$

and $\left.\dfrac{\partial\xi(x,t)}{\partial t}\right|_{t=0}=0$ for zero initial velocity.

This gives

$$\sum_{n=1}^{\infty}a_n\sin\frac{\omega_n}{c_q}x=A_0\cos\left(\frac{\pi x}{2l}\right) \qquad 4.3$$

and

$$\frac{\partial\xi(x,t)}{\partial t}=\sum_{n=1}^{\infty}(-a_n\omega_n\sin\omega_n t+b_n\omega_n\cos\omega_n t)\sin\frac{\omega_n}{c_q}x$$

$$\left.\frac{\partial\xi(x,t)}{\partial t}\right|_{t=0}=\sum_{n=1}^{\infty}b_n\omega_n\sin\frac{\omega_n}{c_q}x=0 \qquad 4.4$$

To evaluate a_n and b_n, we multiply both side of equation (4.3) and (4.4) by the m^{th} eigen function $X_m(x)=\sin\frac{\omega_m}{c_q}x$ (where m is an arbitrary integer) and integrate the resulting expression with respect to x from 0 to l, which gives

$$\sum_{n=1}^{\infty}a_n\int_0^l\sin\left\{\left(n-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx=\int_0^l A_0\cos\frac{\pi}{2l}x\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx$$

Applying the orthogonally condition, that

$$\int_0^l X_n(x)X_m(x)dx = \begin{cases} 0, n \neq m \\ c, n = m \end{cases}$$

We have

$$\sum_{n=1}^{\infty} a_n \int_0^l \sin^2\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \int_0^l A_0 \cos\frac{\pi}{2l}x \sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx \qquad 4.5$$

Starting with the left hand side, we have

$$\int_0^l \sin^2\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \frac{1}{2}\int_0^l\left[1-\cos\left\{2\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}\right]dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}}\sin\left\{2\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}\right]_0^l$$

$$= \frac{1}{2}[l-0] = \frac{l}{2}$$

For the right-hand side, we have

Since $\sin C + \sin D = 2\sin\frac{1}{2}(C+D)\cos\frac{1}{2}(C-D)$

$$\int_0^l \cos\frac{\pi x}{2l}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \frac{1}{2}\int_0^l\left[\sin\frac{m\pi}{l}x + \sin\left\{(m-1)\frac{\pi}{l}x\right\}\right]dx$$

For $\quad m = 1$

$$\int_0^l \cos\frac{\pi x}{2l}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \frac{1}{2}\int_0^l \sin\frac{\pi}{l}x dx = -\frac{l}{2\pi}\cos\frac{\pi}{l}x\Big|_0^l$$

$$= \frac{l}{\pi}$$

For $m = 2,3,4,\ldots$

$$\int_0^l \cos\frac{\pi x}{2l}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \frac{1}{2}\int_0^l\left\{\sin\frac{m\pi}{l}x + \sin\frac{(m-1)\pi}{l}x\right\}dx$$

$$= -\frac{1}{2}\left[\frac{l}{m\pi}\cos\frac{m\pi}{l}x + \frac{l}{(m-1)\pi}\cos\frac{(m-1)\pi}{l}x\right]_0^l$$

$$= -\frac{1}{2}\left[\frac{l}{m\pi}\left\{(-)^m - 1\right\} + \frac{l}{(m-1)\pi}\left\{(-1)^{m-1} - 1\right\}\right]$$

If m is odd

$$\int_0^l \cos\frac{\pi x}{2l}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \frac{l}{\pi} - \frac{l}{2\pi}\left\{\frac{-2}{3},\frac{-2}{5},\frac{-2}{7},\ldots\right\}$$

$$= \frac{l}{\pi}\left\{1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\ldots\right\}$$

$$= \frac{l}{\pi}.\frac{1}{2k-1} \qquad \text{for } k = 1, 2, 3, \ldots$$

If m is even

$$\int_0^l \cos\frac{\pi x}{2l}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = -\frac{l}{2\pi}\left\{-2,\frac{-2}{3},\frac{-2}{5},\ldots\right\}$$

$$= \frac{l}{\pi}.\frac{1}{2k-1}$$

For m = 1, 2, 3, …

$$\int_0^l \cos\frac{\pi x}{2l}\sin\left\{\left(m-\tfrac{1}{2}\right)\frac{\pi}{l}x\right\}dx = \frac{l}{\pi}.\frac{1}{2k-1} + \frac{l}{\pi}.\frac{1}{2k-1}$$

$$= \frac{2l}{\pi(2k-1)} \quad \text{, for} \quad k = 1, 2, 3, \ldots$$

Equation (4.5) becomes

$$a_m \frac{l}{2} = A_0 \frac{2l}{\pi(2k-1)}$$

$$a_m = \frac{4A_0 l}{\pi(2k-1)}, \qquad \text{for} \qquad k = 1, 2, 3, \ldots$$

Multiplying m[th] eigen function $X_m(x) = \sin \frac{\omega_m}{c_q} x$ by equation (4.4) and integrate with respect to x from 0 to l, we have

$$\int_0^l b_n \omega_n \sin \frac{\omega_n}{c_q} x \sin \frac{\omega_n}{c_q} x dx = 0$$

$$b_n \frac{\omega_n}{c_q} = 0 \Rightarrow b_n = 0$$

Substituting a_n and b_n in to equation (4.2), we have

$$\xi(x,t) = \sum_{n,k=1}^{\infty} \frac{4A_0 l}{\pi(2k-1)} \cos \omega_n t \sin \frac{\omega_n}{c_q} x$$

$$= \sum_{n,k=1}^{\infty} \frac{4A_0 l}{\pi(2k-1)} \cos\left\{\left(n - \frac{1}{2}\right)\frac{\pi}{l} c_q t\right\} \sin\left\{\left(n - \frac{1}{2}\right)\frac{\pi}{l} x\right\}$$

where $c_q = \sqrt{\frac{E}{\rho}}$ is the wave propagation speed.

CHAPTER FIVE

5.1 NUMERICAL SIMULATION

If the body characteristics appear or the same throughout the energy supply or disturbance at any point on/with the body will give the same effect (or disturbance) or waves. That is, the same displacement properties will be transmitted throughout the body with the first excited particle vibrating and transmitting (elastically) same energy to its immediate neighboring particle.

The difference in the displacement form in figures (1,2,3,4,5) is due the fact that the disturbance (energy or force) applied was applied at different phase angle, which is as a result of time lag or lead for individual displacement generated at a particular spot of strike and transmitted at different frequencies. Similarly, if the phase difference is the same, for a homogenous median having the same wavelength, the same waveform of the same amplitude will be generated. For an elastic medium, same energy will be transmitted from the point of energy supply throughout the body except that its amplitude will decrease as a result of frequency changes only that the wave form will be maintained.

In this analysis, we assume that cross sections of a rod remain plane and that the particles in every cross section move only in the axial direction of the rod. The longitudinal extensions and compressions during such vibration of the rod is accompanied by some lateral deformation. From fig. (1,2,3,4,5) the values of cross sectional area (A_o) was varied, at each case increase in the value of the cross sectional area (A_o) give rise to corresponding increase in the amplitude length. The system response to the wave propagation speed was also analyzed.

Three different values were used for the analysis. It is observed that the highest value of speed at which the disturbance propagates gives the highest amplitude length. The analysis was carried out with small step time.

This periodic change of displacement is the propagation of the disturbance, due to the removal of the right-hand wall, along the rod and its reflection at the ends of the rod. The duration L/C_q is the time required for a disturbance to travel the distance L, where C_q is the speed at which the disturbance propagates.

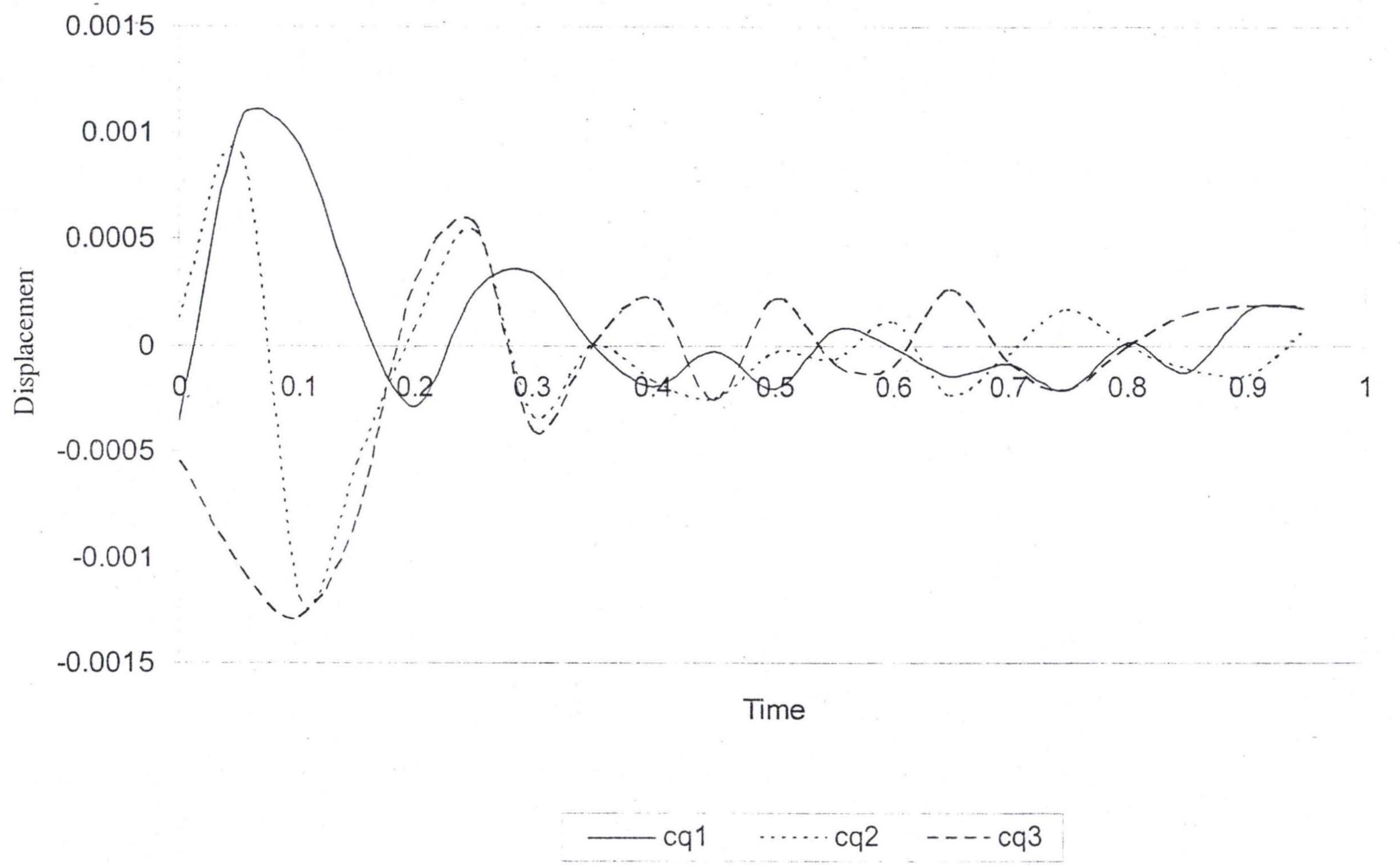


Fig 1. Transverse displacement throughout rod subject to initial compression

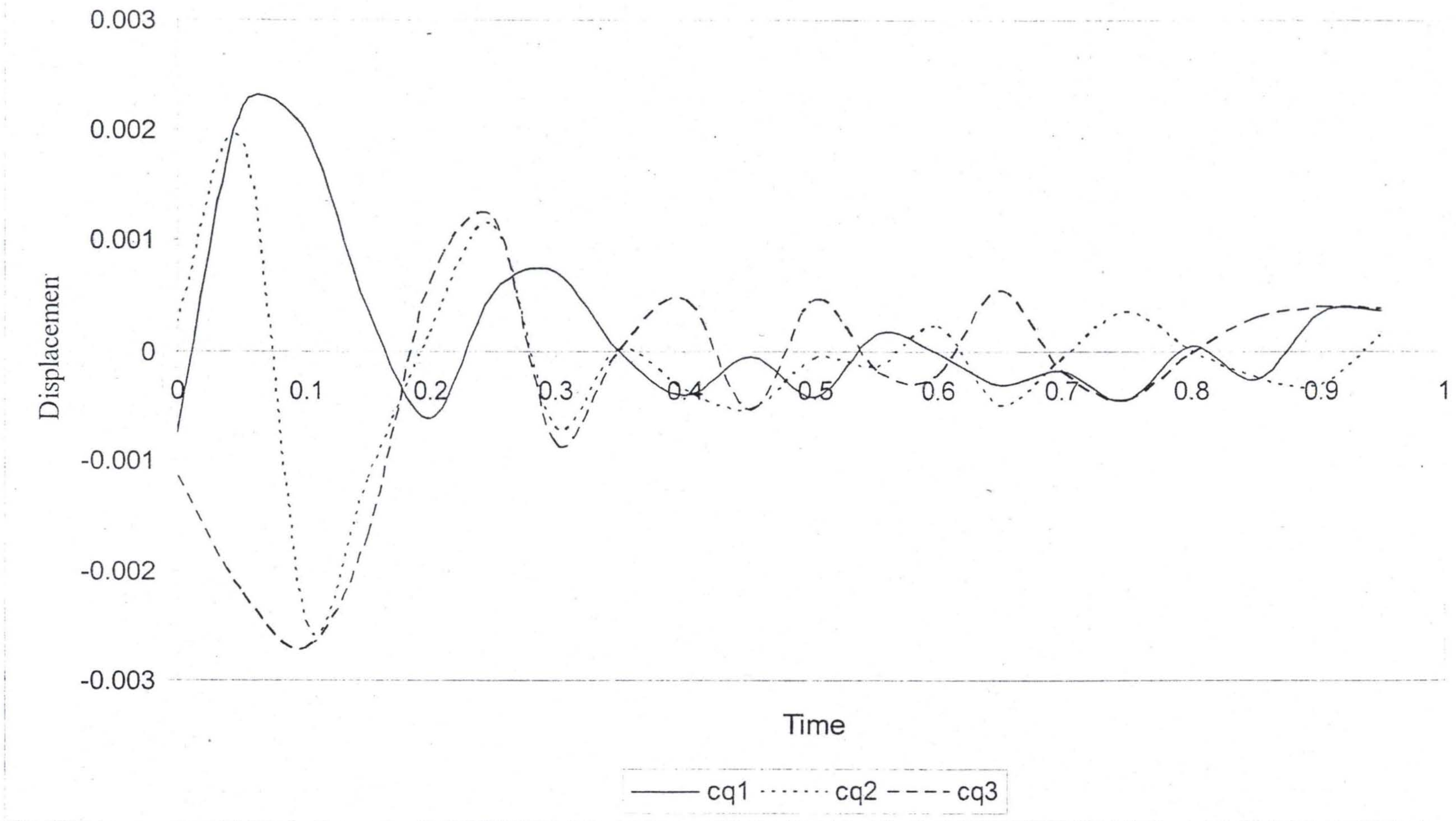


Fig 5. Transverse displacement throughout rod subject to initial compression

5.2 CONCLUSIONS

Based on the above analysis and discussion the following conclusions are obtained:-

(i) The amplitude of the vibration of the rod rises when the cross sectional area of the rod increases.

(ii) The amplitude of the response for the rod is higher when the speed at which the wave propagate increases.

5.3 RECOMMENDATION

The presentation in this dissertation was to obtain and solve the equation of motion for system containing a longitudinal force distribution f(x, t) only when f(x, t) is separable into spatial and temporal functions and the temporal functions is harmonic with simply-supported ends condition. A more general problem could be solved with other forms of ends condition.

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