A FIXED-POINT ITERATIVE METHOD FOR THE SOLUTION OF THREE – POINT BOUNDARY VALUE PROBLEMS

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL, FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF DEGREE OF MASTER OF TECHNOLOGY (M. TECH) IN MATHEMATICS

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NIGER STATE, NIGERIA

OCTOBER, 2011

DECLARATION

I hereby declare that this thesis has been written by me and that it is a record of the own research work. It has not been presented before in any previous application for a higher degree.

All sources of information are acknowledged by means of references.

.... 11/11/2011 ....

AL-MUSTAPHA ADAMU KULUWA DATE

CERTIFICATION

This thesis titled: "A FIXED-POINT ITERATIVE METHOD FOR THE SOLUTION OF THREE – POINT BOUNDARY VALUE PROBLEMS", by Al-Mustapha Adamu Kuluwa (M.Tech/SSSE/2008/1877) meets the regulations governing the award of the degree of Master of Technology of Federal University of Technology, Minna, Niger State and is approved for its contribution to scientific knowledge and literary presentation.

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DEDICATION

Dedicated to my children.

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ABSTRACT

Given a three – point, fourth order boundary value problem of the form:

$$Lu = y^{iv} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = f(x), \ a \leq x \leq b$$

$$y(a) = y''(a) = y''(b) = y(\alpha) = 0, \ a \leq \alpha \leq b$$

where $p, q, r, s, f \subset [a, b]$, we used a fixed-point process to construct an iterative scheme that approximates the solution. While the success of the variational or weighted residual method of approximation from a practical point of view, depends on selection of suitable coordinate functions (basis functions), this method is a self – correcting one and leads to fast convergence. Problems were experimented on to show the effectiveness and accuracy of the method. Thus in this work, we have successfully extended the use of fixed-point iterative method to the solution of three-point boundary value problems.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background of Study

A differential equation can be defined as an equation which contains a derivative. In other words, it is a relation involving an independent variable x, a dependent variable y and one or more differential co-efficients of y with respect to x. An example of a differential equation is

$$y'^{v} + xy = 0 \quad (1.1)$$

Differential equations occur in connection with most real life problems which arise in various fields of study be it engineering or science. Some of such problems are:

(i) The conduction of heat in a rod or in a slab.

(ii) The charge of a current in an electric.

(iii) The motion of a projectile, rocket, satellites or planet.

(iv) The determination of curves that have certain geometrical properties.

(v) The reactions of chemicals

(vi) The rate of decomposition of a radioactive substance or the rate of growth of population.

The mathematical formulation of such problems is often too complicated to solve exactly. Even if an exact solution is obtained, the required calculations may be too complicated to carry out or the resulting solutions may be difficult to translate.

Today, mathematics affects almost every discipline such as medicine, social sciences, engineering and so on. Therefore, there is a need to develop numerical methods for obtaining good approximate solutions for finding a closed form solution of differential equations by mathematical manipulations.

There are three principal methods for analyzing and solving differential equations. These are:

Qualitative Analysis

Analytic (exact) Analysis

Numerical Approximations

However, Qualitative analysis can not give accurate results while Analytic solution can only be derived for a limited class of problems. Also, Analytic solutions are of limited practical value because most real life problems are non-linear and involve complex shapes and processes. Numerical approximation on the other hand, provides solution to complex problems where the differential equation defies solution analytically. For example, in solving fourth order linear three-point boundary value problems of differential equation, various methods are used to approximate the solution numerically. Some of these methods are:

Collocation Method, Galerkin Method, Rayleigh-Ritz Method and so on. The goal is to develop methods which give solutions that are very close to the exact solutions.

But, the less rigorous the computation is, the more desirable the method.

It is always advisable to check existence of solution before any numerical scheme is applied; for example, Banach fixed point theory is a well known theory that guarantees the existence and uniqueness of solution of an initial- value problems. The fixed point iteration method is used in this research to approximate the solution of the three-point boundary value problems for ordinary linear fourth order differential equations.

1.2 Aim and Objectives of the Thesis

Aim: The main aim of this research is to develop suitable numerical method for the solution of three-point boundary value problems.

Objectives: The Objectives of the research work are:

* To explore the use of fixed – point iteration method for the solution of three – point boundary value problems.
* To analyse the local error and error constant and convergence of the method.
* To make suggestions for improvement on the fixed – point iteration method for the solution of three – point boundary value problems

1.2 The Significance of the Study

This research will be valuable not only to students, teachers and school in order to comprehend, explain, control and predict how we can have a break-through in this modern age of technology with the three - point boundary value problems associated with systems of linear or nonlinear ordinary differential equations satisfying three different points within an interval of a given boundary conditions as:

$$Lu = y'^{v} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = f(x),\ a \leq x \leq b \quad (1.2)$$

Satisfying the conditions

$$y(a) = y''(a) = y''(b) = y(\alpha) = 0,\ a \leq \alpha \leq b \quad (1.3)$$

In the development of a new scheme, the analysis of convergence of the fixed point iteration process also produces additional information that can be used to examine the rate of convergence and a mechanism to detect convergence to within a certain degree of accuracy. Future researchers will find this work useful for further advancement.

1.4 Scope and Limitations

The study is limited to the fixed-point iteration method for the solution of three – point boundary value problems.

1.5 Definition of Terms

1.5.1 Three – Point Boundary Value Problems

The three – point boundary value problem consists of the following;

Consider the boundary – value problem:

$$Lu = y^{iv} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = f(x), \; a \leq x \leq b \quad (1.4)$$

Satisfying three different points within the interval of a given boundary conditions as

$$y(a) = y''(a) = y''(b) = y(\alpha) = 0, \; a \leq \alpha \leq b \quad (1.5)$$

The boundary conditions (1.5) specify a linear relationship between the values of the desired solution and its derivative at the end point $[a,b]$. As indicated above, we usually consider problems posed on the interval $0 \leq x \leq 1$. Corresponding results hold for problems posed on an arbitrary interval. Indeed, if an interval is originally $\alpha \leq t \leq \beta$, it can be transformed into $0 \leq x \leq 1$ by change of variable $x = (t-\alpha)/(\beta-\alpha)$. Boundary value problems with higher order differential equations can also occur; so long as the number of boundary conditions is equal to the order of the differential equation.

1.5.2 Initial Value Problems

An initial value problem is an ordinary differential equation with specified value, called the initial condition of the unknown function. In physics or other sciences, modelling a system frequently amounts to solving an initial value problem. In the instance, the differential equation is an evolution specifying that with given initial conditions, the system will evolve with time.

1.5.3 Boundary Value Problems

A boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. The differential equation must also satisfy the boundary conditions.

Boundary value problems arise in several branches of physics. Problems involving the wave equation, such as the determination of normal modes, are stated as boundary value problems. A large class of important boundary value problems are Sturn – Liouville problems. To be useful in applications, a boundary value problem should be well posed. It shows that given the input to the problem there exists a unique solution, which depends on the input. Much theoretical work in the field of partial differential equations is devoted to proving that boundary value problems arising from scientific and engineering applications are in fact well-posed.

1.5.4 Numerical Method (Lambert, 1991)

A numerical method is a difference equation involving a number of consecutive approximations $y_{n+i,}$ $i = 0,1,2,\cdots,z$ from which it will be possible to compute sequentially the sequence $\{ y_n, \ n = 0,1,2,\cdots,N\}$; naturally this difference equation will involve the function f. The integer z is called the step number. If $z = 1$, it is a 1 – step method, while if $z > 1$, it is called a multistep or z – step method.

1.5.5 Numerical Solution (Mackenzie, 2000)

A numerical solution of

$$y' = f(x, y), \qquad y(a) = A \tag{1.6}$$

consists of a set of discrete approximations $\{ y_n \}|_{n=0}^{N}$.

According to Lambert (1991), it can also be defined as a sequence of values $\{y_n\}$ which approximates the solution of initial value problem (1.4) on the discrete point set.

1.5.6 Metric Space

Let X be non – empty set and let $\Re$, denote the set of real numbers.

A metric d on X is a real – valued function $d : X \times X \rightarrow \Re$, which satisfies the following conditions for $x, y, z \in X$,

$M_1 : d(x, y) \geq 0$

$M_2 : d(x, y) = 0$

$M_3 : d(x, y) = d(y, x)$

$M_4 : d(x, y) \leq d(x, z) + d(z, y)$

E.g. Consider real line ($\Re$, |.|) and let $d(x, y) = |x - y| \forall x, y \in \Re$, then d is metric on $\Re$ and it is referred to as the usual metric.

1.5.7 Convex Set

Let $Y \subseteq X$. The subspace Y is a convex set if for each pair of points $a, b \in Y$ such that $a<b$, the interval $(a,b) = \{ x \in X / \ a < x < b\}$ is contained in Y. That is Y is convex if and only if $\forall$, $a<b$, implies $(a,b) \subseteq Y$.

According to Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object. For example, a solid cube is convex, but anything that is hollow or has a dent in it, example, a crescent shape is not convex.

1.5.8 Cauchy Sequence

A sequence $\{ y_n \}|_{n=0}^{\infty}$ of points in a metric space (x, d) is a Cauchy sequence if for any real number $\varepsilon > 0,$, there exists an integer $N_0 > 0$ such that for all n, $m, \geq N_0,\ d(x_n, x_m) < \varepsilon$.

Example: Let $X = (R, d)$ be real line with usual metric and let $x_n = \left(\frac{3}{4}\right)^n$. Then

$$\{y_n\}|_{n=0}^{\infty} = \left\{\frac{3}{4}, \left(\frac{3}{4}\right)^2, \left(\frac{3}{4}\right)^3, \ldots, \left(\frac{3}{4}\right)^n\right\}$$

is a Cauchy sequence.

1.5.9 Contraction

Let (X, d) be any arbitrary metric space. A mapping $T: X \to X$ is a strict contraction (or simply a contraction) if there exists a constant $\alpha \in (0,1)$ such that

$$d\,(Tx, Ty) \leq \alpha\, d\,(x, y)\, for\, all\, x, y \in X.$$

1.4.10 A Fixed Point

A fixed point of a mapping $T: X \to X$ of a set into itself is as $x \in X$ which is mapped onto itself (kept fixed by T) i.e. $T\,x = x$.

Example: $T: [0,1] \to [0,1]$, $T: X \to X$ any point $x \in \{0,1\}$ is a fixed point of T.

1.4.11 The Mann Iteration Process

Let D be a non – empty convex subset of X and $T: D \to X$ a mapping. The sequence $\{x_n\} \subset D$ defined by $x_0 \in D, x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n$, for all $n \geq 1$, where $\{y_n\}|_{n=0}^{\infty}$ is a real sequence satisfying $c_0 = 1, 0 \leq \lambda \leq 1$ for all $n \geq 1$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

The condition $\sum_{n=0}^{\infty} \lambda_n = \infty$ in some applications is replaced by $\sum_{n=0}^{\infty} \lambda_n (1-\lambda_n) = \infty$.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Some Previous Related Works

In the past and recent years, many methods have been developed and others are still being developed for the solution of differential equations. Some of these methods are grouped into classes and further subclasses. Speed and accuracy are two major characteristics that give a method an upper hand over another method. Many have undergone changes either to shed more light on their behaviour, improve on their accuracies or error control strategies.

As Henri Poincare once remark "solution of the mathematical problem" is a phase of indefinite meaning, pure mathematicians sometimes are satisfied with showing that the non existence of a solution implies a logical contradiction, while engineers may consider a numerical method result as the only reasonable goal. Such one-sided views seen to reflect human limitation rather than objective values. In itself, mathematics is an indivisible organism uniting theoretical contemplation and active application.

The approximate methods of solving boundary value probably existed long before the appearance of computers and are still of particular importance, some of these methods are least squares, collocation, sub domain and also the sufficiently universal Galerkin methods.

In the finite difference approximation of a differential equation, the derivatives in the equation are replaced by difference quotients which involve the values of the solution at the mesh point. Although, the finite difference method is simple in concept, it suffers several disadvantages. The most notable are the inaccuracy of the derivatives of the approximate solution, the difficulty in accurately representing geometrically complex domains and inability to employ non uniform and irregular meshes.

The finite element method was introduced first in 1960 by Clough (1965). The method was proposed and mathematically analyzed in early 1960s independently in the United States and former USSR. The finite element method is a discretization method based on the variational principle. Hence, it can also be understood as a derivation of special finite scheme based on the variational approach (variational difference method). The collocation and least squares methods have received very little attention, primarily, because of the increased inter – element smoothness requirements placed on the basis functions. The difficulty is removed, however, if the boundary value problem is first recast as a lower order system. That is instead of standard formulation for a boundary value problem of order $2m$, we introduce new variables to develop a mixed formulation of order m.

Yildiz (1998),defined an efficient method for the solution of the general linear boundary value problem of order $2n$. As it is known linear boundary value problems are transformed into equivalent linear integral equations.

Onumanyi el al. (1999), shows that continuous finite difference formulae aside providing dense accurate solutions and global error estimates economically, can also be used to provide uniform treatment for both Initial and Boundary Value Problems without using the shooting method for the latter.

Adeboye (1999), first defined an H^1 Galerkin method on boundary value problem and then an iterative process that leads to supper convergence.

Ibiejugba et al. (1992), used the Lanczos – Chebyshew reduction method. The main idea behind the reduction method is to solve a boundary value problem by reducing it to system of ordinary differential equations using a finite sum of products of two functions. The Lanczos - Chebyshev reduction method involves the use of the popular Chebyshev polynomials as the trial function, which results in a solution that can be differentiated or integrated making it easily adaptable to any type of boundary conditions.

He J. H. (1999), originally proposed variational iteration method which has been proved by many authors to be a powerful mathematical tool for various kinds of nonlinear and linear problems. Adomian et al. (2005), solved a generalization of Airy's equation by decomposition method; Ravi and Reddy (2005), dealt with singular two-point boundary value problems by cubic spline; Caglar et al. (2006), applied B-spline interpolation to two-point boundary value problems compared results with finite difference, finite element and finite volume method.

Ndanusa (2007), once remarked that "One of the main objectives of numerical analysis is to solve complex numerical problems using only the simple operations of arithmetic to develop and evaluate methods for computing numerical results. Another goal of finding numerical solutions to differential equations is to get a method that will give an answer that will be (if possible) the same as the exact solution".

Zheyan and Jianhe, (2010) remark that "Boundary Value Problems of ordinary differential equations can be used to describe a large number of mechanical, physical, biological, and chemical phenomena". The works relating approximation of solutions are relatively rare. In recent years, some approximate methods, such as shooting method, monotone iterative technique, homotopy analysis method, and general quasilinearization method have been applied to boundary value problems for obtaining approximation of solutions. Among these methods, the general quasilinearization becomes more and more popular.

The quasilinearization method was originally proposed by Bellman and Kalaba, (1965). It is a very powerful approximation technique and unlike perturbation methods, it is not dependent on the existence of a small or large parameter. The method, whose sequence of solutions of linear problems converge to the solution of the original nonlinear problem, is quadratic and monotone, which is one of the reasons for the popularity of this technique.

This method was generalized by Lakshmikantham and Vatsala (1998), in which the convexity or concavity assumption on the nonlinear functions involved in the problems is relaxed.

So far, the general quasilinearization method, coupled with the method of upper and lower solutions, have been applied to obtain approximation of solutions for a large number of nonlinear problems. For example boundary value problems of ordinary differential equations, such as first-order boundary value problems with nonlinear boundary condition and second-order boundary value three-point boundary condition.

Zengqin (2007), remarked that "The Green's function method for solving the boundary value problem is an effective tool in numerical experiments". Some boundary value problems for nonlinear differential equations can be transformed into the nonlinear integral equations the kernels of which are the Green's functions of corresponding linear differential equations. The integral equations can be solved by investigating the property of the Green's functions.

2.2 Three – Point Boundary Value Problem

"Three - point boundary value problems associated with systems of linear or nonlinear ordinary differential equations occur in many branches of mathematics, engineering and sciences", (P.W. Meyer, 1987).

Recently, three-point boundary value problem of the differential equations of the form (2.1) were presented and studied.

$$y'^{v}(t) = a(t)f(y(t), t \in (0,1); = y(0) = y(1) = y''(0) = y''(\alpha) = 0,$$

$$0 < \alpha < 1 \tag{2.1}$$

Boundary conditions model supports, but they can also model point loads, distributed loads and moments. The support or displacement conditions are used to fix values of displacement (w) and rotation (dw/dx) on the boundary. Such conditions are called Dirichlet conditions. Flux boundary conditions are called Neumann boundary conditions.

Furthermore, a rigid body mode may arise due to the collocation nature of satisfying the boundary conditions. The point values of the applied load at the collocation point may not satisfy equilibrium or the point values of the specified displacements may not satisfy the condition of zero translation and rotation. For bodies under pure traction, we know that the analytical solution can contain an arbitrary amount of rigid body mode. Numerically, however, some unknown value is assigned to this rigid body mode. (Madhukar,(2005).

According to Ogunfiditimi and Adeboye (2004), 'The fixed-point iteration frame work allows for greater freedom in formulation of iterative schemes and provides a more rigorous framework for the analysis of convergence.'

Another example is a fourth-order system which has the form

$$y'^{v}(x) + ky(x) = q \tag{2.2}$$

with boundary conditions

$$y(0) = y'(0) = 0 \text{ and } y(L) = y''(\alpha) = 0, \quad 0 < \alpha < L \tag{2.3}$$

This arises in the contexts of beam bending, here, $y(x)$ represents the beam defection at point x along its length, q represents a uniform load, and L is the beam's length. The first two boundary conditions say that one end of the beam $(x=0)$ is rigidly attached. The second two boundary conditions say that the order end of the beam $(x=L)$ is simply supported.

This thesis discusses and proves the existence and uniqueness of positive solutions for three-point boundary value problems for a linear fourth-order equation with three-point boundary conditions.

Numerical examples given confirm the theoretical rate of convergence.

2.3 Numerical Methods for Solving Boundary Value Problems

There are different types of numerical methods that can be used to solve boundary value problems, but few will be discussed here:

* Galerkin Method
  - Standard Galerkin
  - H^{-1}- Galerkin
  - H^{1}- Galerkin
  - H^{2} - Galerkin by Adeboye
* Raleigh – Ritz Method

* Collocation Method

* Least Square Method

The above listed methods have been shown to solve effectively, easily and accurately a large class of nonlinear and linear problems, (Bello, 2004).

2.3.1 Standard Galerkin Method

This is one of the methods for the solution of boundary value problems which was suggested by B. G. Galerkin and which is based on the requirement that the basis functions

$\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_N$ be orthogonal to the residual.

That is

$$\int \Psi(x_1, a_0, a_1, a_2 \cdots, a_n)\varphi_1 dx = 0, \; i = 0, 2, 3, \cdots, n.$$

This gives rise to the following system of linear algebraic equations for the coefficients of the approximate solution

$$y_n(x) = \varphi_0 + a_1\,\varphi_1(x) + a_2\,\varphi_2(x) + \cdots, + a_n\,\varphi_n(x) \tag{2.4}$$

of the boundary value problem,

$$Lu = y^{iv} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = f(x), \; 0 \le x \le 1 \tag{2.5}$$

$$y(0) = y''(0) = y''(1) = y(\alpha) = 0, \; 0 \le \alpha \le 1 \tag{2.6}$$

Therefore,

$$a_1\,L(\varphi_1, \varphi_1) + a_2\,L(\varphi_2, \varphi_1) + a_3 L(\varphi_3, \varphi_1) + \cdots + a_n L(\varphi_n, \varphi_1) = (f - L(\varphi_0, \varphi_1))$$

$$a_1L(\varphi_{1,}\varphi_2) + a_2L(\varphi_{2,}\varphi_2) + a_3L(\varphi_{3,}\varphi_2) + . . . + a_n L(\varphi_{n,}\varphi_2) = (f - L(\varphi_{0,}\varphi_2))$$

. . . .

. . . .

$$a_1L(\varphi_{1,}\varphi_n) + a_2L(\varphi_{2,}\varphi_n) + a_3L(\varphi_{3,}\varphi_n) + . . + a_nL(\varphi_{n,}\varphi_n) = (f - L(\varphi_{0,}\varphi_n))$$

(2.7)

Where,

$$(f,g) = \int_a^b f(x)g(x)dx \tag{2.8}$$

2.3.2 The Collocation Method

This is a numerical method for the solution of boundary value problems. Here, in the interval $[a,b]$, n points $x_1, x_2, x_3, x_4, . . . , x_n$ are fixed and are called the collocation points at which the discrepancy ($\psi(x; a_1, a_2, . . . ,a_n)$) is equated to zero, at the collocation points. That is:

$$\psi\ (x_1, a_1, a_2, a_3, . . . , a_n) = 0$$

$$\psi\ (x_2, a_1\ a_2, a_3, . . . , a_n) = 0$$

. . . . . .

$$\psi\ (x_{n,}, a_1, a_2, a_3, . . . , a_n) = 0$$

This is system of linear algebraic equations with respect to $a_1, a_2, a_3, . . . , a_n$

and has the form:

$$a_1 L\varphi_1 (x_1) + a_2 L \varphi_2(x_1) + a_3 L \varphi_3(x_1) + . \quad . + a_n L\varphi_n(x_1) = f(x_1) - L\varphi_0(x_1)$$

$$a_1 L \varphi_1 (x_2) + a_2 L \varphi_2(x_2) + a_3 L \varphi_3(x_2) + . \quad . + a_n L \varphi_n(x_2) = f(x_2) - L\varphi_0(x_2)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$a_1 L\varphi_1 (x_n) + a_2 L \varphi_2(x_n) + a_3 L \varphi_3(x_n)) + . \quad . + a_n L \varphi_n(x_n) = f(x_n) - L\varphi_0(x_n) \tag{2.9}$$

If the system is uniquely solvable, the coefficients $a_1, a_2, a_3, . \quad . , a_n$ are substituted into

$$y_n(x) = \varphi_0 + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots\ldots + a_n\varphi_n(x) \tag{2.10}$$

2.3.3 Variational Iteration Method

Variational Iteration Method has been favourably applied to various kinds of nonlinear problems. The main property of the method is in its flexibility and ability to solve nonlinear equations accurately and conveniently. To illustrate the basic concept of the technique, we consider the following general differential equation.

$$Lu + Nu = g(x) \tag{2.11}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method (He, 1999, 2000, 2006), He and Wu, 2006; Inokuti et al.1978), we can construct a functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s)) + N\tilde{u}(s) - g(s)ds \tag{2.12}$$

(where λ is a Lagrange multiplier), which can be identified optimally via variational iteration method. The subscript n denotes the nth iteration, u_n is considered as a restricted variation i.e

$$\delta \tilde{u}n = 0. \tag{2.13}$$

is called a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principle of variational iteration and its applicability to various kinds of differential equations are given in He (1999, 2000, 2006), He and Wu (2006), Inokuti et al. (1978). In this method, it is required to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0. Consequently, the solution is given by $u = \lim_{n \to \infty} u_n$.

2.3.4 Fixed Point Iteration Method

In numerical analysis, fixed point iteration is a method for computing fixed points of iteration functions (Sniedovich, 2010).

More specifically, given a function f defined on the real numbers with real values and given a point x_0 in the domain of f, the fixed point iteration is

$$x_{n+1} = f(x_n) \quad n = 0,1,2, \ldots, z \tag{2.14}$$

which gives rise to the sequence $x_0, x_1, \ldots, x_n$ which is hoped to converge to a point x, if f is continuous. Then one can prove that, the obtained x is a fixed point of f i.e

$$f(x) = x \tag{2.15}$$

More generally, the function f can be defined on any metric space with values in that same space.

Bello (2004), Sniedovich (2010) and kumar (2010) described the fixed point iteration method as a process by which the student of calculus is able to determine the fixed point of iterated functions. When solving limits such as this, the student must have a function, denoted in $f's$ domain and also the fixed point iteration. Wikipedia (2009), the free encyclopaedia further supports this procedure and explains, often times, the elusive solution to a nonlinear equation can be elicited from forming iterative schemes. These iteration schemes seek to produce sequences which will converge to fixed points. Two definitions are given below:

(a) A fixed point of a function $g(x)$ is a real number z; such that $z=g(z)$

(b) The iteration $x_{n+1} = g(x_n)$ for $n = 0, 1, 2, \ldots$, is called a fixed point iteration.

A first simple and useful example is the Babylonian method for computing the square root of $z > 0$, which consists of taking $f(x)=\frac{1}{2}\left(\frac{z}{x}+x\right)$; i.e the mean value of x as, $\frac{z}{x}$ approach the limit $x = \sqrt{z}$ (Kumar 2010).

The fixed point iteration framework allows for greater freedom in formulation of iterative schemes and provides a more rigorous framework for the analysis of convergence. The analysis of convergence of the fixed point iterations also produces additional information that can be used to examine the rate of convergence and a mechanism to detect convergence to within a certain amount of accuracy. See Ogunfiditimi and Adeboye (2004).

Obviously, fixed point iteration method is a complex matter. In fact, it is often considered as one of the most difficult type of problems in calculus study. However, with dedication and hard work for those who are willing to do more research and to put in practice, they can have a much easier time solving and even excelling at such problems.

2.4 Contraction Mapping Theorem and its use

Theorem 2.4.1: Contraction mapping theorem (Banach fixed point theorem)

If $T:X \to X$ is a contraction of a closed subset of x of a Banach space, then there is exactly one $x \in X$, such that $Tx = X. For\ any\ x_0 \ \in X,$ the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to X.

Proof:

For any $x_0 \ \in X, set\ x_n = \ T^n x_{0.}$ Let β be positive number; $0 < \beta < 1$. Then,

$$\|x_{n+1} - x_n \| \leq \ \beta \| \ x_n - x_{n+1} \| \leq \beta^2 \| \ x_{n+1} - \ x_{n-2} \| \leq . \ \ . \ \ . \leq \beta^n \ \| \ x_1 - \ x_0 \ \|$$

Hence for any $m > n$ the triangle inequality gives

$$\| \ x_m - \ x_n \| \leq \ \| \ x_m - \ x_{m-1} \| + \ \| \ x_{m-1} - \ x_{m-2} \| + . \ \ . \ \ . + \| \ x_{n+1} - \ x_n \|$$

$$\leq \beta^{m-1} + \ \beta^{m-2} + . \ \ . \ \ . + \beta^n \ \ \| \ x_1 - \ x_0 \ \|$$

$$\leq \left[\frac{\beta^n}{1-\beta}\right] \ \| \ x_1 - \ x_0 \ \ \| \to 0 \text{ as } n \ \to \ \infty \tag{2.16}$$

Hence for any $\varepsilon > 0$ there is an N such that $\| \ x_m - \ x_n \| \leq \ \varepsilon$ whenever $m > n > N$, which means that $\{x_n\}$ is a Cauchy sequence, since the space is complete $\{x_n\}$ is convergent; call its limit $\bar{x}$. Since X is closed, $\bar{x} \in X$, we must to show that $T\bar{x} = \bar{x}$. We have for any n

$$\|T\bar{x} - \bar{x}\| \leq \| T\bar{x} - Tx_n \| + \| Tx_n - \bar{x} \|$$

$$\leq \beta\| \bar{x} - x_n \| + \| \bar{x} - x_{n-1} \| \qquad (2.17)$$

But (2.17) tends to zero as $n \to \infty$; $\| T\bar{x} - \bar{x} \|$ is less than every member of a sequence which tends to zero, hence $T\bar{x} - \bar{x} = 0$ as required.

Finally, to prove uniqueness:

Suppose that $T\bar{x} = \bar{x}$ and $T\bar{y} = \bar{y}$.

Then,

$\| \bar{x} - \bar{y} \| = \| T\bar{x} - T\bar{y} \| \leq \beta \| \bar{x} - \bar{y} \|$, a contradiction unless $\|\bar{x} - \bar{y} \| = 0$,

that is, $\bar{x} = \bar{y}$.

Lemma 2.4.2 (Fixed Point)

Let $T: X \to X$ be a mapping on a complete metric space $X = (X, d)$ and suppose that T^m is a contraction on X for some positive m. Then T has a unique fixed point.

Proof:

By assumption $B = T^m$ is a contradiction on X. By Banach's fixed point theorem, this mapping B has a unique fixed point, x^* that is $Bx^* = x^*$. Hence, $B^n x^* = x^*$. Banach's theorem also implies that for every $x \in X$, $B^n x$ tends to x^* as $n \to \infty$. In the particular $x = Tx^*$ since $B^n = T^{nm}$. We thus obtain

$$x^* = \lim_{n \to \infty} B^n Tx^* = \lim_{n \to \infty} TB^n x^* = \lim_{n \to \infty} Tx^* = Tx^*$$

This shows that x^* is a fixed point of T, since every fixed point of T is also a fixed point of B. Thus, T cannot have more than one fixed point.

2.5 Existence and Uniqueness Theorems

Augustine Louis Cauchy (1789 – 1857) was the first mathematician who proved the existence theorem for general types of differential equations, for which no explicit solution is available. His strategy was to consider the various methods introduced earlier for the purpose of numerical computations; and to show that under certain conditions, these methods usually gave convergent approximation processes having a solution as limit. In particular, in a paper published in 1835 in Prag, he took the method outlined earlier not for an ordinary differential equation, but linear partial differential equation of first order

$$U_1 = AU_x + BU_y \tag{2.18}$$

and Cauchy transformed this into the equivalent integro – differential equation:

$$U = +\int_0^1 (AU_x + BU_x)dt \tag{2.19}$$

which he resolved by successive approximations. In order to apply a fixed point theorem to establish the existence of a solution of boundary value problem, we must determine a suitable mapping of a function space into itself which is such that a fixed point of the mapping is a solution of the boundary value problems.

Iyase (2010) investigated existence and uniqueness theorems for a class of three point fourth-order boundary value problem where Caratheodory conditions were used.

Also, Tejumola et al. (2010) obtained existence and uniqueness results for a wider class of fourth-order equations subject to varied boundary conditions. According to them, the following are some of the ways the method can be applied.

Theorem 2.5.1: (Caratheodory Condition)

We consider,

$$y^{iv} + f(y'')y''' = g(t, y, y', y'', y''') \qquad (2.20)$$

$$y(0) = y''(x) = y''(1) = y(\eta) = 0, \; 0 \leq \eta \leq 1 \qquad (2.21)$$

A function $g : [0,1] \times \Re^4 \rightarrow \Re$ is said to satisfy the Caratheodory condition if

(i) for each $(x, y, w, z) \in \Re^4$ the function $g(., x, y, w, z) : t \in [0,1] \rightarrow g(x, y, w, z) \rightarrow \Re$ is measurable on $[0,1]$

(ii) for a, e $t \in [0,1]$, the function $g(t., ., ., ..) : (x, y, w, z) \in \Re^4 \rightarrow g(t, x, y, w, z) \in \Re$ is continuous on $\Re^4$ and

(iii) for each $r \geq 0$ there exists $a_r \in L^1[0,1]$ such that $|g(t, x, y, w, z)| \leq a(t)$ for each $t \in [0,1]$ and all $(x, y, w, z) \in \Re^4$ with $\sqrt{x^2 + y^2 + w^2 + z^2} \leq r$

Example 2.5.1: Let $u \in c^1([0,1])$, if

(i) $u(0) = u(1) = 0$, then $\|y\|_2^2 \leq \frac{1}{\pi^2} \|y^1\|_2^2$

(ii) $u(0) = u(1) = 0$, then $\|y\|_\infty^2 \leq \frac{1}{\pi^2} \|y^1\|_2^2$

Theorem 2.5.2 (Existence Theorem): Let $g : [0,1] \times \Re^4 \rightarrow \Re$ be a function satisfying the Caratheodory conditions, $f, h \in (\Re, \Re)$.

Assume that:

(i) There exists $k \in \Re$ such that $h(x^l) \leq k$

(ii) There exist functions $a(t) \in c^1([0,1]); b(t), e(t) \in c\,(0,1); d(t) \in L^1([0,1])$, $a_0, b_0, c_0, and\ e_0$ are real numbers such that $a^1(t) \leq a_0$, $b(t) \geq -b_0$ $c(t) \geq -c_0$, $e(t) \geq e_0$; $a, e, t \in [0,1]$, and for every $y,\ y', y'', y''' \in \Re$; $a, e, t \in [0,1]$, we have

$$y''g(t, ,y,\ y', y'', y''') \geq a(t)y''y''' + b(t)(y'')^2 + c(t)|y''y''| + d(t)|y''|$$

$$+ e(t)|yy'|$$

(iii) There exists $\propto\ :\ [0,1]\ x\ \Re^4 \rightarrow \Re$ and $\beta \in L^l([0,1])$ such that $g(t, y, y', y'', y''') \mid \leq |a(t, y, y', y'', y''')|^2 + \beta(t)$ for every

$y,\ y', y'', y''' \in \Re;\ a, e, t \in [0,1]$

Then, for every $p(t) \in L^1[0,1] \in L^l\ [0,1]$, the boundary value problem (2.20) with (2.21) has at least one solution provided:

$$\pi^3 a_0 + 2\pi^3 b_0 + 4\pi^2 c_0 + 16e_0 + 2\pi^3|k| \leq 2\pi^5$$

Proof:

It suffices to verify that the set of all possible solutions of the family of equations

$$(y^{iv}) + \lambda f(y'')y''' + \lambda h(y')y'' + \lambda g(t, y, y', y''y''') + \lambda p(t) \qquad (2.22)$$

$$y(0) = y''(0) = y''(1) = y(\eta) = 0, \quad 0 \leq \eta \leq 1 \qquad (2.23)$$

is priori bounded in $C^3[0,1]$ independent of solution $y(t)$and λ. Multiplying

(2.22) by y'' and integrating over $[0,1]$ we have

$$\int_0^1 y^{iv} y'' dt + \lambda \int_0^1 f(y'') y''' y'' dt + \lambda \int_0^1 h(y')(y'')^2 dt$$

$$-\lambda \int_0^1 g(t, y, y', y'', y''') y'' dt + \lambda \int_0^1 p(t) y'' dt$$

Since $y''(0) = y'(1)$, it follows that $\lambda \int_0^1 f(y'') y''' y'' dt = 0$

and from condition (ii) we have

$$-\int_0^1 ||y'''|| dt \geq \lambda \int_0^1 a(t) y'' y''' dt + \lambda \int_0^1 b(t)(y'')^2 dt$$

$$+\lambda \int_0^1 c(t)|y'y''| dt + \lambda \int_0^1 d(t)|y''| dt$$

$$+\lambda \int_0^1 e(t)|yy'| dt - \lambda \int_0^1 h(y')(y'')^2 dt + \lambda \int_0^1 p(t)(y'') dt - ||y'''||_2^2$$

$$\geq \frac{\lambda}{2} a_0^0 ||y''||_2^2 - \lambda b_0 ||y''||_2^2 - \lambda c_0 ||y''|| 2 ||y''||_2 - \lambda ||d|| 1 ||y||_\infty$$

$$-e_0 ||y|| 2 ||y''||_2 - \lambda k ||y''||_2^2 - \lambda ||p|| 1 ||y||_\infty .$$

We observe that $y(0) = y(\eta) = 0$ and there exists $t_1 \in (0,1)$ with $t_1 < \eta < 1$ such that $y_1'(t_1) = 0$. It follows that

$$||y'||_\infty \leq ||y'||_2 , ||y||_2^2 \leq \frac{4}{\pi^2} ||x||_2^2 \qquad (2.24)$$

$$||y'||_\infty \leq ||y''||_2, ||y||_2^2 \leq \frac{4}{\pi^2} ||y''||_2^2 \qquad (2.25)$$

Since $y''(0) = y''(1)$ we have from example (2.18) that

$$||y''||_2^2 \leq \frac{1}{\pi^2} ||y'''||_2^2 \qquad (2.26)$$

$$||y''||_\infty \leq \frac{1}{2} ||y'||_2 \qquad (2.27)$$

Using the inequalities (2.24) to (2.27) we get,

$$||y''||_2^2 \leq \left(\frac{a_0}{2\pi^2} + \frac{b_0}{\pi^2} + \frac{c_0}{\pi^3} + \frac{8e_0}{\pi^5} + \frac{|k|}{x^2}\right) ||y'''||_2^2 + \frac{1}{2}\,||d||1||y'''||_2$$

$$+\frac{1}{2}\,||p||1||y'''||_2$$

$$||y'''||_2^2 \leq \pi^5(||d||_1 + ||p||_1) \,/\, 2\pi^5 - (\pi^2 a_0 + 2\pi^3 b_0 + 4\pi^2 c_0$$

$$+ 16e_0 + \; 2\pi^3|k|_1 \tag{2.28}$$

Hence,

$$||y||_\infty \leq ||y''||_\infty \leq ||y'''||_\infty \leq \rho \tag{2.29}$$

Let

$$M_\rho = \max f(v),\; v \in (-\rho, \rho)$$

$$M_\rho = \max h(z),\; z \in (-\rho, \rho)$$

Then,

$$|y^{iv}| \leq |f(y'')||y'''|| + ||h(y')||x''| + |g(t, y, y', y'', y''')| + |p(t)|$$

and using condition (ii) we have,

$$||y^{iv}||_1 \leq M_\rho\, ||y'''||_2 + N_\rho ||y''||_2 + \int_0^1 |\alpha(t, y, y', y'')||y'''\,|^2 dt + ||\beta||_1$$

$$+||p||_1$$

$$\leq \rho M\rho + \frac{N_\rho}{\pi} + \rho^2 E_\rho + \|\beta\|_1 + \|p\|_1 - \rho_1$$

$a'(\mathrm{t}) \leq a_o$, $\mathrm{b(t)} \geq -b_0$, $\mathrm{c(t)} \geq -c_0$, $\mathrm{e(t)} \geq -e_0$,

$t \in [0,1]$ and for every $y_{i,}\ y_i', y_i'', y_i''' \in \Re$

for $i = 1, 2. \ . \ .$

$$g(t, y_1, y_1', y_1'', y_1''') - g(t, y_2, y_2', y_2'', y_2''')(y_1 - y_2)''$$

$$\geq a(t)((y_1 - y_2)'' + b(t)|((y_1 - y_2)''|^2 + c(t)|(y_1 - y_2)'((y_1 - y_2)''|$$

$$+e(t)|(y_1 - y_2)(y_1 - y_2)'$$

Then, for every $p(t) \in L^1([0,1])$, boundary value problem (2.32) with (2.33) has a unique solution provided

$$\pi^2 a_0 + 2\pi^3 b_0 + 4\pi^2 c_0 + 16e_0 + 2\pi^3 |B| < 2\pi^5 \tag{2.34}$$

Proof:

Existence of solution follows from Theorem (2.5.2)

Suppose u_1, u_2 are two solutions of (2.32), then,

$$u_1^{IV} - u_2^{IV} + A(u_1''' - u_2''') + B(u_1'' - u_2'')$$

$$-g(t, u_1, u_1', u_1'', u_1''') - g(t, u_1, u_2', u_2'', u_2''') \tag{2.35}$$

$$(u_1 - u_2)(0) = (u_1 - u_2)''(0) = (u_1 - u_2)''(1) = (u_1 - u_2)(\eta) = 0 \tag{2.36}$$

On multiplying (2.36) by$(u_1 - u_2)''$and integrating from 0 to1we have,

$$-\int_0^1 [(u_1 - u_2)''']^2 \, dt + B\int_0^1 [u_1'' - u_2'']^2 dt$$

$$-\int_0^1(g(t,u_1,u_1',u_1'',u_1''')(u_1+u_2)'')dt$$

$$-\int_0^1(g(t,u_2,u_2',u_2'',u_2''')(u_1+u_2)'')dt \tag{2.37}$$

Let $v = u_1 - u_2$, then,

$$\int_0^1(v''')^2dt \geq \int_0^1 a(t)v''v'''dt + \int_0^1 b(t)(v'')^2\,dt + c(t)|v'v''|dt$$

$$+\int_0^1 e(t)|vv'|dt - B\int_0^1(v'')^2dt$$

$$\geq -\frac{a_0}{2}\int_0^1 |v''|^2dt - b_0\int_0^1 |v\,''|^2dt - c_0\int_0^1|v'||v''|dt$$

$$-e_0\int_0^1 |v|v'dt - B\int_0^1 |v''|^2dt$$

$$||v'''||_2^2 \leq \frac{a_0}{2}||v'''|| + b_0||v'''||_2^2 + c_0|v'|_2|v''||_2 + e_0|v|_2|v'|_2 + |B|||v''$$

$$\leq \left(\frac{a_0}{2\pi^2}+\frac{b_0}{\pi^2}+\frac{2c_0}{\pi^3}+\frac{8e_0}{\pi^5}+\frac{B}{\pi^2}\right)||v'''||_2^2$$

$$[2\pi^5-(\pi^2a_0+2\pi^3b_0+4\pi^2c_0+16e_0+2\pi^3|B|)]||v'''||_2^2 \leq 0$$

From condition (2.34) we derive $||v'''||_2^2 \leq 0$ and hence $u_1(t)-u_2(t)$, $a,e,t\in[0,1]$ by the continuity of $u_1(t)$ and $u_2(t)$.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 The Residual

Consider the boundary – value problem:

$$Lu = y'^{v} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = f(x), \quad a \le x \le b \tag{3.1}$$

satisfying the three boundary conditions

$$u(a) = u''(a) = y''(b) = u(\propto) = 0, \quad a \le \propto b \tag{3.2}$$

$p,q,r,s,f \in C[a,b]$ are given functions $\propto, a, b$ are given numbers. To determine the approximate solution of the boundary – value problem (3.1) to (3.2), following steps are considered:

Assign on the interval $[a,b]$ a linearly independent system of fourth continuously differentiable function, $\varphi_0, \varphi_1\ \varphi_2, \ldots, \varphi_n$, such that φ_0 satisfies the boundary conditions (3.2) that is $l_0\ \varphi_i = y_0$, $l_1\varphi_1 = y_1$ and the homogeneous boundary conditions:

$$l_0\varphi_i = 0, \quad l_1\varphi_i = 0, \quad i = 1, 2, \ldots, n$$

The assigned system of functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, is referred to as the base system.

Then, a linear combination of $n + 1$ base functions

$$y_n(x) = \varphi_0 + a_1\varphi_1(x) + a_2\varphi_2(x) + \cdots + a_n\varphi_n(x) \tag{3.3}$$

represents the approximate solution being sought for.

The function

$$\psi(x; a_1, a_2, a_3, \ldots, a_n) = Ly_n(x) - f(x) \tag{3.4}$$

resulting from the substitution of $y_N(x)$ into the boundary value problem is referred to as the discrepancy or residual. The function (3.3) is taken as the approximate solution of the boundary - value problem (3.1) to (3.2) and in the techniques for finding the parameters; $a_1, a_2, a_3, \ldots, a_n$ varies.

3.2 The Collocation Method

This is a method of solution of boundary value problems. In the interval $[a,b]$, n points $x_1, x_2, x_3, \ldots x_n$ are fixed and are called the collocation point at which the discrepancy (3.4) is equated to zero, that is,

$$\psi\ (x_1, a_1, a_2, a_3, \ \ldots, a_n) = 0$$

$$\psi\ (x_2, a_1 a_2, a_3, \ \ldots, a_n) = 0$$

$$\ldots \quad \ldots \quad \ldots$$

$$\psi\ (x_{n,}, a_1, a_2, a_3, \ \ldots, a_n) = 0$$

This is system of linear algebraic equations with respect to $a_{1,} a_{2,} a_3, \ldots, a_n$ has the form:

$$a_1 L\varphi_1 (x_1) + a_2 L\,\varphi_2(x_1) + a_3 L\,\varphi_3(x_1) + \ldots + a_n\ L\varphi_n(x_1) = f\,(x_1) - L\varphi_0(x_1)$$

$$a_1 L\,\varphi_1 (x_2) + a_2 L\,\varphi_2(x_2) + a_3 L\,\varphi_3(x_2) + \ldots + a_n\ L\,\varphi_n(x_2) = f\,(x_2) - L\varphi_0(x_2)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_1 L\varphi_1 (x_n) + a_2 L \varphi_2(x_n) + a_3 L \varphi_3(x_n)) + \ . \ . + a_n L \varphi_n(x_n) = f(x_n) - L\varphi_0(x_n)$$

If the system is uniquely solvable, the coefficients $a_{1,}a_{2,}a_3,. \ . , a_n$ are substituted into (3.3).

Example 3.1.1

$$y'^{v} + y'' = 2 \tag{3.5}$$

Subject to boundary conditions

$$y(0) = y''(0) = y'''(\tfrac{1}{2}) = 0, \quad y(1) = 1$$

The exact solution is given by

$$y_E(x) = 2\cos x + 1.092082623\sin(x) + x^2 - 2$$

Solution

Consider,

$$y(x) = u_0(x) + c_1(x - x^3) + c_2(x - x^4) \tag{3.6}$$

Let $u_0(x) = x$

Therefore, equation (3.6) becomes,

$$y(x) = x + c_1(x - x^3) + c_1(x - x^4) \tag{3.7}$$

Substituting y(x) into (3.5), we obtain:

$$-6xc_1 - c_2(24 - 12x^2) = 2$$

Let the collocation points be $x = \frac{1}{3}$, and $x = \frac{2}{3}$

Then,

$$-2c_1 - 76/3c_2 = 2$$

Thus,

$$c_1 = 0.1875, \quad c_2 = -0.09375$$

Substitute the values of c_1 and c_2 into equation (3.6), we have

$$y(x) = x + 0.1875\,(x - x^3) - 0.09375\,(x - x^4)$$

Therefore,

$$y\left(\frac{1}{2}\right) = 0.529296875$$

Note that $y_E(\frac{1}{2}) = 0.528737468$

Example 3.2.2

$$y^{iv} - y'' = -12x^2 \tag{3.8}$$

Subject to boundary conditions

$$y(0) = y''(0) = y'''\left(\tfrac{1}{2}\right) = 0, \quad y(1) = 13$$

The exact solution is:

$$y_E(x) = 24 + \frac{12(e-1)x}{e^{1/2}} - \frac{\left(24 + 12e^{1/2}\right)e^{x}}{e+1} + \frac{\left(12e^{1/2} - 24e\right)e^{-x}}{e+1} + x^4 + 12x^2$$

Solution

Consider,

$$y(x) = u_0(x) = c_1(x - x^4) + c_2(x - x^5) \tag{3.9}$$

Let $u_0 = 13x$

Therefore, equation (3.9) becomes,

$$y(x) = 13x + c_1(x - x^4) + c_2(x - x^5) \tag{3.10}$$

Substituting (3.10) into (3.8), we have

$$-c_1(-24 + 12x^2) - c_2(120 - 20x^3) = -12x^2$$

Let the collocation points be $x = \frac{1}{3}$, and $x = \frac{2}{3}$

Then,

$$-68/3 - c_1 \, 1060/27 \, c_2 = -4/3$$

$$612c_1 + 1060c_2 = 36$$

Therefore, $c_1 = -0.1169102297$, $c_2 = 0.1014613779$

Therefore, substitute the values of c_1 and c_2 into equation (3.10), we have

$$y(x) = 13x - 0.1169102297(x - x^4) + 0.1014613779(x - x^5)$$

Then,

$$y(\tfrac{1}{2}) = 6.496411795$$

$$y_E(1/2) = 6.486584557$$

3.3 Standard Galerkin Method

This is also one of the method for solution of boundary value problems which was suggested by B. G. Galerkin. It is based on the requirement that the base functions

$\varphi_0, \varphi_1, \varphi_2, . . . , \varphi_N$ be orthogonal to (3.4).

That is

$$\int \psi(x_1, a_0, a_1, a_2, . . ., a_n)\, \varphi_i dx = 0, \quad i = 0, 1, 2, 3, . . ., n$$

This gives rise to the following system of linear algebraic equations for the coefficients of the approximate solution (3.3) of the boundary value problem (3.1) with (3.2):

$$a_1 L(\varphi_1, \varphi_1) + a_2 L(\varphi_2, \varphi_1) + a_3 L(\varphi_3, \varphi_1) + . \quad . + a_n L(\varphi_n, \varphi_1) = (f - \ L(\varphi_0, \varphi_1))$$

$$a_1 L(\varphi_1, \varphi_2) + a_2 (L \varphi_2, \varphi_2) + \ a_3 L(\varphi_3, \varphi_2) + . \quad . + a_n L(\varphi_n, \varphi_2) = (f - L(\varphi_0, \varphi_2))$$

. . . .

$$a_1 L(\varphi_1, \varphi_n) + a_2 L(\varphi_2, \varphi_n) + a_3 L(\varphi_3, \varphi_n) + . \quad . + a_n L(\varphi_n, \varphi_n) = (f - L(\varphi_0, \varphi_n))$$

Where,

$$(f, g) = \int_a^b f(x) g(x) dx$$

Example 3.3.1

$$y'^{v} + y'' = 2, \quad 0 < x < 1 \tag{3.12}$$

Subject to boundary conditions

$$y(0) = y''(0) = y'''\left(\tfrac{1}{2}\right) = 0, \; y(0) = 1$$

The exact solution is given by:

$$y_E(x) = 2\cos x + 1.092082623\sin(x) + x^2 - 2$$

Solution

Consider,

$$y(x) = u_0(x) + c_1\,(x - x^3) + c_2(x - x^4) \tag{3.13}$$

Let $u_0(x) = x$

Therefore, equation (3.13) becomes,

$$y(x) = \mathrm{x} + c_1\;(x - x^3) + \;c_2\,(x - x^4) \tag{3.14}$$

Substituting (3.14) into (3.12) we obtain:

$$-6x\,c_1 - \;c_2\,(24 + 12x^2) = 2$$

The residual $R(x;c_1,c_2) = -6x\,c_1 - \;c_2\,(24 + 12x^2) - 2$

$$\int_0^1 \left(x - x^3\right) R(x;\,c_1,c_2)dx = 0$$

$$\int_0^1 \left(x - x^3\right)[-6xc_1 - c_2(24 + 112x^2) - 2]dx = 0$$

$$\int_0^1 c_1\left(-6x^2 + 6x^4\right)dx - \int_0^1 c_2\left(24x + 12x^3 - 12x^5\right)dx = \int_0^1 \left(2x - 2x^3\right)dx$$

$$4\,c_1 + \;35c_2 \;= 2.5 \tag{3.15}$$

Also,

$$\int_0^1 \left(x-x^4\right)R\left(x;c_1,c_2\right)dx = 0$$

$$\int_0^1 c_1\left(-6x^2+6x^5\right)dx - \int_0^1 c_1\left(24x+12x^3-24x^4-12x^6\right)dx$$

$$=\int_0^1 \left(2x-2x^4\right)dx$$

$$35c_1 - 543c_2 = -21 \tag{3.16}$$

From equation (3.15) and (3.16) we have,

$$4\,c_1 + 35c_2 = 2.5$$

$$35c_{1,} - 543c_2 = -21$$

Thus:

$$c_1 = 0.18352499264, \quad c_2 = 0.0504857227$$

$$y(x) = x - 0.18352499264((x - x^3) + 0.0504857227\,(x - x^4)$$

$$y\left(\tfrac{1}{2}\right) = 0.5908062261$$

$$y_E\left(\tfrac{1}{2}\right) = 0.528737281$$

Example 3.3.2

$$y'^{v} - y'' = -12x^2 \tag{3.17}$$

Subject to boundary conditions

$$y(0) = y''(0) = y'''\left(\frac{1}{2}\right) = 0, \;\; y(1) = 13$$

The exact solution is given by:

$$y_E(x) = 24 + \frac{12(e-1)x}{e^{1/2}} - \frac{\left(24+12e^{1/2}\right)e^{x}}{e+1} + \frac{\left(12e^{1/2}-24e\right)e^{-x}}{e+1} + x^4 + 12x^2$$

Solution

Consider,

$$y(x) = u_0(x) = c_1u_1(x) + c_2u_2(x) \tag{3.18}$$

Let the base functions be

$$u_0(x) = x, \qquad u_1(x) = (x - x^3), \;\; u_2(x) = (x^2 - x^4)$$

Therefore, equation (3.18) becomes,

$$y(x) = 13x + c_{1,}\,(x - x^3) + c_2\,(x^2 - x^4) \tag{3.19}$$

Substituting this into (3.17), we obtain:

$$6xc_1 \;\; + c_2\,(-\,26 + 12x^2) = -\,12x^2$$

The residual $R(x;c_1,c_2) = 6xc_1 + c_2\,(-24 + 12x^2) + 12x^2$

$$\int_0^1 \left(x - x^3\right)R(x;c_1,c_2)dx = 0$$

$$\int_0^1 \left(x - x^3\right)[-6xc_1 + c_2\left(-24 + 12x^2\right) + 12x^2]dx = 0$$

This gives:

$$8c_1 + 205c_2 = -10 \tag{3.20}$$

Also,

$$\int_0^1 (x^2 - x^4) R(x; c_1, c_2) dx = 0$$

$$\int_0^1 c_1 (6x^3 - 6x^5) dx - \int_0^1 c_2 (-26x^2 + 38x^4 - 12x^6) dx$$

$$= \int_0^1 (-12x^3 + 12x^6) dx$$

which gives:

$$105c_1 - 292c_2 = -135 \tag{3.21}$$

From equation (3.20) and (3.21) we have,

$c_1 = -1.266692724, \quad c_2 = 0.00683994528$

Therefore, substitute the value of c_1, c_2 into equation (3.18), we get

$$y(x) = 13x - 1.266692724(x - x^3) + 0.00683994528(x^2 - x^4)$$

Then,

$$y\left(\tfrac{1}{2}\right) = 6.026272718$$

$$y_E\left(\tfrac{1}{2}\right) = 6.486584557$$

3.4 The New Scheme

3.4.1 The Development of Fixed – Point Iteration Process

Let (E,d) be a complete metric space and let T be self- map of E. Let $y_0 \in E$

and let $y_{n+1} = f(T,y_n)$ denote an iteration procedure which gives a sequence $\{y_n\}_{n=1}^{\infty}$.

T is iteration process defined for arbitrary $y_0 \in E$ by:

$$y_{n+1} = f(T,y_n) = (1-\alpha_n)y_n + \alpha_n Ty_n,\ n \geq 0 \tag{3.22}$$

Where, $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence satisfying $\alpha_n = 1$, $0 \leq \alpha_n \leq 1$ for $n \geq 0$ and

$$\sum_{n=0}^{\infty} \alpha_n = \infty$$

Let

$$y'^{v} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = t(x)$$

where, $p,q,r,s,t \in C[a,b]$. Then, the scheme

$$y_n'^{v} = (1-\lambda_n)y_n'^{v} + \lambda_n(1-\alpha_n)y_n'^{v}$$

$$= \lambda_n(t(x) - p(x)y_n''' - q(x)y_n'' - r(x)y_n' - s(x)y_n) + (1-\lambda_n)y_n'^{v} \tag{3.23}$$

converges for $0 \leq \lambda_n \leq 1$

Let,

$$y'^{v} = f(x,y,y',y'',y''') \tag{3.24}$$

Therefore, for any $y(x)$ solution of the integral equation on $[a,b]$:

$$y(x) = \int_0^1 G(x,t)f(t,y(t),y'(t),y''(t),y'''(t))dt + v(x) \tag{3.25}$$

where $G(x,t)$ is a green function of the associated boundary – value problem

Let

T : $c^{(1)}[a,b] \to c^{(1)}[a,b]$ be defined by:

$$(Ty)(x) = \int_{x_1}^{x_2} G(x,t) f\big(t, y(t), y'(t), y''(t), y'''(t)\big)dt + v(x) \quad (3.26)$$

T is an operator such that any $y(x)$ is a solution of (3.24) at fixed point of T. T can be referred to as a fixed point operator. For convergence of (3.23) and (3.22):

Let

$$y_{n+1} = (1 - \propto_n)\, y_n + \propto_n Ty_n\ ,$$

then,

$$y'_{n+1} = (1 - \propto_n) y'_n + \propto_n Ty'_n \quad (3.27a)$$

$$y''_{n+1} = (1 - \propto_n) y''_n + \propto_n Ty''_n \quad (3.27b)$$

$$y'''_{n+1} = (1 - \propto_n) y'''_n + \propto_n Ty'''_n \quad (3.27c)$$

$$y'^{v}_{n+1} = (1 - \propto_n) y'^{v}_n + \propto_n Ty'^{v}_n \quad (3.27d)$$

From equation (3.26)

$$(Ty_n)' = \int_{x_1}^{x_2} \frac{\partial}{\partial x} G(x,t) f\big(t, y(t), y'(t), y''(t), y'''(t)\big)dt + v'(x) \quad (3.28a)$$

$$(Ty_n)'' = \int_{x_1}^{x_2} \frac{\partial^2}{\partial x^2} G(x,t) f\big(t, y(t), y'(t), y''(t), y'''(t)\big)dt + v''(x) \quad (3.28b)$$

$$(Ty_n)''' = \int_{x_1}^{x_2} \frac{\partial^3}{\partial x^3} G(x,t) f\big(t, y(t), y'(t), y''(t), y'''(t)\big)dt + v'''(x) \quad (3.28c)$$

$$(Ty_n)'^{v} = \int_{x_1}^{x_2} \frac{\partial^4}{\partial x^4} G(x,t) f\big(t, y(t), y'(t), y''(t), y'''(t)\big)dt + v'^{v}(x) \quad (3.28d)$$

Therefore, equation (3.27d) becomes

$$y_{n+1}^{\prime iv} = (1 - \propto_n) y_n^{\prime iv} + \propto_n \int_{x_1}^{x_2} \frac{\partial^4}{\partial x^4} G(x,t) f\big(t, y(t), y'(t), y''(t), y'''(t)\big) dt$$

$$+ v^{\prime iv}(x) \qquad (3.29)$$

From equation (3.25), equation (3.29) becomes:

$$y_{n+1}^{\prime iv} = (1 - \propto_n)\, y_n^{\prime iv} + \propto_n\, y_n^{\prime iv} \qquad (3.30)$$

Therefore, the scheme (3.23) and (3.22) convergent are equivalent. Therefore, the theorem holds.

Theorem 3.4.2: Assume that:

(i) The sequence $\{\propto_n\}_{n\geq 0}$ of real number satisfies the conditions

(a) $0 \leq \alpha_n \leq 0$

(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$

(ii) The operator T defined by equation (3.26) is Lipschitz constant $L \in (0,1)$.

Then, the sequence $\{y_n\}_{n\geq 0}$ in $c^{(1)}[a,b]$ defined iteratively a linear function $y_0 \in c^{(1)}[a,b]$ that satisfies the boundary conditions (3.24).

Proof

Let

$$y_{n+1} = (1 - \propto_n)\, y_n + \propto_n\, T\, y_n$$

and $y_n \rightarrow y^*$ as $n \rightarrow \infty$.

Let $\rho = \|y_n - y^*\|$

$$\rho_{n+1} = \|y_{n+1} - y^*\| = \|(1-\alpha_n)(y_n - y^*) + \alpha_n(Ty_n - y^*)\|$$

$$\leq \|(1-\alpha_n)(y_n - y^*)\| + \|\alpha_n(Ty_n - y^*)\| \tag{3.30}$$

Recall that y^* is a solution to (3.24) implies $T\,y^* = y^*$

Therefore, (3.30) becomes:

$$\rho_{n+1} = \|y_{n+1} - y^*\| \leq \|(1-\alpha_n)(y_n - y^*) + \alpha_n(Ty_n - y^*)\| \tag{3.31}$$

T is contractive and (3.31) becomes,

$$\rho_{n+1} = \|y_{n+1} - y^*\| = \|(1-\alpha_n)(y_n - y^*)\| + \|\alpha_n L(Ty_n - y^*)\|$$

$$\leq (\|1-\alpha_n| + |\alpha_n L|)\|y_n - y^*\|$$

$$\leq (1-\alpha_n + \alpha_n L)\|y_n - y^*\|$$

From (a) in (i)

$$\rho_{n+1} = \|y_{n+1} - y^*\| \leq (1-\alpha_n + |\alpha_n L|)\|y_n - y^*\|$$

$$\leq e^{-\alpha_{n(l-L)}}\,\rho_n$$

$$\rho_1 \leq e^{-\alpha_{0(l-L)}}\,\rho_0$$

$$\rho_2 \leq e^{-\alpha_{1(l-L)}}\,\rho_1 \leq e^{-\alpha_{1(l-L)}}\,e^{-\alpha_{0(l-L)}}\rho_0 = e^{-(l-L)(a_0 + a_1)}\,\rho_0$$

$$\rho_3 \leq e^{-\alpha_{1(1-L)}}\,\rho_2 \leq e^{-\alpha_{1(l-L)}}e^{-(l-L)(a_0 + a_1)}\,\rho_0$$

$$= e^{-(l-L)(a_0+ a_1 + a_2)} \rho_0$$

.

$$\rho_{n+1} \le e^{-\alpha_0 (1-L)} \rho_n \le e^{-(1-L)\sum_{i=0}^{n} \alpha_i} \rho_0$$

Therefore, from (b) in (i),

$$\sum_{n=0}^{\infty} \alpha_n = \infty$$

This implies that $\rho_{n+1} \to y^*$ as $n \to 0$.

3.4.2 Application of the New Scheme

Example 3.4.2.1

Solve the problem

$$y'^{v} + y'' = 2, \quad 0 \le x \le 1$$

subject to the boundary conditions

$$y(0) = y^{II}(0) = y^{III}\left(\tfrac{1}{2}\right) = 0; \quad y(1) = 1.$$

The exact solution is given by:

$$y_E(x) = 2\cos x + 1.09208262\sin(x) + x^2 - 2$$

Solution:

Consider,

$$y'^{v}_{n+1} + y''_n = 2, \quad n = 0,1,2,. \ . \ . \ .$$

Let, $y'' = 0$

Then

$$y_1^{iv} = 2$$

Therefore

$$y_1(x) = \frac{x^4}{12} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$

Using the boundary conditions, we have

$$y_1(x) = 1.08333333337x - 0.1666666667x^3 + 0.08333333333x^4$$

Then,

$$y_1(1/2) = 0.5260416665$$

Again,

$$y_2^{iv}(x) = -y_1'' + 2$$

$$y_2^{iv}(x) = 2 + x - x^2$$

Integrating successively four times, we obtain:

$$y_2(x) = \frac{x^5}{120} - \frac{x^5}{120} + \frac{x^4}{12} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$

Using the boundary conditions, we have,

$$y_2(x) = 1.091666667x - 0.180555556x^3 + 0.08333333333x^4$$

$$+ 0.008333333333x^5 - 0.002777777778x^6$$

$$y_4\left(\tfrac{1}{2}\right) = 0.528958871$$

Example 3.4.2.2

Solve the problem

$$y^{iv} - y'' = -12x^2, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0; \; y(1) = 13$$

The exact solution is given by:

$$y_E(x) = 24 + \frac{12(e-1)x}{e^{1/2}} - \frac{\left(24+12e^{1/2}\right)e^{x}}{e+1} + \frac{\left(12e^{1/2}-24e\right)e^{-x}}{e+1} + x^4 + 12x^2$$

Solution:

Consider the scheme,

$$y_{n+1}^{iv} = y_n'' - 12x^2, \quad n = 0,1,2. . .$$

Let, $y''= 0$,

Then,

$$y_1^{iv}(x) = -12x^2$$

Successive integration gives:

$$y_1(x) = -\frac{x^6}{30} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$

Using the boundary conditions we have,

$$y_1(x) = 12.95x - 0.1820811505x^3 + 0.08333333333x^3 - 0.03333333333x^6$$

Then,

$$y_1\left(\tfrac{1}{2}\right) = 6.4640625$$

Further application of the method, gives:

$$y_2(x) = 12.95580357x + 0.07395833333x^3 + 0.004166666667x^5$$

$$- 0.03333333333x^6 - 0.0005952380952x^8$$

Then,

$$y_2\left(\tfrac{1}{2}\right) = 6.486753628$$

Repeating the process for $n = 2,3,4$, we obtained respectively:

$$y_3(x) = 12.95472264x + 0.07491939484x^3 + 0.003697916667x^5$$

$$- 0.03333333333x^6 - 0.0005952380943x^7 - 0.0005952380952x^8$$

$$- 0.000006613756612x^{10}$$

$$y_4(x) = 12.95527134x + 0.0748218893x^3 + 0.00374569742x^5$$

$$- 0.03333333333x^6 + 0.00008804563492x^7 - 0.0005952380952x^8$$

$$+ 0.0000013778659598x^9 - 0.000006613756613x^{10}$$

$$- 0.00000005010421676x^{12}$$

$$y_5(x) = 12.95547436x + 0.07462447937x^3 + 0.003745969742x^5$$

$$- 0.03333333333x^6 + 0.00008918975575x^7 - 0.0005952380952x^8$$

$$+ 0.000001222856041x^9 - 0.000006613756613x^{10}$$

$$+ 0.00000001252605417x^{11} - 0.00000005010421676x^{12}$$

$$- 0.0000000002752978943x^{14}$$

Check:

$$y_E\left(\tfrac{1}{2}\right) = 6.486584557$$

$$y_4\left(\tfrac{1}{2}\right) = 6.486582992$$

$$y_5\left(\tfrac{1}{2}\right) = 6.486659836$$

Example 3.4.2.3

Solve the problem

$$y^{iv} - y'' = -2, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y'''(0) = y''(1/2) = 0; \quad y(1) = 1$$

The exact solution is given by:

$$y_E(x) = \frac{4e^{1/2}}{1+e} + \frac{\left(2e^{3/2} + 2e^{-1/2} - 4e^{1/2}\right)x}{1+e} + x^2 - \frac{2e^{1/2}\left(e^{-x} + e^{x}\right)}{1+e}$$

Solution:

Let consider the scheme,

$$y_{n+1}^{iv} = y_n'' - 2, \quad n = 0,1,2, \ldots \quad y_{n+1}^{iv} = y_n'' - 2, \quad n = 0,1,2, \ldots$$

Let,

$$y_1'' = 3x^2 - 2x^3 ,$$

Integrating successively four times, we have:

$$y_2(x) = -\frac{x^5}{10} + \frac{x^4}{6} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$

Using the boundary conditions, we have,

$$y_2(x) = 1.058333333\,x - 0.125x^2 - 0.1666666667x^4 - 0.1x^5$$

Then

$$y_2\left(\tfrac{1}{2}\right) = 0.5052083334$$

Further iterations, give:

$$y_3(x) = 0.9535962302\,x + 0.2739583333x^2 - 0.09375x^4 + 0.005555555556x^6$$

$$- 0.002380952381x^7$$

$$y_4(x) = 0.9572453702\,x + 0.221546379x^2 - 0.003125x^6$$

$$+ 0.0000992063492x^8 - 0.00003306878307x^9$$

$$y_5(x) = 0.9631294622\,x + 0.113425054x^2 - 0.07410223421x^4$$

$$- 0.002397280093x^6 - 0.00005580357143x^8$$

$$+ 0.00000110229277x^{10} - 0.0000003006253006x^{11}$$

$$y_6(x) = 0.9632383423\,x + 0.1131563999x^2 - 0.0738812455x^4$$

$$- 0.002470074474x^6 - 0.00004280857309x^8$$

$$- 0.0000006200396825x^{10} + 0.000000008350702802x^{12}$$

$$- 0.00000000192708526x^{13}$$

Check:

$$y_E\left(\tfrac{1}{2}\right) = 0.505251934$$

$$y_5\left(\tfrac{1}{2}\right) = 0.5052519304$$

$$y_6\left(\tfrac{1}{2}\right) = 0.5052519305$$

Example 3.4.2.4

Solve the problem

$$y'^v - y'' = -6x, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0; y(1) = 1$$

The exact solution is given by:

$$y_E(x) = 6x\left(e^{\frac{1}{2}} - e^{-\frac{1}{2}}\right) + \frac{6\left(e^{-x} - e^{x}\right)}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} + x^3$$

Solution

Let,

$$y_1(x) = 3x^2 - 2x^3 - x^4 - x^5$$

Substituting this into the differential equation, we have:

Therefore,

$$y_2^{\prime v}(x) = 6 - 18x - 12x^2 - 20x^3$$

Integrating successively, we obtain:

$$y_2(x) = -\frac{x^7}{42} - \frac{x^6}{30} - \frac{3x^5}{20} + \frac{x^4}{4} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} c_3x + c_4$$

Using the boundary conditions, we have,

$$y_2(x) = 0.9467261905x + 0.01041666667x^3 + 0.25x^4 - 0.15x^5$$

$$- 0.3333333333x^6 - 0.0238095238x^7$$

Then,

$$y_2\left(\tfrac{1}{2}\right) = 0.4848958333$$

Following the same process, we obtain:

$$y_3(x) = 0.93349041x + 0.1121527778x^3 - 0.04947916667x^5$$

$$+ 0.008333333333x^6 - 0.003571428571x^7 - 0.00059238095x^8$$

$$- 0.0003306878307x^9$$

$$y_4(x) = 0.9321100819x + 0.1133707683x^3 - 0.04439236112x^5$$

$$- 0.001178075397x^7 + 0.0001488095238x^8$$

$$- 0.0000496031746x^9 - 0.000006613756613x^{10}$$

$$- 0.000003006253006x^{11}$$

$$y_5(x) = 0.9322426523x + 0.1131609989x^3 - 0.04433146158x^5$$

$$- 0.001056960979x^7 - 0.00001636215829x^9$$

$$+ 0.000001653439153x^{10} - 0.0000004509379509x^{11}$$

$$- 0.00000005010421677x^{12} - 0.0000000192708526x^{13}$$

Check:

$$y_E\left(\frac{1}{2}\right) = 0.47886889$$

$$y_4\left(\frac{1}{2}\right) = 0.4790789336$$

$$y_5\left(\frac{1}{2}\right) = 0.4788728018$$

CHAPTER FOUR

4.0 RESULTS

4.1 Numerical Analysis and Results

In this chapter, Collocation method, Standard Galerkin method and exact solution will be compare with the Fixed- Point Iteration Method directly.

Experiment I:

Comparison with Collocation method.

Let consider,

$$y'^{v} + y'' = 2, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0; \quad y(1) = 1$$

The exact solution is given by:

$$y_E(x) = 2\cos x + 1.09208262\sin(x) + x^2 - 2$$

Table 4.1 gives summary result of the above problem when fixed point iteration method is compared with collocation method at $0 \le x \le 1$.

Table 4.1 For experiment $y'^{v} + y'' = 2,\ 0 \leq x \leq 1;$

$$y(x) = y''(x) = y'''\left(\frac{1}{2}\right) = 0, y(1) = 1$$

X	$Y_C(x)$	$Y_N(x)$	$Y_E(x)$	$\lvert Y_C(x) - Y_E(x)\rvert$	$\lvert Y_N(x) - Y_E(x)\rvert$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1091	0.1091	0.1090	1.6221×10^{-4}	4.1288×10^{-5}
0.2	0.2174	0.2172	0.2171	3.0320×10^{-4}	8.2853×10^{-5}
0.3	0.3238	0.3235	0.3234	4.1641×10^{-4}	1.2615×10^{-4}
0.4	0.4279	0.4276	0.4274	5.0101×10^{-4}	1.7056×10^{-4}
0.5	0.5293	0.5290	0.5287	5.5959×10^{-4}	2.1661×10^{-4}
0.6	0.6279	0.6276	0.6273	5.9255×10^{-4}	2.7113×10^{-4}
0.7	0.7238	0.7235	0.7232	5.9856×10^{-4}	3.1122×10^{-4}
0.8	0.8174	0.8172	0.8168	5.7446×10^{-4}	3.5800×10^{-4}
0.9	0.9092	0.9091	0.9087	5.1923×10^{-4}	4.0186×10^{-4}
1.0	1	1	0.9996	4.3955×10^{-4}	4.3955×10^{-4}

Keys:

$Y_E(x)$ represents the exact solution.

$Y_C(x)$ represents the collocation method.

$Y_N(x)$ represents the approximate for the new scheme, y_{n+1}, $n=3$.

Similarly, consider,

$$y^{iv} - y'' = -12x^2, \quad 0 \leq x \leq 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0, \; y(1) = 13$$

The exact solution is given by:

$$y_E(x) = 24 + \frac{12(e-1)x}{e^{1/2}} - \frac{\left(24+12e^{1/2}\right)e^{x}}{e+1} + \frac{\left(12e^{1/2}-24e\right)e^{-x}}{e+1} + x^4 + 12x^2$$

Table 4.2 gives summary result of the above problem when fixed point iteration method is compared with collocation method at $0 \leq x \leq 1$.

Table 4.2: For experiment $y'^{v} - y'' = -12x^2 \ , \ 0 \le x \le 1$

$$y(0) = y''(0) = y'''\left(\frac{1}{2}\right) = 13$$

X	$Y_C(x)$	$Y_N(x)$	$Y_E(x)$	$\lvert Y_C(x) - Y_E(x)\rvert$	$\lvert Y_N(x) - Y_E(x)\rvert$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	1.2985	1.2956	1.2960	2.8637×10^{-3}	1.23×10^{-7}
0.2	2.5971	2.5917	2.5917	5.4126×10^{-3}	2.97×10^{-7}
0.3	3.8961	3.8886	3.8886	7.4789×10^{-3}	5.7×10^{-7}
0.4	5.1958	5.1868	5.1868	8.9747×10^{-3}	9.85×10^{-7}
0.5	6.4964	6.4866	6.4866	9.8288×10^{-3}	1.565×10^{-6}
0.6	7.7980	7.7881	7.7881	9.9374×10^{-3}	2.311×10^{-6}
0.7	9.0992	9.0910	9.0910	8.1576×10^{-3}	3.223×10^{-6}
0.8	10.4023	10.3949	10.3949	7.3566×10^{-3}	9.4×10^{-6}
0.9	11.7029	11.6986	11.6817	2.1188×10^{-2}	1.6870×10^{-2}
1.0	1	12.99999	13	12	6.61×10^{-6}

Keys:

$Y_E(x)$ represents the exact solution.

$Y_C(x)$ represents the collocation method.

$Y_N(x)$ represents the approximate for the new scheme, y_{n+1}, n = 3.

Experiment II:

Comparison with Standard Galerkin Method:

Let consider,

$$y^{iv} + y'' = 2, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0; \quad y(1) = 1$$

The exact solution is given by:

$$y_E(x) = 2\cos x + 1.09208262\sin(x) + x^2 - 2$$

Table 4.3 gives summary result of the above problem when fixed point iteration method is compared with standard Galerkin method at $0 \le x \le 1$.

Table 4.3: For experiment $y^{iv} + y'' = 2,\ 0 \leq x \leq 1;$

$$y(x) = y''(x) = y'''\left(\frac{1}{2}\right),\ y(1) = 1$$

X	$Y_G(x)$	$Y_N(x)$	$Y_E(x)$	$\lvert Y_G(x) - Y_E(x)\rvert$	$\lvert Y_N(x) - Y_E(x)\rvert$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1232	0.1091	0.10903	1.4151×10^{-2}	4.1288×10^{-5}
0.2	0.2452	0.2172	0.2171	2.8104×10^{-2}	8.2853×10^{-5}
0.3	0.3648	0.3235	0.3234	4.1359×10^{-2}	1.2615×10^{-4}
0.4	0.4805	0.4276	0.4274	5.3075×10^{-2}	1.7056×10^{-4}
0.5	0.5908	0.52895	0.52874	6.2069×10^{-2}	2.1661×10^{-4}
0.6	0.6941	0.6276	0.6273	6.2069×10^{-2}	2.7113×10^{-4}
0.7	0.7886	0.7235	0.7232	6.5415×10^{-2}	3.1122×10^{-4}
0.8	0.8725	0.8172	0.8168	5.5661×10^{-2}	3.5801×10^{-4}
0.9	0.9436	0.9091	0.9087	3.4972×10^{-2}	4.0186×10^{-4}
1.0	0.5	1	0.9996	4.9956×10^{-2}	4.3955×10^{-4}

Keys:

$Y_E(x)$ represents the exact solution.

$Y_G(x)$ represents the Galerkin method.

$Y_N(x)$ represents the approximate for the new scheme, y_{n+1}, n = 3

Similarly, let consider another example as,

$$y^{iv} - y'' = -12x^2 \ , \ 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0, y(1) = 13$$

The exact solution is given by:

$$y_E(x) = 24 + \frac{12(e-1)x}{e^{1/2}} - \frac{\left(24 + 12e^{1/2}\right)e^{x}}{e+1} + \frac{\left(12e^{1/2} - 24e\right)e^{-x}}{e+1} + x^4 + 12x^2$$

Table gives 4.4 summary result of the above problem when fixed point iteration method compare with standard Galerkin method at $0 \le x \le 1$.

Table 4.4: For experiment $y^{iv} - y'' = -12x^2,\ 0 \leq x \leq 1$

$$y(0) = y''(0) = y'''\left(\frac{1}{2}\right) = 0,\ y(1) = 13$$

X	$Y_G(x)$	$Y_N(x)$	$Y_E(x)$	$\lvert Y_G(x)-Y_E(x)\rvert$	$\lvert Y_N(x)-Y_E(x)\rvert$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	1.1747	1.2956	1.2960	1.2094×10^{-1}	1.23×10^{-7}
0.2	2.3571	2.5917	2.5917	2.3459×10^{-1}	2.97×10^{-7}
0.3	3.5548	3.8886	3.8886	3.3383×10^{-1}	5.7×10^{-7}
0.4	4.7753	5.1868	5.1868	4.1149×10^{-1}	9.85×10^{-7}
0.5	6.0262	6.4866	6.4866	4.6031×10^{-1}	1.565×10^{-6}
0.6	7.3152	7.7881	7.7881	4.7289×10^{-1}	2.311×10^{-6}
0.7	8.6495	9.0910	9.09103	4.4154×10^{-1}	3.223×10^{-6}
0.8	10. 0368	10.3949	10.3949	3.5816×10^{-1}	9.4×10^{-6}
0.9	11.4844	11.6986	11.6817	1.9725×10^{-1}	1.687×10^{-2}
1.0	13	12.9999	13	0.00	6.61×10^{-6}

<u>Keys:</u>

$Y_E(x)$ represents the exact solution.

$Y_G(x)$ represents the Galerkin method.

$Y_N(x)$ represents the approximate for the new scheme, y_{n+1}, $n = 3$.

Experiment lll:

Comparison with Exact Solution.

Let consider,

$$y^{iv} - y'' = -2, \quad 0 \leq x \leq 1$$

Subject to the boundary conditions

$$(y) = y'''(0) = y''(1/2) = 0; y(1) = 1$$

The exact solution is given by:

$$y_E(x) = \frac{4e^{1/2}}{1+e} + \frac{\left(2e^{3/2} + 2e^{-1/2} - 4e^{1/2}\right)x}{1+e} + x^2 - \frac{2e^{1/2}\left(e^{-x} + e^{x}\right)}{1+e}$$

Table 4.5 gives summary result of the above problem when fixed point iteration method is compared with exact solution at $0 \leq x \leq 1$.

Table 4.5: For experiment $y^{iv} - y'' = -2, \quad 0 \le x \le 1$

$$y(0) = y'''(0) = y''(\tfrac{1}{2}) = 0; \; y(1) = 1$$

X	$Y_N(x)$	$Y_E(x)$	$\lvert Y_N(x)-Y_E(x)\rvert$
0.0	0.0	0.0	0.0
0.1	0.0974	0.0974	7.5663×10^{-7}
0.2	0.1971	0.1971	1.0473×10^{-6}
0.3	0.2986	0.2986	9.436×10^{-7}
0.4	0.4015	0.4015	5.465×10^{-7}
0.5	0.5053	0.5053	3.5×10^{-9}
0.6	0.6099	0.6090	5.466×10^{-7}
0.7	0.7117	0.7117	6.115×10^{-7}
0.8	0.8121	0.8121	1.0468×10^{-6}
0.9	0.9088	0.9088	7.565×10^{-7}
1.0	1.0	1.0	0.0

Keys:

$Y_E(x)$ represents exact solution.

$Y_N(x)$ represents approximate solution for new scheme, y_{n+1}, $n = 5$.

Similarly, let consider,

$$y^{iv} - y'' = -6x, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$$y(0) = y''(0) = y'''(1/2) = 0; \quad y(1) = 1$$

The exact solution is given by

$$y_E(x) = 6x\left(e^{1/2} - e^{-1/2}\right) + \frac{6\left(e^{-x} - e^{x}\right)}{e^{1/2} + e^{-1/2}} + x^3$$

Table 4.6 gives summary result of the above problem when fixed point iteration method is compared with exact solution at $0 \le x \le 1$.

Table 4.6: For experiment $y'^{v} - y'' = -6x, \qquad 0 \le x \le 0$

$$y(0) = y''(0) = y'''(\tfrac{1}{2}) = 0; \; y(1) = 1$$

X	$Y_N(x)$	$Y_E(x)$	$\lvert Y_N(x) - Y_E(x) \rvert$
0.0	0.0	0.0	0.0
0.1	0.0933	0.0933	1.2083×10^{-7}
0.2	0.1873	0.1873	2.2996×10^{-6}
0.3	0.2826	0.2826	3.1678×10^{-6}
0.4	0.3797	0.3797	3.7244×10^{-6}
0.5	0.4789	0.4789	3.9118×10^{-6}
0.6	0.5803	0.5803	3.7213×10^{-6}
0.7	0.6838	0.6838	3.1648×10^{-6}
0.8	0.78898	0.78898	2.2966×10^{-6}
0.9	0.8948	0.8948	1.2089×10^{-6}
1.0	1.0	1.0	3.3911×10^{-5}

Keys:

$Y_E(x)$ represents exact solution.

$Y_N(x)$ represents approximate solution for new scheme, y_{n+1}, $n = 4$.

CHAPTER FIVE

5.0 DISCUSSION OF RESULTS, CONCLUSION AND RECOMMENDATION

5.1 Discussion of Results

The accuracy and convergence of the method are of great importance in a numerical experiment as this. Accuracy measures the degree of closeness of the numerical solution to the theoretical solution while convergence measures the even closer approach of successive iterations to the exact solution as the number of iteration increased. To assess the success of our method, the scheme was tested with some numerical examples whose results presented as tables were discussed as follows:

Tables 4.1 – 4.2: These tables show the comparison with collocation method as well as, exact method and fixed-point approximate solution. It is to be noted that even with three iterations, the order of the error is at least five and seven respectively, which indicates fast rate of convergence. Again, as the iteration proceeds, the error decreases and convergence is achieved.

Tables 4.3 – 4.4: Show the results from the Galerkin method as well as, exact method and approximate solutions obtained by fixed-point iterative method. Again, with only three iterations, the order of the error is at least five and seven respectively, which indicate fast rate of convergence. Again, as the iterations proceeds, the error decreases and convergence is assured.

Tables 4.5 – 4.6: Show the exact values and approximate values obtained by fixed-point iterative method. Here, the 5^{th} and 4^{th} iteration results respectively were used in evaluating the approximate solution. The order of the error as shown in the table is least seven, which is very high.

5.1.1 Summary of Results

The proposed method competes favourably with other methods like Collocation and Galerkin. Numerical experiments of the fixed-point iterative method for the solution of three-point boundary value problems were carried out. Results obtained for the problems solved by our present method were tabulated, and accuracy and fast convergence were achieved.

5.2 Conclusion

From the numerical experiments carried out and summary of tables shown, we can conclude that the fixed-point iterative method is very powerful and it is an efficient technique for finding approximate solutions to both linear and nonlinear boundary value problems. The method is more accurate and converge fester than many well-known methods in use.

5.3 Recommendation

Research in Numerical Analysis is a continuous process. Thus, future research work can be carried out in the following area:

The choice of y_1 is a major factor. It is recommended that how y_1 is to be chosen to obtain faster convergence of the scheme is a factor to be worked upon. That is to say more work should be done on how to choose the starting value y_1.

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