AN OPTIMAL CONTROL PROBLEM OF ENVIRONMENTAL POLLUTION WITH UNCONTROLLABLE SOURCES

BY

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CERTIFICATION

This thesis titled "AN OPTIMAL CONTROL OF ENVIRONMENTAL POLLUTION WITH UNCONTROLLABLE SOURCES" by Ogwuche Otache Innocent, meets the regulations governing the award of the degree of Masters of Technology in Mathematics, Federal University of Technology Minna and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to the memory of my late Brother, **Igoche Ogwuche** and to my good friend **Sis. Rose Odeje** who I meet through God's leading at the course of this work.

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A good number of people contributed to the success of this programme to which I remain indebted.

My thanks go to God who made this programme possible in the face of all odds. To Him be praise forever Amen.

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May God reward all that contributed to this programme greatly, Amen.

ABSTRACT

We treat the problem of the control of environmental pollution with uncontrollable sources as a semi-infinite optimization problem with a system of linear constraints and developed a computer program to compute the solution of the problem taking cognizance of the relationship between approximation and optimization.

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CHAPTER ONE

BASIC CONCEPT OF OPTIMAL CONTROL.

1.1 INTRODUCTION

Optimal control is a part of mathematics in which a study is made of ways of formalising and solving problems of choosing the best way of realising the control of dynamical process. This dynamical process can be described using differential, integral functional and finite difference equation or other formalised relation depending on the input function called the control and usually subject to constraints.

The term 'theory of optimal control' is applied to mathematics theory in which methods are studied for solving non-classical variational problem of optimal control (as a rule, with differential constraint), which permit the examination of non-smooth functional and arbitrary constraint on the control parameter or on other dependent variable.

The concept of mathematical theory of optimal control is sometimes used in a broader sense to cover the theory which studies mathematical method of investigating problem whose solution include a process of statistical dynamical optimisation, while the corresponding model situation permits interpretation in terms of some applied procedure for adopting an optimal solution. Mathematical theory of optimal control therefore contains an element of operation research, mathematical programming and game theory.

Although particular problem of optimal control and non-classical variational problem were encountered earlier, the foundations of the general mathematical theory of optimal control were laid in 1956 - 1961. The key point of the theory was the Pontryagin maximum principle formulated by LS Pontryagin in 1956. The main stimuli in the formulation of this theory were the discovery of the theory of Dynamic Programming, the explanation of the role of functional analysis in the theory of optimal system, the discovery of the relationship between solutions of problems of optimal control and result of the theory of the Lyapunov stability and appearance of works relating to the concept of controllability and observability of dynamical systems.

The result of the mathematical theory of optimal control has found broad applications in the construction of control process relating to diverse process of modern technology in the study of economics, dynamics and in the solutions of problems in the field of biology, medicine, ecology, demography etc.

Problems of optimal control can be described in general term in any of the following ways:

1. A controllable system S whose position at the instant of a time t is represented by a value x e.g. by vector of a generalised coordinate and impulse of a mechanical system, or by a function in the spital coordinate of a distributed system; by a probability distribution which characterises the current state of the stochastic system or by a vector of production output in a dynamic model of an economy etc.
2. Optimal control problem can also be described by an equation, which connects the variable x, u, t and describe the dynamics of a system. An instant of time is indicated in which the system is considered. Foe example, the ordinary differential equation of the form:

$$X^{I} = f(t, x, u),\ t_0 \leq t \leq t_1\ x\varepsilon R^n,\ u\varepsilon R^{m.} \qquad 1.1.0$$

with previously stipulated properties of the function (e.g. continuity of f in u, t, differentiability are often required).

3. Optimal control problem can further be described if information is available which can be used to construct the controls (e.g. at any instant time or at previously described instances). A class of function describing the control which can be considered is stipulated, e.g. a set of pair-wise continuous function in x of the form:

$U = U(x, t) = p^{T}(t)$ 1.1.1

with continuous coefficient etc.

4 Optimal control problems can also be defined by imposing constraint on a process to be realised. At this point in particular, the conditions defining the aim of the control comes into consideration (e.g. a system to hit a given set of the phase space R^{n}, the demand for stabilisation of the solution around a given motion etc.). Furthermore constraint can be imposed on the value of the controls U or the coordinates of the position x in the variable or functional in their realisation etc. In the system 1.1.1 for example, constraint on the control parameters

$u \in U \leq R^{P}$ or $\phi(u)$; $\phi : R^{P} \rightarrow R^{k}$ 1.1.3

and on the coordinates

$x \in X \subseteq R^{n}$ or $\varphi(x) \leq 0, R^{P} \rightarrow R^{1}$ 1.1.4

are possible. Here, U, X are closed set ϕ, φ are differentiable functions.

5. An index (a criterion) is given of the quantity of the process to be realised. It can take the form of a function J (x (.), u (.)) in the realisation of the variable x, u over the period of time under consideration. Conditions 1 - 4 above can then be supplemented by the requirement of the optimality process (i.e., the minimum or maximum) of the criterion J (x (.), u (.)). In this way, for a given class of control for a system, a control u must be chosen which optimises the index J(x (.), u (.)) such that the aim of the control and the constraint are both satisfied.

1.2 THE THEORY OF OPTIMAL CONTROL

The basic concept of optimal control problem is that of finding the control vector $U = (u_1, u_2, \ldots, u_n)^T$ which maximises the functional called the performance index (criterion)

$$J = \int_{t_0}^{t_1} f_0 (x, u, t) \, dt \text{ where}$$

$X = (x_1, x_2, \ldots, x_n)^T$ is called the state vector, t is the time parameter and f_0 is the function of x, u and t. The state variable x_i and the control variable are related as follows:

$$\frac{d x_i}{dt} = f_i (x_1, x_2 \ldots x_n \, u_i, u_i, \ldots u_n)$$

$$X' = f (x, u, t).$$

In many problems, the system is linear and $X' = f (x, u, t)$ can be stated as follows:

$X' = Ax + Bu$ where A is an nxn matrix and B is an nxm matrix. In finding the control vector u, the state vector is to be transformed from a known initial vector x_0 at $t = 0$ to a terminal vector at $t = T$ where some (all or non) of the state variable are specified.

1.3 NECESSARY CONDITIONS FOR OPTIMAL CONTROL

In order to derive the generalised necessary condition for optimal control, we consider the following specific problem:

Find u, which maximise

$$J = \int_{to}^{t1} [f_0 (x, u, t) \, dt \qquad 1.3.1$$

Subject to

$$X = f(x, u, t)$$

With the boundary condition x (0) = k. Let λ be the Langrange multiplier. Let

$$J^* = \int_{to}^{t1} \{f_0 (x, u, t) + + \lambda[f (x, u, t) - x']\} \, dt \qquad 1.3.2$$

Now, the integrand

$$F = f_0 + \lambda f (-x) \qquad 1.3.3$$

is a function of two variable x and u. The Euler -Langrange equation with

$$u_1 = x, u_1' = \frac{\partial x}{\partial t} = x'$$

$$u_2 = u \text{ and } u_2 = \frac{\partial u}{\partial t} = x' \qquad 1.3.4$$

as

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \cdot \frac{\partial F}{\partial x} = 0 \qquad 1.3.5$$

$$\frac{\partial F}{\partial u} - \frac{d}{dt} \cdot \frac{\partial F}{\partial u} = 0 \qquad 1.3.6$$

In view of equation 1.3.4, equation 1.3.5 and 1.5.6 can be expressed as

$$\partial F/\partial x + \lambda \, \partial F/\partial u + \lambda' = 0 \qquad 1.3.7$$

$$\partial F/\partial u - \lambda \partial F/\partial u = 0 \qquad 1.3.8$$

Now, let H be the Hamiltonian function defined as $H = f_0 + M$

Then equation 1.3.7 and 1.3.8 can be written as

$$\frac{\partial H}{\partial x} = \lambda'$$

$$1.3.9$$

$$\frac{\partial H}{\partial x} = 0 \qquad 1.3.10$$

Equation 1.3.9 and 1.3.10 represent two first order differential equations the integration of whose values can be found from the known boundary condition of the problem. If two boundary conditions are specified as x (0) = k and x (T) = kT, the two integration constant can be evaluated without any difficulty. On the other hand, if only one boundary condition is specified as say, x (0)= k, the free end condition is used as

$$\frac{\partial H}{\partial x.} = 0 \text{ or } \lambda = 0 \text{ at } t = T$$

This specific approach can now be used to derive the general necessary condition for optimal control problems.

A general optimal control problem can be stated as:

$$\int_{to}^{t1} f_0 + (\vec{x}, \vec{u}, t)\, dt \qquad 1.3.11$$

Subject to

$$X_i = f_i \quad (\vec{x}, \vec{u}, t), I = 1,2,3 \ldots n. \qquad 1.3.12$$

Now let p_i be Langrange multiplier also known as the adjoint variable for the i^{tn} constraint equation in 1.3.12 above. Then J* an augmented functional can be defined as

$$J^* = \int_{to}^{t1} f_0 + \sum_{I=0}^{n} p_i (f_i - x_i)\, dt \qquad 1.3.13$$

The Hamiltonian functional H is defined as

$$H = f_0 + \sum_{i=0}^{n} p_i f_i \qquad 1.3.14$$

Such that

$$J = \int_{to}^{t1} (H - \sum_{I=0}^{n} p_i (f_i)\, dt \qquad 1.3.15$$

Since the integrand

$F = H - \sum_{i=0}^{m} p_i f_i$ depends on x, u, t, there are m + n dependent variables (x and u)

and hence, the Euler Langrange equation becomes

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\frac{F}{\partial x} = 0, I = 1,2 \ldots n. \qquad 1.3.17$$

$$\frac{\partial F}{\partial u_j} - \frac{d}{dt} \cdot \frac{F}{\partial u_j} = 0; j = 1,2 \ldots n. \qquad 1.3.18$$

In view of the relation 1.3.16 above, equation 1.3.17 and 1.3.18 can be re-written as

$$- \frac{\partial H}{\partial x_i} = p_i, I=1,2, \ldots n \qquad 1.3.19$$

$$\frac{\partial H}{\partial u_j} = 0, j=1,2, \ldots m \qquad 1.3.20$$

Equation 1.3.19 and 1.3.20 above are called the adjoint equation.

The optimum solution of x, u and p can be obtained by solving equation 1.3.7, 1.3.9 and 1.3.10. There are totally 2n +m equations with nx_i and mu_j unknown. If we now know the initial condition x_i (0), I= 1,2, ...n and the terminal condition x_j (T), j=1,2, ...m with l < m we will have the terminal values of the remaining variables, namely x_j (T), j = i+1, I+2, ...n free. Hence the free end condition will have to be

$$p_j (T) = 0, j = I+1, I+2, \ldots m \qquad 1.3.21$$

Equation 1.3.21 above is called transversality conditions.

Below is an example that involves the direct application of the theory.

EXAMPLE 1.3.1 (Application Of Optimal Control Theory)

Find the optimal control of u, which makes the functional

$J = \int (x^2 + u^2)\, dt$ stationary with

$x^{ll} = u$ and $x(0) = 1$

It is necessary to note that x is not specified at t.

Solution:

The Hamiltonian is defined as

$H = f_0 + \lambda u$

$\quad = x^2 + u^2 + \lambda u$

From equation 1.3.9 and 1.3.10, we have

$- 2x = \lambda^l$ (i)

$2u + \lambda = 0$ (ii)

Differentiating (I) above we have:

$2u^l = - \lambda^l$

Therefore, $\lambda^l = - 2u^l$

But from (i), $\lambda^l = - 2x$, hence, $x = u^l$

Since $x^{ll} = u$, we have that $x^{ll} = u^l = x$

$x^{ll} - x^l = 0$ (iii)

Equation (iii) has a solution of the form

$X(t) = c_1 \text{Sinh } t + c_2 \text{Cosh } t$ (iv)

where c_1 and c_2 are constants. By using the initial condition $x(0) = 1$, we obtain

$x(0) = c_2 = 1$

Since x is not fixed at the terminal point, $t = T = 1$. We use the condition $\lambda = 0$ at $t = 1$. Now, from

$x(t) = c_1 \text{Sinh } t + c_2 \text{Cosh } t$

$x^l(t) = c_1 \text{Cosh } t + c_2 \text{Sinh } t$

Hence, $u = x^l = x(t) = c_1 \text{Cosh } t + c_2 \text{Sinh } t$

$$u(1) = 0 = c_1 \text{ Cosh } 1 + \text{ Sinh } 1$$

So that c1 $= c_1 = \dfrac{\text{Sinh t}}{\text{Cosh 1}}$

and hence the optimal control is

$$u(t) = \frac{\text{Sinh t Cosht}}{\text{Cosh 1.}} + \text{Sinh t}$$

$$\frac{1.}{\text{Cosh 1}} [\text{Sinh 1Cosh t} + \text{Cosh 1 Sinh t}]$$

$$= \frac{- \text{Sinh } (1\text{-t})}{\text{Cosh t}}$$

The corresponding state trajectory is given by

$$x(t) = u^{l} = \frac{\text{Cosh } (1\text{-t})}{\text{Cosh t}}$$

We shall now state in a general way the general control problem.

1.4 CLASSICAL CONTROL PROBLEM

In order to define a classical control problem, we will impose some conditions on the function and sets. These conditions are conditions that are usually met when considering classical problems and also allows for modification of the classical problem into others that appears to have some advantages over classical formulations.

So, let X be a vector space in n - space R^n, U a vector space in m -space R^m and t a real variable. Consider:

i. a closed interval $J = [t_a, t_b]$ with $t_{a,} < t_b$. Let $J^0 \cdot J^0 = (t_{a,} t_b)$ be the interior of this interval, i.e., the time interval in which the control will evolve.

ii. a bonded , closed path-wise connected set A in R^n . The trajectory of the control system is constrained to be in this set for $t \in J$

iii. Two elements of A, x_a, x_b which are the initial and final state of the trajectory of the control system

iii. Two elements of A, x_a, x_b which are the initial and final state of the trajectory of the control system

iv. A bounded closed subset U of R^m, U is the set in which the control function takes values

v. Let φ = JxAxU, and g: $\varphi \rightarrow R^n$ a continuous function. We consider the differential equation

$$x^I(t) = g[t, x(t), u(t)], \; t \in J^0 \qquad 1.4.1$$

Where the trajectory $t \in J \rightarrow x(t) \in A$ is a function $t \in J \rightarrow u(t) \in U$ is Lebesque measurable. The differential equation describes the control system and must be satisfied in the sense of caratheodory.

vi. Let f_0: $\varphi \rightarrow R$ be continuous function where f_0 is the integrand of performance criterion for the problem.

A trajectory (control pair) is said to be admissible if the following conditions holds

i. X (.) (The trajectory function satisfy x $(t) \in A$, $t \in J$ and is absolutely continuous on J).

ii. U (.), the control function, takes value in the set U and is Lebesque measurable on J

iii. The boundary conditions $x(t_a) = x_a$, $x(t_b) = x_b$ are satisfied.

iv. The pair p satisfied the differential equation in the sense of caratheodory.

Now, let W denote the admissible pairs. A classical control problem does not have a solution unless W is non-empty.

Consider the functional I: W→R defined by:

$$I(p) = \int_j f_0[t, x(t), u(t)]\,dt \qquad 1.4.2$$

where P is the control pair i.e.

P = [x (.), u(.)]

Therefore the classical control problem seeks P that will maximise the functional over the set W.

To analyse the classical problem further, it is necessary to establish the characteristics of W, the set of the admissible pairs. Now, lets consider the boundary conditions of 1.4.1. Let p = [x (.), u (.)] be an admissible pair, and B an open ball in R^{n+1} containing J x A. Let c^1 (B) denote the space of real valued continuously differentiable functions on B such that they and their first derivative are bounded on B. Now, for $\omega \in B$, let $\omega^g (t, x, u) = \omega_x (t, x)\, g (t, x, u) + \omega_t (t, x)$ for all t, x, u $\in \omega$ where $\omega_x (t, x)$ and g (t, x, u) are vectors and ω^x is in the space C (ω) where ω = JxAxU. Since P = [x (.), u (.)] is an admissible pair.

$$\int_j \omega^g (t, x (t), u (t))\, dt = \int_j \omega_x \{[t, x (t)]\, x(t) + \omega_x [t, x (t)]\} \qquad 1.4.3$$

$$\int_j [\omega_x'(t)\, x (t)]\, dt = \omega (t_b , x_b) - \omega (t_a , x_a) \equiv \triangle \omega \qquad 1.4.4$$

for all $\omega \in c^1$ (B).

Note that it was necessary to introduce the set B and the space c^1 (B) because A may have an empty interior in R^n.

1.5 DEFINITION OF TERMS

Definition 1.5.1 (Mathematical programming)

A mathematical programme is an optimization problem subject to constraint in R^n of the form

Minimize f (x)

Subject to $g_i (x) \leq 0$, I = 1,2, …m, $s \in R^n$

The vector $x \in R^n$ has component $x_1\ x_2, \ldots x_n$ which are called the unknown of the problem. The function f is called the objective function also called the economic function and the set of conditions $g_i (x) \leq 0$, I = 1,2,.., m and $s \in S$ is the set of constraint of the problem.

Every vector x which satisfies the constraint such that $g_i (x) \leq 0$ and $x \in S$ is said to be the solution of the problem (p). We say that x* is a global solution of problem (p) if and only if there exist a neighbourhood v (x*) of x* such that x* is a global optimum of the problem

Minimize f (x)

Subject to $g_i (x) \leq 0$, I = 1,2, …m, $s \in R^n$

and $x \in S \cap V (x^*) R^n$

A mathematical programming problem is said to be convex if it comprise of minimizing a convex function (or maximizing a concave function) on a convex domain. Problem (p) above is a convex problem if

i. f is a convex function

ii. the function g_i, I= 1,2, …m are convex

iii. $s \in R^n$ is convex

Definition 1.5.2 (Interior point)

Let $S \subset R^n$ be a subset of R^n. Then we say that $y \in S$ is an interior point if there exist ε such that $| x - y| < \varepsilon$; $x \in S$. In other words there is a ball with centre y contained in S. The set of all interior points of S is called the interior of S.

Definition 1.5.3 (Convex Set)

If E is a vector space over R, then a subset C of R is said to be convex if x, y ∈C, $0 \leq \lambda \leq 1$

$\Rightarrow \lambda x + (1-\lambda)y \in C$ i.e. if the closed line segment connecting any two points in C also belongs to C. This line segment is denoted as

$[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$

For example:

i. the empty set is a convex set

ii. for any two points x, y ∈ E, the closed segment [x, y] and for any $x \neq y$, the open line segment

$[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1$ connecting x and y is convex.

Definition 1.5.4 (Convex and Concave Function)

The function f(x) is said to be Convex over a convex Set X in En if for any two points $x_1, x_2 \in E$ and for all $\lambda \in [0, 1]$,

$$f(\lambda x_2 + (1-\lambda)\, x_1\,) \leq \lambda f(x_2\,) + (1-\lambda)f(x_1) \qquad 1.5.1$$

As a special case, the function f(x) of the scalar x is convex in the domain X of x if $PN \leq QN$ in all triads A, N, B as shown in the figure below:

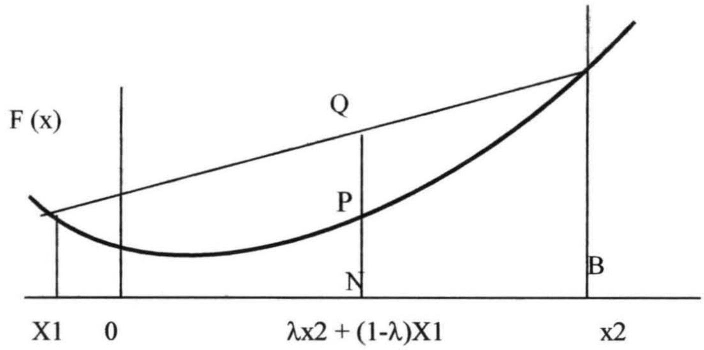


Figure 1.5.1(Convex function)

The function f(x)is said to be concave over the set $X \in E^n$ if for any two points $x_1, x_2 \in E$ and for all $\lambda \in [0, 1]$,

$$f(\lambda x_2 + (1-\lambda)\, x_1\,) \geq \lambda f(x_2\,) + (1-\lambda)f(x_1) \qquad 1.5.2$$

This is indicated in figure 1.5.2 below:

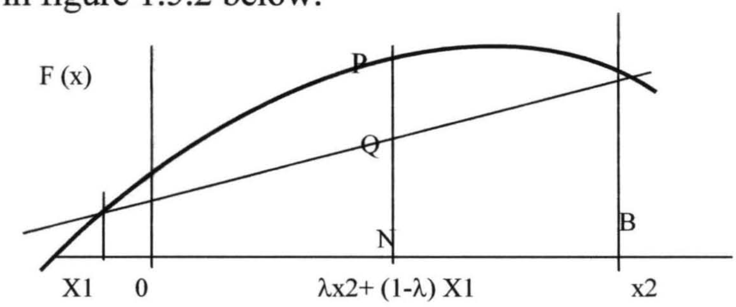


Figure 1.5.2 (Concave function)

The following are some elementary results of convex and concave functions of considerable importance.

i. If f(x) is convex, the - f(x) is concave and vice versa
ii. The linear function $Z = C^l x$ is both convex and concave throughout E^n
iii. A concave (convex) function has the property that its value at an interpolated points is greater than (less than) or equal to the value that would be obtained by linear interpolation
iv. The sum of a finite number of convex (concave) functions is itself a convex (concave) function.

The following theorems are vital.

Theorem 1.5.1

Let f(x) be a convex function over a closed convex set X in E^n . Then any local minimum of f(x)is also the global minimum of f(x) over X.

Proof:

The proof is by contradiction. Assume that f(x) takes on a local minimum at $x_l \in X$, that its global minimum is at $x^* \in X$ and that $f(x^*) < f(x_l)$.

Now,

$$\begin{aligned} f(\lambda x^* + (1-\lambda)\, x_l) &\leq \lambda f(x^*) + (1-\lambda) f(x_1) \\ &< \lambda f(x_1) + (1-\lambda) f(x_1) \\ &= f(x_1) \end{aligned} \qquad 1.5.3$$

for all $\lambda \in [0, 1]$. But for sufficiently small λ, the point

$x = \lambda x^* + (1-\lambda)x_1$ lies in the neighbourhood of x_1 and equation 1.5.3 then shows that $f(x) < f(x_1)$ in this neighbourhood, which contradicts the fact that f(x) has a local minimum at $x = x_1$. Thus x_1 and x^* cannot be distinct and that ends the proof.

Theorem 1.5.2

Let f (x) be a convex function over the closed convex X in E^n . Then the set of points at which f (x) takes on its global minimum is a convex set.

Proof:

The case where f (x) takes on its global minimum at a single point is trivial. Otherwise, suppose the global minimum is taken at x_1 and x_2 , $x_1 \neq x_2$ and let

$$x = (\lambda x_1 + (1-\lambda)\, x_1),\ \lambda \in [0, 1] \qquad 1.5.4$$

Then,

$$f(x) = f(\lambda x_2 + (1-\lambda)x_1) \leq \lambda f x_2) + (1-\lambda)\, f(x_1)$$

But $fx_2) = f(x_1) = f(x^*)$, the global minimum. Hence, f (x) = f(x*) for all points x defined by equation 1.5.3.

Definition 1.5.5 (Convex Cone)

Let E be a linear vector space over R. Let C be a subset of E such that C is convex. C is said to be a cone (with the vertex Θ_E, the zero element of E) if $x \in$ E

$\lambda > 0 \rightarrow \lambda x \in C$. The subset C of E is called a convex cone if C is a cone and is also convex.

Lemma1.5.1

The cone K is convex if and only if $x, y \in K \rightarrow x + y \in K$ 1.5.5

Proof:

1. Suppose K is a convex cone. Given $x, y \in K$, if follows that $½(x + y) = ½(x) + ½(y) \in K$ since K is convex and $x + y = 2(1/2(x + y)) \in K$ since K is a cone.

2. Suppose K is a cone satisfying equation 1.5.5. If $x, y \in K$ and $\lambda \in [0, 1]$ are given, then $\lambda x + (1-\lambda)y \in K$ since K is a cone.

Now equation 1.5.5 implies

$\lambda x + (1-\lambda)y \in K$.

If K is non-empty convex cone in E, then let $x \geq y$ ($\Leftrightarrow y \leq x$) $\Leftrightarrow x - y \in E$ be an ordering relation on E. Since $\Theta_E \in K$ we have (reflexivity).

Also, $x \geq , y \geq z \Rightarrow x \geq z$ (transivity).

Since $x \geq y$ ($\Leftrightarrow y \leq x$) $\Leftrightarrow x - y \in E$ gives $x - y \in E$ and $y - z \in K$ Lemma 1.5.1 above implies $x - z = (x - y) + (y - z) \in K$. Hence $x \geq \Theta_E$, $\lambda \geq 0 \Rightarrow \lambda x \geq \Theta_E$, $x \geq y$, $z \in E \Rightarrow x + z \geq y + z$. Hence the proof of the Lemma.

Definition 1.5.6 (Topological Dual space)

Suppose E is a unitary space. If $y \in E$, then setting $C(x) <x, y>$ for $x \in E$ a linear form is defined on E which satisfies

$|C(x)| \leq ||y|| \, ||x||$ for all $x \in E$. By Cauchy Schwarz inequality. This is continuous and $||c|| \leq ||y||$.

If $y = \Theta_E$, then C is the zero mapping and $||c|| = ||y|| = 0$. Otherwise, $||y|| > 0$ and $|C(z)| \leq ||c||$ for $z = y$ which implies $||y|| \leq ||c||$. Hence, $||y|| = ||c||$.

Definition 1.5.7 (Linear Mapping)

Let E and F be two normed vector spaces. A mapping:

$A: E \rightarrow F$ is said to be linear if

$A(\lambda x + \mu y) = \lambda A(x) + \mu A(y)$ for all $x, y \in E$ and $\lambda, \mu \in R$.

A mapping $A: E \rightarrow F$ is called continuous if $x_n \rightarrow x$, x_n, $x_n \in E \Rightarrow A(x_n) \rightarrow A(x)$. Below are some relvant theorems in linear mapping.

Theorem 1.5.3

A linear mapping A of E into f is continuous if and only if there is $\alpha \geq$ with

$||A(x)||_f \leq \alpha \, ||x||_E$ for all $x \in E$

Theorem 1.5.4

For every linear mapping AxF → G there are unique linear mappings $A_1 : E \rightarrow G$ and $A_2 : F \rightarrow F$ satisfying

$A(x, y) = A_1(x) + A_2(y)$ for all $x \in E, y \in F$.

For example, let E and F be two hilbert space and $A \in L(E, F)$. If for each fixed $y^* \in F^*$ we define

$h^*(x) = y^*(A(x))$ for all $x \in E$, then $h^* \in E^*$. Since E and F are hilbert space, there must be unique element $y \in F$, $h \in E$ with

$$<h, x>_E = h^*(x) = y^*(A(x)) = <y, A(x)_F \qquad 1.5.6$$

Since the mappings $y \rightarrow y^*$, A and h^* are continuous and linear, a continuous linear mapping $A^l: F \rightarrow E, y \rightarrow h$ is defined by equation 1.5.6 with

$A^*(y^*)(x) = y^*(A(x)) = <y, A(x)> = < A^l(y), x >_E$ for all $x \in E$.

Suppose E, F and G are linear vector spaces. The cartesian product E x F is also a linear vector space, with componentwise addition and scalar multiplication.

Definition 1.5.8 (Positive Cones and Covex Mappings)

Let P be a convex cone in a vector space X. For cx, $y \in X$, we write $x \geq y$ (with respect to P) if $x - y \in P$. The cone P defining this relation is called the positive cone in X. The cone N = -P is called the negative cone in X and we write $y \leq x$ for $y - x \in N$. For example, in E^n, the convex cone

$$P = \{ x \in E^n : x = k_1, k_2, \ldots k_n\} \; k_i \geq 0 \text{ for all I} \qquad 1.5.7$$

defines the ordinary positive orthart.

In a normed vector space it is important to define positive cone by closed convex cone. For example, in E^n, the cone defined 1.5.7 is closed. If one or more of the ineqaulities is changed to strict inequality, the resulting cone is not closed.

Let X be a vector space and let Z be a vector space having the cone spacified as a positive cone. A mapping G: X → Z is said to be convex if the domain φ of G is convex set and if G $(\alpha x_1 + (1-\alpha)x_1) \leq \alpha G(x_1) + (1-\alpha)G(x_1)$ for all x_1, $x_2 \in \varphi$ and ll α, $0 < \alpha < 1$.

Proposition 1.5.1

Leg G be a convex mapping as in the last definition. Then for every $z \in Z$, the set $\{x : x \in \varphi , G(x) \leq z \}$ is covex.

Definition 1.5.9 (Linear Manifold)

Let X be a normed vector space over R. A subset A of X is called a linear manifold if

$$x, y \in A, \lambda \in R \Rightarrow \lambda x + (1-\lambda)y \in A.$$

If x^*: X →R is non trivail form and $\alpha \in R$ then,

$$H = \{x \in X : x^*(x) = \alpha\} \qquad 1.5.8$$

is alinrear manifold and is called a *hyperplane*

Assertion:

It can be shown that every hyperplane is a maximal linear manifold, i.e. If A is a manifold, $A \supseteq H \Rightarrow A = H$ or $A = X$. Furthermore, a hyperplane H of the form of equation 1.5.8 is closed if x is continuous. In other words, a linear manifold H in X is maximal and closed if and only it is of the form 1.5.8 with $x^* \in X^*$, the topological dual of X and $x^* \neq \Theta_{x^*}$

CHAPTER TWO

LINEAR OPTIMIZATION IN FUNCTION SPACES

2.1 INTRODUCTION

It is worth starting with in this chapter to mention that there exists a close relationship between approximation and optimization.

The introduction of optimization theory to the discipline of mathematics about the time of the advent of computers brought about revolution to approximation theory.

The connection between the two (approximation and optimization) is even clearer in case of discrete approximation problem. Here, a given real value function **f** is to be approximated at finite number of points x_1, x_2 ... x_m of a set X (e.g., real numbers). Most of the times the given function values $f_i = f(x_i)$, i = 1,2,3...m are the result of measurement so that an expression of **f** in algebraic or analytic form is not known.

In the case of a linear approximation problem a finite dimensional vector space V is constructed consisting of linear combinations of the functions V_1 V_2.... V_n, whose values are easily computed and $v \in V$ is given by

$$V(t) = \sum_{I=1}^{n} x_i V_i(t), \text{ for } s \in S \text{ where } x_{i,} {}_{x2} \ldots x_n \qquad 2.1.1$$

are real numbers and $V(t_i)$, ...$V(t_m)$ are vectors in R^m

The main focus in this work however is the so-called semi-infinite optimization, which results directly to continuous approximation problems. In this case, infinitely many free parameters can occur. This leads to such an infinite optimization problem especially when one seeks to approximate real valued function f on a compact set S (e.g., real interval) of continuous functions on S as on S as well as possible in the sense of maximum norm.

This type of problem occurs, for example, when one seeks representation of function for evaluation with computer. It also occur in the approximate solution of boundary and initial boundary value problems for ordinary and partial differential equations as well as in other area of application e.g., the control of environmental pollution to which this volume is dedicated to.

2.2. THE GENERAL LINEAR OPTIMIZATION PROBLEM.

2.2.1 Statement of the problem and the weak Quality Theorem

Let E and F be two partially ordered normed Vector space whose order relation is denoted by

$\geq$ or $\leq$. Let K_E and K_F be the associated cones for the ordering. Let a continuous linear mapping defined by:

A: E → F: a continuous linear form

C: E → R and a fixed element be F be given. Then, the linear optimization problem can be stated as:

Maximize C (X) Subject to the side conditions.

$A(x) \geq b,\ x \geq Q_E$

$\Rightarrow A(x) - b \in K_f,\ x \in K_E.$ 2.2.1

To this problem, a dual problem can be associated. For the dual problem, we consider the mapping

A*: F → E* which is adjoint to A. We define the following problem:

Maximize Fb (y*) = y* (b) subject to the side conditions

$A^*(y^*) \leq C,\ y^* \geq q_F$

$A^*(y^*)(x) = y^*(A(x)) \leq C(x)$ for all $x \in K_E$ and

$y^*(y) \geq 0$ for all $Y_E\ K_F$ 2.2.2

Now, let

$M = \{x \in F: A(x) \geq b, x \geq q_f.$ 2.2.3

$N = \{y^* \in F^*: A^*(y^*) \leq C, y^* \geq q_{F^*}$ 2.2.4

let a be defined as

$$\alpha = \begin{cases} \inf_{x \in M} C(x), & \text{if M is man non-empty} \\ +\infty, & \text{if M is empty} \end{cases} \qquad 2.2.5$$

and
$$\beta = \begin{cases} \sup_{y^* \in N} y^*(b), & \text{if N is non empty} \\ -\infty, & \text{if N is empty} \end{cases} \qquad 2.2.6.$$

We then have the following theorem:

Theorem 2.2.1

α and β defined as:

$$\alpha = \begin{cases} \inf_{x \in M} C(x), & \text{if M is man non empty} \\ +\infty\ \infty, & \text{if M is empty} \end{cases}$$

and

$$\beta = \begin{cases} \sup_{y^* \in N} y^*(b), & \text{if N is non empty} \\ -\infty\ \infty, & \text{if N is empty} \end{cases}$$

satisfy $\beta \leq \alpha$ ... 2.2.7

Theorem 1.2.2

If x and y* are admissible, respectively, for the dual problem, and if C (x) = y* (β) holds, then x and y* are optimal and β = α The two theorems so stated above are referred to as the ***Weak Duality Theorem*** whose proofs are shown below:

Proof (Theorem 2.2.1)

If M or N is empty, the inequality in 2.2.7 follows immediately from the definitions of α and β in 2.2.3 and 2.2.4 above.

If M and N are not empty, then for each x∈M and y* ∈ N, it follows that

$$y^* (\beta) \leq y^* (A (x)) = A^* (y^*) (x) \leq C (x)$$

If one keeps y* ∈N fixed and varies x in M, then it follows that

$$y^* (\beta) \leq \alpha = \inf_{x \in M} C (x)$$

This inequality holds for all y* ∈N and hence, the Proof.

Theorem 2.2.3 **(Slackness Theorem)**

Let x∈M and y* ∈N be given. The following two statements are equivalent:

a. X and Y are optimal and β= α

b. $Y^* (A (\hat{x}) - b) = 0$ and $(A^*(\hat{y}^*) - C (\hat{x}) = 0$ 2.2.8

Proof:

Suppose (a) is satisfied. Then $C (\hat{x}) = y^*(b) \leq \hat{y}^* (A (\hat{x})) = A^*(\hat{y}^*) (\hat{x}) \leq C (\hat{x})$ from which 2.2.8 follows.

Now, let (b) be satisfied. Then, $C (\hat{x}) = A^*(\hat{y}^*) (\hat{x}) = y^*(A (\hat{x}) = \hat{y}^*(b)$ and by theorem 2.2.2, (a) follows.

2.3 LINEAR OPTIMIZATION PROBLEM IN SEMI INFINITE SPACE.

As in the general case, let E be a normed vector space in R^n equipped with some norm and partially ordered by means of the cone.

$K_E = K^n_r = \{x = (x_1, x_{2,\ldots\ldots,} x^n)^T : x_1 \geq 0$ for i = 1, 2,..... $_r$ where $0 \leq_r \leq n$ and $K_0 = R^n$

If $\{e_1, e_2, \ldots., e_n\}$ is only basis of R^n (e.g., let $e^i_j = M_{i,j}$ be the Kronecker delta for i,j=1,2, ..,n then every (continuous linear mapping)

A: $R^n \rightarrow F$ can be represented in the form

$$A(x) = \sum_{J=i}^{n} f x \quad \text{for } x = (x_1, x_2, \ldots., x_n)^T$$

with $f_i = A(c_i)\, 0\, F$, i = 1, 2, ..., n.

Then, by the definition of Topological dual space, every $C \in F^*$ can be represented in the form

$$C(x) = \sum_{I=1}^{n} C_i x_i \quad \text{for } x = (x_1, x_2, \ldots., x_n)^T$$

for $x = (x_1, x_2 \ldots x_n)^T$ with $C_i = C(e_j) \in R$, j = 1, 2, ..., n.

With this, the linear optimization problem for the semi - infinite case can be stated as:

Maximize

$$C(x) = \sum_{I=1}^{n} C_i X_i$$

Subject to the side conditions

$$A(x) = \sum_{I=1}^{n} F_i X_i \geq b \text{ for } x \in K^n_r \qquad 2.3.1$$

The mapping

$A^*: F^*$, E^* ($=R^{n)}$ which is adjoint to A is defined by

$$A^*(y^*)(x) = y^*(A(x)) = \sum_{I=1}^{n} y^*(f_i)\, xi$$

for all $y^* \in F$ and $x \in E$.

The statement $A^*(y^*) \leq C$ is equivalent to

$$\sum_{i=1}^{n} y^*(f_i) - C_i)x_i \leq 0 \text{ for } x \in K^n_r \qquad \text{or all } x \in K^n_r \qquad 2.3.2$$

i.e., for all $x \in R^n$ with $x_i \geq 0$, $i = 1,2,......r(\leq n)$

The statement in 2.3.2 is equivalent to

$y^*(f_i) \leq C_i$; for $i = 1,2, ... , n$

$y^*(f_i) = C_i$; for $i = r+1, ..., n$

The dual problem for the problem of linear optimization (semi-infinite case is therefore equivalent to the problem of maximizing

$F_B (y^*) = y^* (b)$

Subject to the side conditions

$y^* (f_i) \leq Ci$, for $i= 1, 2, ..., r$

$y^* (f_i) = Ci$, for $j = r+1, ...,n$

$y^* \geq q_F^* (A\ y^* (y) \geq 0)$, for all y 0 K_F 2.3.3

2.4 SEMI-INFINITE OPTIMIZATION PROBLEM IN FUNCTION SPACE.

As in the previous section, let $E = R^n$ equipped with some norm and partially ordered by means of the cone $K^n_r \leq r \leq n$. Further, let M be a compact metric space of continuous real valued functions on M equipped with maximum norm i.e.

$$V (t) = \sum_{I=1}^{n} x_i V_i (t), \text{ for all } t \in S \text{ as in } 2 .1,$$

and the ordering relation

$y \geq Z\ A\ y(t) \geq Z(t)$ for all $t \in M$ 2 . 4. 1.

Then the corresponding cone for F is given by

$K_F = \{y \in C (M): y (t) \geq 0$ for all $t \in M$ 2. 4. 2

and the associate semi-infinite problem reads

Maximize:

$$\sum_{I=1}^{n} C_i x_i \qquad 2.4.3$$

Subject to the side conditions

$$\sum_{i=1}^{n} f_i(t)\, xi \geq b(t)$$

for all $t \in M$. $x_i \geq 0$, for $i = 1,2, \ldots\ldots\ldots, r$

If one chooses a finite number of points $t_{1,\, t2,}.,\, ,t_m$ and defines for enemy

$y \in F = C(M)$,

$$y^*(y) = \sum_{I=1}^{n} y_i^*(t_i) \qquad 2.4.4$$

where $y_1^*, \ldots y_m^* \in R$ are fixed non-negative numbers, then y^* is a linear functional on F.

Now, a linear functional for C on E say, is continuous if and only if there is a constant $a \geq 0$ such that

$|C(x)| \leq \alpha \, \|x\|$ for all $x \in X$. For this reason,

$$|y^*(y)| \leq \Sigma'^{n}_{i=1} |y_i^*| \, \|y\|_\infty , \quad \text{for } y \in F$$

is continuous and consequently, $y^* \in F$. Furthermore, we have $y^*(y) \geq 0$ for all $y \in K_F$, i.e., $y^* \geq F_F$

For every $y^* \in F^*$ of the form of 2.4.4 i.e.

$$y^*(y) = \sum_{i=1}^{n} y_i^* y_i(t_i)$$

the side conditions

$y^*(f_i) \leq Ci$, for $i=1,2,\ldots\ldots\ldots, r$

$y^*(f_i) = Ci$, for $i=r+1, \ldots\ldots\ldots, n$

$y^* \geq q_F$ ($\Rightarrow y^*(y) \geq 0$) for all $y \in K_F$ as in 2.3.3 take the form

for $i=1,2, \ldots\ldots\ldots, r$

$$\text{for } j = r+1, \ldots., n$$

$y^*, \geq 0, t_1 \in M$ for $i=1, \ldots\ldots, m.$ 2.4.5

as in 2.2.3 take the form

$$\sum_{i=1}^{m} y_i * f_j (t_i) \leq C_i ; \text{ for } I=1,2, \ldots\ldots., r$$

$$\sum_{j=1}^{m} y_j * fj (t_j) = C_i ; \quad \text{for } j= r+1, \ldots\ldots\ldots, n$$

$$y^*_1 \geq 0, t \in M, \quad \text{for } I=1,2, \ldots\ldots., m$$

If conditions 2.4.5 above is satisfied, the it follows from the first weak duality theorem (theorem 2 .2.1) that

$$\sum_{I=1}^{n} y_i * b (t_i) \leq \alpha = \operatorname{Inf}_{x \in M} C(m)$$

and these exist $x \in R^n$ satisfying the side constrain

$$\sum_{I=1}^{n} f_i *(t_i) x_i \geq b(t), \text{ for all } t \in M$$

$$x_i \geq 0, \quad \text{for all } i=1, \ldots\ldots\ldots.., r \text{ and}$$

$$\sum_{j=1}^{'n} y_i * b (t_i) = \sum_{j=1}^{n} C_i x_i$$

Then, by the second duality theorem (theorem 2.2.2), x and y defined by

$$y^*(y) = \sum_{I=1}^{n} y_i * y(t_i)$$

are optimal.

The complimentary slackness theorem (theorem 2.2.3) yields the following assertion.

Corollary 2.4.1

Let $x \in R^n$ with the side condition

$$\sum_{I=1}^{n} y_i * b (t_i) \leq b(t), \text{ for all } t \in M$$

$$xj \geq 0, \quad \text{for } I=1, \ldots\ldots., r$$

and $y^* \in F^*$ be given according to

$$y^*(y) = \sum_{J=1}^{m} y_i^* \, y(t_i)$$

with the side condition of the form 2.2.5. Then, the following two assertions are equivalent:

a. x and y* are optimal and $\beta = \alpha$

b. The following two implications hold:

$$yi > 0 \Rightarrow \quad \sum_{I=1}^{n} f_i{}^*(t_i)x_i \leq b(t_i), \; (I=1,2, \ldots\ldots, m) \qquad 2.4.6$$

$$xi > 0 \Rightarrow \quad \sum_{I=1} y_i^* \, f_i(t_i) = C_i \, , \; (j = 1,2, \ldots\ldots\ldots\ldots, r) \qquad 2.4.7$$

]The proof is a direct implication of the slackness theorem in theorem 2 .2.3 and the fact that in this special case, the conditions.

(i). x and y are optimal and f_0 =a

(ii). $y^* (A(\hat{x}) - b) = 0$ and $(A^*(y^*) - c)(\hat{x}) = 0$ as stated in the theorem becomes

$$\sum_{i=1}^{m} y_i^*(\sum_{i=1}^{n} y_i^* \, f_i(t_i) \, x_j \; - b\,(ti)) = 0$$

and

$$\sum_{j=1}^{\prime m} (\sum_{j=1}^{\prime m} y_i^* \, f_i(t_i) - C_j) \, x_i) = 0$$

which however are equivalent to (a) and (b) above so stated.

2.5. COUNTER EXAMPLES TO THE VALIDITY OF THE GENERAL EXISTENCE AND DUALITY STATEMENTS

In order to guarantee the solvability of the control problems (P) and its dual (D) with the side conditions

$A(x) \geq b, x \geq q_F$

$() A(x)-b \in K_F, x \in K_E$

and

$A^*(y^*) \leq C, y^* \geq q_F$

$\Leftrightarrow A^*(y^*)(X) = y^*(A(X)) \leq C(X)$, for all $y \in K_F$ respectively, it is not sufficient to assume that the sets M and N as defined in 2 .2.3 and 2.2.4 respectively of the consistent element for the dual are not empty. We shall consider with a particular reference to the semi-infinite case.

2.5.1 Insolvability of a Semi- infinite Problem

Let $E = R^2$ and let it be equipped with some norm and the trivial partial ordering with a positive cone $K_0^2 = R^2$. Further, let B = [0,1] and F = C (B), equipped with the maximum norm

$K_0^2 = R^2$

$$\|g\|_\infty = \max_{T \in B} {}^*g(t) \qquad \text{for } g \in C(B)$$

with the partial ordering

$y \geq Z \Leftrightarrow y(t) \geq 0$ for all $t \in B$.

Finally, let b (t) = t for $t \in [0,1]$ and $A: E \rightarrow F$ be defined by

$A(x)(t) = t^2 X_1,+X_2$ for $t \in [0,1]$, $x = (x_1, x_2)$.

We consider the problem (P) of maximizing the continuous linear functional

$C(x) = 0.\, x_1 + x_2$ subject to the side conditions.

$x_1, x_2 \in R,\ t^2 x_1 + x_2 \geq t$ for all $t \in [0,1]$.

One can see easily that the set M of consistent element is non-empty and that $\alpha = 0$ (α is defined as in the weak duality theorem 2 .2.5. There is however no $x \in M$ with $C(x) = 0$. The problem therefore is not solvable.

Now, referring to the weak duality theorem, the dual problem is equivalent to the problem of maximizing the linear functional $y^*(b)$ subject to the side conditions

$y^*(f_1) = 0$

$y^*(f_2) = 1, \quad y^* \geq q_F$

Hence, $f_1(t) = t^2$ and $f_2(t) = 1$ for all $t \in [0,1]$.

If one defines

$y^*(y) = y(0)$ for all $y \in F = C[0,1]$,

then $y^* \in F^* \geq q_F^*$, $y^*(f_1) = 0$ and $y^*(f_2) = 1$. Thus y^* is consistent for the dual problem. Furthermore $y^*(b) = 0 = \alpha$, from which it follows by theorem 2 .2..1 that $y^*(b) = \beta = \alpha = 0$. The dual problem is therefore solvable and the extreme values of both problems coincide as demonstrated in the diagram below.

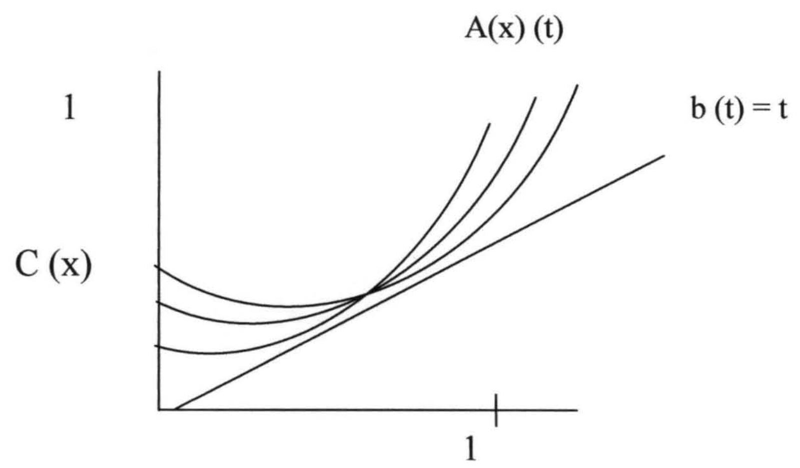


Fig 2.5.1 (Insolvability of semi-infinite problem)

2.5.2 Occurrence of Duality Gap.

Another reason why the general existence and duality statement may not hold is that of occurrence of duality gap. This occurs when the problem and it dual are solvable but the extrema do not coincide.

As an example, let E be a set of all infinite real sequence $X=\{x_n\}$, $n = 0,1, \ldots n$ in which only a finite number of terms Xn are non zero. If one defines addition and a scalar multiplication in E component wise, then E becomes a linear vector space over R. Further, let E be equipped with the norm.

$$\| X\| = | Xo | + \sum_{n=1}^{\infty} n |x_n| \qquad 1.5.1$$

and partially ordered by the relation $x \geq \hat{x} \geq x_n$ for all n. The positive cone of E is then.

$K_E = \{x = \{x_n\}\ n = 0, 1, 2, \ldots 0\ E \geq 0$ for all n

Let $F = R^2$ be equipped with some norm and partially ordered by means of the positive cone $K_F = \{q_2\}$. From this it follows that $K_F^* = F^*$ and for every $y^* \geq q_F^*$, (i.e. $y^* \in K_F^*$), there is a unique $y \in F$ with

$y^*(y) = (y, y) = \hat{y}\, y_1 + \hat{y}_2 y_2$ for all $y \in F$.

Let A: E → F be defined by

$$A(X) = \begin{cases} x_0 + \sum_{n=1}^{\infty} n\, x_n ; \\ \sum_{n=1}^{\infty} x_n ; \end{cases} \quad \text{for } x = \{x_n\} n = 0, 1, \ldots\ldots \in E \qquad 1.5.2$$

A is linear and continuous. Further, let b = (1,0) and C (x) = x_0, $x \in E$. Problem (P) reads:

Minimize C (x) = x_0 subject to the side conditions

$$x_0 + \sum_{n=1}^{\infty} n\, x_n = 1, \text{ for } x = 1,2, \ldots\ldots \in E,\ x_n \geq 0 \text{ for all } n;$$

$$\sum_{n=1}^{\infty} x_n = 0 \qquad 1.5.3$$

Obviously $x \in E$ is admissible if and only if $x_0 = 1$and $x_n = 0$ for all $n \geq 1$. Consequently, $\alpha = 1$ and C(x) equals a for this single x0M, i.e. problem (P) is solvable.

The adjoint mapping:

A*: F*→ E is given by

$$A^* (y^*) (x) = y^* (A (x)) = y_1^{\wedge} \{x_0 + \sum_{n=1}^{\infty} n\, x_n\} + y_2^{\wedge}\{\sum_{n=1}^{\infty} x_n\}$$

$$= y_1^{\wedge} x_0 + \sum_{n=1}^{\infty} (y_1 n + y_2 n). \qquad 1.5.4$$

The statement

$A^*(y^*) \leq C \quad y^* \geq q\, F^*$ therefore is equivalent to.

$$A^*(y^*) (x) = y^{\wedge} x_0 + \sum_{n=1}^{\infty} (y_1 n + y_2 n)\, x_n \leq C (x) = x_0$$

for all $\{x_n\}$, $n = 0,1,\ldots\ldots \in E$ with $x_n \geq 0$ for all n.

This statement is equivalent to, $y \leq 1$and $y_1 n + y_2 \leq 0$ for all $n \geq 1$ which equals

$y^{\wedge} \leq 0$ and $y_1^{\wedge} + y_2^{\wedge} \leq 0$

The dual problem (1) is therefore equivalent to the problem of maximizing the linear functional

$y^*(b) = \langle y, b\rangle = y^{\wedge}$, Subject to the side conditions

$y^{\wedge}_1 \leq 0$ and $y_1^{\wedge} + y^{\wedge}_2 \leq 0$

Obviously, $\beta = 0$ and $y_1^{\wedge} = y^{\wedge}_2 = 0$ is a solution of problem (D). Thus, both the problem and its dual are solvable.

Example 2.5.1

This example explains further the case of exemption to the duality theorem.

Let L_2 [0,1] x R. Here L_2 [0,1] is the Hilbert space of equivalent classes of measurable and Lebesque square integral functions on [0,1] with the norm.

$$\| f \|_2 = (\int_0^1 *(t)*^2 \, dt)$$

and the partial ordering

$$(f, r) \geq (g, s) <=> f \geq g \text{ and } r \geq s.$$

Furthermore, let F = L_2 [0,1] be equipped with the above norm and partial ordering. For every $y^* \in F$, there exist a unique $y \in F$ with

$$y^*(f) = \int_0^1 y(t)f(t)$$

for all $f \in F$

2.6 EXISTENCE AND DUALIZATION OF AN OPTIMIZATION PROBLEM.

Let E and F be two partially ordered linear vector spaces. Let A: E→ F be a contentious linear mapping, C: E → R be a continuous linear functional, and be F. The general optimization problem consist of minimizing the linear functional C(x) subject to the side conditions.

$$A (x) \geq b \; x \geq q_E \qquad 2.6.1$$

The associated dual problem (D) is then: Maximize the linear functional

$$F_b (y^*) = y^*(b)$$

Subject to the side conditions $A^* (y^*) \leq C$, $y^* \geq F_{F^*}$ 2.6.2

Here, A*: F* → E* is the adjoint mapping of the topological space F* and due to E. A* is therefore linear and continuous if one norms E* and F* in the natural way. The symbol "≤" and "≥" above refers to the partial ordering imposed on E*, F* respectively.

The dual problem therefore is equivalent to the problem of minimizing the linear form.

$-\varphi_b (y^*) = - y^*(b)$

subject to the side conditions

$- A^*(y^*) \geq - C, y^* \geq F_F^*.$

This problem again has the form of the original one and therefore can be dualize. The problem arising is equivalent to the problem (P*) of minimizing the linear functional C**(X**) subject to the side conditions.

$A^{**}(X^{**}) \geq F_b, X^{**} \geq O_E^{**}$ 2 .6.3

Here $A^{**} = E^{**} \rightarrow F^{**}$ is the mapping which is adjoint to $A^*: E^* \rightarrow E^*$, $C^{**}: E^{**} \rightarrow R$ is defined by $C^{**} (X^{**}) = X^{**}(C)$ for all $X^{**} \in E$ and the symbol "$\geq$" denote the partial ordering induced in F**, E** respectfully. If for an $x \in E$ the side condition $A (x) \geq b$, $X \geq O_E$ and satisfied, the we define $X^{**} \in E^{**}$ by $X^{**} = F_{x,}$ that is, by

$x^{**} (x^*) = X^*(x)$ for all $x^* \in E^*$.

Since $x \geq O_E$, it follows that $X^{**}(x^*) \geq 0$ for all $X^* \geq O_E \Rightarrow X^{**} \geq O_E^{**}$.

Furthermore,

$A^{**}(X^{**})(y^*) = X^{**}(A^*(y^*)) = F_x (A^*(y^*))$

$= A^*(y^*) (X) = y^* (A (X))$ and since $A (x) \geq b$, one has

$A^{**}(X^{**}) y^* \geq y^*(b) = F_b (y^*)$ for all $y^* \geq \Theta_F$ when $A^{**}(X^{**}) \geq F_b$,

Finally, one obtains $C^{**}(X^{**}) = X^{**}(C) = C (X)$

2.6.2 Sub-consistency and Normality of an Optimization Problem.

Let R x F be the Cartesian product normed by $||\lambda,y|| = ||\lambda|| + ||y||$, $\lambda \in R$, $y \in F$ and let R x F be a convex cone (with $(0, \Theta_F)$ as vertex defined by.

$K (A, C) = \{C (x) + r, A (x) -y): r \geq 0, X 0 K_E, y \in K_F$ 2.6.4

Further let

$L_b = \{(\alpha, b): \alpha \in R\} = R \times \{b\}$ 2.6.2

Hence, the set

$M = \{x \in E: A(x) \geq b, x \geq \Theta_E\}$ of the consistent elements of problems (P) is non empty if and only if $L_b \cap K(A, C)$ is non-empty and problem (P) is equivalent to the problem of finding the element $(\hat{\alpha}, \hat{z}) \in 0\ L_b \cap K(A, C)$ such that $\hat{\alpha} \leq \alpha$ for all

$(\alpha, z) \in L_b \cap K(A, C)$.

Therefore, we can find the extremal value of problem (P) as

$$V(A, C, b) = \begin{cases} \inf\limits_{(\alpha, b) \in K(A, C)} & L_b \cap K(A, C), \text{ is non empty} \\ +\infty, & \text{If } L_b \cap K(A, C), \text{ is empty} \end{cases} \quad 2.6.3$$

The problem (P) is called sub-consistent if the intersection $L_b \cap K(A, C)$ is not empty. Here, K (A, C) is the closed hull of K (A, C) in R x F.

The sub value of the problem (P) is defined by:

$$Vs(A, C, b) = \begin{cases} \inf\limits_{(\alpha, b) \in K(A, C)} L_b \cap K(A, C), \text{ is none empty} \\ +\infty, \quad \text{If } L_b \cap K(A, C), \text{ is empty.} \end{cases} \quad 2.6.4$$

The problem (P) is said to be normed if

$$L_b \cap K(A, C) = L_b \cap K(A,C) \quad 2.6.5$$

Consequent of these definitions, we have the following lemma.

Lemma 2.6.1

If the problem (P) is consistent, i.e. L_b n K (A, C) = f, then (P) is also sub-consistent and the following inequalities holds. $Vs(a, c, b) \leq V(A, c, b) < +\infty$

Lemma 2.6.2

Let the problem (P) be normal, then the following two assertions holds:

a. (P) is consistent if and only if (P) is sub-consistent.

b. V (a, c, b) = Vs (A, c, b)

Let A: E → F be defined by

$A (f, r) (t) = \int_0^1 f(s)ds + r$, for $t \in [0,1]$

Then A is linear and continuous.

Finally, let $b \equiv 1$ and

$C (f, r) = \int_0^1 t\, f(t)\, dt + 2r$

Then C is a continuous linear functional and for problem (P) we obtain the problem of minimizing.

$\int_0^1 t\, f(t)\, dt + 2r$

subject to the side conditions

$\int_0^1 f(s)\, ds + r \geq 1$, for almost all $t \in [0,1]$ t

$f(t) \geq 0$, for almost all $t \in [0,1]$ and $r \geq 0$

2.6.3 The General Existence and Duality Theorems

Before stating the general existence theorem, it is necessary to state the following decisive theorem of the theory.

Theorem 2.6.1

Let N be non-empty, i.e., the dual problem (D) is consistent.

Then for $(a, b) \in R \times F$, the assertion:

$(a, b) \in K(A,C) \Leftrightarrow a, \geq V^* (A^*, b, c)$ 2.6.3.

with

$$V^* (A^*, b, c) = \underset{y^* \in N}{\text{Sup.}}\ y^* (b) \text{ holds}$$

Before the proof of this theorem, the following lemma is necessary.

Lemma 2.6.3

If the problem (P) is sub-consistent and the dual problem (D) is consistent, then we have:

$$-\infty < V^*(A^*, b, c) \leq Vs\ (A, b, c) < +\infty \qquad 2.6.3$$

Proof:

Let (, b)$\in$ K(A ,c) and $y^* \in N$. Then there are sequences $\{r_n\}$, $r_n \geq 0$, $\{x_n\}$,

$x_n \in K_e$ and $\{y_n\}$, $y_n \in K_F$ with

$$= \underset{n \to \infty}{\text{Lim.}} \{C(x_n) + r_n\},\ b = \underset{n \to \infty}{\text{Lim.}} \{A(x_n) - y_n) \text{ holds.}$$

By the continuity of y^*, it follows that $\alpha \geq y^* (b)$, which implies equation 2 .6. 2.

Lemma 2. 6. 4.

If the problems (p) and (D) are both constant, then (P) is also sub consistent and

$$-\infty < V^* (A^*, b, c) \leq Vs\ (A, c, b,) \leq V (A, c, b) < +\infty$$

The proof of theorem 2.6.1 now follows.

Proof (Theorem 2.6.1)

2. Let $(\alpha, b) \in K (A, C)$

For every $y^* \in N$ it follows then, from the proof of lemma 2.6.1 that

$\alpha \geq y^*(b)$ which yields $\alpha \geq V^*(A^*, b, c)$

2. Let $(\alpha , b) \notin K(A,C)$

Since K (A,C) is convex closed come, then there exist $(,y^*) \in R \times F$ with

$\lambda \alpha + y^*(b) < 0 \leq \lambda\beta + y^*(z)$ for all $(\beta, z \in K(A,C)$

In particular, one has

$\lambda\{C(x) + r\} + y^* (A(x) -y) \geq q_F$ for all $y \geq q_F$.

If we choose $x = q_E$ and $r = 0$, there it follows that $(-y^*) (y) \geq 0$ for all $y \geq q_F$ which implies $y^* \geq q_F$.

If $x = q_E$ and $y = q_F$ then it follows that $\lambda r \geq 0$ for all $r \geq 0$ which implies $\lambda=0$.

Two cases are possible: (a) $\lambda \geq 0$ and (α) $\lambda = 0$.

Then for $y = q_F$, $y^*(A (x) \geq 0$ for all $x \geq q_F$ and consequently, $A^*(-y^*) \leq q_F$. Since N is not empty, there exist $y^* \geq q_F$ with $A^*(y^*) \leq C$. If we defines for every $\rho \geq 0$,

$Y_\rho^* = y^* - \rho y^*$

Then $y^* \geq q_F$ and $A^* (y_D^*) \leq C$ implies y_D^*, N and $y_D^* (b) = y^*(b) - \rho y^* (b)$ with

$y^*(b) < 0$. Therefore $V^*(A^*, b, c) \geq y_\rho^* (b)$ for all $\rho \geq 0$.

Since $\lim\limits_{\rho \to \infty} y_\rho^*(b) = + \infty$, then necessarily $V^*(A^*, b, c) = +\infty$ which implies

$\alpha < V^*(A, b, c)$. $\beta\lambda > 0$

If we set $y^* = -y^*/\lambda$, then $y^* \geq q_F$ and $\alpha < y^* (b) \leq V^*(A^*, b, c)$ follows from 2.6.3

Theorem 2.6.2 (Existence Theorem)

Suppose the convex cone:

K (A, C) = $\{C(X) + r, A(X) - y: r \geq 0$; K is closed. Therefore every $b \in F$, the following assertion holds:

a.. Problem (P) is consistent and its value is finite if and only if the dual problem (D) is consistent and its value is finite. In both cases, the problem (P) is solvable, i.e. the infimum is assumed and $-\infty < V(A\ b, c) = V^*(A^*, b, c) < +\infty$

b. If the problem (P) is consistent and the dual (D) is not consistent then we have

$$V(A, c, b) = V^*(A^*, b, c) = -\infty$$

Proof:

Since K (A, C) is closed, problem (P) is normal and all assertions except the solvability of (P) in (a) are guaranteed. Because of the fact that K (A, C) is closed, however, we have.

$$V(A, c, b) = \underset{(\alpha,\, b) \in K(A,\, C)\ x \in M}{\text{Min.}} \in K(A, c),\ \alpha = C(x)$$

when problem (P) is consistent and $V(A, c, b) > -\infty$

For variable $b \in F$, the closedness of K (A, c) is also characteristical for the solvability of the problem under the assumption that the dual problem (D) is consistent and its value is finite.

2.6.1 Application of the Duality Theorem to the Semi-infinite Problem

Consider the Semi-infinite problem (P) of minimizing the linear form

$$C(X) = \sum_{J=1}^{n} C_i x_i$$

Subject to the side condition

$$A(x) = \sum_{J=1}^{n} f_i x_j \geq b \text{ for } X \in Kr \qquad 2.6.4$$

Here, $E = R^n$ is reflexive and $K_E = K^n{}_r$ is closed. The dual problem is equivalent to the problem of minimizing the linear form $- y^*(b)$ subject to the side condition.

$- y^* (fi) \geq -Cj, j = 1, -----r$

$- y^* (fi) = - Cj, j = 1, ----,r$

$y^* \geq q_{F*}$

$K (A^*, b) \subseteq R^{n+1}$ is given by

$K (A^*, b) = \{-y^*(b) + S_0 -y^*(fi) -S_1,$

$- y^*(f_r) -S_r - y^*(f_{r+1}), ---------y^*(f_n):So \geq o,$

$S \geq o, ... S_r \geq 0, y^* \geq q_F^*\}$

If one defines $f_0 = b$, then one can also represent $K (A^*, b)$ in the form.

$K (A^*, b) = K^* - K$

With $K \{-y^*(f_0), ... -y^*(f_n): y^* \geq q_F^*\}$ and $K = \{X \in R^{n+1}: X_0 \leq, \geq 0, ...X_r \geq X_{r+1} = ... = ...= 0\}$ 2.6.5

k is obviously a closed convex cone in R^{n+1}.

CHAPTER THREE
CONTROL MODEL FOR ENVIRONMENTAL POLLUTION

3.1 THE GENERAL CONVEX OPTIMIZATION PROBLEM

3.1.1 INTRODUCTION

Linear optimization problem in function space has been considered in chapter two with its duality and existence theorems with particular reference to the semi -infinite optimization problem. In this chapter convex optimization problem in function space, its duality and optimality is considered so as to construct a control model for the environmental pollution problem. Some terminologies defined in chapter one especially those that relates to convex function and functional are used.

3.1.2 Convex optimization Problem

Let E be a vector space and X a non empty convex subset of E. Let F be partially ordered normed vector space and Y a positive cone. Furthermore, let $f: X \rightarrow R$ be a convex functional and $g: X \rightarrow F$ be a concave mapping and $b \in F$ a fixed element of F .

Suppose that the set

$$S(X, g, b) = \{ x \in X : g(x) \geq b \} \qquad 3.1.1$$

is not empty. Hence $g(x) \geq b$ is equivalent to $g(x) - b \in Y$ or $g(x) \in (Y + b)$. The concavity of g refers to the partial ordering in F induced by Y and the set S(X, g, b) is convex.

The general convex optimization is to seek an $x^{-} \in S(X, g, b)$ such that

$$f(x^{-}) \leq f(x) \text{ for all } x \in S(X, g, b) \qquad 3.1.2$$

Every $x \in S(X, g, b)$ is called consistent and every $x^{-} \in S(X, g, b)$satisfying 3.1.2 is said to be optimal.

3.1.3 The Dual Problem (D)

Let F* be a topological dual space of F (F is defined as in 3.1.1 above) We define in F* x R the set

$S^* = \{ (\beta, y^*) \in R \times F^* : F(x) - y^* \geq \beta - y^*(y) \text{ for all } x \in X, y \in Y\}$

3.1.3

The dual problem (D) is to seek a pair $(\beta, y^*) \in S^*$ such that $\beta + \bar{y}^*(b) \geq \beta + y^*(b)$ for all

$(\beta, y^*) \in S^*$ 3.1.4

Each pair (β, y^*) is said to be dually consistent and each pair $(\bar{\beta}, \bar{y}^*) \in S$ satisfying 3.1.4 is said to be dually optimal. The problem (D) is called consistent if the set S* is not empty.

The extremal value of the dual problem is defined by :

$$v^*(D) = \begin{cases} \sup\limits_{(\beta, y^*) \in S^*} \beta + y^*(b) & \text{if } S^* \neq 0 \\ -\infty & \text{: Otherwise} \end{cases}$$

3.2 CONVEX OPTIMIZATION IN FUNCTION SPACE

3.2.1 Posing The Problem and Characterising The Optimality

Consider the linear vector space E, a non empty subset X of E. Let $f: X \to R$ be a functional and $g: X \to C(T)$ a linear mapping where C(T) is the vector space of continuous real valued function defined on a compact Housdorf space T. We imagine C(T) to be equipped with the maximum norm

$$\| g \|_\infty = \max_{t \in M} |(t)| \quad \text{for all } g \in C(T) \qquad 3.2.1$$

and partially ordered in the natural way. Let Y denote the positive cone of C (T)

Let

$$S = \{x \in X: g(x) \in Y\} \qquad 3.2.2$$

be a non empty set. We seek for an $x^{-} \in S$ such that

$$f(x^{-}) \leq f(x) \text{ for all } x \in S \qquad 3.2.3$$

To be able to give sufficient condition for the existence of such optimal element

$x^{-} \in S$, we associate with every $x^{-} \in X$ the value

$$\delta(x) = \operatorname*{Inf.}_{t \in T} g(x, t) \qquad 3.2.4$$

_and the non empty set

$$I(x) = \{t \in T: g(x, t) = \delta(x)\} \qquad 3.2.5$$

Then we have the following theorems.

Theorem 3.2.1

A element $x^{-} \in S$ is optimal, i.e. for any $x^{-} \in X$, there exist an $x \in S$ such that $f(x^{-}) \leq f(x)$ if for all $x \in X$ the following implication is true.

$$g(x, t) \geq \text{ for all } t \in I(x) \Rightarrow f(x^{-}) \leq f(x) \qquad 3.2.6$$

i.e. it maximize the functional f on the set

$$S(x^{-}) = \{ x \in X: g(x, t) \geq 0 \text{ for all } t \in I(x^{-}) \qquad 3.2.7$$

The proof of this follows immediately from the fact that $S \subseteq S(x^{-})$ and the equivalence of the implication 3.2.6

The issue now is under what requirement is 3.2.6 necessary for the optimality of $x^{-} \in S$. To answer this question, we refer to the definition of star shaped set, Convex and Concave functional in chapter one and state the following theorem.

Theorem 3.2.2

Suppose $\bar{x} \in S$ is optimal, i.e. $f(\bar{x}) \leq f(x)$ for all $x \in S$. If T is finite, X star shaped, f convex and g concave with respect to $\bar{x}$, then the implication g $(x, t) \geq 0$ for all $t \in I(x) \Rightarrow f(\bar{x}) \leq f(x)$ holds for all $x \in X$.

Proof:

Let assume that there is an $x^* \in X$ such that

$g(x, t) \geq 0$ for all $t \in I(x)$ and $f(x^*) \leq f(\bar{x})$ 3.2.8

Let a set B be defined thus

$$B = \begin{cases} t \in T: g(x^*, t) - g((\bar{x}, t) < 0 \text{ and the set} \\ \min_{t \in B} g((\bar{x}, t): \quad \text{if B is empty} \end{cases}$$

$$\bar{\lambda} = g((\bar{x}, t) - g(x^*, t); \text{ if B is non empty}$$

Clearly, $\bar{\lambda} > 0$, for in the case that B is not empty, we have $B \cap I(\bar{x}) = \phi$ on the basis of the assumption of 3.2.8 which implies that $g(\bar{x}, t) > 0$ for all $t \in B$. For $\lambda = \text{Min}(\bar{\lambda}, 1)$ we have that $\lambda \in (0, 1)$ and since X is star shaped with respect $\bar{x}$, it follows that $x_\lambda = \lambda x^* + (1-\lambda)\bar{x} \in X$. That g is concave with respect to $\bar{x}$ follows from the definition of λ

$$g(x_\lambda, t) \geq \lambda g(x^*, t) + (1+\lambda)g(\bar{x}, t)$$

$$= g(\bar{x}, t) + \lambda[g(x^*, t) - g(\bar{x}, t) \geq 0$$

for all $t \in T$, i.e. $x_\lambda \in S$.

From the concavity of f with respect to $\bar{x}$ and the assumption in 3.2.8 follows finally because of $\lambda > 0$.

$$f(x_\lambda) \leq \lambda f(x^*) + (1+\lambda)f(\bar{x}) = f(\bar{x}) + [f(x^*) - f(\bar{x}) < f(\bar{x})$$

This clearly contradict the assumption of 3.2.8 and hence the assumption in 3.2.8 is false.

If we however assume that T is not infinite, then the following theorems (though weak) can be proved.

Theorem 3.2.3

Let $x^* \in X$ be such that $g(x^*, t) \geq$ for all $t \in I(x^-)$ and $f(x) \leq f(x^-)$ then the optimality of $x^- \in S$ for all $x \in X$ implies

$g(x, t) > 0$ for all $t \in I(x) \Rightarrow f(x^-) \leq f(x)$ 3.2.9

Proof:

Let $x^* \in X$ with

$g(x^*, t) > 0$ for all $t \in I(x)$ and $f(x^-) > f(x^*)$ 3.2.10

Let δ de defined as

$$\delta = \underset{t \in I(x)}{\text{Min}}\, g(x^*, t)$$

Then $\delta > 0$ and the set

$I = \{t \in T: g)(x^*, t) > 1/2\delta\}$ is open and contains $I(x)$.

If $I = T$, then $x^* \in S$ and 3.2.10 is a contradiction to the optimality of x^-

If $I \neq T$, then the complement of I is the non empty, closed subset of T and

$$\mu_1 = \underset{t \in I(x)}{\text{Min}}\, g(x, t)$$

If however,

$g(x^*, t) \geq g(x^-, t)$ for all $t \in T$ 3.2.11

then again $x^* \in S$ and 3.2.10 is a contradiction to the optimality of x^-. If 3.2.11 is not satisfied, then

$$\mu_2 = \underset{t\in T}{\text{Min}}\ g(x, t)\ [\ g(x^*, t) - g(x^-, t)\ ] < 0$$

If one chooses $\lambda = \text{Min}\ (1-\lambda^-)$ with $\lambda^- = \mu_1 / (-\mu_2)$, then $x_\lambda = \lambda x^* + (1-\lambda)x^- \in X$ and

$$g(x_\lambda, t) \geq \lambda g(x^*, t) + (1-\lambda)g(x^-, t)$$

$$= g(x_\lambda, t) + \lambda(g(x^*, t) - g(x^-, t) > \lambda\delta/2, \text{ for all } t\in I$$

$$= g(x^-, t) + \lambda g(x^*, t) - g(x^-, t) \quad > \lambda\delta/2, \geq \mu_1 + \mu_2 \geq 0 \text{ for all } t\in B$$

where ever $x_\lambda \in S$. Furthermore, because $\lambda\in[0, 1]$, we have

$$f(x_\lambda) + \lambda f(x^*) + (1-\lambda)f(x^-) + \lambda[f(x^*) - f(x^-)] < f(x^-)$$

which contradict the optimality of x^-. Thus the assumption 3.2.10 is false.

The theorem below shows the relationship between the implication 3.3.6 and 3.2.9 i.e. $g(x, t) \geq 0$ for all $t \in I(x^-) \Rightarrow f(x^-) \leq f(x)$ and

$g(x, t) > 0$ for all $t\in I(x^-) \Rightarrow f(x^-) < f(x)$

Theorem 3.2.4

Let E be a normed vector space and f: X → R a continuous functional. If for a given $x^- \in S$ the set

$$S_0(x^-) = \{\ x\in X: g(x, t) > 0 \text{ for all } t\in I(x^-)\ \} \qquad 3.2.12$$

is non empty and if $S(x^-)$ defined by

$g(\lambda x + (1-\lambda)x^- \geq \lambda g(x) + (1-\lambda)g(x^-)$, then the statement $S(x^-) \subseteq S_0$ (S_0 (x^-)) (i.e. the closure of S_0 (x^-)) holds and then the implication of 3.2.6 and 3.2.9 is equivalent.

Proof:

Proof:

If for some $x \in X$, $g(x, t) \geq 0$ for all $t \in I(S_0(x^-)$ and there is a sequence $\{S_k\}$ of points $x_k \in S_0(x^-)$ with $\lim_{x \to \infty} = f(x_k)$

From 3.2.9 it follows that $f(x^-) \leq f(x_k)$ for all k and from this $f(x^-) \leq f(x_k)$ because of the continuity of f which proves the implication 3.2.6.

Lemma 3.2.1

If E is a normed vector space and X a non empty subset of E, g concave on X and the set

$$S_0 = \{x \in X: g(x, t) > 0 \text{ for all } t \in T\} \qquad 3.2.13$$

is not empty, then for very $x^- \in S$, the set $S_0(x^-)$ defined by

$S_0(x^-) = \{x \in X: g(x, t) > 0 \text{ for all } t \in T\}$ is not empty and $S_0(x^-) \subseteq S_0(x^-)$ where $S_0(x^-)$ is defined by $S_0(x^-) = \{x \in X: g(x, t) > 0 \text{ for all } t \in T\}$.

Proof:

By assumption, there exist $x_0 \in X$ such that $g(x, t) > 0$ for all $t \in T$ when ever $x_0 \in S_0(x^-)$ for all $x^- \in S$.

Now, let an $x \in S(x^-)$ be given. Then for all $k \geq$ we have

$x_k = 1/k\, x_0 + (1-1/k)x \in X$ and $g(x_k, t) \geq 1/k\, g(x_0, t) + (1-1/k)g(x, t) > 0$ for all $t \in I(x^-)$ i.e. $x_k \in S_0(x^-)$. Furthermore, $x = \lim_{k \to \infty} x_k$

which implies that $x \in S_0(x^-)$.

In summery, theorem 3.2.1 - 3.2.4 and Lemma 3.2.1 can be put together in one as in theorem below.

Theorem 3.2.5

Let E be a normed vector space, X a non empty convex subset of E, f a continuous and convex on X, g concave on X and the S_0 defined by is non empty. Then an element $x^{-} \in$ is optimal if and only if for all $x \in X$, the implication

$$g(x, t) \geq 0 \text{for all } t \in I(x^{-}) \Rightarrow f(x^{-}) \leq f(x)$$

holds and this is equivalent to the implication

$$g(x, t) > 0 \text{for all } t \in I(x^{-}) \Rightarrow f(x^{-}) \leq f(x)$$

If T is finite, then the assumption $S_0 \neq \phi$ is superfluous.

This theorem is of particular interest when

$I(x^{-}) = \{t \in T: g(x, t) = 0\}$, i.e. $\delta(x^{-}) = 0$ where

$$\delta(x^{-}) = \operatorname*{Inf.}_{t \in T} G(x, t)$$

On the other hand, if $\delta(x^{-}) > 0$ then we have the next theorem.

Theorem 3.2.6

Suppose $x^{-} \in S$ is optimal and $\delta(x^{-}) > 0$. If X is star shaped, f convex and g concave with respect to x^{-} then $f(x^{-}) \leq f(x)$ for all $x \in X$ i.e. x^{-} is in fact, a minimal point of f on the set X.

Proof:

Let $x^* \in X$ with $f(x^*) \leq f(x^{-})$. If $g(x^*, t) \geq g(x^{-}, t)$ for all $t \in T$, then $x^* \in S$, in contradiction to the optimality of x^{-}. Therefore we have

$$\mu = \operatorname*{Min}_{t \in T} [g(x^*, t) - g(x^{-}, t)] < 0$$

If we choose $\lambda \in [0, 1]$ and $x_\lambda = \lambda x^* + (1-\lambda)x^{-} \in X$ and

$g(x_\lambda, t) \geq g(x^{-}, t) + g(x^*, t) + [g(x^*, t) - g(x^{-}, t)]$ or all $t \in T$ i.e. $x_\lambda \in S$

Finally, $f(x_\lambda) \leq f((x^{-}) + \lambda f(x^*) - f(x^{-}) < f(x^{-})$ which is a contradiction to the optimality of x^{-}. Therefore the assumption is false.

3.3 A mixed Linear - Convex Problem

3.3 A mixed Linear - Convex Problem

Let E = R^n equipped with any norm and let X be a non-empty convex subset of E. Let T also be a compact Housdorf space and let v: T → R^n be a continuos mapping . Let α: T → R be a functional and let c∈R be a given vector, for every x∈R^n and every t∈T we define

$$g(x, t) = \langle v(t), x \rangle - \alpha(t) \qquad 3.3.1$$

where <., .>denote the ordinary inner product in R^n . The g: E → C (T) is an afine linear and hence concave mapping.

A mixed linear convex problem (P) can be stated as:

Maximize the continuous linear functional f(x) = <c, x> subject to the side conditions x∈X and g(x, t) ≥ 0 for all t∈T. For every x^- ∈E, we define

$$T(X, x^-) = \bigcup_{\lambda>0} \{ \lambda(x - x^-): x\in X$$

where T(X, x) is a close convex cone in E = R^n .

Theorem 3.3.1

An element x^-∈S with S defined by

S = {x∈X: g (x)∈Y} where δ (x^-) > 0 and δ is defined by

$$\delta(x) = \operatorname*{Inf}_{t\in T} g(x, t)$$

is optimal if and only if c∈T(X, x^-) where T(X, x^-) is defined according to

S_0 = {x∈X: g(x, t)> 0 } and T(X, x^-) is a convex cone defined by

$K_0 = K^*$∈R: <K*, k> ≥ 0 for all k∈K.

Proof:

By theorem 3.3.1, x^- is optimal if and only if

< c, x - x^- ≥ 0 for all x∈X and this is equivalent to <c, h> ≥ 0 for all h∈T(X, x^-). That is c∈ $T(X, x^-)^0$

Now consider an x^- ∈S with δ(x^-) = 0. If the set

S_0 = {x∈X: g (x, t) > 0} for all t∈T is not empty or if T is a finite set quipped with discrete topology, then x^- is optimal if and only if

$<c, x - x^- > \geq$ for all x∈X with

$< v(t), x> - \alpha(t) \geq 0$ for all $t \in I(x^-)$ 3.3.3

vhere $I(x^-) = \{t \in T: <v(t), x^-) - \alpha(t) = 0\} \neq \phi$

f one defines $L(E, x^-) = \{h \in R^n: <v(t), h> \geq 0$ for all $t \in I(x^-)$ 3.3.4

Then $L(E, x^-)$ is obviously a closed convex cones in $E = R^n$

3.4 Control Model for Environmental Pollution

We shall now make use of the analysis so far to construct a control model for typical environmental pollution problem. We shall consider the problem as a semi -infinite optimization problem in which there are infinitely many variables and side conditions.

Consider a given (two dimensional) control region S in which a certain environmental conditions is to be guaranteed. That is the yearly contribution of environmental pollution is to be kept below a certain prescribed standard. We shall describe this standard by a real valued function φ on the control region S.

Furthermore, we consider the pollution arising from the various sources into two categories namely:

3.4.1 Controllable Sources

By controllable sources, we mean those sources of pollution that can be regulated while the uncontrollable sources refer to those sources that can not be regulated. Let n controllable sources be present in the region S and U_1 , U_2 , ...,U_n be the average yearly contribution from the n - controllable sources. Then the total contribution from the controllable sources is

$$\sum_{j=1}^{n} U_j \; ; j = 1, 2, \ldots, n \qquad 3.4.1$$

i. Where U_1, ..., U_n are real valued functions on the control region S.

3.4.2 Uncontrollable Sources

Let m controllable sources be present in the region S and V_1 , V_2 , ...,U_m be the average yearly contribution from the m - uncontrollable sources. Let the total contribution from the uncontrollable sources be U_0 i.e.

$$U_0 = \sum_{i=1}^{n} V_i; \; i = 1, 2, \ldots, m \qquad 3.4.2$$

where V_1 , ..., V_n are real valued functions on the control region S.

Now, it is required that the that the average concentration should not be exceeded hence, we have the side condition

$$\sum_{j=1}^{n} U_j(s) + U_0(s) \leq \varphi(s) \text{ for all } s \in S \qquad 3.4.3$$

Since U_0 is uncontrollable, in any of these conditions are not satisfied, then the uncontrollable sources (U_1 , U_2 , ...U_n) is regulated by multiplying it by a factor. Let x_j be the factor of the contribution from the controllable sources. Then the jth contribution is reduced by the factor x_j where $0 \leq x_j \leq 1$ for j = 1, 2, ..., n so that the side conditions

$$\sum_{j=1}^{n} U_j(s) - x_j U_j(s) + U_0(s) \leq \varphi(s) \text{ for all } s \in S$$

for all $s \in S$ are satisfied i.e.

$$\sum_{j=1}^{n} (1-x_j) U_j(s) + U_0(s) \leq \varphi(s) \text{ for all } s \in S \qquad 3.4.4$$

Clearly, this will be satisfied if for every $s \in S$ the condition $U_0(s) \leq \varphi(s)$ is satisfied.

Now, if an $x_j \neq 0$ must be chosen then cost must come in, e.g. to enforce plans and policies, to introduce air purifiers (in case of air pollution) etc. where cost, in this case is proportional to x_j .

Let c_j be the constant of proportionality. Then the total cost of reduction of factors x_1 , x_2 , ..., $x_n \in [0, 1]$ can be expressed as

$$C(x_1, \ldots x_n) = \sum_{j=1}^{n} C_j x_j \qquad 3.4.5$$

The factors $x_1, x_2, \ldots, x_n$ must be chosen subject to the side condition

$$\sum_{j=1}^{n} (1-x_j)U_j(s) + U_0(s) \leq \varphi(s) \text{ for all } s\in S$$

so that the cost $C(x_1, \ldots x_n$ is as small as possible.

The control problem can now be stated as that of minimizing the linear functional

$$C(x_1, \ldots x_n) = \sum_{j=1}^{n} C_j x_j$$

subject to the side condition

$$\sum_{j=1}^{n} U_j(s) x_j \geq \sum_{j=1}^{n} U_j(s) - \varphi(s) \text{ ; where } 0 \leq x_j \leq 1 \text{ for } j = 1, 2, \ldots, n, \ldots \; 3.4.6$$

This is a typical semi - infinite optimization problem.

Let $V(s) = (U_1(s), \ldots, U_n(s))^T$

$$\alpha(s) = \sum_{j=1}^{n} U_j(s) - \varphi(s) \; ; s\in S \qquad 3.4.7$$

Then,

$g(x, s) = < V(s), x> - \alpha(s)>$, $x\in R^n$ where $<x, x>$ denotes the scalar product in R^n. Furthermore, Let $X = \{x\in R^n : 0 \leq x_j \leq 1 \text{ for } j = 1, \ldots, n\}$ and let $T = S$. Then, the problem (P) is that of finding an $x^-\in S$ such that $f(x^-) \leq f(x)$ for all $x\in S$ where

$S = \{x\in X: g(x)\in Y\}$ is a non empty set. Y denotes the positive cone of $C(T)$.

We assume that the pollution arising from the uncontrollable sources lies in the whole region of S under the prescribed standard i.e.

$$U_0(s) \leq \varphi(s) \text{ for all } s\in S \qquad 3.4.8$$

The set (x, g) is closed and bounded, hence the linear and continuous functional assumes its minimum on S (x, g). Consequently, there exist an x^- $\in$S(X, g) with

$$<c, x^-) \leq <c, x> \text{ for all } x\in S(X, g) \qquad 3.4.9$$

If we define $\delta(x) = \inf_{t\in T} G(x, t)$

and g according to 3.4.7 above then two cases are possible

1. $\delta(x) > 0$

By theorem 3.3.1, that is the case if $c\in T(X, x^-)^o$ with $T(X, x^-)$ given by

$T(X, x^-) = \cup \{(x - x^-) : x\in X$ and

$T(X, x^-)^o = k^*\in R^n : <K^*, k> \geq 0$ for all $k\in K$ in R^n

For this case at hand, we have

$T(X, x^-) \{x_j\in R$ for all $x_j \in (0, 1)$, $x_j \geq 0$ for $x_j^- = 0$, $x_j \leq 0$ for $x_j^- = 1$and

$T(X, x^-)^o = \{x\in R^n$ $x_j = 0$ for $x_j \in (0, 1)$,

$$x_j \geq 0 \text{ for } x_j^- = 0 , x_j \leq 0 \text{ for } x_j^- = 1 \qquad 3.4.10$$

Since $c_j > 0$ by assumption, $C\in T(X, x)^o$ is possible only in the case when $x^- = \Theta_n$ which implies

$$\sum_{j=1}^{n} Uj(s) \leq \varphi(s) \text{ for all } s\in S$$

so that no reduction is necessary in order not to exceed the standard and consequently no cost arise. Conversely, 3.4.9 is naturally sufficient in order that $x^- = \Theta_n$ belongs to S(X, g) and is optimal i.e. $<c, x^- > \leq <c, x>$ for all $x\in S(X, g)$ is fulfilled.

CHAPTER FOUR

COMPUTATIONAL METHOD & ALGORITHM

4.1 INTRODUCTION

There are a number of optimal control problems that can be resolved completely analytically or reduced to simple finite dimensional problem. However, a great majority of problems arising from large industrial, aerospace or governmental systems must ultimately be treated by computational methods. This is not because the necessary conditions for optimality are too difficult to derive but rather, the solutions of the resulting non-linear equations are beyond analytical tractability.

There are two basic approaches for resolving complex optimization problems by numerical techniques:

(i) By formulating the necessary conditions describing the optimal solutions and solving these equations numerically (usually by iterative scheme) or

(ii) By bypassing the formulation of the necessary conditions and implement a direct iterative search for the optimum

Though both methods has their merit and demerit, but the second method appears to be more effective since progress during the iterations can be measured by monitoring the corresponding values of the objective functional.

In this chapter, the basic concept of dealing with both procedures is considered mainly, method for solving system linear equations. There are several computational methods of handling this. The Gauss Elimination method is used n this work to solve the system of linear constraint.

EXAMPLES 4.1 (Linear Environmental Pollution Problem)

Consider a hypothetical air shed with a single cement manufacturing industry. The annual production is 2,500 barrels of cement. Although the industry is equipped with mechanical collectors for air control, they are still emitting two pounds of dust for every barrel of cement produced. The industry can be required to replace the with four field electrostatic precipitator which will reduce emission to 0.5 pounds of dust per barrel or with five field electrostatic precipitator that could reduce emission to 0.2 pound per barrel. If the capital and the operating cost of the four wheel precipitator are N 0.4 million per barrel of cement produced and if the five field precipitator are N 0.18 million per barrel of cement, what control methods should required of this industry? Assume that in this hypothetical air shed, it has been determined that particulate emission (which now total 5, 000, 000 pound per day) should be reduced by 4, 200, 000 pounds.

SOLUTION

Now, if C represent the cost control, x is the number of barrel of annual cement production subject to the four field electrostatic precipitator (cost of N 0.4 million per barrel of cement produced) and y is the number of barrels of annual cement production subject to the five field electrostatic precipitator (cost of N 0.8 million per barrel of cement and pollution reduction is 5 – 4.2 =0.8 million pounds per barrel produced), then the problem can be stated as

Minimize

$$C(x, y) = 0.4x + 0.18y$$

Subject to

$$x + y \leq 2,500,000$$

$$1.5x + 1.8y \geq 4,200,000$$

$$x \geq 0, y \geq 0$$

This problem can be resolved by the linear programming method.

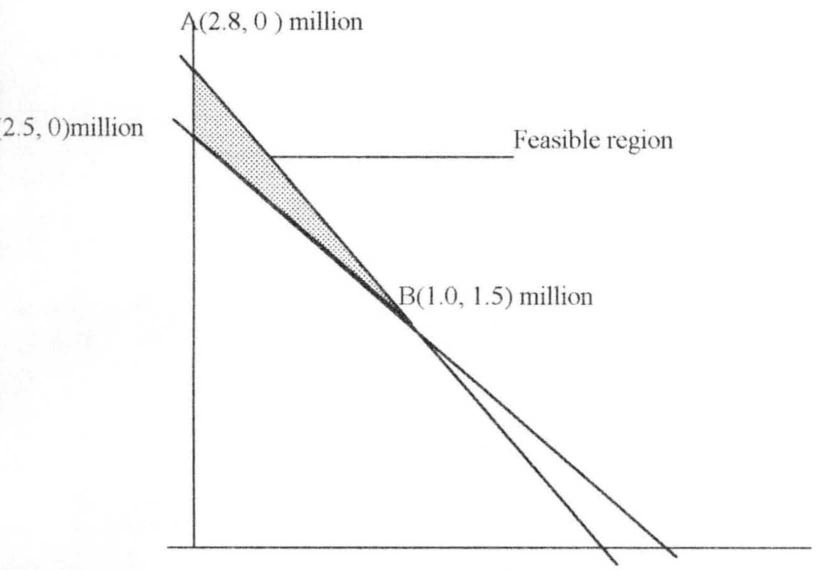


Fig. 3.5.(Graphical solution of pr5oblem 3.5.1)

Now,

X + y = 2, 500, 000

When x = 0, y = 2,500,000;

When y = 0, x = 2, 500,000.

Also, for

1.5x +1.8y = 4,200,000,

x = 0 => y = 4200000/1.8 = 233333.33

and y = 0 => x = 4200000 / 1.5 = 2800000

Solving x + y = 2500000 and 1.5x +1.8y = 4200000 simultaneously, we

X= 1000000

Y = 1500000

To determine the optimal cost of control (i.e. the minimum cost), we evaluate C(x, y) at (2800000,0), (2500000,0) and (1000000, 1500000).

Now, C (x, y) = 0.14x + 0.18y

C (2500000,0) = 0.14 x 2500000 +0.18 x 0 = N 35000

C(2800000, 0) = 0.14 x 2800000 + 0.18 x 0 = N 392000

C(1000000, 1500000) = 0.4 x 1000000 + 0.18 x 1500000 = N 410000

The least cost solution therefore is to install the four field electrostatic precipitator on the industry producing 1000000 (x = 1000000) barrels and five-field precipitator producing 1500000 (y = 1500000) barrels of cement at the cost of N 410000 (C (x, y) = 410000)

Example 4.2.

An environmental pollution control agency has N 30 million to use in the control of pollution in a given year. The money is to be appropriate among air pollution, chemical pollution and domestic waste. The rules for the administration of the fund require that at least N 3 million be invested in the control of each type of the pollutants, at least half the money be spent on chemical pollution and pollution arising from domestic waste, and the amount spent in the control of chemical pollution must not exceed twice the amount spent in the control of air pollution. The annual concentrations of pollutants are 7% from air pollution, 8% from chemical pollution, and 9% domestic waste. How should the money be allocated among the various pollutants to produce a minimal concentration of pollutants?

In millions of naira, let x = the amount spent on the control of air pollution, y = the amount spent on the control of chemical. Then the amount spent the control of domestic waste is 30- (x + y). The constraints are:

$$\begin{cases} x, y > 3 \\ 30-(x+y) > 3 \\ x + y > 15 \\ y < 2x \end{cases}$$

and the objective function is:

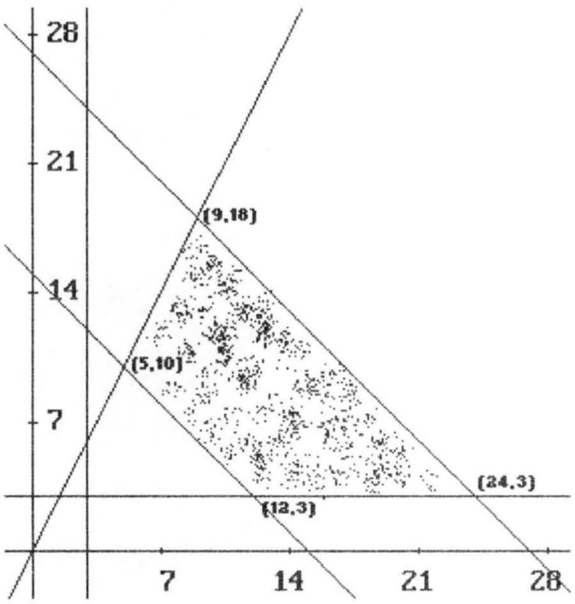
$$f(x, y) = 0.7x + 0.8y + 0.9[30 - (x + y)]$$

$$f(x, y) = 27 - 0.2x - 0.1y$$

The corresponding system of linear equation is:

$$\begin{cases} x, y = 3 \\ x + y = 27 \\ x + y = 15 \\ y = 2x \end{cases}$$

The feasible region is:



y = 2x and y = -x + 27 => x=9, y = 18 hence we (9, 18)

y =3 and y= -x + 27 => x = 24, i.e. (24, 3)

EXAMPLE 4.3

In a certain region S the environmental pollution problem there arises from accumulation of domestic waste, improper sewage disposal, chemical pollution arising from the use of chemical in fishing and oil spillage. The table below shows the concentration of each pollutant at each location in the region.

Locat-ion i	Domestic waste U1(s)	Sewage U1(s)	Chemical U1(s)	Oil Spillage U1(s)	Combustion $U_1(s)$
1	800	600	250	300	1000
2	650	900	750	1000	700
3	900	350	780	950	600
4	500	800	250	600	900
5	250	650	800	250	300

To bring the level of pollution to a comfortable level, the annual concentration at the locations should be reduced to 1259, 1343, 1564, 1145 and 660 tonnes respectively. Determine the reduction factor and the cost of reducing the pollution level in the region to desired tolerable level if contribution to the annual concentration are 20 %, 15%, 30%, 25 % and 10 % from Domestic waste, Sewage, Chemical , Oil Spillage and Combustion respectively?

Solution:

Now the control model is:

Minimize :

$$C(x_1 , ...x_n) = \sum_{j=1}^{n} C_j x_j$$

Subject to the side condition:

$$\sum_{j=1}^{n} U_j (s) x_j \geq \sum_{j=1}^{n} U_j (s) - \varphi(s)$$

$C(x_1, ...x_n) = 0.2\,x_1 + 0.15\,x_2 + 0.3\,x_3 + 0.25\,x_4 + 0.10\,x_5$ and

$$\sum_{j=1}^{n} U_j(s)\,x_j \geq \sum_{j=1}^{n} U_j(s) - \varphi(s)$$

for each of the locations I = 1, 2, ...5 is

$$\left.\begin{array}{l} 800x_1 + 600\,x_2 + 250\,x_3 + 300\,x_4 + 1000\,x_5 \\ 650x_1 + 900\,x_2 + 750\,x_3 + 1000x_4 + 700\,x_5 \\ 900x_1 + 350\,x_2 + 780\,x_3 + 950\,x_4 + 600\,x_5 \\ 500x_1 + 800\,x_2 + 250x_3 + 600\,x_4 + 900\,x_5 \\ 250x_1 + 650\,x_2 + 800\,x_3 + 250\,x_4 + 300\,x_5 \end{array}\right\} \quad 4.3.1$$

$\sum_{j=1}^{n} U_j(s)$ for each of the locations are:

Location	1:	800 + 600 + 250 +300 + 1000 = 2950
	2:	650+ 900 + 750 +1000+ 700 = 3950
	3:	900 + 350 + 780 + 950 + 600 = 2980
	4	500+ 800 + 250 +600 + 900 = 3050
	5:	250+ 650 + 800 +250 + 300 = 1550

φ(s) for each of the locations as given are: 1259, 1343, 1564, 1145 and 660

Hence $\sum_{j=1}^{n} U_j(s) - \varphi(s)$ = 1259, 1343, 1564, 1145 and 660 respectively.

Equation 4.3.1 above is the system of linear constraint giving rise to the augmented matrix:

$$\begin{pmatrix} 800 & 600 & 250 & 300 & 1000 \\ 650 & 900 & 750 & 1000 & 700 \\ 900 & 350 & 780 & 950 & 600 \\ 500 & 800 & 250 & 600 & 900 \\ 250 & 650 & 800 & 250 & 300 \end{pmatrix} = \begin{pmatrix} 1259 \\ 1343 \\ 1564 \\ 1145 \\ 890 \end{pmatrix} \quad --- \quad 4.3.2$$

Equation 4.3.2 can easily be computed by the Gussian elimination method. The

$y = 15 - x$ and $y = 3 \Rightarrow x = 12$, i.e. (12, 3)

$y = 15 - x$ and $y = 2x \Rightarrow x = 5$, hence , (5, 10)

$$f(9, 18) = 27 - 0.2 * 9 - 0.1 * 18$$

$$= 27 - 3.6$$

$$= 23.4$$

$$f(24, 3) = 27 - 0.2 * 24 - 0.1 * 3$$

$$= 27 - 5.1$$

$$= 21.9$$

$$f(12, 3) = 27 - 0.2 * 12 - 0.1 * 3$$

$$= 27 - 2.7$$

$$= 24.3$$

$$f(5, 10) = 27 - 0.2 * 5 - 0.1 * 10$$

$$= 27 - 2.0$$

$$= 25$$

The minimum cost of control will be incurred by spending N24 million on the control of air pollution, N3 on the control of chemical pollution and N 3 on the control of pollution from domestic waste.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1 CONCLUSION

So far, a good number of concepts in optimal control with particular reference to control of environmental pollution with uncontrollable sources have been considered. It is important to emphasize that the interest in this work has been to exploit the intimate relationship that exist between approximation theory and optimization theory to solve optimization problems. The fact that approximation problem can be considered as an optimization problem is clear. Therefore, if approximation theory is included under the more general concept of optimization theory, then several optimization theories can be treated computationally be means of approximation theories. This is the principle imbibed in this work.

It is noticeable however that when approximation theory is included in the more general theories of optimization; some special properties of approximation are lost. Hence, one would no longer be able to answer all theoretical problems of approximation by means of optimization.

The use of optimization theories in treating approximation problems is very fruitful in terms of characterizing the best approximations and calculation and estimation of minimal deviation. Furthermore, various methods for solution of approximation problems may be applied successively to optimization problems.

A number of computational examples in this work are hypothetical. The methods of solutions are based on age long computational procedures involving iterative search for solution until convergence is attained.

There must be a number of other suitable approaches that may not have been treated in this work. It all depends on problem formulation, however the fundamental idea remains the same. A problem with linear multivariable function subject to a number of constraints could be treated with the popular linear programming algorithm known as the Simplex Method credited to Danzig (1963).

Classical optimization technique can be applied to problems for which there is no general procedure base on the work of Kuhn and Tucker (1951).

I will like to mention on a concluding note that this work is limited to some extent by unavailability of suitable literature that deals directly with the subject matter.

There are other alternative methods of computation of the solution to the constraint equation such as Jacobi iteration method, Gauss elimination and Sidel iteration method.

5.2 RECOMMENDATIONS

The topic "Optimal Control of Environmental Pollution with Uncontrollable Sources" is topic of immense practical application. Hence, it will be worthwhile to carry out further research in this area of study. I recommend that non-linear constraint or differential constraint be considered instead of the linear constraint used in this work. This will also form a prospective area of study.

4.4 DISCUSSION OF RESULT

The computation below shows the step – by- step computation of the cost factor for the control of environmental pollution. The program philosophy is based of the popular Gussian method of solving system of linear equation. The solution is iteratively substituted to the cost function until the system converges.

The cost reduction factor is the factor by which the concentration of the pollutants concerned must be reduced in order to guarantee the desired level of concentration. Substituting the cost reduction factor x_i to the cost function gives the cost of controlling the pollutants concerned the total cost of control is the summation of the cost of controlling of the pollutants

Below is the result of generated by the computer program developed to implement the algorithm.

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