# HIGH ORDER QUASI-RUNGE-KUTTA METHODS BY REFINEMENT PROCESS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS

By

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# CERTIFICATION

This thesis titled "HIGH ORDER QUASI-RUNGE-KUTTA METHODS BY REFINEMENT PROCESS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS" by Salisu Abdullahi, meets the regulations governing the award of the degree of Masters of Technology in Mathematics, Federal University of Technology Minna and is approved for its contribution to knowledge and literary presentation.

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# DEDICATION

This project work is dedicated to my parents, Alhaji Salisu Saidu , Hajiya Hadiza Salisu and to the entire members of Mai Rago Family.

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# ABSTRACT

One of the major tasks of numerical analysis is that of solving differential equations. The goal, target and objective of numerical analysis are to replicate the exact solutions or at least produce solutions that are very close to exact solutions. Hence, the closer such solutions are to the exact solutions, the better the method. In this research work, we examine the existing processes, how they are derived and their limitations. Based on such analysis, we derived, some Quasi-Runge-Kutta methods, through a refinement process, for the solution of initial value problems. For acceptability, the schemes so derived are tested for consistency, zero-stability, and convergence. Also provided, is an example of initial value problem solved with the new methods and their results help to establish their high degree of accuracy.

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# CHAPTER ONE

## 1.0 INTRODUCTION

### 1.1 Background Of The Study:

The use of simple operations to find approximate solutions to complex problems constitutes the main goal of numerical analysis. One of the major tasks of numerical analysis is that of solving differential equations, which are just relationships involving an independent variable x, a dependent variable y, and one or more differential coefficients of y with respect to x – An example of differential equation is

$$y'' + 2y' + y = 0 \quad \text{..............................(1.1)}$$

Differential equations represent dynamic relationships, i.e quantities that change, and are thus frequently occurring in scientific, engineering, as well as social problems. The solution of a differential equation thus provides solution to the physical problem it represents.

Solutions to differential equations were derived using analytical or exact methods. Those solutions are often useful as they provide excellent insight into the behavior of some systems. However, analytical solutions can be derived for only a limited class of problems. These include those that can be approximated with linear models and those that have simple geometry and low dimensionality. Consequently, analytical solutions are of limited practical value because most real life problems are non-linear and involve complex shapes and processes.

In such cases, where differential equation defies solution analytically, an approximate solution is often obtainable by the application of numerical methods. Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations. This means that the relevant particular solution is obtained as a set of function values for the range of values of the independent variable. This set of points is an approximation of exact solution at these points.

A variety of methods have been derived for solving differential equations. These methods can be classified into two:

One-step and multistep methods.

One-step methods permit the calculation of $y_{i+1}$, given the differential equation and $y_i$. They utilize information at a single point $x_i$ to predict a value of the dependent variable $y_{i+1}$ at a future point $x_{i+1}$. Runge-Kutta methods are members of this family.

Multi-step methods require additional values of y other than at i. Multi-step methods are based on the insight that, once the computation has began, valuable information from previous points is at our disposal. Some famous sub-classes are Adam-Moulton and Adam-Bash forth methods. Various reasons determine the choice of one method over another, two obvious criteria being speed and accuracy. However, the advent of fast and efficient digital computers has increased dramatically the role of numerical methods in solving scientific, engineering as well as social problems. Scheid (1998).

## 1.2 Definitions:

### 1.2.1 Differential Equation

A differential equation is an equation involving an unknown function and one or more of its derivatives. It is a relationship between an independent variable x, a dependent variable y, and one or more differential coefficients of y with respect to x, e.g $\frac{dy}{dx} = x + y$

### 1.2.2 Initial value Problem (IVPs)

A first order differential equation, $y' = f(x + y)$, together with an initial condition, $y(x_0) = y_0$ constitutes an initial value problem,

$y' = f(x, y), y(x_0), x > x_0$ .............(1.2)

1.2.3 **Numerical Solution Techniques:**

We wish to solve the standard initial value problem given by equation (1.2) above. Since analytical or exact solutions are not always possible to find, it is essential to work with techniques which work without then. One approach is the numerical analysis, which tries to find good algorithms to approximate solutions. This simply means finding procedures by which computers can do the solving for us.

1.2.4 **Numerical Method:**

A numerical methods can be defined as a differential equation that involves a number of consecutive approximations $y_{n+j}$, j=0,1...k, from which it will be possible to compute sequentially the sequence $y_n$; {n=0,1,2...n}. Lambert (1991). Although numerical methods for IVPs can take many forms, all of them can be written in the general form.

$$\sum_{j=0}^{k} a_j y_{n+j} = h\Phi_f\left(y_{n+k}, y_{n+k-1}, \ldots\ldots y_n, x_n, h\right) \ldots\ldots\ldots (1.3)$$

$$y_i = \mu_i(h),\ i=0,1,\ \ldots\ldots k-1 \qquad \ldots\ldots\ldots\ldots (1.4)$$

Where subscript f indicates that the dependence of $\phi$ on $y_{n+k}$, $y_{n+k-1}$.........,$y_n$, $x_n$ is through the function f (x,y) and [$\mu$; (h)] i=0,1......k-1 are the initial points.

Patrizia (2001).

1.2.5 **Convergence:**

A numerical method is convergence if

$$\lim_{h\to 0} y_n = y(x)$$

For all x over the finite interval $[x_0, x_n]$ i.e if the sequence of improved values converge to the true value of y. A method is not convergent is said to be DIVERGENT.

Patrizia (2001).

### 1.2.6 Local Truncation Error

The local truncation Error $t_{n+1}$ of the one-step scheme is given by

$$t_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n); h)$$

Where y(x) is the true solution to the IVP.

The local truncation error simply put, is the amount by which the true solution of the IVP fails to satisfy the first order differential equation, under the simplying assumption that the previous solutions are exact. i.e $(y_n = y(x_n))$.

### 1.2.7 Total Truncation Error

The total truncation error is the difference between the solution $y(x_{n+1})$ and $y_{n+1}$ (The solution calculated after n+1 steps).

$e_{n+1}$: $//y(x_{n+1})-y_{n+1}//$……………(1.5)

## 1.3 Aims And Objectives:

The aim of this study is to derive High Order Quasi-Runge-Kutta methods by a Refinement Process for the solution of initial value problems.

The objectives of the study include the following:

i) To derive new methods which have less compùtation steps than the Runge-Kutta methods.

ii) To verify the accuracy of the methods by making comparison with the exact solutions and reference method (Runge-Kutta methods).

iii) To use the methods to solve some differential equations.

# CHAPTER TWO

## 2.0 LITERATURE REVIEW

### 2.1 Overview

The family of One-step methods for solving initial value problems offers a wide range of methods which are further grouped into subclasses. A great many methods have been developed in this direction, and yet other are still being developed. Many have undergone changes to improve their accuracies, or their error control strategies or shed more light on their behaviors in general.

**DAVID RUNGE (1895)**, in his paper on the numerical solutions of differential equations put forward a method for solving first order differential equations (Specifically, IVP), that achieved a higher order than the linear multistep methods (LMM), by sacrificing the linearity of the algorithm while preserving its one–step nature. His method involves extending the approximations of the improved Euler method further, so as to obtain a one-step method having a higher order of accuracy. This is because one-step methods, have the advantage of permitting a change of mesh length at any step, since no starting process is required. Since the time of Runge, many researchers have taken advantage of the flexibility of the method to derive schemes either to improve accuracy or error control strategies.

**HEUN (1900)**, put forward the following third-order formula for a three stage method.

$$y_{n+1} - y_n = \frac{h}{4}(k_1 + 3k_3)$$

$$k_1 = f(x_n y_n)$$

$$k_2 = f(x_n + \frac{h}{3}, y_n + \frac{h}{3}k_1)$$

$$k_3 = f(x_n + \frac{2h}{3}; y_n + \frac{2h}{3}k_2)$$

He reckoned that Runge's work could be further extended to include terms up to order $h^3$ previously ignored by Runge.

**WILHELM KUTTA (1901)**, extended the method of Runge further to systems of equations. Thus, this method has come to be known as the Runge-Kutta method. Kutta's third order rule is given by

$$y_{n+1}-y_n=\frac{h}{4}(k_1+4k_2+k_3)$$

$$k_1=f(x_n,y_n)$$

$$k_2=f(x_n+\frac{h}{4};y_n+\frac{h}{2}K_1)$$

$$k_3=f(x_n+h;y_n-2hk_2)$$

According to Lambert (1973); "it is most popular third order Runge-Kutta Method, for desk computations (largely because the coefficient ½ is preferable to ⅓, which appears frequently in Heun's method)".

**MERSON (1957)**, was the first to propose the idea of deriving a special R-K method, which would admit an easily calculated error estimation, which does not depend on quantities calculated at previous steps. Merson's method is

$$y_{n+1}-y_n=\frac{h}{6}(K_1+4k_4+k_5)$$

$$k_1=f(x_n,y_n)$$
$$k_2=f(x_n+\frac{h}{3};y_n+\frac{h}{3}k_1)$$
$$k_3=f(x_n+\frac{h}{3};y_n+\frac{h}{6}(k_1+k_2)$$
$$k_4=f(x_n+\frac{h}{2};y_n+\frac{h}{8}k_1+\frac{3}{8}hk_3)$$
$$k_5=f(x_n+h,y_n-hk_1+2hk_2)$$

and it is defined by the butcher tableau below:

| | | | | | |
|---|---|---|---|---|---|
| 0 | | | | | |
| $^1/_3$ | $^1/_3$ | | | | |
| $^1/_3$ | $^1/_6$ | $^1/_6$ | | | |
| ½ | $^1/_8$ | $^3/_8$ | | | |
| 1 | ½ | 0 | $^{-3}/_2$ | 2 | |
| | $^1/_6$ | 0 | 0 | $^2/_3$ | $^1/_6$ |

The above method, has order four and an estimate for the local truncation error given by

$30\ T_{n+1} = h\ (-2k_1 + 9k_3+8k_4+k_5)$

This method has been widely used for non-linear problems, although the error estimate is valid only when the differential equation is linear in both x and y, that is of the form:

$y' = a\ x + by + c$

Merson's idea, is to derive R-K methods of order r and r+1, which share the same set of vectors ($k_j$).

**BUTCHER J.C (1963, 1976)**, in a long series of papers starting in the mid-sixties, has developed various theories out of the Ruge-Kutta method. Notable among his theories are,

(i) An S-stage explicit R-K method, cannot have order great than s,

(ii) There exists no five-stage explicit R-K method of order five he also established the order condition for all class of Runge-Kutta method. Below is the representation of Runge-Kutta scheme, in Matrix notation, a form know as the Butcher Tableau. Recall the general S-stage Runge-Kutta method.

$$y_{n+1}-y_n=h\sum_{i=1}^{s}b_{i},k_i$$

$$k_i= f(x_n+c_ih,y_n+h\sum_{i=1}^{s}a_{i\ j}k_j),i=1,2,3,.....s$$

Call the $b_i$'s the weights, the $c_i$'s the abscissas, and the $k_i$'s the slopes. Butcher defined the s-dimensional vectors c and b and the s x s matrix A, by c > $[c_1, c_2, \ldots\ldots c_s]^T$ and b = $[b_1, b_2, .. b_s]^T$ and A = $[a_{ij}]$. Then method expressed conveniently as Butcher tableau

$$\begin{array}{c|c} C & A \\ \hline & b^1 \end{array} \quad = \quad \begin{array}{c|llll} c_1 & a_{11} & a_{12} & a_{23} & \ldots\ldots a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \ldots\ldots\ldots a_{2s} \\ c_3 & a_{31} & a_{32} & a_{33} & \ldots\ldots\ldots a_{3s} \\ c_r & a_{s1} & a_{s2} & a_{s3} & \ldots\ldots\ldots a_{ss} \\ \hline & b_1 & b_2 & b_3 & \ldots\ldots\ldots b_s \end{array}$$

Will assume

$$c_i = \sum_{j=1}^{s-1} a_{ij}, i = 1,2,\ldots s$$

One important use to which the Butcher tableau could be put, is in determining the type of the method (i.e explicit, implicit, and semi-implicit).

i) If "A" is strictly lower triangular => explicit method; calculate $k_1$ explicitly; then $k_2$, e.t.c, up to $k_s$,

ii) If $\exists$ $a_{ij} = 0$, j>1 => implicit method:

Requires a system of sxs (non-linear) equations be solved per step.

iii) If $a_{ij} = 0$, J>1 and $\exists$ $a_{ij} \neq 0$ => semi-implicit;

Requires s scalar (non-linear) equations be solved per step.

**SCRATON (1964)**, derived a fourth-order estimate which admits an error which is valid for a non-linear differential equation, unlike merson's, the method is as below:

$$y_{n+1} - y_n = h(\frac{17}{162}k_1 + \frac{81}{170}k_3 + \frac{32}{135}k_4 + \frac{250}{1377}k_5)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{2h}{9}, y_n + \frac{2h}{9}k_1)$$

$$k_3 = f(x_n + \frac{h}{3}, y_n + \frac{h}{12}k_1 + \frac{h}{4}k_2)$$

$$k_4 = f[x_n + \frac{3h}{4}, y_n + \frac{3h}{128}(23k_1 - 81k_2 + 90k_3)]$$

$$k_5 = f[x_n + \frac{9h}{10}, y_n + \frac{9h}{10000}[-345k_1 + 2025k_2 - 1220k_3 + 544k_4]$$

He gave the estimate for the local truncation error as:

$$T_{n+1} = {hqr}/{s}$$

Where

$$q = \frac{-1}{18}k_1 + \frac{27}{170\,k_3} - \frac{4}{15}k_4 + \frac{25}{153}k_5$$

$$r = \frac{19}{24}k_1 - \frac{27}{8}k_2 + \frac{57}{20}k_3 - \frac{4}{15}k_4$$

$s = k_4 - k_1$

Although, Scraton's estimate was more realistic than merson's when applied to a general non-linear differential equation, it has the disadvantage that it is not linear in the krs. As a result, it is applicable only to a single differential equation and does not extend to a system equations. As noted by Lambert (1973);

"in order to find a method which admits an error estimate which is linear in the $k_r$, and this holds for a general non-linear differential equation, or system of equation, it is necessary to make a sacrifice in the form of additional functions evaluations".

**SHAMPINE and ALLEN (1973)**, developed a subroutine for solving the fourth-order R-K method which was different from Ralston's fourth order R-K method.

**HAIRER and WANNER (1981)**, showed that R-K method could be extended to orders five and six which have the properties of order, stability and efficiency of implementation to high extent. These authors classified all algebraically stable methods of an arbitrary order and give various relationships between contractivity and order of implicit methods.

**ONUMANYI, et al (1981)**, developed software for a method of finites approximations for the numerical solution of differential equation, which was based on the Tau method. According to them, problems with complex initial boundary conditions or mixed conditions involving combinations of functions and derivative values, can be dealt with by means of their program. Accordingly, encouraging results have been obtained in the solutions of problems with regions of rapid variation, oscillatory behavior and in the presence of stiffness.

**BURRAGE (1987)**, examine the stability properties of some special class of multi-valued methods known as multi-step R-K methods. He further constructed some families of algebraically stable methods of arbitrarily high order for the solution of the first order initial value problems. In particular, Burrage has studied the order conditions of these methods, and has shown that one can always construct methods of order, $2s + r-1$, where $2s$ denotes the highest order possible, and $r-1$, the number free parameters existing in the methods.

**DORMAND, et al (1989)**, considered the application of Runge-Kutta interpolation to global error estimation. They brought some special formulae of orders two, four, and six and went on to show that a pseudo-problem, which is based on dense output values within any one step and reliable global error estimates could be mesh-points, by using the special R-K formulae.

**KEELING, (1989)**, constructed an implicit Runge-Kutta method with a stability function having distinct real poles. Such methods offer computational speed up when used on parallel machines (multiprocessor computers) with a modest number of processor. Sometime, the method is called multiple implicit Runge-Kutta (MIRK) and hence due to the so-called order reduction phenomenon, the poles of the (MIRK) are required to be real.

**BUTCHER and CASH (1990)**, derived a special class of implicit R-K methods of the numerical solution of stiff IVP. They derived the formulae from simple implicit methods by adding one or more extra diagonally implicit stages for the derivation they considered singly implicit methods and in particular diagonally implicit methods.

They established that each class of methods offers some advantages over the methods as well as some disadvantage for diagonally implicit methods, their limitation of the stage order to 1, and the difficulty of finding high order for the methods as whole, or of constructing realistic local error estimates, makes these methods unlikely candidates for incorporating into highly accurate and efficient software.

**JULYAN and PIRO (1992)**, investigates the dynamics of a continuous time system, described by an ordinary differential equation. They attempted to elucidate the dynamics of the Runge-Kutta methods, by the application of the techniques of dynamical system theory to the maps produced in the numerical analysis. Their aim was to investigate what pitfalls there may be in the integration of non-linear and chaotic systems.

**ADEWALE (1998)**, derived a new five-stage explicit one-step R-K method of order four for the numerical solution of IVPs. The new methods aid computation through the use of whole numbers instead of fractions as observed in existing methods of this form. This is helpful, when the computations are performed manually as it reduces the number of operations involved in the evaluation of the krs. He also provided a computer program that uses the new schemes, to solve IVPs.

The new method with its corresponding Butcher tableau is as below:

$$y_{n+1}-y_n=\frac{h}{12}(2k_1+8k_3+k_4+k_5)$$

$$k_1=f(x_n,y_n)$$

$$k_2=f\left(x_n+\frac{h}{3},y_n+\frac{h}{3}k_1\right)$$

$$k_3=f\left(x_n+\frac{h}{2}k_1+y_n+\frac{h}{2}k_2\right)$$

$$k_4=f(x_n+h,y_n+h(-3k_1+5k_2-k_3))$$

$$k_5=f(x_n+h,y_n+h(3k_1-3k_3+k_4))$$

| | | | | | |
|---|---|---|---|---|---|
| 0 | | | | | |
| ⅓ | ⅓ | | | | |
| ½ | 0 | ½ | | | |
| 1 | -3 | 5 | -1 | | |
| 1 | 3 | 0 | -3 | 1 | |
| | 1/6 | 0 | 2/3 | 1/12 | 1/12 |

**GARBA and YAKUBU (1999)**, derived a new R-K formulae of order five, which does not require the use of error control strategy, but has better approximations than some existing R-K formulae.

**ADEBOYE, K.R and OCHOCHE, A (2006)**, developed a fifth order six-stage Runge-Kutta method for solving initial value problem. The strength of the new scheme is that, it gives solutions that are very close to the exact solution, even closer than some popular existing methods which are known to be highly efficient. Some initial value problems were solved using the new scheme and the results help to establish its very high degree of accuracy.

The new six-stage Runge-Kutta methods of order five is

$$y_{n+1}=y_n+\frac{h}{90}[7k_1+7k_2+12k_3+32k_5+32k_6]$$

Where

$$k_1=f(x_n,y_n)$$

$$k_2=f(x_n+h,y_n+hk_1)$$

$$k_3=f\left(x_n+\frac{h}{2},y_n+h(1.249655737k_1-0.749655737k_2)\right)$$

$$k_4=f\left(x_n+\frac{h}{5},y_n-h(0.0.7015442631k_1-0.5600588106k_2-0.341486157k_3)\right)$$

$$k_5=f\left(x_n+\frac{h}{4},y_n-h(0.25710705k_1+0.045073568k_2+0.353037791k_3-0.341486157k_3)\right)$$

$$k_6=f\left(x_n+\frac{3h}{4},y_n-h(0.754830268k_1-0.290909052k_2-0.331676697k_3-1.359792241k_4)\right)$$

## 2.2 Numerical Methods For Initial Value Problem (IVPs)

In the previous chapter, we made an introduction into what numerical methods (solution techniques) for solving (IVPS) are all about. A great many of such methods have been developed, and yet many more are still being produced. Although all the methods have certain fundamental properties common to them all of them are classified into different sub-classes, with specific characteristics peculiar to each class. It is this classification of numerical method (solution techniques) we shall discuss in this chapter.

## 2.3 One-step Methods

One-step methods are numerical methods that determine the solution at the support times through the reccursive formula

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h); n \in N \text{.................}(2.0)$$

i. e

k=1 in the formula

$$\sum_{j=o}^{k} a_i y_{n+j} = h\phi_f(y_{n+k}, y_{n+k-1}, \ldots\ldots, y_n, t_n; h)$$

There are two families of one-step methods:

Method of Taylor and method of Runge-Kutta, i.e method of Taylor type are further classified into Euler method and method of Taylor of greater order. However, Euler method, if we take the first two terms of the Taylor series, which describes the exact solution at $x_{n+1}$

$$y(x_{n+1}) = \sum_{r=o}^{\infty} \frac{h}{r!} y^{(r)}(x_n)$$

to compute

$y(x_1) \approx y(x_0) + hy'(x_0) = y_0 + hf_0 \equiv y_1$

After n steps it yields

$y_{n+1} = y_n + hf_n$..........(2.1)

Equation (2.1) above is called Euler's formula or the Euler method, the simplest of the numerical methods for solving first order differential equations.

Although Euler method is simple in procedure, it is lacking in accuracy especially away from the starter value of the initial condition. And it is of use only for very small values of the interval h Stroud (1996).

Similarly, methods of Taylor of greater order, in order to obtain a numerical method with greater order of accuracy than the Euler method, we could just as well take more terms of the Taylor's series expansion. A method of second order like this:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n$$

Since $y'_n = f(x_n, y_n) = f_n$ then

$$y_{n+1} = y_n + hy_n + \frac{h^2}{2} f'_n$$

This implies the truncation error

$$E_2 = y(x_1) - y_1 = \frac{h^3}{6} y'''(\xi) \approx 0(h^3)$$

More generally, a k-order numerical method is $y_{n+1} = y_n + hf_n + \frac{h^2}{2} y''_n + - - - - - \frac{h^k}{k!} y_n^k$

With a truncation error

$$E_k = y(x_1) - y_1 = \frac{h^{k+1}}{(k+1)!} y^{k+1}(\xi), \xi \in [x_o, x_1]$$

### 2.3.1 Runge-Kutta Methods

The idea of extending the Euler method, by allowing for a multiplicity of evaluations of the function f within each steps, was originally proposed by Runge (1895), further contributions were made by Heun (1900) and by Kutta (1901).

Given $y_n$ as an approximation to $y(x_n)$, where y satisfies the differential equation system.

$$y'(x) = f(x, y), y(x_0) = y_0, f : RxR^m \to R^m$$

The approximation $y_{n+1}$ to $y(y_{n+1})$ is computed by evaluating

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2)$$

where

$$k_i = f\left(x_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij} k_i\right), i = 1,2,\ldots\ldots,s \quad \ldots\ldots (2.3)$$

Lambert (1991)

An alternative term of the above, is,

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i); i = 1,2,,,,,s$$

where

$$Y_i = y_n + h\sum_{i=1}^{s} a_{ij} f(x_n + c_i h, y_i), i = 1,,2,\ldots.,s \quad \ldots\ldots\ldots\ldots (2.4)$$

The two forms are equivalent by making the interpretation $k_i = f(x_n + c_i h, Y_i); i = 1,2,\ldots\ldots\ldots s$

Lambert (1991)

The integer S is the number of stages of the method and measure it's complexity, since the number of the evaluations of f per step equals s. The set $(a_{ij},b_i)$ i=1, .. ,s of constants characterize a particular method of this type.

The quantities $Y_i$ are approximations to solutions values y(x) to x ranging through various values near $x_n$. Also $f(Y_i)$ are approximations to $y_i'(x)$ at the same values x.

Patrizia (2001).

Runge-Kutta methods are often represented using the Butcher array as follows:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ c_s & a_{1s} & a_{2s} & \cdots & a_{5s} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

An S-stage Runge-Kutta method is completely specified by its butcher array as

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array}$$

$C = [c_1, c_2, \ldots, c_s]^T$, $b = \{b_1, b_2, \ldots, b_s]^T$, $A = (a_{ij})$

The components of C are the row sums of A

Lambert (1991)

From the definition a Runge-Kutta is consistent when

$$\sum_{i=1}^{s} b_i = 1$$ Lambert (1991)

And when $y_{n+1}$ depends only upon the evaluations of the previous points $f(y_i)$; i=0,1,.....n, if $a_{ij} = o$ for all $1 \le i \le j \le s$, it is called EXPLICIT. Otherwise it is said be IMPLICIT.

We present below some explicit Runge-Kutta methods:

**One Stage**:

The general s-stage Runge-Kutta method (1.9) becomes 1-stage if we set $b_2 = b_3 = 0$

then

$$y_{n+1} = y(x_n) + hb_1 f \dots\dots\dots\dots\dots\dots(2.5)$$

From the Taylor expansion follows that the best one can do is set $b_1=1$, hence

$E_{n+1} = o(h^2)$.

Thus there exists only one explicit one-stage Runge-Kutta method of order 1, namely Euler's Rule. Lambert (1991)

**Two Stage**:

If we set $b_3 = 0$, the method becomes two stage

$$y(x_{n+1}) = y(x_n) + h(b_1 + b_2)f + h^2 b_2 c_2 F + \frac{1}{2}h^3 b_2 c_2^2 G + 0(h^4) \dots\dots\dots\dots\dots\dots(2.6)$$

Where

$F := f_x + ff_y$, $G := f_{xx} + 2ff_{yy} + f^2 f_{yy}$ Lambert (1991)

On comparison with the expansion for $y(x_{n+1})$,

$$y(x_{n+1}) = y(x_n) + hf + \frac{1}{2}h^2 F + \frac{1}{6}h^3 (Ff_y + G) + 0(h^4) \dots\dots\dots\dots\dots\dots(2.7)$$

we see that order 2 can be achieved by choosing

$b_1 + b_2 = 1$, $b_2 c_2 = {}^1/_2$

There exist an infinite family of explicit two-stage

Runge-Kutta method of order 2

Two solutions yield well-known methods:

(i) The modified Euler (or improved polygon) method

$b_1 = 0$, $b_2 = 1$, $c_2 = {}^1/_2$.

it Butcher array is

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

(ii) The improved Euler (or Heun) method

$b_1 = b_2 = ½, c_2 = 1$

It Butcher array is

| | | |
|---|---|---|
| 0 | | |
| 1 | 1 | |
| | ½ | ½ |

**Three stage:**

By satisfying the following coefficient conditions one can achieve order 3

$b_1+b_2+b_3 = 1$

$b_2c_2 + b_3c_3 = {}^1/_2$

$b_2 c_2^2 + b_3 c_3^2 = {}^1/_3$

$b_3b_2a_{32} = {}^1/_6$

Two particular solutions lead to well known methods

(i) Heun's third order formula.

it Butcher array is

| | | | |
|---|---|---|---|
| 0 | | | |
| $^1/_3$ | $^1/_3$ | | |
| $^2/_3$ | 0 | $^2/_3$ | |
| | ¼ | 0 | ¾ |

(ii) Kutta's third order formula

it has the Butcher array

| | | | |
|---|---|---|---|
| 0 | | | |
| ½ | ½ | | |
| 1 | -1 | 2 | |
| | $^1/_6$ | $^2/_3$ | $^1/_6$ |

**Four Stage:**

The most popular Runge-Kutta scheme is the classical Runge-Kutta method of order four (4), so popular is this method that when one sees a reference to a problem having been solved by the Runge-Kutta method, it is most certainly the classical Runge-Kutta method that has been used.

It has the following Butcher array.

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

The classical Runge-Kutta scheme is as follows:

$$y_{n+1}=y_n+\frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

Where

$$k_1=f(x_n,y_n)$$

$$k_2=f\left(x_n+\frac{h}{2},y_n+\frac{h}{2}k_1\right)$$

$$k_3=f\left(x_n+\frac{h}{2},y_n+\frac{h}{2}k_2\right)$$

$$k_4=f(x_n+h,y_n+hk_3)$$

The absence of $k_1$ in the evaluation of $k_3$, and absence of $k_1$ $k_2$ in the evaluation of $k_4$ may have played a role in making this method popular. However, Lambert (1991) suggests another reason for the popularity of the method:

"in the pre-computer days, computations were performed on purely mechanical devices. Multiplication or division was tiresome business on such machines. Since the main computation is in the evaluation of the functions to produce the $k_i$'s. That the $c_i$'s and $a_i$'s are always either 1 or ½ increased the chances of any division in the evaluation of f terminating quickly".

## 2.4 Multistep Methods:

As stated in the previous chapter, we can write a numerical method for solving IVPs in the general form:

$$\sum_{j=o}^{k} a_j y_{n+j} = h\phi_f(y_{n+k}, y_{n+k-1}, \ldots\ldots; y_n, t_n; h) \ldots\ldots\ldots\ldots(2.8)$$

$y_i = \mu_i(h)$ for $i = o, \ldots, k-1$

If k>1 in the above formula then the numerical method is called multistep, because it determines the solution at the support times using k values. Patrizia (2001).

**Linear multistep method. (LMM)**:

Let $y_n$ be an approximation to the theoretical solution at $x_n$, that is, to $y(x_n)$, and let $f_n = f(x_n, y_n,)$. Then, we say a linear multistep method of step number k, or a linear k-step method is a computational method for determining, the sequence $[y_n]$ that takes the form of linear relationship between $y_{n+j}$, $f_{n+j}$, $j = 0, 1, \ldots\ldots, k$

Thus the general LMM may be written

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \ldots\ldots\ldots\ldots(2.9)$$

Where $\alpha_j$ and $\beta_j$ are constants; we assume $\alpha_k = 1$ and that not both $\alpha_o$ and $\beta_o$ are zero.

### 2.4.1 Adam-Bashforth Method

We consider the general Adams multi-step method

$$y_{n+1} = y_n + h\sum_{j=0}^{k} \beta_j f(x_{n+1-j}, y_{n+1-j}) \ldots\ldots\ldots\ldots(2.10)$$

to approximate the solution y at $y_{n+1}$, the equation (2.10) represent k-step method, since if uses the k-previous values of the computation in order to compute the next value.

If the coefficient $\beta_k$ is zero i.e $\beta_k = 0$, then equation (2.10) define $y_{n+1}$ explicity interms of previous k values and such a method is called an explicit method or Adam-Bashforth method.

An k-step Adam-Bashforth method has global truncation error of order 0 ($h^k$).

Examples are

(i) 1 – step Adam-Bashforth method which is Euler's rule

$$y_{n+1} = y_n + hf_n$$

(ii) 2- step Adam-Bashforth method which is midpoint rule

$$y_{n+2} - y_n = 2hf_{n+1}$$

i.e $y_{n+2} = y_n + 2hf_{n+1}$

(iii) 2-step Adam-Bashforth method

$$y_{n+2} = y_{n+1} + \frac{h}{2}(3f_{n+1} - f_n)$$

### 2.4.2 **Adam-Moulton Method:**

If the coefficient $\beta_k$ is non-zero i.e $\beta_k \neq 0$; the equation (2.10) define $y_{n+1}$ implicitly, thus is called implicit method or Adam-Moulton Method.

An k-step Adam-Moulton Method is of order $0(h^{k+1})$

Examples

(i) 1-step Adam-Moulton Method is the Trapezoidal Rule

$$y_{n+2} = y_n + \frac{h}{2}(f_{n+1} + f_n)$$

(ii) 2-step Adam-Moulton Method is the Simpson's rule

$$y_{n+2} = y_n + \frac{h}{3}(f_{n+2} + 4f_{n+1} + f_n)$$

# CHAPTER THREE

## 3.0 DERIVATION OF QUASI-RUNGE-KUTTA METHODS

### 3.1 Finite Difference Method Error Term

A finite Difference is a mathematical expression of the form $f(x+b)-f(x+a)$. If a finite difference is divided by b-a one gets a differential quotient.

The approximation of the derivatives by finite difference plays a central role in finite difference methods for the numerical solution of differential equations, especially boundary value problems (IVPs).

In mathematical analysis, operators involving finite differences are studied. A difference operator is an operator which maps a function F to a function whose values are the corresponding finite differences only three forms are currently considered, forward, backward and central differences.

A forward difference is an expression of the form.

$$\Delta_h[f](x)=f(x+h)-f(x)$$

Depending on the application, the spacing "h" it may be variable or held constant.

A backward difference uses the function values at x and x-h, instead of the values at x+h and x

$$\nabla_h[f](x)=f(x)-f(x-h)$$

Finally, the central difference is given by

$$\partial_h[f](x)=f(x+\tfrac{1}{2}h)-f(x-\tfrac{1}{2}h)$$

An important application of finite difference is in numerical analysis, especially in numerical ordinary differential equations which aim at numerical solution of ordinary equations. The idea is to replace the derivatives appearing in the differential equation by finite difference that approximate them. The resulting methods are called finite difference methods.

The derivatives of a function $F$ at a point x is defined by the limit.

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

If $h$ has a fixed (non-zero) value, instead of approaching zero then the right hand side is

$$\frac{f(x+h) - f(x)}{h} = \frac{\Delta_h[f](x)}{h}$$

hence, the forward difference divided by $h$ approximate the derivatives when $h$ is small. The error in this approximation can be derived from Taylor's theorem.

Assuming that $F$ is continuously differentiable, the error is

$$\frac{\Delta_h[f](x)}{h} - f'(x) = o(h)\ (h \to 0)$$

The same formula holds for the backward difference

$$\frac{\nabla_h[f](x)}{h} - f'(x) = o(h)$$

However, the central difference yields a more accurate approximation. It errors is proportional to square of the spacing (if $F$ is twice continuously differentiable)

$$\frac{\partial_h[f](x)}{h} - f'(x) = o(h^2)$$

**Higher – Order Difference**

In an analogous way one can obtain finite difference approximations to higher order derivatives and differential operators.

For example, by using the above central difference formula for $f\left(x+\frac{h}{2}\right) and\ f\left(x-\frac{h}{2}\right)$ and applying a central difference formula for the derivative of $f$ at $x$, we obtain the central difference approximation of the second derivative of $f$.

$$F'_{(x)} \approx \frac{\partial_h^2[f](x)}{h} = \frac{f(x+h)-2f(x)+f(x-h).}{h^2}$$

More generally, the n[th] – order forward, backward and central differences are respectively given by

$$\Delta_h^n[f](x) = \sum_{i=0}^{n}(-1)^i \binom{n}{i} f(x+(n-i)h),$$

$$\nabla_h^n[f](x) = \sum_{i=0}^{n}(-1)^i f(x-ih),$$

$$\partial_h^n[f](x) = \sum_{i=0}^{n}(-1)^i f\left(x+\left(\frac{n}{2}-i\right)h\right).$$

$$\therefore \frac{d^n f}{dx^n}(x) = \frac{\Delta_h^n[f](x)}{h^n} + 0(h) = \frac{\nabla_h^n[f](x)}{h^n} + 0(h) = \frac{\partial_h^n[f](x)}{h^n} + 0(h^2)$$

## 3.2 Refinement Process For Euler Method

We consider the Euler method

$y_{n+1} = y_n + hf_n$,...... ...............(3.0)

The Error term is

$y_{n+1} - y_n - hf_n$..................(3.1)

We expand $y_{n+1}$ in Taylor's series

i.e $y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n$...........................(3.2)

Note $y'_n = f_n$

Then substitute equation (3.2) into equation (3.1)

We have

$$y_{n+1} - y_n - hf_n \approx y_n + hy'_n + \frac{h^2 y''_n}{2} + \frac{h^3 y'''_n}{6} + \text{--------} y_n - hf_n$$

$$= \frac{h^2 y_n''}{2} + \frac{h^3 y_n'''}{6} + - - - - - - - - -$$

$$=\rangle \frac{h^2 y_n''}{2} + 0(h^3)$$

Now, representing $y_n'$ in terms of $f_n$

i.e $y_n'' = f_n'$

Therefore

$$\frac{h^2 y_n''}{2} = \frac{h^2 [f_n']}{2} = \frac{h^2}{2}\left[\frac{f_{n+1/2} - f_n}{h/2}\right]$$

$$= h[f_{n+1/2} - f_n] \text{..........................} (3.3)$$

Add equation (3.3) to equation (3.0) above, we have

$$y_{n+1} = y_n + hf_n + h[f_{n+1/2} - f_n]$$

$$y_{n+1} = y_n + hf_n + hf_{n+1/2} - hf_n$$

$$\therefore y_{n+1} = y_n + hf_{n+1/2} \text{.....................} (3.4)$$

Again, we find the Error term of equation (3.4)

$$\therefore y_{n+1} - y_n - hf_{n+1/2} \text{.....................} (3.5)$$

We now, expand $y_{n+1}$ and $f_{n+1/2}$ in Taylor's series

i.e

$$y_{n+1} - y_n + hy_n' + \frac{h^2 y_n''}{2} + \frac{h^3 y_n'''}{8} + - - - - - - - (3.6)$$

$$f_{n+1/2} = f_n + \frac{h^2 f_n'}{2} + \frac{h^3 f_n''}{8} + - - - - - - - (3.7)$$

Then substitute equation (3.6) and (3.7) into equation (3.5)

$$y_{n+1} - y_n - hf_{n+1/2} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + ---- y_n - hf_n - \frac{h^2}{2}f'_n - \frac{h^3}{8}f''_n + --$$

$$\therefore y_{n+1} - y_n - hf_{n+1/2} = \left[\frac{h^3}{6} - \frac{h^3}{8}\right]f''_n = \frac{h^3}{24}f''_n$$

$$\frac{h^3}{24}f''_n = \frac{h^3}{24}\left[\frac{f'_{n+1/2} - f'_n}{h/2}\right] = \frac{h^2}{12}\left[f'_{n+1/2} - f_n'\right]$$

$$= \frac{h^2}{12}\left[\frac{f_{n+1} - 2f_{n+1/2} + f_n}{h^2}\right] = \frac{h}{6}\left[f_{n+1} - 2f_{n+1/2} + f_n\right]$$

$$\therefore \frac{h^3}{24}f'_n = \frac{h}{6}\left[f_{n+1} - 2f_{n+1/2} + f_n\right] \text{......................} (3.8)$$

Again, we add equation (3.8) to equation (3,5) above,

We have

$$y_{n+1} = y_n + hf_{n+1/2} + \frac{h}{6}\left[f_{n+1} - 2f_{n+1/2} + f_n\right]$$

$$\therefore y_{n+1} = y_n + \frac{h}{6}\left[f_{n+1} + 4f_{n+1/2} + f_n\right] \text{.................} (3.9)$$

Now, we consider

$$y_{n+1} = y_n + hf_n \text{.............} (3.10)$$

has the Error term $\frac{h^2 f'_n}{2} + 0(h^3)$

but

$$\frac{h^2 f'_n}{2} = \frac{h^2}{2}\left[\frac{f_{n+1} - f_n}{h}\right] \equiv \frac{h}{2}\left[f_{n+1} - f_n\right] \text{.........................} (3.11)$$

Substitute equation (3.11) into equation (3.10), we have.

$$y_{n+1} = y_n + hf_n + \frac{h}{2}\left[f_{n+1} + f_n\right]$$

$$y_{n+1}=y_n+\frac{h}{2}[f_n+f_{n+1}]\text{..............................}(3.12)$$

To find the Error term of equation (3.12), we expand $y_{n+1}$ and $f_{n+1}$ into Taylor's series.

i.e

$$y_n+hy'_n+\frac{h^2}{2}y''_n+\frac{h^3}{6}y'''_n+-----y_n-\frac{h^2}{2}f_n-\frac{h^2}{2}\left[f_n+hf'_n+\frac{h2}{2}f''_n+---\right]$$

$$\therefore y_n+hy'_n+\frac{h^2}{2}y''_n+\frac{h^3}{6}y'''_n-y_n-\frac{2h}{2}f_n-\frac{h^2}{2}f_n-\frac{h^3}{4}f''_n+$$

$$\therefore \frac{h^3}{6}y'''_n-\frac{h^3}{4}f''_n=\left[\frac{h^3}{6}-\frac{h^3}{4}\right]f''_n=\frac{-h^3}{12}f''_n$$

Note $y'''_n=f''_n$

Therefore equation (3.12) becomes

$$y_{n+1}=y_n+\frac{h}{2}[f_n+f_{n+1}]-\frac{h^3}{12}f''_n\text{ ..............................}(3.13)$$

$$\text{But } \frac{h^3}{12}f''_n=\frac{h^3}{12}\left[\frac{f'_{n+1/3}-f'_n}{h/3}\right]=\frac{h^2}{4}\left[f'_{n+1/3}+f'_n\right]$$

$$=\frac{h^2}{4}\left[\frac{f_{n+1}-f_{n+1/2}}{h/3}\right]-\frac{h^2}{4}\left[\frac{f_{n+1/3}-f_n}{h/3}\right]$$

$$=\frac{3h}{4}(2f_{n+1}-2f_{n+1/3})-\frac{3h}{4}(f_{n+1/3}-f_n)$$

$$=\frac{h}{4}[6f_{n+1}-6f_{n+1/3}-3f_{n+1/3}+3f_n]$$

$$\therefore \frac{h^3}{12}f''_n=\frac{h}{4}[6f_{n+1}-9f_{n+1/3}+3f_n]------------------(3.14)$$

Substitutes equation (3.14) into equation (3.13) above.

We have,

$$y_{n+1}=y_n+\frac{h}{2}[f_n+f_{n+1}]-\frac{h}{4}[6f_{n+1}-9f_{n+1/3}+3f_n]$$

$$\therefore y_{n+1} = y_n + \frac{h}{4}[-4f_{n+1} + 9f_{n+1/3} - f_n] \text{........................}(3.15)$$

## 3.3 Refinement Process For Mid-Point Method

**We consider the mid point method**

$$\therefore y_{n+2} = y_n + 2hf_{n+1} \text{........................}(3.16)$$

We expand $y_{n+2}$ and $f_{n+1}$ to find the Error term

$$\therefore y_n + 2hy'_n + \frac{4h^2}{2}y''_n + \frac{8h^3}{6}y'''_n + ----y_n - 2h\left[f_n + hf'_n + \frac{h^2}{2}f''_n\right]$$

$$\Rightarrow y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n --- y_n - 2hf_n - 2h^2f'_n - h^3f''_n$$

$$= \left[\frac{4h^3}{3} - h^3\right]f''_n \Rightarrow \frac{h^3}{3}f''_n$$

$$\therefore \textit{The Error term is } \frac{h^3}{3}f''_n$$

$$\frac{h^3}{12}f''_n = \frac{h^3}{3}\left[\frac{f'_{n+1/2} - f'_n}{h/2}\right] = \frac{2h^2}{3}[f'_{n+1/2} + f'_n]$$

$$\frac{4h}{3}[f_{n+1} + f_{n+1/2}] - \frac{2h}{3}[f_{n+1} - f_n]$$

$$= \frac{h}{3}[2f_{n+1} + 4f_{n+1/2} + 2f_n] \text{..............................}(3.17)$$

We substitute equation (3.17) into equation (3.16) above.

$$\therefore y_{n+2} = y_n + 2hf_{n+1} + \frac{h}{3}[2f_{n+1} - 4f_{n+1/2} + 2f_n]$$

$$\therefore y_{n+2} = y_n + \frac{h}{3}[8f_{n+1} - 4f_{n+1/2} + 2f_n] \text{......................}(3.18)$$

Again, we consider

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h^3}{3}f_n^{11} \text{........................}(3.19)$$

$$\frac{h^3}{3}f_n'' = \frac{h^3}{3}\left[\frac{f_{n+1}-2f_n+f_{n-1}}{h^2}\right]$$

$$= \frac{h}{3}\left[f_{n+1}-2f_n+f_{n-1}\right] \qquad (3.20)$$

Substitute equation (3.20) into equation(3.19) above.

i.e

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h}{3}\left[f_{n+1} - 2f_n + f_{n-1}\right]$$

$$\Rightarrow y_{n+2} = y_n + \frac{h}{3}\left[7f_{n+1} - 2f_n + f_{n-1}\right] \qquad (3.21)$$

Again, consider

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h^3}{3}f_n'' \qquad (3.22)$$

$$\frac{h^3}{3}f_n'' = \frac{h^3}{3}\left[\frac{f'_{n+7/12} - f'_n}{7h/12}\right]\frac{12}{7h}$$

$$= \frac{4h^2}{7}\left[f'_{n+7/12} - f'_n\right]$$

$$= \frac{4h^2}{7}\left[\frac{f_{n+1} - f_{n+7/12}}{5h/12}\right] - \frac{4h^2}{7}\left[\frac{f_{n+1} - f_n}{h}\right]$$

$$= \frac{h}{35}\left[48f_{n+1} - 48f_{n+7/12} - 20f_{n+1} + 20f_n\right]$$

$$= \frac{h}{35}\left[28f_{n+1} - 48f_{n+7/12} + 20f_n\right]$$

$$\therefore \frac{h^3 f_n''}{3} = \frac{h}{35}\left[28f_{n+1} - 48f_{n+7/12} + 20f_n\right] \qquad (3.23)$$

Substitute (3.23) into equation(3.22) above.

i.e

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h}{35}\left[28f_{n+1} - 48f_{n+7/12} + 20f_n\right]$$

$$\therefore y_{n+2} = y_n + \frac{h}{35}\left[98f_{n+1} - 48f_{n+7/12} + 20f_n\right] \text{.................(3.24)}$$

Again, we consider

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h^3}{3}f_n'' \text{ ..............................................(3.25)}$$

$$\frac{h^3}{3}f_n'' = \frac{h^3}{3}\left[f'_{n+3/5} - f'_n\right] = \frac{5h^2}{9}\left[f'_{n+3/5} - f'_n\right]$$

$$= \frac{5h^2}{9}\left[\frac{f_{n+1} - f_{n+3/5}}{2h/5}\right] - \frac{5h^2}{9}\left[\frac{f_{n+1} - f_n}{h}\right]$$

$$= \frac{25h}{18}\left[f_{n+1} - f_{n+3/5}\right] - \frac{10h}{18}\left[f_{n+1} - f_n\right]$$

$$\therefore \frac{h^3 f_n''}{3} = \frac{h}{18}\left[15f_{n+1} - 25f_{n+3/5} + 10f_n\right] \text{.......................(3.26)}$$

Therefore, Equation (3.25) becomes

$$y_{n+2} = y_n + 2hf_{n+1} + \frac{h}{18}\left[15f_{n+1} - 25f_{n+3/5} + 10f_n\right]$$

$$y_{n+2} = y_n + \frac{h}{18}\left[51f_{n+1} + 10f_n - 25f_{n+3/5}\right] \text{.............(3.27)}$$

## 3.4 Refinement Process For Multi-Step Method

we consider the scheme:

$$y_{n+2} = y_{n+1} + \frac{h}{2}\left[3f_{n+1} - f_n\right] \text{.................................(3.28)}$$

The Error term is

$$y_{n+2} - y_{n+1} - \frac{h}{2}\left[3f_{n+1} - f_n\right] \text{.................................(3.29)}$$

We expand $y_{n+2}, y_{n+1}$ and $f_{n+1}$ in Taylor's series

i.e

$$y_n + 2hy'_n + 2h^2 y''_n + \frac{4}{3}h^3 y'''_n - y_n - hy'_n - \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n - \frac{3h}{2}\left[f_n + hf'_n + \frac{h}{2}f''_n\right] + \frac{h}{2}f_n$$

$$\Rightarrow y_n + 2hy'_n + 2h^2 y'_n + \frac{4}{3}h^3 y'''_n - y_n - hy'_n - \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n - \frac{3h}{2}f_n - \frac{3h^2}{2}f'_n - \frac{3h^3}{4}f''_n + \frac{h}{2}f_n$$

$$\therefore \left[\frac{4h^3}{3} - \frac{h^3}{6}\right]y'''_n - \frac{3h^3}{4}f''_n = \frac{5h^3}{12}f''_n$$

The Error term is $\frac{5h^3}{12}f''_n$

$$\frac{5h^3 f''_n}{12} = \frac{5h^3}{12}\left[\frac{f_{n+1} - 2f_n + f_{n-1}}{h^2}\right] = \frac{5h}{12}\left[f_{n+1} - 2f_n + f_{n-1}\right] \text{..............(3.30)}$$

We substitute (3.30) into equation (3.29)

i.e

$$y_{n+2} = y_{n+1} + \frac{h}{2}\left[3f_{n+1} - f_n\right] + \frac{5h}{12}\left[f_{n+1} - 2f_n + f_{n-1}\right]$$

$$y_{n+2} = y_{n+1} + \frac{h}{12}\left[18f_{n+1} - 6f_n + 5f_{n+1} - 10f_n + 5f_{n-1}\right]$$

$$y_{n+2} = y_{n+1} + \frac{h}{12}\left[23f_{n+1} - 16f_n + 5f_{n-1}\right] \text{........................(3.31)}$$

Again, consider

$$y_{n+2} = y_{n+1} + \frac{h}{2}\left[3f_{n+1} - f_n\right] + \frac{5h^3}{12}f''_n \text{ ............(3.32)}$$

$$\frac{5h^3}{12}f''_n = \frac{5h^3}{12}\left[\frac{f'_{n+7/12} - f'_n}{7h/12}\right] = \frac{5h^2}{7}\left[f'_{n+7/12} - f'_n\right]$$

$$\left[\frac{5h^2/7\left[f_{n+1} - f_{n+7/12}\right]}{5h/12}\right] - \left[\frac{5h^2/7\left[f_{n+1} - f_n\right]}{h}\right]$$

$$= \frac{12h}{7}\left[f_{n+1} - f_{n+7/12}\right] - \frac{5h}{7}\left[f_{n+1} - f_n\right]$$

$$= \frac{h}{7}[12f_{n+1} - 12f_{n+7/12} - 5f_{n+1} + 5f_n]$$

$$\frac{5h^3 f_n^{11}}{12} = \frac{h}{7}\left[7f_{n+1} - 12f_{n+7/12} + 5f_n\right] \quad (3.33)$$

Substitute (3.33) into eqn (3.32)

$$\therefore y_{n+2} = y_{n+1} + \frac{h}{2}[3f_{n+1} - f_n] + \frac{h}{7}[7f_{n+1} - 12f_{n+7/12} + 5f_n]$$

$$y_{n+2} = y_{n+1} + \frac{h}{14}[21f_{n+1} - 7f_n + 14f_{n+1} - 24f_{n+7/12} + 10f_n]$$

$$\therefore y_{n+2} = y_{n+1} + \frac{h}{14}[35f_{n+1} + 3f_n - 24f_{n+7/12}] \quad (3.34)$$

## 3.5 Convergence Of The Methods

Numerical method is convergent if

$$\lim_{h \to 0} \max_{n=0,1\ldots x} \|e_n\| = 0 \quad (3.35)$$

To prove that a linear multistep method is convergence, it is sufficient for us to show that the method is consistent as well as zero-stable.

**CONSISTENCY**

A numerical method is consistent if the local truncation error satisfies.

$$\lim_{h \to 0} \theta_{n+k} = 0 \quad (3.36)$$

The necessary and sufficient conditions, which must be satisfied by a numerical method to be consistent are.

$$\sum_{j=0}^{k} \alpha_j = 0$$

And

$$\frac{\varphi_f[y(t_{n+k}), y(t_{n+k-1}), \ldots\ldots, y(t_n), t_n; 0] = f(y(t_n))}{\sum_{j=0}^{k} j\alpha_j = 0} \quad (3.37)$$

Which using the first characteristics polynomial $\rho(\xi)=\sum_{j=0}^{k}\alpha_j.\xi_j$

$\xi \in c$, it is possible to write the two qualities in the usual form

$\rho(1)=0$..........(3.38) and

$$\frac{\Phi f[y(t_{n+k}),y(t_{n+k-1}),............,y(t_n),t_n;0]}{\rho^1(1)}=f(y(t_n))...........(3.39)$$

Patrizia (2001)

for linear multistep methods, consistency demands that

(i) $\rho(1)=0$

(ii) $\rho'(1)=\sigma(1)$; where $\sigma(\xi)=\sum_{j=0}^{k}\beta_j\xi^j$ ...............(3.40)

**ZERO-STABILITY**

A numerical method is said to be zero-stability when it satisfies the ROOT CONDITIONS.

ROOT CONDITIONS:

A numerical method is said to satisfy the ROOT CONDITION if all of the roots of the first characteristic polynomial have modulus less then or equal to unity and those with modulus unity are simple. Lambert (1991)

From the above facts, we conclude that the necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and Zero-stable. That is, it must satisfy the following conditions;

(i) $\rho(1)=0$

(ii) $$\frac{\phi_f(y(t_{n+k}),y(t_{n+k-1}),......y(t_n),t_n;0)}{\rho'(1)}=f(y(t_n))$$

(iii) No root of the equation: $\rho(\xi)=0$ has modulus greater than 1 and every root with modulus 1 is simple.

Where $\rho(\xi)=\sum_{j=0}^{k}\alpha_j\xi^j$.................(3.41)

**Scheme 1**

$$y_{n+2}=y_n+\frac{h}{3}[7f_{n+1}-2f_n+f_{n-1}]..................(3.42)$$

The error term is

$$y_{n+2}-y_n=\frac{h}{3}[7f_{n+1}-2f_n+f_{n-1}]...............(3.43)$$

**Convergence**

To prove that scheme 1, given by equation (3.42) converges, it is sufficient for us to show that it is consistent as well as zero-stable.

**Consistency**

From equation (3.43), the first characteristic polynomial $\rho(\xi)$ is given by

$$p(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j...............(3.44)$$

$$\rho'(\xi)=\sum_{j=0}^{2}j\alpha_j\xi^{j-1}$$

$$\rho(1)=\sum_{j=0}^{2}\alpha_j=1-1=0.............(3.45)$$

$$p'(1)=\sum_{j-0}^{2}j\alpha_j=2(1)-0(1)=2.............(3.46)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{2}\beta_j\xi^j$$

$$\sigma(1)=\sum_{j=0}^{2}\beta_j=\frac{7}{3}-\frac{2}{3}+\frac{1}{3}=\frac{1}{6}=2................(3.47)$$

From equations (3.45), (3.46) and (3.47), we have

(i) $\rho(1)=0$

(ii) $\rho'(1) = \sigma(1)$..............(3.48)

hence, scheme 1 is consistent

**Zero-stability**

For scheme 1 to be zero stable it must satisfy the zero-stability condition.

Zero-stability conditions are

(i) Each root of the first characteristic polynomial must be of modulus not greater than 1

(ii) Any root with modulus 1 must be simple.

$\rho(\xi) = \xi^2 - 1$

i.e $(\xi + 1)(\xi - 1)$, we have two real roots

Either $\xi = -1$ or $\xi = 1$, thus $\xi = -1,1$ which satisfy the zero stability condition.

Hence, since scheme 1 satisfied these conditions,

(i) $\rho(1) = 0$

(ii) $\rho'(1) = \sigma(1)$

(iii) Zero –stability condition

We conclude that scheme 1 is convergent.

**Scheme 2**

$$y_{n+2} = y_n + \frac{h}{3}[8f_{n+1} - 4f_{n+1/2} + 2f_n] \text{.................}(3.49)$$

The error term is

$$y_{n+2} - y_n = \frac{h}{3}[8f_{n+1} - 4f_{n+1/2} + 2f_n] \text{.............}(3.50)$$

**Consistency**

From equation (3.50), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j \ldots\ldots\ldots\ldots\ldots(3.51)$$

$$\rho(\xi)=\xi^2-1$$

$$\rho(1)=1-1=0\ldots\ldots\ldots\ldots\ldots(3.52)$$

$$\rho'(\xi)=2\xi$$

$$\rho'(1)=2(1)-0(1)=2\ldots\ldots\ldots\ldots(3.53)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j$$

$$\sigma(\xi)=\frac{8}{3}-\frac{4}{3}-\frac{2}{3}=\frac{6}{3}=2\ldots\ldots\ldots\ldots(3.54)$$

From equations (3.52), (3.53), and (3.54), we have

(i) $\rho(1)=0$

(ii) $\rho'=\sigma(1)$

Hence scheme 2 is consistent.

**Zero-stability**

The roots of the first characteristics polynomial are $\rho(\xi)=\xi^2-1$

i.e $(\xi+1)(\xi-1)$, we have two real roots

Either $\xi=-1$ or $\xi=1$

Thus $\xi=-1,1$ which satisfy the zero stability condition

We conclude that scheme 2 is convergent.

**Scheme 3**

$$y_{n+2}=y_n+\frac{h}{18}[51f_{n+1}+20f_n-25f_{n+3/5}]\ldots\ldots\ldots\ldots\ldots(3.55)$$

The Error term is

$$y_{n+2}-y_n=\frac{h}{18}[51f_{n+1}+20f_n-25f_{n+3/5}]\ldots\ldots\ldots\ldots\ldots(3.56)$$

**Consistency**

From equation (3.56), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j$$

$$\rho(\xi)=\xi^2-1$$
$$\rho(1)=1-1=0\ldots\ldots\ldots\ldots\ldots(3.57)$$
$$\rho'(1)=2\xi$$
$$\rho'(1)=2(1)=2\ldots\ldots\ldots\ldots\ldots\ldots(3.58)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{1}\beta_j\xi^j$$

$$\sigma(1)=\frac{51}{18}+\frac{20}{18}-\frac{25}{18}=\frac{46}{18}=2.555556\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.59)$$

From equations (3.57),(3.58) and (3.59), we have

(i) $\rho(1)=0$

(ii) $\rho'(1)\neq\sigma(1)\ldots\ldots\ldots\ldots\ldots\ldots(3.60)$

Hence scheme 3 is inconsistent

**Zero-stability**

The roots of the first characteristics polynomial of scheme 3

$$\rho(\xi)=\xi^2-1$$
$$i.e(\xi+1)(\xi-1),$$

We have the two real root either $\xi=-1$ or 1

Since scheme 3 satisfies the zero stability conditions, but is inconsistent, we conclude that scheme 3 is divergent.

**Scheme 4**

$$y_{n+2} = y_n + \frac{h}{35}[98f_{n+1} - 48f_{n+7/12} + 20f_n] \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.61)$$

The Error term is

$$y_{n+2} - y_n = \frac{h}{35}[98f_{n+1} - 48f_{n+7/12} + 20f_n] \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.62)$$

**Consistency**

From equation (3.62), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi) = \sum_{j=0}^{2} \alpha_j \xi^j$$
$$\rho(\xi) = \xi^2 - 1$$
$$\rho(1) = 1 - 1 = 0 \quad \ldots\ldots\ldots\ldots(3.63)$$
$$\rho'(\xi) = 2\xi$$
$$\rho'(1) = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.64)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi) = \sum_{j=0}^{1} \beta_j \xi^j$$
$$\sigma(\xi) = \frac{98}{35} - \frac{48}{35} + \frac{20}{35} = \frac{70}{35} = 2 \quad \ldots\ldots\ldots\ldots(3.65)$$

From equations (3.63), (3.64) and (3.65) we have

(i) $\rho(1) = 0$

(ii) $\rho'(1) = \sigma(1) \quad \ldots\ldots\ldots\ldots\ldots(3.66)$

Hence scheme 4 is consistent

**Zero-stability**

The roots of the first characteristic polynomial $\rho(\xi)$ are

$$\rho(\xi) = \xi^2 - 1$$

i.e $(\xi + 1)(\xi - 1)$ we have two real roots either $\xi = -1$ or 1

Thus $\xi$ satisfies the zero-stability condition, hence scheme 4 is convergent.

**Scheme 5**

$$y_{n+2}=y_{n+1}+\frac{h}{12}[23f_{n+1}-16f_{n}+5f_{n-1}]......................(3.67)$$

The error term is

$$y_{n+2}-y_{n+1}=\frac{h}{12}[23f_{n+1}-16f_{n}+5f_{n-1}]................(3.68)$$

**Consistency**

From equation (3.68), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j$$
$$=\rangle\rho(\xi)=\xi^2-\xi$$
$$\rho(1)=1-1=0............(3.69)$$
$$\rho'(\xi)=2\xi-1$$
$$\rho'(1)=2(1)-1=1...........(3.70)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j-0}^{1}\beta_j\xi^j$$
$$\sigma(1)=\frac{23}{12}-\frac{16}{12}+\frac{5}{12}=\frac{12}{12}=1....................(3.71)$$

From equations (3.69), (3.70) and (3.71), we have

(i) $\rho(1)=0$

(ii) $\rho'(1)=\sigma(1)$...............(3.72)

Hence scheme 5 is consistent

**Zero-stability**

The roots of first characteristic polynomial $\rho(\xi)$ are

$$\rho(\xi)=\xi^2-\xi$$
$$i.e\,\xi(\xi-1)$$

Either $\xi=0$ or $\xi=1$

Since $\xi$ satisfies the zero stability condition we conclude that scheme 5 is convergent.

**Scheme 6**

$$y_{n+2}=y_{n+1}+\frac{h}{14}[35f_{n+1}+3f_n-24f_{n+7/12}]\ldots\ldots\ldots\ldots..(3.73)$$

The Error term is

$$y_{n+2}-y_{n+1}=\frac{h}{14}[35f_{n+1}+3f_n-24f_{n+7/12}]\ldots\ldots\ldots\ldots\ldots..(3.74)$$

**Consistency**:

From equation (3.74), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi)=\sum_{j=0}^{2}\alpha_j\xi^j$$
$$\rho(\xi)=\xi^2-\xi$$
$$\rho(1)=1-1=0\ldots\ldots\ldots\ldots\ldots..(3.75)$$
$$\rho'(\xi)=2\xi-1$$
$$\rho'(1)=2(1)-1=1\ldots\ldots\ldots\ldots\ldots..(3.76)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{1}\beta_j\xi^j$$
$$\sigma(1)=\frac{35}{14}+\frac{3}{14}-\frac{24}{14}=\frac{14}{14}=1\ldots\ldots\ldots\ldots\ldots..(3.77)$$

From equations (3.75),(3.76) and (3.77), we have

(i) $\rho(1)=0$

(ii) $\rho'(1)=\sigma(1)\ldots\ldots\ldots\ldots\ldots..(3.78)$

Hence since scheme satisfies the above condition, we said that scheme 6 is consistent

**Zero-stability**

The roots of the first characteristic polynomial $\rho(\xi)$

$$\rho(\xi)=\xi^2-\xi$$
$$\xi(\xi-1)$$

Either $\xi = 0$ or $\xi = 1$

Since $\xi$ satisfies the zero-stability condition

We conclude that scheme 6 is convergent.

# CHAPTER FOUR

## 4.0 NUMERICAL APPLICATION AND COMPARISON OF RESULTS

### 4.1 Numerical Applications

We use the 6 derived methods to solve differential equation. To start, we solve the following differential equation.

$y' = x + y; y(0) = 1, h = 0.1$ ..................(4.1)

**Starting values**

As with all k-step methods (k>1) we face the problem of generating additional starting values. Also, we demand that these starting values should be calcuiated to accuracy at least as high as the local accuracy of the main method. This means that any method we use to calculate the starting values must itself require no starting values other than $y_0$.

In this work, we decide to use the exact solution to evaluate the starting values $y_n, n = 0,1,2$ as the case may be.

Given the initial values problem (IVP) as

$y' = x + y; y(x_0) = 1, h = 0.1$

Let consider

$y_{n+1} = y_n + hf_{n+1/2}$ ..............(4.2)

Note

$f_{n+\alpha} = f[x_n + \alpha h, y_n + \alpha hf_n], n = 0,1,2$..........

$y_1 = y_0 + hf_{1/2}$

$f_{1/2} = f[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_0]$

$f_0 = 1$

$\therefore f[0.05; 1 + 0.05(1)]$

$\Rightarrow 0.05 + 1.05 = 1.10$

$f_{1/2} = 1.10$

$\therefore y_1 = 1 + 0.1(1.10) = 1.11$

$y_1 = 1.11$

$y_2 = y_1 + hf_{3/2}$

$f_{3/2} = f\left(x_1 + \frac{h}{2}; y_1 + \frac{h}{2} f_1\right)$

$f_1 = x_1 + y_1 = 0.1 + 1.11 = 1.21$

$\therefore f_{3/2} = f[0.1 + 0.05; 1.11 + 0.05(1.21)]$

$f_{3/2} = 1.3205$

$\therefore y_2 = 1.11 + 0.1(1.3205)$

$y_2 = 1.24205$

$y_3 = y_2 + hf_{5/2}$

$f_{5/2} = f\left(x_2 + \frac{h}{2}; y_2 + \frac{5}{2} f_2\right)$

$f_2 = x_2 + y_2 = 0.2 + 1.24205 = 1.44209$

$f_{5/2} = f(0.2 + 0.05; 1.24205 + 0.05(1.44209))$

$\therefore y_3 = 1.24205 + 0.1(1.564525)$

$y_3 = 1.39846545$

$y_4 = y_3 + hf_{7/2}$

$f_{7/2} = f\left(x_3 + \frac{h}{2}; y_3 + \frac{h}{2} f_3\right)$

$f_3 = x_3 + y_3 = 0.3 + 1.39846545 = 1.69846545$

$f_{7/2} = [0.3 + 0.05; 1.39846545 + 0.05(1.69846545)]$

$f_{7/2} = 1.833388723$

$\therefore y_4 = 1.39846545 + 0.1(1.833388723)$

$y_4 = 1.581804322$

$y_5 = y_4 + hf_{9/2}$

$f_{9/2} = f\left(x_4 + \frac{h}{2}; y_4 + \frac{h}{2} f_4\right)$

$f_4 = x_4 + y_4 = 0.4 + 1.581804322 = 1.981804322$

$f_{9/2} = f(0.4 + 0.05; 1.581804322 + 0.05(1.981804322))$

$f_{9/2} = 2.130894538$

$\therefore y_5 = 1.581804322 + 0.1(2.130894538)$

$y_5 = 1.794893775$

$$y_6 = y_5 + hf_{11/2}$$

$$f_{11/2} = f\left[x_2 + \frac{h}{2}; y_5 + \frac{h}{2}f_5\right]$$

$$f_5 = x_5 + y_5 = 0.5 + 17.94893775 = 2.294893725$$

$$f_{11/2} = 2.459638464$$

$$\therefore y_6 = 1.794893775 + 0.1(2.459638464)$$

$$\therefore y_6 = 2.040857621$$

$$y_7 = y_6 + hf_{13/2}$$

$$f_{13/2} = f\left[x_6 + \frac{h}{2}; y_6 + \frac{h}{2}f_6\right]$$

$$f_6 = x_6 + y_6 = 0.6 + 2.040857621 = 2.640857621$$

$$f_{13/2} = f[0.6 + 0.05; 2.040857621 + 0.05(2.640857621)]$$

$$f_{13/2} = 2.822900502$$

$$\therefore y_7 = 2.040857621 + 0.1(2.822900502)$$

$$y_7 = 2.323147671$$

$$y_8 = y_7 + hf_{15/2}$$

$$f_{15/2} = f\left[x_7 + \frac{h}{2}; y_7 + \frac{h}{2}f_7\right]$$

$$f_7 = x_7 + y_7 = 0.7 + 2.323147671 = 3.023147671$$

$$f_{15/2} = f[0.7 + 0.05; 2.323147671 + 0.05(3.023147671)]$$

$$\therefore y_8 = 2.323147671 + 0.1(3.224305055)$$

$$y_8 = 2.645578176$$

$$y_9 = y_8 + hf_{17/2}$$

$$f_{17/2} = f\left(x_8 + \frac{h}{2}; y_8 + \frac{h}{2}f_8\right)$$

$$f_8 = x_8 + y_8 = 0.8 + 2.645578176 = 3.445578176$$

$$f_{17/2} = [0.8 + 0.05; 2.645578176 + 0.05(3.445578176)]$$

$$f_{17/2} = 3.667857055$$

$$\therefore y_9 = 2.645578176 + 0.1(3.667857055)$$

$$y_9 = 3.012363884$$

$$y_{10} = y_9 + hf_{19/2}$$

$$f_{19/2} = f\left(x_9 + \frac{h}{2}; y_9 + \frac{h}{2} f_9\right)$$

$$f_9 = x_9 + y_9 = 0.9 + 3.012363884 = 3.912363884$$

$$f_{19/2} = f(0.9 + 0.05; 3.012363884 + 0.05(3.912363884))$$

$$f_{19/2} = 4.157982078$$

$$\therefore y_{10} = 3.012363884 + 0.1(4.157982078)$$

$$y_{10} = 3.428162092$$

**Scheme 1**

$$y_{n+2} = y_n + \frac{h}{3}[7f_{n+1} - 2f_n + f_{n-1}], n = 1,2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3)$$

$$y_3 = y_1 + \frac{0.1}{3}[7f_2 - 2f_1 + f_0]$$

Let $y_1 = 1.1103, y_2 = 1.2428$

$$y_3 = 1.1103 + \frac{0.1}{3}[7f_2 - 2f_1 + f_0]$$

$$f_0 = x_0 + y_0 = 1$$

$$f_1 = x_1 + y_1 = 0.1 + 1.1103; 2f_1 = 2.4206$$

$$f_2 = x_2 + y_2 = 0.2 + 1.2428 = 1.4428; 7f_2 = 10.0996$$

$$\therefore y_3 = 1.1103 + \frac{0.1}{3}[10.0996 - 2.4206 + 1]$$

$$y_3 = 1.39960$$

$$y_4 = y_3 + \frac{0.1}{3}[(7f_3 - 2f_2 + f_1)]$$

$$f_1 = 1.2103$$

$$f_2 = 1.4428; 2f_2 = 2.8856$$

$$f_3 = x_3 + y_3 = 0.3 + 1.39960 = 1.69960; 7f3 = 11.89720$$

$$\therefore y_4 = 1.2428 + \frac{0.1}{3}[11.89720 - 2.8856 + 1.2103]$$

$$= 1.2428 + 0.34073$$

$$y_4 = 1.58353$$

$$y_5 = y_4 + \frac{0.1}{3}[(7f_4 - 2f_3 + f_2)]$$

$f_2 = 1.4428$

$f_3 = 1.69960; 2f_2 = 3.3992$

$f_4 = x_4 + y_4 = 0.4 + 1.58353 = 1.98353; 7f4 = 13.88471$

$$\therefore 1.3996 + \frac{0.1}{3}[13.88471 - 3.3992 + 1.4428]$$

$= 1.3996 + 0.397610333$

$y_5 = 1.797210333$

$$y_6 = y_4 + \frac{0.1}{3}[7f_5 - 2f_4 + f_3]$$

$f_3 = 1.69960$

$f_4 = 1.98353; 2f_4 = 3.96706$

$f_5 = x_5 + y_5; 0.5 + 1.797210333 = 2.297210333; 7f_5 = 16.08047233$

$$\therefore y_6 = 1.58353 + \frac{0.1}{3}[16.08047233 - 3.96706 + 1.69960]$$

$= 1.58353 + 0.43963744$

$y_6 = 2.043963744$

$$y_7 = y_5 + \frac{0.1}{3}[7f_6 - 2f_5 + f_4]$$

$f_4 = 1.98353$

$f_5 = 2.297210333; 2f_5 = 4.594420666$

$f6 = x6 + y6 = 0.6 + 2.04393744 = 2.643963744; 7f_6 = 18.50774621$

$$\therefore y_7 = 1.797210333 + \frac{0.1}{3}[18.50774621 - 4.594420666 + 1.98353]$$

$= 1.797210333 + 0.529894984$

$y_7 = 2.327105318$

$$y_8 = y_6 + \frac{0.1}{3}[7f_7 - 2f_6 + f_5]$$

$f_5 = 2.297210333$

$f_6 = 2.643963744; 2f_6 = 5.287927488$

$f_7 = x_7 + y_7 = 0.7 + 2.327105318 = 3.027105318; 7f_7 = 21.18973722$

$$y_8 = 2.043963744 + \frac{0.1}{3}[21.18973722 - 5.287927488 + 2.297210333]$$

$= 2.043963744 + 0.606634002$

$y_8 = 2.650597746$

$$y_9 = y_7 + \frac{0.1}{3}[7f_8 - 2f_7 + f_6]$$

$f_6 = 2.643963744$

$f_7 = 3.027105318; 2f7 = 6.054210636$

$f_8 = x_8 + y_8 = 0.8 + 2.650597746 = 3.450597746; 7f_8 = 24.15418422$

$$\therefore y_9 = 2.327105318 + \frac{0.1}{3}[24.15418422 - 6.054210636 + 2.643963744]$$

$= 2.327105318 + 0.691464577$

$y_9 = 3.018571088$

$$y_{10} = y8 + \frac{0.1}{3}[7f_9 - 2f_8 + f_7]$$

$f_7 = 3.027105318$

$f_8 = 3.4505597746; 2f_8 = 6.901195492$

$f_9 = x_9 + y_9 = 0.9 + 3.018571088 = 3.918571088; 7f_9 = 27.42999762$

$$\therefore y_{10} = 2.650597746 + \frac{0.1}{3}[27.42999762 - 6.901195492 + 3.027105318]$$

$= 2.650597746 + 0.7851969$

$y_{10} = 3.435794647$

**Scheme 2**

$$y_{n+2} = y_n + \frac{0.1}{3}[8f_{n+1} - 4f_{n+1/2} + f_n], n = 0,1,2 \ldots\ldots\ldots\ldots\ldots (4.4)$$

$$y_2 = y_0 + \frac{h}{3}[8f_1 - 4f_{1/2} + 2f_0]$$

Let $y_1 = 1.11$

$$2f_0 = 2$$
$$f_1 = 0.1 + 1.11 = 1.21$$
$$8f_1 9.68$$
$$f_{1/2} = f[0 + 0.05;1 + 0.05(1)] = 1.1;4f_{1/2} = 4.4$$
$$\therefore y_2 = 1 + \frac{0.1}{3}[9.68 + 2 - 4.4]$$
$$y_2 = 1.2426667$$

$$y_3 = y_1 + \frac{h}{3}[8f_2 - 2f_1 - 4f_{3/2}]$$
$$2f_1 = 2(1.21) = 2.42$$
$$f_2 = x_2 + y_2 = 0.2 + 1.2426667 = 1.4426667;8f_2 = 11.5413336$$
$$f_{3/2} = f[0.1 + 0.05,1.11 + 0.05(1.21)] = 1.3205;4f_{3/2} = 5.282$$
$$\therefore y_3 = 1.11 + \frac{0.1}{3}[11.5413336 + 2.42 - 5.282]$$
$$= 1.11 + 0.28931111$$
$$y_3 = 1.39931111$$

$$y_4 = y_2 + \frac{0.1}{3}[8f_3 + 2f_2 - 4f_{5/2}]$$
$$8f_3 = 8(0.3 + 1.39931111) = 13.59448889$$
$$2f_2 = 2(0.2 + 1.2426667) = 2.8855554$$
$$4f_{5/2} = 4f[0.25,1.2426667 + 0.05(1.442777778)] = 6.2592222356$$
$$\therefore y_4 = 1.2426666 + \frac{0.1}{3}[13.59448889 + 2.8855554 - 6.2592222356]$$
$$y_4 = 1.583454074$$

$$y_5 = y_3 + \frac{0.1}{3}[8f_4 + 2f_3 - 4f_{7/2}]$$
$$8f_4 = 8(0.4 + 1.583354078) = 15.86683259$$
$$2f_3 = 2(1.69931111) = 3.39862222$$
$$4f_{7/2} = 4f(0.35,1.39931111 + 0.05(1.69931111)) = 7.337106666$$
$$\therefore y_5 = 1.39931111 + \frac{0.1}{3}[15.86683259 + 3.39862222 - 7.337106666]$$
$$y_5 = 1.7969227170$$

$$y_6 = y_4 + \frac{0.1}{3}[8f_5 + 2f_4 - 4f_{9/2}]$$

$$8f_5 = 8(0.5 + 1.7969227170) = 18.37538174$$

$$2f_4 = 2(0.4 + 1.583354078) = 3.966708148$$

$$4f_{9/2} = 4f[0.45, 1.583354078 + 0.05(1.983354407)] = 8.530087128$$

$$\therefore y_6 = 1.583354078 + \frac{0.1}{3}[18.37538174 + 3.966708148 - 8.530087128]$$

$$y_6 = 2.04375417$$

$$y_7 = y_5 + \frac{0.1}{3}[8f_5 + 2f_5 - 4f_{11/2}]$$

$$8f_6 = 8(0.6 + 2.04575417) = 21.15003336$$

$$2f_5 = 2(0.5 + 1.7969227170) = 4.593845434$$

$$4f_{11/5} = 4f[0.55, 1.7969227170 + 0.5(2.296922717)] = 9.847075411$$

$$\therefore y_7 = 1.7969227170 + \frac{0.1}{3}[21.1500336 + 4.593845434 - 9.847075411]$$

$$y_7 = 2.326816163$$

$$y_8 = y_6 + \frac{0.1}{3}[8f_7 + 2f_6 - 4f_{13/2}]$$

$$8f_7 = 8(0.7 + 2.326816163) = 24.2145293$$

$$2f_6 = 2(0.6 + 2.04375417) = 5.2870834$$

$$4f_{13/2} = 4f[0.65, 2.04375417 + 0.05(2.64375417)] = 11.30376751$$

$$\therefore y_8 = 2.04375417 + \frac{0.1}{3}[24.2145293 + 5.2870834 - 11.30376751]$$

$$y_8 = 2.6503631714$$

$$y_9 = y_7 + \frac{0.1}{3}[8f_8 + 2f_7 - 4f_{15/2}]$$

$$8f_8 = 8(0.8 + 2.6503631714) = 27.60290537$$

$$2f_2 = 2(0.7 + 2.326816163) = 6.053632326$$

$$4f_{15/2} = 4f[0.75; 2.326816163 + 0.05(3.026816163)] = 12.91262788$$

$$\therefore y_9 = 2.326816163 + \frac{0.1}{3}[27.60290537 + 6.053632326 - 12.91262788]$$

$$y_9 = 3.018279824$$

$$y_{10} = y_8 + \frac{0.1}{3}[8f_9 + 2f_8 - 4f_{17/2}]$$

$$8f_9 = 8(0.9 + 3.018279824) = 31.34623859$$

$$2f_8 = 2(0.8 + 2.6503631714) = 6.900726343$$

$$4f_{17/2} = 4f[0.85; 2.6503631714 + 0.05(3.450363171)] = 14.69152532$$

$$\therefore y_{10} = 2.6503631714 + \frac{0.1}{3}[31.34623859 + 6.900726343 - 14.69152532]$$

$$y_{10} = 3.45544494$$

**Scheme 3**:

$$y_{n+2} = y_n + \frac{h}{35}[98f_{n+1} - 48f_{n+7/12} + 20f_n], n = 0,1,2............(4.5)$$

$$y_2 = y_0 + \frac{0.1}{35}[98f_1 + 20f_0 - 48f_{7/12}]$$

$$98f_1 = 98(0.1 + 1.11) = 118.58$$

$$20f_0 = 20(1) = 20$$

$$48f_{7/12} = 48(1.1166666667) = 53.60000042$$

$$y_2 = 1 + \frac{0.1}{35}[118.58 + 20 - 53.60000042]$$

$$y_2 = 1.24280$$

$$y_3 = y_1 + \frac{0.1}{3}[98f_2 + 20f_1 - 48f_{19/12}]$$

$$98f_2 = 98(0.2 + 1.24280) = 141.3944$$

$$20f_1 = 20(1.21) = 24.2$$

$$48f_{19/12} = 48f[0.158333; 1.11 + 0.058333(1.21)] = 64.26799996$$

$$\therefore y_3 = 1.11 + \frac{0.1}{3}[141.3944 + 24.2 - 64.26799996]$$

$$y_3 = 1.399504$$

$$y_4 = y_2 + \frac{0.1}{35}[98f_3 + 20f_2 - 48f_{31/12}]$$

$$98f_3 = 98(1.699504) = 166.551392$$

$$20f_2 = 20(1.44280) = 28.856$$

$$48f_{13/12} = 48[0.258333; 1.24280 + 0.583333(1.44280)] = 76.09423996$$

$$\therefore y_4 = 1.24280 + \frac{0.1}{35}[166.551392 + 28.856 - 76.09423996]$$

$$y_4 = 1.58369472$$

$$y_5 = y_3 + \frac{0.1}{35}[98f_4 + 20f_3 - 48f_{43/12}]$$

$$98f_4 = 98(1.98369472) = 194.4020826$$

$$20f_3 = 20(1.699504) = 33.99068$$

$$48f_{43/12} = 48[0.358333;1.399504 + 0.0583333(1.69504)] = 89.13480291$$

$$\therefore y_5 = 1.399504 + \frac{0.1}{35}[194.4020826 + 33.99008 - 89.13480291]$$

$$y_5 = 1.79738217$$

$$y_6 = y_4 + \frac{0.1}{35}[98f_5 + 20f_4 - 48f_{55/12}]$$

$$98f_5 = 98(2.29738217) = 225.1434527$$

$$20f_4 = 20(1.98369472) = 39.6738944$$

$$48f_{55/12} = 48f[0.45833333;1.58369472 + 0.05833333(1.98369472)] = 103.571687$$

$$\therefore y_6 = 1.58369472 + \frac{0.1}{35}[225.1434527 + 39.6738944 - 103.571687]$$

$$y_6 = 2.044396595$$

$$y_7 = y_5 + \frac{0.1}{35}[98f_6 + 20f_5 - 48f_{67/12}]$$

$$98f_6 = 98(2.644396595) = 259.1508663$$

$$20f_5 = 20(2.29738217) = 45.9476434$$

$$48f_{67/12} = 48f[0.5583333;1.79738217 + 0.0583333(2.29738217)] = 119.5070391$$

$$\therefore y_7 = 1.79738217 + \frac{0.1}{35}[259.1508663 + 45.9476434 - 119.5070391]$$

$$y_7 = 2.327643585$$

$$y_8 = y_6 + \frac{0.1}{35}[98f7 + 20f6 - 48f89/12]$$

$$98f_7 = 98(3.027643585) = 296.7090713$$

$$20f_6 = 20(2.644396595) = 52.8879319$$

$$48f_{89/12} = 48f[0.65833333;2.044396595 + 0.05833333(2.644396595)] = 137.1353412$$

$$\therefore y_8 = 2.044396595 + \frac{0.1}{35}[296.7090713 + 52.8879319 - 137.1353412]$$

$$y_8 = 2.651429897$$

$$y_9 = y_7 + \frac{0.1}{35}[98f_8 + 20f_7 - 48f_{101/12}]$$

$$98f_8 = 98(3.35142987) = 328.4401273$$

$$20f_7 = 20(3.027643585) = 60.5528717$$

$$48f_{101/12} = 48f[0.75833333;2.327643585 + 0.05833333(3.027643585)] = 156.600906$$

$$\therefore y_9 = 2.327643585 + \frac{0.1}{35}[328.4401273 + 60.5528717 - 156.600906]$$

$$y_9 = 3.019611321$$

$$y_{10} = y_8 + \frac{0.1}{35}[98f_9 + 20f_8 - 48f_{113/12}]$$

$$98f_9 = 98(3.919611321) = 384.1219095$$

$$20f_8 = (3.35142987) = 67.0285979$$

$$48f_{113/12} = 48f[0.85833333;2.651429897 + 0.05833333(3.35142987)] = 177.8526317$$

$$\therefore y_{10} = 2.651429897 + \frac{0.1}{35}[384.1219095 + 67.0285979 - 177.8526317]$$

$$y_{10} = 3.437195236$$

**Scheme 4**:

$$y_{n+2} = y_n + \frac{h}{18}[51f_{n+1} + 10f_n - 25f_{n-3/5}], n = 0,1,2 \ldots\ldots\ldots\ldots\ldots (4.6)$$

$$y_2 = y_0 + \frac{0.1}{18}[51f_1 + 10f_0 - 25f_{3/5}]$$

$$51f_1 = 51(1.21) = 61.71$$

$$10f_0 = 10(1) = 10$$

$$25f_{3/5} = 25f[0.06;1 + 0.06(1)] = 28$$

$$\therefore y_2 = 1 + \frac{0.1}{18}[61.71 + 10 - 28]$$

$$y_2 = 1.24283333$$

$$y_3 = y_1 + \frac{0.1}{18}[51f_2 + 10f_1 - 25f_{8/5}]$$

$$51f_2 = 51(1.44283333) = 73.58449983$$

$$10f_1 = 10(1.21) = 12.1$$

$$25f_{8/5} = 25f[0.16;1.11 + 0.06(1.21)] = 33.565$$

$$\therefore y_3 = 1.11 + \frac{0.1}{18}[73.58449983 + 12.1 - 33.565]$$

$$y_3 = 1.399552778$$

$$y_4 = y_2 + \frac{0.1}{18}[51f_3 + 10f_2 - 25f_{13/5}]$$

$$51f_3 = 51(1.699552778) = 86.67719168$$

$$10f_2 = 10(1.44283333) = 14.4283333$$

$$25f_{13/5} = 25f[0.26;1.24283333 + 0.06(1.44283333)] = 39.73508325$$

$$\therefore y_4 = 1.24283333 + \frac{0.1}{18}[86.67719168 + 14.4283333 - 39.73508325]$$

$$y_3 = 1.583780231$$

$$y_5 = y_3 + \frac{0.1}{18}[51f_4 + 10f_3 - 25f_{18/5}]$$

$$51f_4 = 51(1.983780231) = 101.1727918$$

$$10f_3 = 10(1.699552778) = 16.99552778$$

$$25f_{18/5} = 25f[0.36;1.399552778 + 0.06(1.699552778)] = 46.5314862$$

$$\therefore y_5 = 1.399552778 + \frac{0.1}{18}[101.1727918 + 16.99552778 - 46.5314862]$$

$$y_5 = 1.797498172$$

$$y_6 = y_4 + \frac{0.1}{18}[51f_5 + 10f_4 - 25f_{23/5}]$$

$$51f_5 = 51(2.297498172) = 117.1724068$$

$$10f_4 = 10(1.983780231) = 19.83780231$$

$$25f_{23/5} = 25f[0.46;1.583780231 + 0.06(1.983780231)] = 54.07017612$$

$$\therefore y_6 = 1.583780231 + \frac{0.1}{18}[117.1724068 + 19.83780231 - 54.07017612]$$

$$y_6 = 2.044558192$$

$$y_7 = y_5 + \frac{0.1}{18}[51f_6 + 10f_5 - 25f_{28/5}]$$

$$51f_6 = 51(2.644558192) = 134.8724678$$

$$10f_5 = 10(2.297498172) = 22.97498172$$

$$25f_{28/5} = 25f[0.56;1.797498172 + 0.06(2.297498172)] = 62.38370156$$

$$\therefore y_7 = 1.797498172 + \frac{0.1}{18}[134.8724678 + 22.97498172 - 62.38370156]$$

$$y_7 = 2.327852329$$

$$y_8 = y_6 + \frac{0.1}{18}[51f_7 + 10f_6 - 25f_{33/5}]$$

$$51f_7 = 51(3.027852329) = 154.4204688$$

$$10f_6 = 10(2.644558192) = 26.44558192$$

$$25f_{33/5} = 25f[0.66; 2.044558192 + 0.06(2.644558192)] = 71.58079209$$

$$\therefore y_8 = 2.044558192 + \frac{0.1}{18}[154.4204688 + 26.44558192 - 71.58079209]$$

$$y_8 = 2.651698518$$

$$y_9 = y_7 + \frac{0.1}{18}[51f_8 + 10f_7 - 25f_{38/5}]$$

$$51f_8 = 51(3.451698518) = 176.0366244$$

$$10f_7 = 10(3.027852329) = 30.27852329$$

$$25f_{38/5} = 25f[0.76; 2.327852329 + 0.06(3.027852329)] = 81.73808672$$

$$\therefore y_9 = 2.327852329 + \frac{0.1}{18}[176.0366244 + 30.27852329 - 81.73808672]$$

$$y_9 = 3.019918073$$

$$y_{10} = y_8 + \frac{0.1}{18}[51f_9 + 10f_8 - 25f_{43/5}]$$

$$51f_9 = 51(3.919918073) = 199.9158217$$

$$10f_8 = 10(3.451698518) = 34.51698518$$

$$25f_{43/5} = 25f[0.86; 2.651698518 + 0.06(3.451698518)] = 95.97000228$$

$$\therefore y_{10} = 2.651698518 + \frac{0.1}{18}[199.9158217 + 34.51698518 - 95.97000228]$$

$$y_{10} = 3.437602941$$

**Scheme 5**:

$$y_{n+2} = y_{n+1} + \frac{h}{12}[23f_{n+1} - 16f_n + 5f_{n-1}], n = 1,,2 \qquad (4.7)$$

$$y_3 = y_2 + \frac{0.1}{12}[23f_2 - 16f_1 + 5f_0]$$

Let $y_1 = 1.1103, y_2 = 1.2428$

$$5f_0 = 5(1) = 5$$

$$16f_1 = 16(1.2103) = 19.3648$$

$$23f_2 = 23(1.4428) = 33.1844$$

$$\therefore y_3 = 1.2428 + \frac{0.1}{12}[33.1844 - 19.3645 + 5]$$

$$y_3 = 1.39963$$

$$y_4 = y_3 + \frac{0.1}{12}\left[23f_3 - 16f_2 + 5f_1\right]$$

$$5f_1 = 5(1.2103) = 6.0515$$

$$16f_2 = 16(1.4428) = 23.0848$$

$$23f_3 = 23(1.69963) = 39.09149$$

$$\therefore y_4 = 1.39963 + \frac{0.1}{12}\left[39.09149 - 23.0848 + 6.0515\right]$$

$$y_4 = 1.58344825$$

$$y_5 = y_4 + \frac{0.1}{12}\left[23f_4 - 16f_3 + 5f_2\right]$$

$$5f_2 = 5(1.4428) = 7.214$$

$$16f_3 = 16(1.6993) = 27.1888$$

$$23f_4 = 23(1.98344825) = 45.61930975$$

$$\therefore y_5 = 1.58344825 + \frac{0.1}{12}\left[45.61930975 - 27.1888 + 7.214\right]$$

$$y_5 = 1.797152498$$

$$y_6 = y_5 + \frac{0.1}{12}\left[23f \;- 16f_6 + 5f_5\right]$$

$$5f_3 = 5(1.6993) = 8.4965$$

$$16f_4 = 16(1.98344825) = 31.735172$$

$$23f_5 = 23(2.297152498) = 52.83450715$$

$$y_6 = 1.797152498 + \frac{0.1}{12}\left[52.83450715 - 31.735172 + 8.4965\right]$$

$$y_6 = 2.04378446$$

$$y_7 = y_6 + \frac{0.1}{12}\left[23f_6 - 16f_5 + 5f_4\right]$$

$$5f_4 = 5(1.98344825) = 9.91724125$$

$$16f_5 = 16(2.297152498) = 36.75443997$$

$$23f_6 = 23(2.64378446) = 60.80704258$$

$$\therefore y_7 = 2.04378446 + \frac{0.1}{12}\left[60.80704258 - 36.75443997 + 9.91724125\right]$$

$$y_7 = 2.326866492$$

$$y_8 = y_7 + \frac{0.1}{12}[23f_7 - 16f_6 + 5f_5]$$
$$5f_5 = 5(2.297152498) = 11.48576249$$
$$16f_6 = 16(2.64378446) = 42.30055136$$
$$23f_7 = 23(3.026866492) = 69.61792932$$
$$\therefore y_8 = 2.326866492 + \frac{0.1}{12}[69.61792932 - 42.30055136 + 11.48576249]$$
$$y_8 = 2.650225996$$

$$y_9 = y_8 + \frac{0.1}{12}[23f_8 - 16f_7 + 5f_6]$$
$$5f_6 = 5(2.64378446) = 13.2189223$$
$$16f_7 = 16(3.626866492) = 48.42986387$$
$$23f_8 = 23(3.450225996) = 79.35519791$$

$$\therefore y_9 = 2.650225996 + \frac{0.1}{12}[79.35519791 - 48.42986387 + 13.2189223]$$
$$y_9 = 3.018094799$$

$$y_{10} = y_9 + \frac{0.1}{12}[23f_9 - 16f_8 + 5f_7]$$
$$5f_7 = 5(3.626866492) = 15.13433246$$
$$16f_8 = 16(3.450225996) = 55.20361594$$
$$23f_9 = 23(3.918094799) = 90.11618038$$
$$\therefore y_{10} = 3.018094799 + \frac{0.1}{12}[90.11618038 - 55.20361594 - 15.13433246]$$
$$y_{10} = 3.435152273$$

**Scheme 6**:

$$y_{n+2} = y_{n+1} + \frac{0.1}{14}[35f_{n+1} + 3f_n - 24f_{n+7/12}], n = 1,2 \text{.............................} (4.8)$$

$$y_1 = 1.11$$

$$y_2 = y_1 + \frac{0.1}{14}[35f_1 + 3f_0 - 24f_{n+7/12}]$$
$$35f_1 = 35(1.21) = 42.35$$
$$3f_0 = 3(1) = 3$$
$$24f_{7/12} = 24f[0.0583333; 1 + 0.05833333(1)] = 26.79999984$$
$$\therefore y_2 = 1.11 + \frac{0.1}{14}[42.35 + 3 - 26.79999984]$$
$$y_2 = 1.2425$$

$$y_3 = y_2 + \frac{0.1}{14}\left[35f_2 + 3f_1 - 24f_{19/12}\right]$$

$$35f_2 = 35(1.4425) = 50.4875$$

$$3f_1 = 3(1.21) = 3.63$$

$$24f_{19/12} = 24f\left[(0.15833333;1.11 + 0.0583333(1.21)\right] = 32.13399982$$

$$\therefore y_3 = 1.2425 + \frac{0.1}{14}\left[50.4875 + 3.63 - 32.13399982\right]$$

$$y_3 = 1.399525001$$

$$y_4 = y_3 + \frac{0.1}{14}\left[35f3 + 3f2 - 24f_{31/12}\right]$$

$$35f_3 = 35(1.699525001) = 59.48337504$$

$$3f_2 = 3(1.4425) = 4.3275$$

$$24f_{31/12} = 24f\left[0.2583333;1.2425 + 0.0583333(1.4425)\right] = 38.0394998$$

$$\therefore y_4 = 1.399525001 + \frac{0.1}{14}\left[59.48337504 + 4.3275 - 38.0394998\right]$$

$$y_4 = 1.583606257$$

$$y_5 = y_4 + \frac{0.1}{14}\left[35f_4 + 3f_3 - 24f_{43/12}\right]$$

$$35f_4 = 35(1.983606251) = 69.42621849$$

$$3f_3 = 3(1.699525001) = 5.098575003$$

$$24f_{43/12} = 24f[0.3583333;1.3995205001 + 0.05833333(1.699525001)] = 44.56793481$$

$$\therefore y_5 = 1.58360251 + \frac{0.1}{14}\left[69.42621849 + 5.098575003 - 44.56793481\right]$$

$$y_5 = 1.797583815$$

$$y_6 = y_5 + \frac{0.1}{15}\left[35f_5 + 3f_4 - 24f_{55/12}\right]$$

$$35f_5 = 35(2.297583815) = 80.41543353$$

$$3f_4 = 3(1.983606251) = 5.950818753$$

$$24f_{55/12} = 24f[0.45833333;1.583606251 + 0.05833333(1.983606251)] = 51.78359854$$

$$\therefore y_6 = 1.797583815 + \frac{0.1}{14}\left[80.41543353 + 5.950818753 - 51.78359854\right]$$

$$y_6 = 2.044602694$$

$$y_7 = y_6 + \tfrac{0.1}{14}\left[35f_6 + 3f_5 - 24f_{67/12}\right]$$

$$35f_6 = 35(2.644602694) = 92.56109429$$

$$3f_5 = 3(2.29583815) = 6.89261445$$

$$24f_{67/12} = 24f[0.5583333;1.797583815 + 0.0583333(2.297583815)] = 59.75862864$$

$$y_7 = 2.32813993$$

$$y_8 = y_7 + \frac{0.1}{14}\left[35f_7 + 3f_6 - 24f_{79/12}\right]$$

$$35f_7 = 35(3.02813993) = 105.9848976$$

$$3f_6 = 3(2.644602694) = 7.933808080$$

$$24f_{79/12} = 24f\left[0.6583333)2.044602694 + 0.0583333(2.644602694\right] = 68.57290814$$

$$\therefore y_8 = 2.32813993 + \frac{0.1}{14}\left[105.9848976 + 7.9338082 - 68.57290814\right]$$

$$y_8 = 2.651245337$$

$$y_9 = y_8 + \frac{0.1}{14}\left[35f_8 + 3f_7 - 24f_{91/12}\right]$$

$$35f_8 = 35(3.451245337) = 120.7935868$$

$$3f_7 = 3(3.02813993) = 9.08441979$$

$$25f_{91/12} = 25f[0.7583333;2.32813993 + 0.05833333(3.02813993)] = 78.314751$$

$$\therefore y_9 = 2.651245337 + \frac{0.1}{14}[120.7935868 + 9.08441979 - 78.314751]$$

$$y_9 = 3.019554283$$

$$y_{10} = y_9 + \frac{0.1}{14}\left[35f_9 + 3f_8 - 24f_{103/12}\right]$$

$$35f_9 = 35(3.919554283) = 1.37.1843999$$

$$3f_8 = 3(3.451245337) = 10.35373601$$

$$24f_{103/12} = 24f\left[0.8583333;2.65124337 + 0.0583333(3.451245337)\right] = 89.061628$$

$$\therefore y_{10} = 3.019554283 + \frac{0.1}{14}[137.1843999 + 10.35373601 - 89.061628]$$

$$y_{10} = 3.437104808$$

## 4.0 Comparison of Results

In the same way we solve the following problem $y' = x + y; y(0) = 1, h = 0.1$ using the six new schemes. Their results are obtained and compare for accuracy. The problem is solved on computer using Microsoft excel software package.

The results obtained from the six new schemes are compared with the exact solution and 3–stage Runge-kutta method. Note that Range-Kutta method is taken as a reference method.

**TABLE 4.1**

**PROBLEM**: $y' = x + y; y(0) = 1; h = 0.1$

**EXACT:** $Y_E(x) = 2e^x - x - 1$

| X | Exact solution | $y_{n+2} = y_n + 2hf_{n+1}$ | ERROR | $\dot{y}_{n+1} = y_n + \frac{h}{4}[k_1 + 3k_3]$ | ERROR | $y_{n+2} = y + \frac{h}{3}[7f_{n+1} - 2f_n + f_{n-1}]$ | ERROR |
|---|---|---|---|---|---|---|---|
| 0.1 | 1.110341836 | 1.1103 | 4.0836E-05 | 1.1103333333 | 8.503E-06 | 1.1103 | 4.1836E-05 |
| 0.2 | 1.242805516 | 1.24206 | 7.45516E-04 | 1.242786666 | 1.885E-05 | 1.2428 | 5.516E-06 |
| 0.3 | 1.399717615 | 1398712 | 1.005615E-03 | 1.399643897 | 7.3718E-05 | 1.39960 | 1.17615E-04 |
| 0.4 | 1.583649395 | 1.5818024 | 1.846995E-03 | 1.583556447 | 9.2940E-05 | 1.58353 | 1.19395E-04 |
| 0.5 | 1.597442541 | 1.7950724 | 2.370141E-03 | 1.797327133 | 1.15408E-04 | 1.797210333 | 2.32221E-04 |
| 0.6 | 2.0442347601 | 2.0408168 | 3.420801E-03 | 2.044096037 | 1.41564E-04 | 2.043963744 | 2.73857E-04 |
| 0.7 | 2.327505415 | 2.3232357 | 4.269715E-03 | 2.327333485 | 1.7193E-04 | 2.327105318 | 4.00097E-04 |
| 0.8 | 1.651081857 | 2.6454639 | 5.167957E-03 | 2.650876197 | 2.05373E-04 | 2.650597746 | 4.84111E-04 |
| 0.9 | 3.019206222 | 3.0123284 | 6.877822E-03 | 3.01896004 | 2.4642E-04 | 3.018571088 | 6.35134E-4 |
| 1.0 | 3.436563657 | 3.279295 | 8.634157E-03 | 3.436270638 | 2.93019E-04 | 3.435794647 | 7.6901E-04 |

**TABLE 4.2**

**PROBLEM**: $y' = x + y; y(0) = 1; h = 0.1$

**EXACT:** $Y_E(x) = 2e^x - x - 1$

| | Exact solution | $y_{n+2} = y_n + 2hf_{n+1}$ | ERROR | $y_{n+2} = y_n + \frac{h}{3}\left[8f_{n+1} - 4f_{n+\frac{1}{2}} + 2f_n\right]$ | ERROR |
|---|---|---|---|---|---|
| 0.1 | 1.110341836 | 1.1103 | 4.1836E-05 | 1.11 | 3.41836E-04 |
| 0.2 | 1.242805516 | 1.24206 | 7.45516E-04 | 1.2426667 | 1.3889E-04 |
| 0.3 | 1.399717615 | 1.398712 | 1.005615E-03 | 1.39931111 | 4.06505E-04 |
| 0.4 | 1.583649395 | 1.5818024 | 1.846995E-03 | 1.583454078 | 1.95317E-04 |
| 0.5 | 1.797442541 | 1.7950724 | 2.370141E-03 | 1.796922717 | 5.19824E-04 |
| 0.6 | 2.044234601 | 2.0408168 | 3.420801E-03 | 2.04375417 | 4.83431E-04 |
| 0.7 | 2.327505415 | 2.3232357 | 4.269715E-03 | 2.326816163 | 6.89252E-04 |
| 0.8 | 2.651081857 | 2.6454639 | 5.6217957E-03 | 2.6503631714 | 7.18686E-04 |
| 0.9 | 3.019206222 | 3.0123284 | 6.877822E03 | 3.01827824 | 9.26398E-04 |
| 1.0 | 3.436563657 | 3.4279295 | 8.634157E-03 | 3.4554494 | 1.8885743E-02 |

**TABLE 4.3**
**PROBLEM**: $y' = x + y; y(0) = 1; h = 0.1$
**EXACT**: $Y_E(x) = 2e^x - x - 1$

| X | Exact solution | $y_{n+2} = y_{n+} 2hf_{n+1}$ | ERROR | $y_{n+2} = y_n + \frac{h}{35}[98f_{n+1} - 48f_{n+7/12} + 20f_n]$ | ERROR |
|---|---|---|---|---|---|
| 0.1 | 1.110341836 | 1.1103 | 4.1836E-05 | 1.1 | 1.0341836E-02 |
| 0.2 | 1.242805516 | 1.24206 | 7.45516E-04 | 1.24280 | 5.516E-06 |
| 0.3 | 1.399717615 | 1.398712 | 1.005615E-03 | 1.399504 | 2.13615E-04 |
| 0.4 | 1.583649395 | 1.5818024 | 1.846995E-03 | 1.583689472 | 4.5325E-05 |
| 0.5 | 1.797442541 | 1.7950724 | 2.370141E-03 | 1.79738217 | 6.0371E05 |
| 0.6 | 2.044237601 | 2.0408168 | 3.420801E-03 | 2.044396595 | 1.58994E-04 |
| 0.7 | 2.327505415 | 2.3232357 | 4.269715E-03 | 2.327643585 | 1.3817E-04 |
| 0.8 | 2.651081857 | 2.6454639 | 5.6217957E-03 | 2.651429897 | 3.4804E-04 |
| 0.9 | 3.019206222 | 3.0123184 | 6.877822E03 | 3.019611321 | 3.49101E-04 |
| 1.0 | 3.436563657 | 3.4279295 | 8.634157E-03 | 3.437195236 | 6.31579E-04 |

TABLE 4.4

**PROBLEM**: $y' = x + y; y(0) = 1; h = 0.1$

**EXACT**: $Y_E(x) = 2e^x - x - 1$

| X | Exact solution | $y_{n+2} = y_{n+2} + 2hf_{n+1}$ | ERROR | $y_{n+2} = y_n + \frac{h}{3}[51f_{n+1} + 20f_n - 25f_{n+3/5}$ | ERROR |
|---|---|---|---|---|---|
| 0.1 | 1.110341836 | 1.1103 | 4.836E-05 | 1.1 | 1.0341836E-02 |
| 0.2 | 1.242805516 | 1.24206 | 7.45516E-04 | 1.24283333 | 2.7814E-05 |
| 0.3 | 1.399717615 | 1.398712 | 1.005615E-03 | 1.399552778 | 1.64837E-04 |
| 0.4 | 1.583649395 | 1.5818024 | 1.846995E-03 | 1.583780231 | 1.30836E-04 |
| 0.5 | 1.797442541 | 1.7950724 | 2.370141E-03 | 1.797498172 | 5.5631E-05 |
| 0.6 | 2.044237601 | 2.0408168 | 3.420801E-03 | 2.044558192 | 3.20591E-04 |
| 0.7 | 2.327505415 | 2.3232357 | 4.269715E-03 | 2.327852329 | 3.46914E-04 |
| 0.8 | 2.651081857 | 2.6454639 | 5.6217957E-03 | 2.651698518 | 6.16661E-04 |
| 0.9 | 3.019206222 | 3.0123184 | 6.877822E03 | 3.019918073 | 6.55853E-04 |
| 1.0 | 3.436563657 | 3.4279295 | 8.634157E-03 | 3.437602941 | 1.039284E-03 |

TABLE 4.5

**PROBLEM**: $y' = x + y; y(0) = 1; h = 0.1$

**EXACT**: $Y_E(x) = 2e^x - x - 1$

| X | Exact solution | $y_{n+2} = y_{n+1} + \frac{h}{2}[3f_{n+1} - f_n]$ | ERROR | $y_{n+2} = y_{n+1} + \frac{h}{12}[23f_{n+1} - 16f_n + 5f_{n-1}]$ | ERROR |
|---|---|---|---|---|---|
| 0.1 | 1.110341836 | 1.1103 | 4.1836E-05 | 1.1103 | 4.1836E-05 |
| 0.2 | 1.242805516 | 1.2428 | 5.516E-06 | 1.2428 | 5.516E-06 |
| 0.3 | 1.399717615 | 1.398705 | 1.012615E-03 | 1.39963 | 8.7615E-05 |
| 0.4 | 1.583649395 | 1.58137075 | 2.278645E-03 | 1.58344825 | 2.01145E-04 |
| 0.5 | 1.797442541 | 1.795919758 | 1.522784E-03 | 1.797152498 | 2.90043E-04 |
| 0.6 | 2.044237601 | 2.041239184 | 2.998418E-03 | 2.04378446 | 4.53141E-04 |
| 0.7 | 2.327505415 | 2.322629073 | 1.214683E-03 | 2.326866492 | 6.38923E-04 |
| 0.8 | 2.651081857 | 2.643961475 | 7.120382E-03 | 2.650225996 | 8.55861E-04 |
| 0.9 | 3.019206222 | 3.009424243 | 9.837978E-03 | 3.018094799 | 1.167421E-03 |
| 1.0 | 3.436563657 | 3.423639805 | 1.2923852E-02 | 3.43512273 | 1.440927E-03 |

**TABLE 4.6**
**PROBLEM**: $y' = x + y; y(0) = 1; h = 0.1$
**EXACT**: $Y_E(x) = 2e^x - x - 1$

| X | Exact solution | $y_{n+2} = y_{n+1} + \frac{h}{2}[3f_{n+1} - f_n]$ | ERROR | $y_{n+2} = y_{n+1} + \frac{h}{14}[35f_{n+1} - 3f_n + 20$ | ERROR |
|---|---|---|---|---|---|
| 0.1 | 1.110341836 | 1.1103 | 4.1836E-05 | 1.11 | 1.0341836E-02 |
| 0.2 | 1.242805516 | 1.2428 | 5.516E-06 | 1.2425 | 3.05516E-04 |
| 0.3 | 1.399717615 | 1.398705 | 1.012615E-03 | 1.399525001 | 1.97605E-04 |
| 0.4 | 1.583649395 | 1.58137075 | 2.278645E-03 | 1.583606251 | 4.3144E-05 |
| 0.5 | 1.797442541 | 1.795919758 | 1.522784E-03 | 1.797583815 | 1.41272E-04 |
| 0.6 | 2.044237601 | 2.041239184 | 2.998418E-03 | 2.044602694 | 3.65093E-04 |
| 0.7 | 2.327505415 | 2.322629073 | 1.214683E-03 | 2.32813993 | 6.34515E-04 |
| 0.8 | 2.651081857 | 2.643961475 | 7.120382E-03 | 2.651245337 | 1.6348E-04 |
| 0.9 | 3.019206222 | 3.009424243 | 9.837978E-03 | 3.019554283 | 2.92063E-04 |
| 1.0 | 3.436563657 | 3.423639805 | 1.2923852E-02 | 3.437104808 | 5.41151E-04 |

## 4.1 Analysis Of Results

From the Table above, the 3-stage Runge-Kutta method and the new schemes are more accurate than the old scheme, as they both produces less error [up to 4 decimal places] than the old scheme with error [up to 3 decimal places]. However, the new schemes are better because they are less rigorous in computation and have less computational steps than the 3-stage Runge-kutta method.

## 4.2 Estimation of Error

When solving an initial value problem we can achieve better results by varying the step size, Mathew (1992), stated that one way to guarantee accuracy of an initial value problem is to solve the problem twice using step sizes h and ½ h and compare answers at the mesh points corresponding to the larger sizes.

We solve the differential equation $y' = x + y; y(0) = 1$, using the six new methods at different step sizes: 0.1 and 0.05. The results obtained are as follows:

**TABLE 4.7**

**PROBLEM:** $y' = x + y; y(0) = 1$

**SCHEME 1**

| ERROR | | |
|---|---|---|
| **X** | **h = 0.1** | **h = 0.05** |
| 0.1 | 4.1836E-05 | 0.0000E+00 |
| 0.2 | 5.516E-06 | 0.0000E+00 |
| 0.3 | 1.176150E-04 | 6.369674E-06 |
| 0.4 | 1.193950E-04 | 1.384452E -05 |
| 0.5 | 2.32221E-04 | 2.262239E-05 |
| 0.6 | 2.73857E-04 | 3.292693E-05 |
| 0.7 | 4.00097E-04 | 4.500701E-05 |
| 0.8 | 4.84111E-04 | 5.914550E-05 |
| 0.9 | 6.36134E-04 | 7.566011E-05 |
| 1.0 | 7.6901E-04 | 9.491111E-05 |

**TABLE 4.8**

**PROBLEM:** $y' = x + y; y(0) = 1$

**SCHEME 2**

| ERROR | | |
|---|---|---|
| X | h = 0.1 | h = 0.05 |
| 0.2 | 1.3889E-04 | 3.582787E-06 |
| 0.4 | 1.95317E-04 | 1.297789E-05 |
| 0.6 | 4.83431E-04 | 2.620024E-05 |
| 0.8 | 7.18686E-04 | 4.452837E-05 |
| 1.0 | 1.8885743E-02 | 6.960880E-05 |

**TABLE 4.9**

**PROBLEM:** $y' = x + y; y(0) = 1$

**SCHEME 3**

| ERROR | | |
|---|---|---|
| X | h = 0.1 | h = 0.05 |
| 0.2 | 5.5160E-06 | 4.6658637-05 |
| 0.4 | 4.5325E-05 | 7.1446160E-05 |
| 0.6 | 1.58994E-04 | 7.5592420E-05 |
| 0.8 | 3.4804E-04 | 5.7844768E-05 |
| 1.0 | 6.31579E-04 | 1.4584462E-05 |

**TABLE 4.10**

**PROBLEM:** $y' = x + y; y(0) = 1$

**SCHEME 4**

| ERROR | | |
|---|---|---|
| X | h = 0.1 | h = 0.05 |
| 0.2 | 2.7814E-05 | 4.0834528E-05 |
| 0.4 | 1.30836E-04 | 1.0112339E-04 |
| 0.6 | 3.20591E-04 | 1.8704635E-04 |
| 0.8 | 6.16661E-04 | 3.06706610E-04 |
| 1.0 | 1.03984E-03 | 4.7062378E-04 |

**TABLE 4.11**

**PROBLEM:** $y' = x + y; y(0) = 1$

**SCHEME 5**

| ERROR | | |
|---|---|---|
| X | h = 0.1 | h = 0.05 |
| 0.2 | 5.5160E-06 | 0.0000+00 |
| 0.4 | 2.011450E-04 | 2.6470768E-05 |
| 0.6 | 4.53141E-04 | 6.4716830E-05 |
| 0.8 | 8.55861E-04 | 1.1859986E-04 |
| 1.0 | 1.440927E-03 | 1.9317018E-04 |

**TABLE 4.12**

**PROBLEM:** $y' = x + y; y(0) = 1$

**SCHEME 6**

| ERROR | | |
|---|---|---|
| X | h = 0.1 | h = 0.05 |
| 0.2 | 3.055160E-04 | 4.148045E-05 |
| 0.4 | 4.3144E-05 | 1.559357E-04 |
| 0.6 | 3.65093E-04 | 3.190551E-04 |
| 0.8 | 1.6348E-04 | 5.467672E-04 |
| 1.0 | 1.792493E-03 | 8.596782E-04 |

For the differential equation $y' = x + y; y(0) = 1$, using scheme 1 as example, the error for h = 0.1 at x = 1 (table 4.7) is 7.6901 x $10^{-4}$. This error is reduced to 9.491111 x $10^{-5}$ when h = 0.05 (table 4.8). The same trend is noticed for the other five methods, this further shows us that the rate of convergences increases as step length h decreases.

# CHAPTER FIVE

## 5.0 DISCUSSION, CONCLUSION AND RECOMMENDATION

### 5.1 Discussion

In this work we have been able to derive new Quasi – Runge – Kutta methods by refinement process.

These methods are

1. $$y_{n+2} = y_n + \frac{h}{3}\left[7f_{n+1} - 2f_n + f_{n-1}\right]$$

2. $$y_{n+2} = y_n + \frac{h}{3}\left[8f_{n+1} - 4f_{n+1/2} + 2f_n\right]$$

3. $$y_{n+2} = y_n + \frac{h}{35}\left[98f_{n+1} - 48f_{n+7/12} + 30f_n\right]$$

4. $$y_{n+2} = y_n + \frac{h}{18}\left[51f_{n+1} + 10f_n - 25f_{n+3/5}\right]$$

5. $$y_{n+2} = y_{n+1} + \frac{h}{12}\left[23f_{n+1} - 16f_n + 5f_{n-1}\right]$$

6. $$y_{n+2} = y_{n+1} + \frac{h}{14}\left[35f_{n+1} + 3f_n - 24f_{n+7/12}\right]$$

We have also solved differential equation using the six new methods. To assist us in solving differential equations, a computer implementation program using Microsoft excel software package was used.

### 5.2 Conclusion

Since we have used the methods to solve different equation, we can conclude that the six new Quasi-Runge–Kutta methods are accurate as they produce results which are comparable to those produced by other similar methods (3-stage runge-kutta and linear multi-step methods).

## 5.3 Recommendation

The main business of numerical analysis is to provide us with computational methods for the study and solution of mathematical problems.

However, most numerical methods give answers that are only approximations to the desired solution. Consequently, numerical results are seldom free of errors. It is recommended that further work be done in this direction to develop methods with higher accuracies

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