OPTIMIZATION ALGORITHMS FOR AN UNBOUNDED HORIZON

BY

HAKIMI DANLADI
(M.TECH/SSSE/99/406)

DEPARTMENT OF MATHEMATICS/COMPUTER SCIENCE
FEDERAL UNIVERSITY OF TECHNOLOGY
MINNA, NIGER STATE.

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CERTIFICATION

This thesis titled "OPTIMIZATION ALGORITHMS FOR AN UNBOUNDED HORIZON" by Hakimi Danladi meets the regulations governing the award of the degree of Masters of Technology in Mathematics. Federal University of Minna and is approved for its contribution to knowledge and literary presentation.

Dr. S. A. Reju
Supervisor

Date: 16-12-2003

Mr. L. N. Ezeako
Head of Department

Date: 26-1-2004

Prof. (Mrs) H.O. Akanya
Dean, School of Science
and Science Education

Date: 4th-02-2004

Prof. J. A. Abalaka 23/8/04
Dean, Postgraduate School

Date:.............................

Prof. A. U. Afuwape
External Examiner

Date: 31/7/2003

DEDICATION

This piece of work is highly and humbly dedicated to the Almighty God and to my family.

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ABSTACT

Undoubtedly most, if not all, decision-making is part of an ending history of actions. Earlier choices have affected the present, current decisions will influence the future, and so on. As a result, all models must be viewed as imbedded in an unbounded horizon. Several of the dynamic models we have studied so far simply ignored the future beyond a designated horizon period, and sometimes a planning horizon theorem could be established to demonstrate that such a procedure might yield an optimal current decision.

Other models attempted to account for the future by selecting certain terminal conditions (such as a specified minimum level of work force or productive capacity). In contrast to these models, this project assume that the planning horizon is limitless.

In order to derive definite answers for models with an unbounded horizon, we deemed it necessary to add a restrictive assumption termed **"an assumption of Stationarity"**

The scope of optimization algorithm with unbounded horizon is broad and encompasses many topics. This project selects these portions of the subject, which present a logical and coherent body of knowledge for classroom presentation of self-study. The emphasis on successive approximations in function space (value iteration), successive approximation in policy space (policy iteration) and applications of optimization with unbounded horizon.

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CHAPTER ONE

BRIEF OVERVIEW OF OPTIMIZATION THEORY

1.1. MOTIVATION FOR STUDYING OPTIMIZATION

There exist an enormous variety of activities in the everyday world, which can usefully be described as systems, from actual physical systems such as chemical processing plants to theoretical entities such as economic models. The efficient operation of these systems often requires an attempt at the optimization of various indices, which measure the performance of the system. Sometimes these indices are quantified and represented as algebraic variables. Then values for these variables must be found which maximize the gain or profit of the system and minimize the waste or loss. The variables are assumed to be dependent upon a number of factors. Some of these factors are often under the control, or partial control, of the analyst responsible for the performance of the system.

The process of attempting to manage the limited resources of a system can usually be divided into six phases (i) Observation (ii) definition of the problem (iii) Construction of a mathematical model which reflects the important aspects of the system; (iv) model solution; (v) implementation of the solution selected; and (vi) the introduction of a strategy which monitors the performance of the system after implementation.

(i) **OBSERVATION:**

The first step in the optimization process is the identification of a problem that exists in the system (organisation). The system must be continuously and closely observed so that problems can be identified as soon as they occur or are anticipated. Problems are not always the result of a crises that must be reacted to, but instead frequently involve an anticipatory or planning situation. The person who normally identifies a problem is the manager, since the manager is the one who works in the vicinity of places where the problems might occur. However, problems can often be identified by a management scientist, a person skilled in the techniques of management science and trained to identify problems, who has hired specifically to solve problems using optimization techniques.

(ii) DEFINITION OF THE PROBLEM

Once it has been determined that a problem exists, the problem must be clearly and concisely defined. An improperly defined problem can easily result in no solution or an inappropriate solution. Therefore, the limits of the problem and the degree to which it pervades other units of the organisation must be included as part of the problem definition. Since the existence of a problem implies that the objectives of the firm are not being met in some way, the goals (or objectives) of the organisation must be clearly defined. A stated objective helps to focus attention on what the problem actually is.

(iii) MODEL CONSTRUCTION

An optimization model is an abstract representation of an existing problem situation. It can be in the form of a graph or chart, but most frequently, an optimization model consists of a set mathematical relationship. These mathematical relationships are made up of numbers and symbols.

As an example, consider a business firm that sells a product. The product costs \$5 to produce and sells for \$20. A model that computes the total profit that will accrue from the items sold is

$Z = \$20x - 5x$

In this equation x represents the number of units of the product that are sold, and Z represents the total profits that results from the sale of the product. The symbols x and Z are variables. The term variable is used because no set numerical value has been specified for those items. The number of units sold x, and the profit, Z, can be any amount (within limits); they can vary. These variables can be further distinguished Z is a dependent variable because its value is dependent on the number of units sold. Alternatively x is an independent variable, since the number of units sold is not dependent upon anything else (in this equation).

The numbers \$20 and \$5 in the equation are referred to as parameters. Parameters are constant values that generally coefficients of the variables (symbols) in an equation. Parameters usually remain constant during the process of solving a specific problem. The parameter values are derived from data (i.e., pieces of information) from the problem environment. Sometimes the data are readily available and quote accurate. For example, it would be assumed that selling price of \$20 and product cost of \$5 could be obtained from the firm's accounting department and would be very accurate.

The equation as a whole is known as a functional relationship (also called function and relationship). The term is derived from the fact that profit, Z, is a function of the number of units sold, x. As such, the equation relates profit to units sold.

Since only one functional relationship exists in this example, it is also the model. In this case, the relationship is a model of the determination of profit, for the firm. However, this model does not really replicate a problem. Therefore, we will expand our example to create a problem situation.

Let us assume that the product is made from steel and the business firm has 100 pounds of steel available. If it takes 4 pounds of steel to make each unit of the product, then, we can develop an additional mathematical relationship to represent steel utilization.

4x = 100 pounds

This equation indicates that for every unit produced, 4 of the available 100 pounds of steel will be utilized. Now our model consists of two relationships.

Z = \$20x - 5x

4x = 100

In this new model we say that the profit equation is an objective function, and the resource equation is a constraint. In other words the objective of the firm is to achieve as much profit, Z, as possible, but the firm is constrained from achieving an infinite profit because it is limited to the amount of steel available. To signify this distinction between the two relationships in this model, we will add the following notation.

Maximize Z = \$20x - 5x

Subject to 4x = 100

This model now represents the manager's problem of determining the number of units to produce. We will recall that we defined the number of units to be produced as x. Thus, when we determine the value of x, it represents a potential (or recommended) decision from the manager. As such, x is also known as a decision variable.

(iv) **MODEL SOLUTION**

Once models are constructed in optimization, they are solved using the optimization techniques that are briefly discussed in this project. An optimization solution technique usually applies to a specific type of model. Thus, the model type and solution method are both part of the optimization technique.

We are able to say that a model is solved, since the model represents a problem. When refer to model solution we also mean problem solution.

Using the example model developed in the previous section,

Maximize $Z = \$20x - 5x$

Subject to $4x = 100$

the solution technique is simple algebra. Solving the constraint equation for x,

$$4x = 100$$

$$\Rightarrow x = \frac{100}{4}$$

$$\therefore x = 25 \text{ units}$$

Substituting the value of 25 for x into the profit function results in the total profits:

$$Z = \$20x - 5x$$

$$= 20(25) - 5(25)$$

$$= \$375$$

Thus, if the manager decides to produce 25 units of the product, the business firm will receive \$375 in profit. Note, however, that the value of the decision variable does not constitute an actual decision; rather it is information that serves as a recommendation or guideline in helping the manager make a decision.

Some optimization techniques do not generate an answer or recommended decision. Instead, they provide descriptive results: results that describe the system being modeled.

(v) IMPLEMENTATION OF RESULTS

The optimization technique provides information that can aid the manager in making decision. Of course, the manager does not rigidly apply the results of the optimization model solution without contemplation. The information obtained must be combined with the manager's own expertise and experience in making the ultimate decision. If the manager does not use the information derived from the optimization technique, then the results are not implemented (i.e., they are not put to use). If the results are not implemented then the effort and resources that went into problem definition, model construction and solution are wasted. As such, this step in optimization process cannot be ignored. An effort must be made to ensure that the results will be used (assuming that the results are applicable).

(vi) OPTIMIZATION AS AN ONGOING PROCESS

Completion of the five steps described above does not necessarily mean that the optimization process has been completed. The model results and the decisions based on the results provide feedback to the original model. The original optimization model can then be modified to test different conditions and decision the manager thinks might occur in the future. Or, the results may indicate that a different problem exists that had not been considered previously, thus the original model can be altered or reconstructed. As such, the optimization process can be continuous rather than simply providing one solution to one problem.

1.2 OPTIMIZATION AS A BRANCH OF MATHEMATICS

It can be seen from the previous section that the theory of optimization is mathematical in nature. Typically it involves the maximization or minimization of a function (sometimes unknown), which represents the performance of some system. This is carried out by the finding of values for those variables (which are both quantifiable and controllable), which cause the function to yield an optimal value. A knowledge of linear algebra and differential multivariable calculus is required in order to understand how the algorithms operate. A second knowledge of analysis is necessary for an understanding of the theory.

Some of the problems of optimization theory can be solved by the classical techniques of advanced calculus – such as Jacobian methods and the use of Lagrange multipliers. However, most optimization problems do not satisfy the conditions necessary for solution in this manner. Of the remaining problems many, although amenable to the classical techniques, are solved more efficiently by methods designed for the purpose. Throughout recorded mathematical history a collection of such techniques has been built up. Some have been forgotten and reinvented; others received little attention until modern-day computers made them feasible.

The bulk of the material of the subject is of recent origin because many of the problems, such as traffic flow, are only now of concern and also because of the large number of people now available to analyze such problems. When the material is catalogued into a meaningful whole the result is a new branch of applied mathematics.

1.3 WHAT IS OPTIMIZATION

The fundamental problem of optimization is to arrive at the best possible decision in any given set of circumstances. Of course, many situation arise where the "best" is unattainable for one reason or another; sometimes what is 'best" for one person is "worst" for another; more often we are not at all sure what is meant by 'best". The first step, therefore, in a mathematical optimization problem is to choose some quantity, typically a function of several variables to be maximized or minimized, subject possible to one or more constraints. The commonest types of constraints are equalities and inequalities which must be satisfied by the variables of the problem, but many other types of constraints are possible; for example, a solution in integers may be required. The next step is to choose a mathematical method to solve the optimization problem; such methods are usually called optimization techniques or algorithms.

The choice of optimization technique is by no means obvious, for the theory and practice of optimization has developed rapidly since the advent of electronic computers in 1945. It came of age as a subject in the mathematical curriculum in the 1950's when the well-established methods of the differential calculus and the calculus of variation were combined with the highly successful new techniques of mathematical programming, which were being developed at that time.

The optimization problems that have been posed and solved in recent years have tended to become more and more elaborate, not to say abstract. Perhaps the most outstanding example of the rapid development of optimization techniques occurred with the introduction of dynamic programming by Bellman in 1957 and the maximum principle 1958. They were designed to solve the problem of the optimal control of dynamical systems. Both dynamic programming and the maximum principles are closely related to the calculus of variations, and hence to each other.

The simply – stated problem of maximizing or minimizing a given function of several variables has attracted the attention of many mathematicians over the past twenty-five years or so. The direct search methods of solution, which involve function evaluations and comparisons only, are usually simpler though less accurate for the same computational effort, than the indirect or gradient methods, which require values of the function and its derivatives. Both types of method are still undergoing development, with the major emphasis being on the search for efficient and reliable algorithms to deal with general non-linear functions.

1.3.1. TYPICAL PRACTICAL EXAMPLES OF OPTIMIZATION PROBLEMS

(i) Statistics

The frequency function of a population is completely determined once its parameters are known. For example, the binomial distribution is completely determined by the parameters n (the probability of success in a single trial). An important problem in statistics is to estimate the population parameters, given a random simple drawn from the population. If the form of the frequency function is assumed, then values for its parameters may be determined by forming the likelihood function, which gives the probability that the given sample came from a population with the assumed frequency function. The likelihood function is thus a function of unknown parameters. The values of the parameters are non-estimated by maximizing the likelihood with respect to these parameters, subject to any constraints that may be present. The resulting optimal values of the parameters are known as maximum likelihood estimates. The method may be applied to function of discrete or continuous variables.

(ii) AERODYNAMICS

There are many optimization problems concerned with the design, performance and flying qualities of aircraft. The aircraft designer must minimize the structural weight, subject to the structure having sufficient strength and stiffness to carry the critical design loads safely. The cruising altitude should be chosen so as to minimize fuel consumption; it often happens that a steady climb is more economical than flight at constant altitude. Aircraft are designed for many different purposes, and in particular cases it may be important to

(i) Minimize the take-off run, (ii) maximize the rate of climb, (iii) maximize the ceiling, (iv) maximize the endurance, (v) minimize the wave drag in supersonic flight. All these problems are subject to various constraints, which, in certain cases, may be so severe that no optimization problem remains.

(3) CHEMICAL ENGINEERING:-

The manager of a chemical plant has to decide on his major objective in mining the plant. Should he maximize output?. Is this consistent with maximizing profit?. To answer these questions requires the solution of at least two optimization problems. The answer to the second question may be 'No', for lower output could mean better quality output, greater than efficiency and more valuable by - products.

(4) OPERATIONAL RESEARCH

The application of optimization techniques to industrial and commercial problems forms part of the subject of operational research.. The fundamental problem of stock control is to choose a stock level and a stock replacement policy, which maximize overall profit. The usual assumptions are that losses are incurred if either too much or too little stock is kept. The demand may be known exactly or its frequency function may be assumed. A related problem is that of renewing obsolescent machinery while maintaining maximum efficiency.

(5) ECONOMICS

How many new power stations should be built in Britain between now and the year 2010?. How many of them should be atomic power stations?. These questions lead to very complicated optimization problems; it is not at all clear which quantities should be maximized or minimized and it is even less clear what constraints should be imposed. Nevertheless, problems of this kind obviously need careful study before the crucial decisions are taken.

1.4. BASIC CONCEPTS OF OPTIMIZATION

The problem of maximizing or minimizing a given function

$$Z = f(x) \qquad ----- \qquad 1.4.1.$$

Subject to the given constraints.

$$g_i(x) \leq, = \text{ or } \geq b \qquad ----- \qquad 1.4.2.$$

is called the general constrained optimization problem. The function Z appearing in (1.4.1) is called the objective function. In (1.4.2.), the number of independent equality constraints must be less than n, the number of variables, otherwise the problem is over specified.

Inequalities of $\leq$ and $\geq$ types can always be converted into equation by introducing slack and surplus variables, respectively. For example, the inequalities.

$$g_1(x) \leq, b_1, \quad g_2(x) \geq b_2 \quad \text{-----} \quad 1.4.3.$$

are respectively equivalent to

$$g_1(x) + x_{n+1} = b_1, g_2(x) - x_{n+2} = b_2 \quad 1.4.4.$$

provided that the slack variable x_{n+1} and the surplus variable x_{n+2} satisfy

$$x_{n+1} \geq 0, x_{n+2} \geq 0 \quad \text{-----} \quad 1.4.5.$$

The variables x_j are called main variables whenever it is necessary to distinguish them from the slack and surplus variables x_{n+1}. Constraints of the type (1.4.5) are called non-negativity restrictions; in some problems they are also imposed on the main variables. Although it is perfectly correct to regard non-negativity restrictions as constraints to be included among those of (1.4.2), it is often found convenient to treat them separately. In general, the effect of substituting (1.4.4) and (1.4.5) for (1.4.3) is to simplify the constraints at the expense of an increased number of variables; this substitution is often extremely useful.

Strict inequality constraints have been omitted from (1.4.2.). This is not a serious limitation in practice, since any constraint of $<$ or $>$ type can be replaced by one of $\leq$, $=$ or $\geq$ type by means of some simple manipulations. For example, the constraint $g_k(x) < b_k$

is for all practical purposes equivalent to

$$g_k(x) \leq b_k - \in$$

Where $\in$ is a suitably small positive constant. The most important reason, however, for restricting the constraints (1.4.2) to the $\leq$, $=$ and $\geq$ a type is a theoretical one: many fundamental results in optimization theory no longer apply when strict inequality constraints are introduced.

There is no essential difference between a maximizing and a minimizing problem, for the values of the x_j which maximize f(x) also minimize –f(x). Thus every maximizing problem can be formulated as a minimizing problem, and vice versa.

Since there is not at present, nor is there ever likely to be a single recommended method for solving every general constrained optimization problem, it is important to take advantage of any special features that a given problem may posses. It is therefore useful to classify the special cases of the general problem.

The most obvious special case is the general unconstrained optimization problem, in which there are no constraints, and the problem is merely to find values of the x_j which maximize f(x). Many modern optimization techniques are designed specifically to solve the general unconstrained optimization problem, for given a constrained optimization; techniques exist which make it possible to write down an equivalent unconstrained problem. Thus the description of an optimization problem as "unconstrained" is a convenient mathematical classification, but may be a misnomer.

When every constraint in (1.4.2) is an equation, we have the classical optimization problem:

$$\text{Maximize} \quad Z = f(x) \quad \text{-----} \quad 1.4.6(a)$$

$$\text{Subject to} \quad g_1(x) = b_n \quad \text{-----} \quad 1.4.6(b)$$

In this problem, the function f and g_i are assumed to possess continuous first – order partial derivatives with respect to all the variables. Functions with this property are said to belong to the class C_1. Necessary conditions for a maximum can be found by the classical analytic method of Lagrange multipliers; if we assume further that the functions f and g_i posses continuous second-order partial derivatives with respect to all the variables, i.e. f, $g_i \in C_2$, then sufficient conditions for a maximum can also be found. An important advance in optimization theory took place in 1951 when Kuhu and Tucker extended the classical method of Lagrange multipliers to problems with inequality constraints and non-negativity restriction.

If both f(x) and all the $g_i(x)$ are linear functions of the x_j, we have a linear programming problem. Linear programming is still one of the two principal reasons for this: first, it has many hundreds of useful applications and, secondly, extremely large problems can now be solved on electronic computers by means of the simplex method. The simplex method, which was devised by George B. Dantzig in 1947, is an algorithm for the solution of the general linear programming problem.

The function f(x) is said to be separable if it is of the form $\sum_j f_j(x_j)$. If both f(x) and all the g_i (x) are separable, we have a separable programming problem.

$$\text{Maximize} \qquad Z = \sum_j f_j(x_j) \qquad \text{-----} \qquad 1.4.7(a)$$

$$\text{Subject to} \quad \sum_i g_i(x) \leq, \ = \ \text{or} \ \geq b_i \qquad \text{-----} \ 1.4.7\ (b)$$

Special methods are available for the solution of this problem. These methods essentially reduce the separable programming problem to a linear programming problem. Also, general methods tend to be more efficient than usual when they are applied to the separable programming problem, owing to the lack of interaction between the variables.

If either f(x) or one or more of the g_i (x) is non linear in any of the variables, we have non linear programming problem. Thus every constrained optimization problem defined by (1.4.1.) and (1.4.2) is either a linear programming problem or a non-linear programming problem.

If f(x) is a quadratic function of the xj where all the g_i (x) are linear in the x_j, we have a quadratic programming problem. Many algorithms have been devised for the solution of this type; most of them rely on an extension of the simplex method.

The first step towards choosing an appropriate optimization technique to solve a given problem is to find out whether the problem belongs to any of the special categories mentioned above. Among other factors affecting the choice of method are the time available for a solution, the accuracy required, the computer facilities available, the relative ease with which f, g, ∇f, ∇g can be evaluated, and whether the variables x_j are continuous or discrete.

1.5. CLASSIFICATION OF OPTIMIZATION PROBLEMS

Optimization problems can be classified in several ways as described below:

(1) CLASSIFICATION BASED ON THE EXISTENCE OF CONSTRAINTS

As indicated earlier in this project, any optimization problem can be classified as a constrained or an unconstrained one depending upon whether the constraints exist or not in the problem.

(ii) CLASSIFICATION BASED ON THE NATURE OF DESIGN VARIABLES

Based on the nature of design variables encountered, optimization problems can be classified into two broad categories. In the first category, the problem is to find values to a set of design parameters, which make some prescribed function of these parameters minimum subject to certain constraint. For example, the problem of minimum weight design of a prismatic beam subject to a limitation on the maximum deflection can be stated as follows:

Find $x = \begin{Bmatrix} b \\ d \end{Bmatrix}$, which minimizes

$$f(x) = \rho lbd$$

Subject to the constraints

$$\delta t_{ip}(x) \leq \delta_{max}$$

$$b \geq 0 \text{ and } d \geq 0$$

where ρ is the density and δt_{ip} is the tip deflection of the beam. Such problems are called parameter or static optimization problems. In the second category of problems, the objective is to find a set of design parameters, which are all continuous functions of some other parameter, which minimizes an objective function subject to the prescribed constraints.

(iii) CLASSIFICATION BASED ON THE PHYSICAL STRUCTURE OF THE PROBLEM

Depending upon the physical structure of the problem, optimization problems can be classified as optimal control and non-optimal control problems. An optimal control (OC) problem is usually described by two types of variables, namely, the control (design) and the state variables. The control variables govern the evolution of the system from one stage to the next and the state variables describe the behaviour of the system in any stage. Explicitly, the optimal control problem is a mathematical programming problem involving a number of stages, where each stage evolves from the previous stage in a prescribed manner. The problem is to find a set of control or design variables such that the total objective function over the I number of stages is minimized subject to certain constraints on the state and control variables. It can be stated as follows:

Find X, which minimizes:

$$f(X) = \sum_{i=1} f_i(x_i, y_i)$$

Subject to the constraints

$$q_i(x_i, y_i) + y_i = y_{i+1}; \quad i = 1, 2, \ldots, 1$$

$$g_j(x_j) \leq 0 \quad ; \; j = 1, 2, \ldots, l$$

$$\text{and} \quad h_k(y_k) \leq 0 \quad ; \; k = 1, 2, \ldots, l$$

where x_i is the ith control variable, y_i is the ith state variable and f_i is the contribution of the ith stage to the total objective function; g_j, h_k and q_i are functions of x_j, y_k and x_i and y_i respectively.

(iv) **CLASSIFICATION BASED ON THE NATURE OF EQUATION INVOLVED**

Another important classification of optimization problems is based on the constraints. According to this classification, optimization problems can be classified as linear, nonlinear, geometric, and quadratic programming problems. This classification is extremely useful from the computational point of view since there are many methods developed solely for the efficient solution of a particular class of problems. Thus the first task of a designer would be to investigate the class of problem encountered. This will, in many cases, dictate the types of solution procedures to be adopted in solving the problem.

(a) **NON – LINEAR PROGRAMMING PROBLEM**

If any of the functions among the objective and constraint functions in eqn (1.4.1) is non-linear, the problem is called a non-linear programming problem. This is the most general programming problem and all other problems can be considered as special cases of non-linear programming problem (NLP)

(b) **GEOMETRIC PROGRAMMING PROBLEM**

Definition 1.5.1

A function h(x) is called a posynomial if h can be expressed as the sum of power terms of the form

$$C_1 x_1^{a11} x_2^{a12} \text{---} \quad x_n^{a1n}$$

Where c_1 and a_{ij} are constants with $c_1 > 0$ and $x_j > 0$. Thus a posynomial can be expresses as:

$$h(x) = C_1 x_1^{a11} x_2^{a12} \text{---} \quad x_n^{a1n} + \text{---} + C_N x_1^{aN2} \text{---} x_n^{aNn} \quad \text{-----}1.5.3$$

Definition 1.5.2

A geometric programming (GMP) problem is one in which the objective function and constraints are expressed as posynomials in X.

Thus the geometric programming problem can be posed as follows:

Find X, which minimizes

$$f(x) = \sum_{i=1}^{N_0} C_1 \left[\prod_{j=1}^{n} x_j^{P_{ij}} \right], \quad c>0, \; x_j>0$$

subject to

$$g_j(x) = \sum_{i=1}^{N_j} a_{ij} \left[\prod_{k=1}^{n} x_k^{q_{ik}} \right] \leq 0, \; a_{ij}>0 \quad \text{------} \quad 1.5.4$$

$$j = 1,2,\text{---},m$$

where N_0 and N_j denote the number of posynomial terms in the objective and jth constraint function, respectively.

© QUADRATIC PROGRAMMING PROBLEM

Definition 1.5.3

A quadratic programming problem is a non-linear programming problem with a quadratic objective function and linear constraints. It is usually formulated as follows:

Find x, which minimizes

$$F(x) = c + \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} x_i x_j \quad \text{-----}1.5.5$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \; j=1,2,\text{---},m$$

$$x_i \geq 0, \; i = 1,2,\text{---},n$$

where c, q_i , Q_{ij}, a_{ij} and b_j are constants.

(d) LINEAR PROGRAMMING PROBLEM

If the objective function and all the constraints in eqn (1.4.1) are linear functions of the design variables, the Mathematical programming problem is called a linear programming (LP) problem. A linear programming problem is often stated in the following form

$$\text{Find } x = \begin{bmatrix} x_1 \\ x_2 \\ M \\ x_n \end{bmatrix} \quad \text{---- } 1.5.6$$

which minimizes $f(x) = \sum_{i=1}^{n} c_i x_j$

subject to the constraints

$$\sum_{k=1}^{n} a_{jk} x_k = bj, \ j = 1,2,\text{---},m \text{ and } x_i \geq 0, \ i = 1,2,\text{---},n$$

where c_i, a_{ik} and b_j are constants.

(V) CLASSIFICATION BASED ON THE PERMISSIBLE VALUES OF THE DESIGN VARIABLES

Depending on the values permitted for the design variables, optimization problems can be classified as integer and real-valued programming problems.

If some or al of the design variables x_1, x_2, ---, x_n of an optimization problem are restricted to take on only integer (or discrete) values, the problem is called an integer programming problem. On the other hand, if all the design variables are permitted to take any real-value, the optimization problem is called a real-valued programming problem.

(vi) **CLASSIFICATION BASED ON THE DETERMINISTIC NATURE OF DESIGN VARIABLES**

Based on the deterministic nature of the variables involved, optimization problems can be classified as deterministic and stochastic

STOSCHASTIC PROGRAMMING PROBLEM

A stochastic programming problem is an optimization problem in which some or all of the parameters (design variables and/or preassigned parameters) are probabilistic (non-deterministic or stochastic)

(vii) **CLASSIFICATION BASED ON SEPARABILITY OF THE FUNCTIONS**

Optimization problems can be classified as separable and non-separable programming problems based on the separability of the objective and constraint functions.

Definition 1.5.4

A function f(x) is said to be separable if it can be expressed as the sum of n single variable functions $f_1(x), f_2(x), ---, f_n(x)$, that is,

$$f(x) = \sum_{i=1}^{n} f_i(x_i) \quad ----- \quad 1.5.7$$

So, a separable programming problem is one in which the objective function and the constraints are separable and can be expressed in standard form as :

Find x, which minimizes $f(x) = \sum_{i=1}^{n} f_i(x_i) \quad ----- \quad 1.5.8$

subject to

$$g_j(x) = \sum_{i=1}^{n} g_{ji}(x_i) \leq b_j, \; j=1,2,---,m$$

where b_j is a constant.

(viii) **CLASSIFICATION BASED ON THE NUMBER OF OBJECTIVE FUNCTIONS**

Depending on the number of objective functions to be minimized, optimization problems can be classified as single and multi-objective programming problems. According to this classification, the previous classifications of optimization problems are single objective programming problems.

MULTI-OBJECTIVE PROGRAMMING PROBLEM

A multi-objective programming problem can be stated as follows: Find x which minimizes $f_1(x), f_2(x), ---, f_k(x)$ subject to $g_j(x) \leq 0; j = 1,2,---,m$

Where $f_1, f_2, ---, f_k$, denote the objective functions to be minimized simultaneously.

1.6 OPTIMIZATION TECHNIQUES

The various techniques available for the solution of optimization problems are so many that we may not be able to list all of them here.

The classical methods of differential calculus can be used to find unconstrained maxima and minima of a function of several variables. These methods assume that the function is differentiable twice with respect to the design variables and derivatives are continuous. The classical methods of optimization are useful in finding the optimum of continuous and differentiable functions. These methods are analytical and make use of the techniques of differential calculus in locating the optimum points. Since some of the practical problems involve objective functions that are not continuous and/or differentiable, the classical optimization techniques have limited scope in practical applications.

For problems with equality constraints, the Lagrange multiplier method is frequently used. But this method, in general, leads to a set of non-linear simultaneous equations, which may be difficult to solve.

When the problem is one of minimization or maximization of an integral, the methods of calculus of variations can be used to solve it. The calculus of variations is concerned with the determination of extreme (maximal and minima) or stationary values of functional. A functional can be used to solve trajectory optimization problems. The calculus of variations is almost as old as calculus itself and is a powerful method for solution of problems in several fields like statics and dynamics of rigid bodies, general elasticity, vibrations, optics, and optimization of orbits and controls.

The techniques of non-linear, linear, geometric quadratic or inter programming can be used for the solution of the particular class of problems indicated by the name of the technique. These are all numerical methods wherein an approximation solution is sought by proceeding in an interactive manner by starting from an initial solution.

The linear programming is an optimization method applicable for the solution of problems in which the objective function and the constrains appear as linear functions of the decision variables. The constraint equations in a linear programming problem may be in the form of equalities or inequalities. In applying the linear programming technique, first, the problem must be identified as being solvable by linear programming, second, the unstructured problem must be formulated as a mathematical model; and, third, the model must be solved using established mathematical techniques.

The geometric programming technique is a relatively new method for solving a class of non-linear programming problems. It is used to minimize functions, which are in the form of polynomials subject to constraints of the same type. It differs from other optimization techniques in the emphasis it places upon the relative magnitudes of the terms of the objective function rather than the variables. Instead of finding the optimal values of the design variables first, geometric programming first finds the optimal value of the objective function. This feature is especially advantageous in situations where the optimal value of the objective function may be all that is of interest. In such cases, the calculation of the optimum design vectors can be omitted. Another advantage of geometric programming is that it often reduces a complicated optimization problem to one involving a set of simultaneous linear algebraic equations. The major disadvantage of the method is that requires the objective function and the constraints in the form of polynomials.

The stochastic programming deals with situations where some or all parameters of the optimization problem are described by stochastic (or random or probabilitistic variables rather than by deterministic quantities. Depending on the nature of equations involved (in terms of random variables) in the problem, a stochastic problem is called a stochastic linear or dynamic or non-linear programming problem. The basic idea used in solving any stochastic programming problem is to convert the stochastic problem into an equivalent deterministic problem. The resulting deterministic problem is then solved by using the familiar techniques like linear, geometric, dynamic and non linear programming.

Another optimization technique is multi objective and is a situation when two or more opponents are competing for the achievement of conflicting goals, a competitive problem exists. Generally, in such problems, the losses of one opponent signify the gains of the others. Naturally, the objective function depends on a set of controlled as well as uncontrolled variables where the uncontrolled variable depends on the strategy of the competitor. The resulting optimization problem can be solved by using the game theory.

The critical path method (CPM) and the programme evaluation and review technique (PERT) are network methods, which are useful in planning, scheduling and controlling a project. These methods, are called network methods since in both the methods, the various operations necessary to complete the project and the order the operations are to be performed are shown in a graph called a network. Critical path method (CPM) is useful for projects in which the durations of the various operations are known exactly whereas PERT is designed to deal with projects in which there is uncertainty regarding the durations of the various operations.

CHAPTER TWO

UNBOUNDED HORIZON OPTIMIZATION ALGORITHMS

2.1 INTRODUCTION TO UNBOUNDED HORIZON OPTIMIZATION

A decision process with an unbounded horizon is one that has infinitely many stages. Although such situations rarely occur in practice, they are convenient models for analyzing processes that have no obvious terminal point. The following condition is generally assumed for such processes.

ASSUMPTION OF STATIONARY

Now states that the decisions, returns, and states associated with the process are the same in every stage.

For processes that conform to this assumption, optimal policies depend only on the states and not on the stages. Whatever decision is optimal for state U in stage 1 will also be optimal for state U in stage 100, since all the underlying conditions remain invariant.

The stationarity assumption is restrictive in that it does not allow interest rates, costs, charges, or any other quantity to change as long as the process continues into the future. An optimal policy, therefore, remains optimal only so long as the stationarity assumptions remain valid.

2.2 MODELS WITH A LIMITLESS VISTA

Unquestionably most, if not all, decision-making is part of an unending history of actions. Earlier choices have affected the present, current decisions will influence the future, and so on. In this light, all models must be viewed as imbedded in an unbounded horizon. Several of the dynamic models we have studied so far simply ignored the future beyond a designated horizon period, and sometimes a planning horizon theorem could be established to demonstrate that such a procedure might yield an optimal current decision. Other models attempted to account for the future by selecting certain 'terminal' conditions (such as a specified minimum level of work force or productive capacity). In contrast to these models, the illustrations in this project assume that the planning horizon is limitless.

In order to derive any definite answers for models with an unbounded horizon, it is necessary to add a restrictive assumption; broadly, the hypothesis is termed an assumption of stationarity. In the simplest cases, you assume that all economic return functions, decision possibilities and external phenomena (like demand requirements) are identified every period.

2.2.1 OPTIMAL STATIONARY POLICIES

Assuming stationarity, we can safely intuit the meaning of "making a current decision in the face of an unbounded horizon". What may surprise us is that our intuition is of limited help in fathoming how to make optimal decision.

To illustrate, consider a finite horizon dynamic programming model. At any period we need to know only the state of the system and number of stages remaining. Optimality of any strategy is judged according to the sum of a finite stream of returns. Now let the horizon be unbounded, so by definition the "number of periods remaining" is always the same, and any strategy employed over the horizons results in an unending stream of returns. Since for every strategy this stream may grow without bound as the horizon lengthens, you need a way to compare the strategies. Of course, if one strategy accumulates more returns than another for every horizon length, no problem of comparison occurs. But typically one strategy looks better for certain finite horizon and worse for others, so the resolution is by no means obvious. Thus, a limitless vista raises two pertinent questions about determining an optimal solution.

(i) What criterion is appropriate for judging the relative desirability of different infinite streams of returns?

(ii) Is it optimal in a limitless vista to consider only stationary strategies, that is, ones depending solely on the current state of the system?

This chapter focuses on answering (i). We will critically examine several frequently used criteria for evaluating infinite streams. We shall also see in chapter three how to apply these criteria to a simple but important regeneration model. In studying the solution of this model, we will discover several numerical methods of successive approximation that can be applied to more general problems of optimization in an unbounded horizon.

2.3 SUBTLETIES OF INFINITE STREAMS

Businessmen, economists, and mathematicians have argued for centuries over how to assess infinite streams of returns. We will easily see why in the example below, and we should try to articulate the economic in sights and managerial significance of these illustrations.

Experience has shown that most decision makers cannot intuitively make consistent judgments streams of returns. As a consequence, most businessmen turn to formulas for providing at least a preliminary screening or ranking of decision alternatives. We too will apply formulas that convert an infinite stream of returns into a single number so as to indicate the relative merit of the associated alternative. But before doing so, we want to make sure that we see some of the substantive issues involved in different methods for choosing among infinite streams. Only with this knowledge in mind can we appreciate both the strengths and limitation of the simple to apply formulas.

This point investigates two questions that are central to optimization in a dynamic setting.

(i) When is an evaluation formula appropriate for comparing different strategies?

(ii) Does such a formula always reduce an infinite stream to a single number that can be used as the basic for comparison?

These points are treated in considerable details below, since the associated problems can be quite subtle.

We investigate three criteria of merit. The first is average return per period. Actually, this criterion arises most often when the economic measure is cost. Then the selection rule recommended is to choose an alternative having the least average cost per period. The second criterion is present discounted value. As we will soon see, these two criteria do not always select the same alternative, and occasionally give rise to some nasty technical problems.

The third criterion is called equivalent average return. The idea is probably new to us, and is important in operations research models because it provides the mathematical connection between the other two criteria. Often we can derive the form of an optimal policy using this criterion, and then with this result we can calculate specific numerical solutions for either the average return per period or the present value criterion.

2.3.1 UTILITY OF MONEY

We often hear it said that a dollar is worth more than a dollar a year from today. Why? This maxim is based on several considerations. The decision- maker may find the sheer utility or personal worth of a current dollar is greater now than later. For example, consider a company that has paid its stockholders a regular quarterly dividend for 25 years. The firm may be very reluctant to forgo paying out a current dividend, even if it can promise to pay it eventually, perhaps with interest added. This difference in the utility of money at district points in time is the heart of the problem of making commensurate several unending streams of returns.

Actually, the same comparability problem exists in finite horizon models. In the dynamic programming models we might have studied simply ignored the difficulty of comparability. Since it was always possible to sum the profits or costs over the bounded horizon, the resultant criterion function attached a unique finite number to each policy and made optimization straightforward. Here you can no longer ignore the comparability difficulty, even if you want to. Since the sum is now over an unbounded horizon, the total returns are infinite for most strategies. As we might imagine, any naïve approach for comparing several infinite streams of returns can succeed in only the simplest of cases. To illustrate, comparing policies, in one way or another for each and every finite horizon length does not always work. The following example shows why.

Example: 2.3.1.

Suppose we must choose one of the profitable alternatives that are described in fig. 2.3.1 by the sequence of returns (= profits) for each period, starting with the present. At the current period we receive a profit of 3 from A and B, 4 from G etc. In the next period, we obtain a profit of 2 from A, a profit of only 1 from B, etc.

It is reasonable that we would rule out F right away, since we can do strictly better each period with A. In other words, A dominates F. It is also plausible to argue we could just as well eliminate E; since D returns a greater communication profit at any period. But we cannot eliminate G. Although its cumulative return after period 2 is not as good as that of A and B, its profit in period 1 is strictly the best among all policies. If we require as large a return as possible in the current period, then G is the optimal choice.

How would we select among A, B, C and D?. In the second period, C looks most attractive. For every even-numbered period, D gives a better cumulative return than B. In the third period, B provides a cumulative return of 7 as against 6 for A and C and $6\,^{2}/_{3}$ for D. What is more, B is "ahead" of A, C and D at every period 3 + 6n, for n = 0, 1, 2, ... choosing among streams such as these is more the rule than the exception when models with unbounded horizon are considered.

PERIOD

Policy	1	2	3	4	5	6	---
A	3	2	1	3	2	1	---
B	3	1	3	1	3	1	---
C	1	6	-1	1	6	-1	---
D	$2\,^2/_3$	2	2	2	2	2	---
E	1	3	1	3	1	3	---
F	1	1	1	1	1	1	---
G	4	0	0	4	0	0	---

Figure 2.3.1. Infinite streams of returns.

2.3.2 AVERAGE RETURN

We must make further assumptions in order to state that either A, B, C, D, or G is best. For example, we could make the additional postulate that a unit of return received in any period is just as good as a unit received in any other period. "Just as good" means there is just no benefit of any sort in having the return earlier instead of later. How would this assumption resolve the problem?

It is reasonable now to look at the average return per period, letting the number of periods grow without limit, and prefer the alternative with the largest average. For A, we would compute 2/1, (3 + 2)/2, (3 + 2 + 1)/3, …; for B we would similarly calculate 3/1, (3 + 1)/2, (3 +1 + 3)/3, …; These calculations are summarized in Fig. 2.3.2 As we show in the advanced material below, the average profit per period tends towards 2 for A, B, C and D, and toward $1\,^1/_3$

for G. In other words, if you let the number of periods be large enough, the average will be arbitrarily close to 2 for A, B, C, and D, and to $1\,{}^{1}/_{3}$ for G. Therefore, if we make the special added assumption that you have no time preference for returns, the policies A, B, C and D look equally attractive, even though they are not equally good for each and every finite horizon, and policy G looks inferior.

The general terms in the sequences for the average return per period are:

(i) A : $\frac{3+6n}{1+3n}, \quad \frac{5+6n}{2+3n}, \quad \frac{6+6n}{3+3n}$ for n = 0, 1, 2,...

B : $\frac{3+4n}{1+2n}, \quad \frac{4+4n}{2+2n}$ for n = 0, 1, 2, ……

C : $\frac{1+6n}{1+3n}, \quad \frac{7+6n}{2+3n}, \quad \frac{6+6n}{3+3n},$ for n = 0, 1, 2, ……

D : $2+\frac{2}{3n}$ for n = 1, 2, 3, ……

G : $\frac{4+4n}{1+3n}, \quad \frac{4+4n}{2+3n}, \quad \frac{4+4n}{3+3n}$ for n = 0, 1, 2, ……

Thus, when $n \rightarrow \infty$, each term approaches 2 for A, B, C and D, and $1^{1}/_{3}$ for G.

Period

Policy	1	2	3	4	5	6	7	8	9	---
A	3	$2\frac{1}{2}$	2	$2^{1}/_{4}$	$2^{1}/_{5}$	2	$2^{1}/_{7}$	$2^{1}/_{8}$	2	---
B	3	2	$2^{1}/_{3}$	2	$2^{1}/_{5}$	2	$2^{1}/_{7}$	2	$2^{1}/_{9}$	---
C	1	$3\frac{1}{2}$	2	$2^{3}/_{4}$	$2^{3}/_{5}$	2	$1^{6}/_{7}$	$2^{3}/_{8}$	2	---
D	$2^{2}/_{3}$	$2^{1}/_{3}$	$2^{2}/_{9}$	$1^{1}/_{6}$	$1^{2}/_{15}$	$2^{1}/_{9}$	$2^{2}/_{21}$	$2^{1}/_{12}$	$2^{2}/_{27}$	---
E	4	2	$1^{1}/_{3}$	2	$1^{1}/_{3}$	$1^{1}/_{3}$	$1^{5}/_{7}$	$1^{1}/_{2}$	$1^{1}/_{3}$	---

Fig. 2.3.2. Average Return per period.

The most obvious drawback of using average return per period as selection criterion is its complete insensitivity to the level of returns over a finite number of periods. To illustrate, suppose we can select a policy that has returns identical to those in policy A except that in period 1 the return from the alternative policy is 100. Using solely the criterion of average return per period, we would judge that policy A and alternative policies are equally desirable, since over an unbounded horizon, the first-period advantage of the alternative policy is rendered inconsequential. There are other limitations to this criterion as well, which we explore next.

Assuming that we want to employ the criterion, can we always be sure that a given stream will have a well-defined average return per period, as the number of periods grows without limit? For example suppose that the two streams for A and B are modified to have a multiplication trend:

Policy A*:	3,2,1,3,2,1	6,4,2,6,4,2	9,6,3,9,6,3..	
Policy B*:	3,1,3,1,3,1	6,2,6,2,6,2	9,3,9,3,9,3..	---2.3.0

Although we may reason by analogy that the two streams should remain equally desirable, the rule of looking at average profit per period falters, since the averages grow beyond bound as the number of periods grows without limit.

Therefore, if we want to rely on average per period criterion for a measure of optimality, we must also assume that there exists a unique finite limiting average for the particular return streams we are comparing. In many applications this assumption is reasonable.

2.3.3 PRESENT DISCOUNTED VALUE

An alternative approach for making different infinite streams commensurate is to deal with the so-called present discounted value (or present worth) of the returns.

If the stream of returns is

$$R_1, R_2, R_3, \ldots, R_n, \ldots; \qquad \text{------} \quad 2.3.1.$$

its merit should be judged, according to the present discounted value criterion, in terms of the sum;

$$\text{Present Value} \quad = \quad R_1 + \alpha R_2 + \alpha^2 R_3 + \ldots + \alpha^{n-1} R_n + \ldots =$$

$$= \quad \sum_{t=1}^{\infty} \alpha^{t-1} R_t \qquad \text{---} \qquad 2.3.2$$

Where i % is the interest rate per period and $\alpha = [1 + (^1/_{100})]^{-1}$ is the single period discount factor. The higher the interest rate i, the smaller the value of α .

There is an environmental assumption we can make to justify the approach. Suppose we can borrow or lend as much money as we desire and whenever we wan at a fixed compound rate of interest i% per period. (of course, we are eventually required to pay any debt we incur). For example, let the annual rate be 5%. Then if we borrow a dollar today, we must pay back either (1 + 0.05) dollars a year from today, or $(1 + 0.05)^2$ dollars two years from today, or $(1 + 0.05)^{-n}$ dollars n years from today. By the same token, a dollar received n years from today is really only worth $(1 + 0.05)^{-n}$ dollars right now. If we presently had $(1 + 0.05)^{-n}$ and lent it at 5% interest compound, we would be paid back a dollar n years from today.

Here is why this environmental assumption justifies employing present value. Consider the choice between two policies with different present values. For he moment, suppose that we selected the policy with the smaller present value. Because our utility of money may differ from period to period, we may want to borrow and lent in various periods to redistribute the returns.

For example, we may wish to have the benefit of R_2 right now rather than a period. Consequently, we can borrow αR_2 at present, and then pay back R_2 when it becomes available. Similarly, we may want to lend for several periods, and later receive the payment with compound interest earned. As we think about it, we will see that the value of the entire stream really is summarized by the number representing how much we could obtain at present by committing the entire proceeds of the stream for payment.

Now suppose instead that we selected the policy with the larger present value. By assumption, we can borrow and lend, committing the resources of this stream, so as to attain the same benefits we desired with the other alterative, and in so doing we could have some additional value left over. In other words, any pattern we can obtain with the smaller present – value policy we can also obtain with the larger. The difference between the two values is a net benefit. To sum up, we do best by selecting a strategy that gives maximum present value, regardless of our personal time preference for money.

Of course, rarely if ever is the environmental assumption about borrowing and leading exactly satisfied, but it is often a fair enough approximation to provide adequate answers. Other kinds of argument can be advanced in support of a present – value criterion. For example, in practical applications, a feature commending formula (2.3.2.) is that returns in the distant feature are weighted by a small factor, and consequently have less impact on the decision. We can see in fig. 2.3.3. below how the values of α^n drop rapidly as the interest rate and n increase. But remember, present value or any other formula can be justified only by making particular assumptions regarding the decision – maker's time value of money.

i % n	5	10	20
1	0.952	0.909	0.833
5	0.783	0.621	0.402
10	0.614	0.385	0.161
15	0.481	0.239	0.065
20	0.377	0.149	0.026
40	0.142	0.022	0.001

Fig 2.3.3.

Now we must examine the present – value formula to see whether any additional assumptions have to be imposed for it to be a workable criterion. Start by checking whether the sum of an infinite number of terms in (2.3.2) always yields a finite value. To begin, suppose all the returns are identical:

$$\text{Present – Value} = R + \alpha R + \alpha^2 R + \alpha^3 R + \text{--------} \quad 2.3.3$$

$$= \frac{R}{\alpha - 1} \text{ For } 0 \le \alpha < 1 \quad \text{-------} \quad 2.3.4.$$

The coefficients in equation 2.3.3. are simply a geometric series giving the value in equation (2.3.4). Notice the restriction $\alpha < 1$. If α is close to 1, then the present-value is a large number, but it is finite. If $\alpha = 1$, then the sum in equation 2.3.3 is unbounded for $R \neq 0$, and equation (2.3.4) is ill defined.

Next evaluate policies A, B, C, D and G in fig. 2.3.1 for $0 \leq \alpha < 1$,

Policy A : $3 + 2\alpha + 1\alpha^2 + 3\alpha^3 + 2\alpha^4 + 1\alpha^5 + \ldots$ ----- 2.3.5.

$$= (3 + 2\alpha + 1\alpha^2)(1 + \alpha^3 + \alpha^6 + \ldots) = \frac{3 + 2\alpha + 1\alpha^2}{1 - \alpha^3} \quad \text{- --2.3.6.}$$

Policy B : $3 + 1\alpha + 3\alpha^2 + 1\alpha^3 + 3\alpha^4 + 1\alpha^5 + \ldots$ -----2.3.7.

$$= (3 + 1\alpha)(1 + \alpha^2 + \alpha^4 + \ldots) = \frac{3 - \alpha}{1 - \alpha^2} \quad \text{----- 2.3.8}$$

Policy C : $1 + 6\alpha - \alpha^2 + \alpha^3 + 6\alpha^4 - \alpha^5 + \ldots$ ------2.3.9.

$$= (1 + 6\alpha - \alpha^2)(1 + \alpha^3 + \alpha^6 + \ldots) = \frac{1 + 6\alpha - \alpha^2}{1 - \alpha^3} \quad \text{---2.3.10.}$$

Policy D : $2^2/_3 + 2\alpha + 2\alpha^2 + \ldots = 2\frac{2}{3} + 2\cdot\frac{\alpha}{1 - \alpha}$ ---2.3.11.

Policy G : $4 + 4\alpha^3 + 4\alpha^6 + \ldots = \frac{4}{1 - \alpha^3}$ ---2.3.12.

To compare A and B we can look at the difference between their present values:

$$\text{P. V. (A) - P.V. (B)} = \frac{3 + 2\alpha + \alpha^2}{1 - \alpha^3} - \frac{3 + \alpha}{1 - \alpha^2}$$

$$= \frac{\alpha}{(1+\alpha)(1+\alpha+\alpha^2)} - \frac{3+\alpha}{1-\alpha^2} > 0 \text{ for } 0<\alpha<1 \quad \text{---2.3.13}$$

Thus, even though B has a greater cumulative return than A in period 3 + 6n, for n = 0, 1, 2, … , the discounted value of A is larger for all $0<\alpha<1$. The same procedure yields

$$\text{P.V. (A)} - \text{P.V. (C)} = \frac{2(1-\alpha)}{1+\alpha+\alpha^2} \quad \text{------ 2.3.14.}$$

$$\text{P.V.(A)} - \text{P.V.(D)} = \frac{(1-\alpha)(1+2\alpha)}{3(1+\alpha+\alpha^2)} \quad \text{------- 2.3.15}$$

Therefore, A is also attractive than C and D for all $0 < \alpha < 1$. If we compare B and D, we obtain

$$\text{P.V.(B)} - \text{P.V.(D)} = -2^2/_3 + \frac{2\alpha+3}{1+\alpha} \quad \text{---------2.3.16.}$$

Hence B is more advantageous when $\alpha < ½$ and D is better when $\alpha > ½$. Consider the situation when $\alpha = ½$. Then according to equation 2.3.16, the two policies are equally good. For any finite horizon n, the present value of policy B is strictly better than that for policy D if n is odd (1, 3, 5, …), and the reverse is true if n is even (2, 4, 6, …). Therefore, we would be indifferent to these policies only when the horizon is unbounded.

When you compare A and G you find

$$\text{P.V.(A)} - \text{P.V.(G)} = \frac{3+2\alpha+\alpha^2}{1-\alpha^3} - \frac{4}{1-\alpha^3}$$

$$= \frac{\alpha^2+2\alpha-1}{1-\alpha^3} \quad > 0 \text{ for } \alpha > \sqrt{2} - 1 \approx 0.414 \quad \text{-------2.3.17}$$

Thus A is better only if $\alpha > \sqrt{2} - 1$, otherwise G is preferred. If the interest rate is very high, so that α is correspondingly small, then receiving a return of 4 in the first period of policy G outweighs the later gains available from policy A. In general, as we let α become small, the early returns are the most important, and when $\alpha = 0$ in the limit, all that matters is first period return R_1.

So far we have seen that using a present-value criterion may occasionally distinguish two streams in a surprising way. But the approach has always given a definite answer, because all the summations in the present value formula equation (2.3.2.) yielded a finite number. Was this merely the result of a felicitous selection of return streams? The answer is yes.

The kinds of difficulties with the average return criterion have their counterparts for present worth. We can see such examples below. On the other hand, troublesome cases for the average-return criterion may not cause difficulties for a discounted stream. Consider the example of the upward trending return stream policy B* in (2.3.0), which did not have a finite average return per period. The present – value calculation can be shown to give

Policy B*: $(3 + \alpha + 3\alpha^2 + \alpha^3 + 3\alpha^4 + \alpha^5) \times [1 + 2(\alpha^6) + 3(\alpha^6)^6 + \ldots]$ ---- 2.3.18.

$$= (3 + \alpha)(1 + \alpha^2 + \alpha^4)\frac{1}{(1-\alpha^6)^2} \qquad \text{-------- 2.3.19.}$$

Which is finite for $0 \leq \alpha < 1$. The present value for policy A* in (2.3.0) is also finite.

2.3.4 EQUIVALENT AVERAGE RETURN

Before concluding the discussion on how to attach a value to an infinite stream of returns, we need to study one other approach that relates the two notions of average and discounted value.

The idea is to construct an infinite stream of returns that has the same present value as the original stream. The return (before discounting) in each period will be identical, so that this constant value can be interpreted as the equivalent average return of the stream. Specifically, suppose $P(\alpha)$ is the present value of policy X for a specified value of α. Then consider a new stream of returns

$$R_n = (1-\alpha)\,P(\alpha) \text{ for all } n \qquad \text{--------- 2.3.20}$$

For this stream

$$R_1 + R_2\alpha + R_3\alpha^2 + \ldots = (1-\alpha)\,P(\alpha)\,(1+\alpha+\alpha^2+\ldots) = P(\alpha) \qquad \text{------2.3.21.}$$

The stream in (2.3.20) has the same present value as policy X, then, and $(1-\alpha)P(\alpha)$ is the equivalent average return always leads to the same decision as does best present value because the equivalent average returns are simply the present values of all the alternatives multiplied by the same constant $(1-\alpha)$.

Applying the idea to the earlier examples, you obtain the equivalent average returns:

$$\text{Policy A}: \frac{(1-\alpha)(3+2\alpha+\alpha^2)}{1-\alpha^3} = \frac{3+2\alpha+\alpha^2}{1+\alpha+\alpha^2}$$

$$\text{Policy B}: \frac{(1-\alpha)(3+\alpha)}{1-\alpha^2} = \frac{3+\alpha}{1+\alpha}$$

$$\text{Policy C}: \frac{(1-\alpha)(1+6\alpha-\alpha^2)}{1-\alpha^3} = \frac{1+6\alpha-\alpha^2}{1+\alpha+\alpha^2} \qquad \text{----2.3.22}$$

Policy D : $2^2/_3\ (1-\alpha) + 2\alpha$

Policy G : $\dfrac{4(1-\alpha)}{1-\alpha^3} \qquad = \qquad \dfrac{4}{1+\alpha+\alpha^2}$

The significant point is that whatever average return per period is well defined, we always obtain it by letting α converge from below to 1 in the formula for equivalent average return.

Thus, letting $\alpha = 1$ in (2.3.22) yields

$$\left.\begin{array}{lll} \text{Policy A : } 2 & \text{Policy B : } 2 & \text{Policy C : } 2 \\ \text{Policy D : } 2 & \text{Policy G : } {}^4/_3 & \end{array}\right] \quad \text{-----2.3.23}$$

Which are the averages values for A, B, C, D and G obtained previously. Sometimes equivalent average return is well defined for $\alpha = 1$ when average return per period is not well defined. Unfortunately, equivalent average return is not always well defined for $\alpha = 1$; an example is policy B*.

The criterion of equivalent average returns ranks A, B, C and D is being equally desirable when $\alpha = 1$. Are they?. This is a question of personal opinion and not scientific fact; the decision-maker alone must provide the answer. A strong case can be made, however, for saying that for $\alpha = 1$, A is optional, C and D are nearly optimal, and B should be eliminated for equations (2.3.13), (2.3.14), and (2.3.15), we can conclude that for α close to 1, the present value of A is greater than that for B, C and D. For $\alpha = 1$ in (2.3.14) and (2.3.15), the difference between the present values is zero, so we might say that C and D are almost as good. But for B, when $\alpha = 1$ in (2.3.13), the difference is 1/6. Thus, even though A and B have the same equivalent average return, their present values differ by 1/6 as α approaches 1. For this reason, we may want to discard B.

In summary, the criterion of equivalent average return gives the same average return per period when the latter is well defined. Equivalent average return frequently will be adopted as a criterion function in the models of this and later chapters; but do not forget that its relevance for selecting an optimal policy in an unbounded horizon is assumed (in the above example, policy B does offer a greater cummulative return every sixth period, starting with the third. The equivalent average returns completely discounts this advantage.) Furthermore, keep in mind that additional postulates are required to ensure that the present value of each policy is always a unique finite number. And finally, remember that if several policies have the same equivalent average return when $\alpha = 1$, there still may be good reason for preferring one of these policies to the others.

2.3.5 REMARKS ON INFINITE RETURNS

The discussion in this section reached the following conclusions about selecting an optimization criterion for dynamic models:

(i) A method of comparing streams of returns must include an assumption about the time value of money.

(ii) A technique that attempts to reduce any infinite stream to a unique finite number may not work for all such streams.

(iii) Even when a technique does reduce two different streams to the same number, the two policies may not be equally desirable if other economic considerations are examined.

At this point, we may wonder how realistic the preceding specific numerical examples really were. Of course, they were contrived, but we should not discredit them on that account. As tax experts and professional investment analysts can assure us, every corporation does face critical decision that in effect are choices among alternative infinite streams exhibiting behaviour similar to that in the examples. Such situations give rise to full-blown versions of the perplexities we studied, and commonly occur in firms undergoing rapid growth or facing steadily rising costs.

2.4 SUCCESSIVE APPROXIMATIONS

In this section, we initiate the discussion of numerical techniques for solving extremal equations that arise in dynamic programming models having unbounded horizon.

Consider the functional equation

$f = \text{minimum}\ [\alpha^k f + R_k]$ for $0 \leq \alpha < 1$ ------- - 2.4.1.

where k is the alternative and R is the cost of alternative k.

Remember, saying that we want a solution to (2.4.1.) really means we want a value for the unknown f that satisfies the equation; in addition, we would like to have an alternative k that yields this value of f. Three solution approaches are frequently suggested.

The first emanates from the dynamic context of the underlying model. The idea is to see whether a policy that is optimal for a very long, but finite, horizon yields a solution value for f, when used over an unbounded horizon. The second idea is to guess a value for f. Then compute the quantity on the right-hand side of (2.4.1.) using this guess, and see whether the equation is satisfied. If not, let the result of the computation be a revised guess, and repeat the process. The third idea is to guess a policy that may be optimal over an unbounded horizon. Then solve for the corresponding present value, and use it as a trial value for f see whether the equation is satisfied. If not, let the new guess be the policy that gives a minimum on the right-hand side of 2.4.1 and repeat the process. We consider the first approach in this section and the other two in the sections and chapter to follow.

In all these methods, each guess can be viewed as an approximation to the solution. If the guess satisfies the extremal equation, we are done. If not, we must guess again. This iterative process is given the label successive approximation.

Perhaps the most obvious approach for finding a policy that yields a solution to the functional equation (2.4.1) is to solve the finite horizon model.

$$f_n = \underset{K = 1, 2, \ldots, N}{\text{minimum}} [\alpha^k f_{n-k} + R_k] \text{ for } 0 \leq \alpha < 1 \quad \text{------- } 2.4.2$$

for a very large value of n. Can we be sure that for any n large enough, a k_n that results from (2.42) will also satisfy (2.4.1)?. As we try each successively larger n, does a single k remain optimal? If the horizon n is long enough, is an optimal unbounded horizon policy also optimal as the initial decision?. It is significant that for the regeneration model these equations have affirmative answers.

THEOREM 2.4.1. REGENERATION MODEL HORIZON

There exists a finite value n* such that for any finite horizon n > n*, if

$$f_n = \alpha^{kn} f_{n-kn} + R_{kn} \quad \text{then} \quad f = \alpha^{kn} f + R_{kn} \quad \text{------- } 2.4.3$$

$$f = \alpha^k f + R_k \quad \text{then} \quad f = \alpha^k f_{k-n} + R_k \quad \text{--------}2.4.4$$

Thus (2.4.3) asserts that any strategy k_n that is optimal for the current decision when the horizon n is large enough (greater than n*) is also an optimal stationary strategy for an unbounded horizon. And (2.4.4) asserts the reverse proposition. By performing the calculations of (2.4.2) according to a certain computation format, we can ascertain n*. The details of the approach are extraneous to the purpose of this project, and therefore are omitted here.

2.5 SUCCESSIVE APPROXIMATIONS IN FUNCTION SPACE (VALUE ITERATIONS)

The guiding idea of the proceeding method was to find an optimal stationary policy, k, for an unbounded horizon by examining an increasing sequence of values of n. In contrast, the notion below is to successively approximate the function value f in the extremal equation. Accordingly the process is termed value iteration.

Let f^0 be an initial guess for f. Then the technique is to compute $f^1, f^2, f^3, \ldots$ according to the recursion .

$$f^{n+1} = \underset{k=1,2,3,\ldots,N}{\text{minimum}}[\alpha^k f^n + R_k] \text{ for } 0 \leq \alpha < 1 \quad \text{-------2.5.1} \quad \text{(value iteration)}$$

where f^n is the trial value for f from iteration n. (If the optimization in the extremal equation indicates "maximum" then the corresponding change is made in (2.5.1). An example of the method is given below.

Although the algorithm (2.5.1) is well specified, three question arise about its application:

(i) Does the value of f^n always approach the value of f that satisfies the extremal equation?

(ii) If so, is there a finite n such that f^n equals f?

(iii) If alternative k is chosen in (2.5.1) for two successive approximation is it optimal ?

To answer these, suppose for the moment all $R_k > 0$. If you let $f^0 = 0$, then it can be proved that $f^{n+1} > f^n$, so that f^n are a monotonically increasing sequences of approximations. And for n sufficiently large, f^n is arbitrarily close to the optimal value f. In general, however, there is no finite n such that f^n equals f, and further, an alternative may be chosen on the right – hand side of (2.5.1) for two or more successive approximations but need not be optimal in an unbounded horizon.

Example 2.5.1

The following illustrates how the approximation method works when $R_k > 0$. let N = 5 and

$$\left.\begin{array}{lll} R_1 = 8.7 & R_2 = 12.7 & R_3 = 14.7 \\ R_4 = 19.7 & R_5 = 28.7 & \alpha = 0.8 \end{array}\right\} \quad 2.5.2$$

Then we can determine that solution is

$$f = \underset{K=1,\Lambda,5}{\text{Minimum}}\left[\frac{R_K}{1-0.8^K}\right] = \text{minimum } [43.50, 35.28, 30, 33.39, 42.84] = 30.00 \quad \text{-----} \; 2.5.3$$

So that k = 3 is optimal

The function space calculation in recursion (2.5.1) yield, for n = 1,2,3, and $f^0 = 0$

$$f^1 = \underset{k=1,\ldots,5}{\text{minimum}} [\alpha^k \,.\, 0 + Rk] = 8.7 \text{ for } k = 1$$

$$f^2 = \text{Minimum } [0.8(8.7) + 8.7,\ 0.64\ (8.7) + 12.7, 0.53(8.7) + 14.7,$$
$$0.41(8.7) + 19.7,\ 0.33\ (8.7) + 28.7]$$
$$= 15.66 \text{ for } k = 1 \quad \text{-------} \; 2.5.4$$

$$f^3 = \text{Minimum } [0.8(15.66) + 8.7,\ 0.64\ (15.66) + 12.7,\ 0.51\ (15.66) +$$

$14.7, 0.41(15.66) + 19.7, 0.33\ (15.66) + 28.7]$

$= 21.33 \quad \text{for } k = 1.$

For iterations n > 3

$$\left.\begin{array}{llll} f^4 = 25.53 & f^5 = 27.57 & f^6 = 28.76 & f^7 = 29.37 \\ f^8 = 29.68 & f^9 = 29.84 & f^{10} = 29.91 & f^{11} = 29.95 \\ f^{12} = 29.97 & f^{13} = 29.98 & f^{14} = 29.99 & f^{15} = 29.99 \end{array}\right\} \quad 2.5.5$$

all for k = 3

2.5.1 COMMENTARY

The example shows in (2.5.4) that a policy (k = 1) can be selected for several successive approximations but not be an optimal solution for the unbounded horizon. You can alter the example so that k = 1 is selected for an arbitrarily large number of approximations by reducing R_1 close enough to 6. The calculations in (2.5.5) indicate that there is a fast rate of convergence of f^n to f, but that f^n does not equal f for any finite n.

Observe that for $\alpha = 1$, the process breaks down. For every n, a k is selected if it produces the minimum R_k, and such a k does not usually agree with the solution that minimizes the average cost per period R_k/k.

The value iteration method given in (2.5.1) actually works for any values of R_k and initial guess f^0. But then the sequence of f^n values is not always monotonic. An alternative approach for selecting f^0 does always result in a monotonically decreasing sequence of approximations, that is $f^{n+1} \leq f^n$. The idea is to guess an optimal policy, and let f^0 be the corresponding present value for this policy.

If the policy we guessed proves optimal in calculating f^1, then $f^0 = f^1 = f$. But if a new policy is strictly better in calculating f^1, then the recursion (2.5.1) proceeds as before and $f^{n+1} < f^n$. The method is illustrated next.

2.5.2 MONOTONE CONVERGENCE

Consider the example in (2.5.2) and assume your initial guess is k = 1. Then

$$f^0 = \frac{R_1}{1-\alpha^1} = 43.50 \qquad \text{------- 2.5.6}$$

and for n = 1,

$$f^1 = \text{Minimum } [0.8(43.50) + 8.7, 0.64(43.50) + 12.7, 0.51(43.50) + 14.7,$$
$$0.41(43.50) + 14.7, 033(43.50) + 28.7]$$

$$= 36.88 \text{ for } k = 3 \qquad \text{---------- 2.5.7.}$$

for iterations n > 1

$f^2 = 33.50$ $\quad f^3 = 31.78$ $\quad f^4 = 30.90$

$f^5 = 30.45$ $\quad f^6 = 30.22$ $\quad f^7 = 30.11$

$f^8 = 30.05$ $\quad f^9 = 30.02$ $\quad f^{10} = 30.01$, all for k = 3.

We can alter the example, by reducing R_4 close to 17.7, so that k = 4 is selected for an arbitrarily large number of approximations. Had we started the process by guessing k = 3 then $f^1 = 30.00$ for k = 3.

The motivation for letting $f^0 = 0$ in the application of recursion (2.5.1.) was mainly numerical convenience. We would not gain much insight from a verbal description of the approximation process with this starting point. However, letting f^0 be the present value of an initially guessed policy does lead to a key idea. The amount f^0 in (2.5.6) represents the present value of adopting the policy k = 1 over an unbounded horizon. Suppose that instead of k = 1, our immediate decision is k = 3, and thereafter we always let k = 1. The present value of this strategy is f^1 in (2.5.7). Analogously, f^2 in (2.5.8) actually represents the present value of letting k=3 for the first two regeneration decision, and letting k = 1 subsequently. This observation suggests another mode of approximation, discussed in the next section.

2.6 SUCCESSIVE APPROXIMATIONS IN POLICY SPACE (POLICY ITERATION)

Suppose in calculating the right-hand side of the recursion (2.5.1) in the previous section, we find a policy that makes a strict improvement over the one associated with f^n . This means that using this policy is an improvement over using the previous policy for the immediate decision. It is plausible, and correct, that using the new policy throughout the entire unbounded horizon would be even better than employing it only for the immediate decision. Then f^{n+1} can be calculated as the present value of repeatedly the new policy. This process is known as approximation in policy space or simply as policy iteration, since each iteration considers a new trial stationary policy for the unbounded horizon.

The resultant sequence of f^n is monotonically decreasing and a strict improvement occurs at every iteration; therefore we never return to a policy once it has been discarded. Since there is a finite number N of distinct stationary policies, the approach must terminate in a finite number of iterations. As soon as a policy remains optimal for two successive approximations, we may stop the calculations, and f^n equals the optimal value f satisfying the extremal equation. As we will see, the price we pay to obtain a finite algorithm is the effort involved in calculating f^{n+1} for a new policy at each iteration.

The algorithm is

Step 1: Select an arbitrary initial policy and let $n = 0$

Step 2: Given the trial policy, calculate the associated

$$f^n = \frac{R_k}{1-\alpha^k} \text{ (present value of trial k over an unbounded horizon)}$$

Step 3: Test for an improvement by calculating

$$\underset{k=1,2,---,N}{\text{Minimum}} [\alpha^k f^n + R_k] = \alpha^{k/} f^n + R_{k/} \qquad - \quad 2.6.2$$

Step 4: Terminates the iterations if $\alpha^{k/} f^n + R_{k/} = f^n$. Otherwise, revise the policy $k^{/}$. Increase n to n+1 and return to step 2 with the new trial policy.

Observe that whereas the very process of approximation in function space leads immediately to successive trial values for f, now these must be computed separately from (2.6.1). Notice also that the test for termination in step 4 is satisfied if k^1 is the same as the trial policy in step 2. That is, the calculations cease whenever k^1 is the same for two successive approximation.

Example 2.6.1.

To illustrate the approach, consider

$R_1 = 8.7$, $R_2 = 12.7$, $R_3 = 14.7$, $R_4 = 19.7$, $R_5 = 28.7$ ------- 2.6.3

Which is the same as example (2.5.1) that is, equation 2.5.1 of the previous section.

As before, take your initial policy guess to be $k = 1$, so that $f^0 = 43.50$. The test calculation in (2.6.2) is the same as (2.5.7) in the previous section. Thus in the value formula (2.6.1) we now find $f^1 = 30.00$ for $k = 3$. The second application of the test quantity (2.6.2) yields

$$\text{Minimum}\ [0.8(30.00) + 8.7, 0.64(30.00) + 12.7, 0.51(30.00) + 14.7, 0.41(30.00) + 19.7, 0.33(30.00) + 28.7] = 30 \text{ for } k^1 = 3 \quad \text{---- } 2.6.4$$

so that the process terminates.

2.6.1 AVERAGE RETURN PER PERIOD

As usual, to obtain the corresponding method for $\alpha = 1$, it is helpful to recast the procedure in terms of equivalent average return. The analogy to the value formula (2.6.1) is simply

$$g^n = \frac{R_{k^1}}{1+\alpha+\Lambda+\alpha^{k^1-1}} = \frac{R_{k^1}}{k^1} \text{ for } \alpha = 1 \quad \text{------- } 2.6.5$$

Note that if R_k depends on α then the value of R_k at $\alpha = 1$ is used in calculating the ratio on the right side of (2.6.5.)

The expression on the left in the test quantity 2.6.2 becomes

$$\underset{k=1,2,\ldots,N}{\text{Minimum}}\ [\alpha^k g^n + (1-\alpha)R_k], \quad \text{-------- } 2.6.6.$$

but when we let $\alpha = 1$, the bracketed expression is independent of k. To rectify the situation, observe the following. If k^1 minimizes a function g(k), then k^1 also minimizes ag(k) + b for $a > 0$. Let g(k) be the expression in the brackets of 2.6.6.

$a = (1-\alpha)^{-1}$, and $b = -g^n(1-\alpha)^{-1}$. Make this transformation in 2.6.6 so that the following optimization, analogous to the test in 2.6.2 is approximate.

$$\underset{K=1,2,---,N}{\text{Minimum}} \quad [-(1 + \alpha + \ldots + \alpha^{k-1})\, g^n + R_k]$$

$$\underset{k = 1,2, \ldots, N}{\text{Minimum}} \quad [-kg^n + R_k] = -k^1 g^n + R_k . \text{ for } \alpha = 1 \qquad \text{------ } 2.6.7$$

Once again, we must use the value of R_k appropriate for $\alpha = 1$, if in fact R_k depends an α.

To summarise, the technique is

Step 1: Select an arbitrary initial policy, and let $n = 0$

Step 2: Given the trial policy, calculate the associated

$$g^n = \frac{R_k}{k} \text{ (average cost per period of k)} \qquad \text{------- } 2.6.8.$$

Step 3: Test for an improvement by calculating

$$\underset{K=1,2,---,N}{\text{Minimum}} \quad [-k^1 g^n + R_k] = -k^1 g^n + R_k^1 \text{ (select } k^1) \text{ ----- } 2.6.9.$$

Step 4: Terminate the iterations if $-k^1 g^n + R_k^1 = 0$. Otherwise revise the trial policy to k^1. Increase n to n + 1, and return to step 2 with the new trial policy.

To see how the method works, apply the algorithm to the example 2.6.1. The sequence of calculations is

(i) $g^0 = R_1/1 = 8.7$ for $k = 1$ as the initial policy

(ii) Minimum $[-1(8.7) + 8.7, -2(8.7) + 12.7, -3(8.7) + 14.7, -4(8.7) + 19.7$
$-5(8.7) + 28.7] = -15.1$ for $k^1 = 4$

(iii) $g^1 = R_4/4 = 4.925$.

(iv) $\underset{k = 1, \ldots, 5}{\text{Minimum}} \quad [-k(4.925) + R_k] = -0.025$ for $k^1 = 3$

(v) $g^2 = R_3/3 = 4.9$

(vi) $\underset{k=1,\ldots,5]}{\text{Minimum}} \quad [-k(4.9) + R_k] = 0 \text{ for } k^1 = 3$

Notice the same policy $k^1 = 3$ is indicated for two successive iterations, thereby causing termination at step 4, and $k^1 = 3$ is optimal.

We may not really be indifferent between two policies that look equally good according to the test in (2.6.9). For example, if we add another decision $k = 6$ with return $R_6 = 2$, $R_3 = 29.4$, this policy has the same value of g when $\alpha = 1$ as does policy $k = 3$, but for $\alpha < 1$.

$$\frac{R_6}{1-\alpha^6} - \frac{R_3}{1-\alpha^3} = \frac{R_3}{1+\alpha^3} > \frac{R_3}{2}$$

So far $0 \leq \alpha < 1$ the present value from $k = 6$ always exceeds the present value for $k = 3$. Thus, although $k = 3$ is optimal according to the test in (2.6.9), we may really prefer $k = 6$.

CHAPTER THREE

APPLICATIONS OF OPTIMIZATION WITH UNBOUNDED HORIZON

3.1. PRACTICAL SIGNIFICANCE OF MODELS ASSUMING STATIONARITY OVER AN UNBOUNDED HORIZON

These introductory comments are meant to help us make a transition from the point of view of the linear programming models we have ever studied. Whereas most industrial applications of the linear programming models we have seen are oriented to planning decisions in the face of large-scale complex situations, dynamic programming models are typically applied to much smaller-scale phenomena. The following illustrations typify dynamic programming decision models:

- Inventory reordering rules indicating when to replenish an item and by what amount.
- Production–scheduling and employment-smoothing doctrines applicable to an environment with fluctuating demand requirements.
- Spare-parts level determination to guarantee high efficiency utilization of expensive equipment.
- Capital budgeting procedures for allocating scarce resources to new ventures.
- Selection of advertising media to promote wide public exposure to a company's product.
- Systematic plan or search to discover the whereabouts of a valuable resource.
- Scheduling methods for routine and major overhauls on complex machinery.
- Long-range strategy for replacing depreciating assets.

In the case of models assuming stationarity over an unbounded horizon, we well may wonder whether models assuming stationary over an unbounded horizon have much practical significance since stationarity rarely exists for an extended period of time. We will distinguish two types of applications to show their importance.

The first type pertains to situations in which dynamic optimization models are used to improve day-to-day operating decision, such as replenishing inventory and scheduling production.

For example, consider a firm that monitors its inventory-level daily and reorders an item every few weeks when the stock level reaches a critical point. Suppose, as is frequently true, that for at least twelve months the items demand rate and the ordering and holding cost functions are stable. Then it is reasonable for the firm to use the same inventory replenishment rule during three to six months, and at the end of that time to revise its inventory policy based on a new twelve –month forecast. (Another important consideration also justifies this mode of operation. If the firm stocks hundreds of different items it would be too time–consuming and disruptive to recompute a new replenishment rule at frequent intervals for every item. The cost of doing so would far outweigh any efficiency savings from the improved decisions.)

How, then, should the company set each specific replenishment rule.? One possibility is to determine for a horizon length N = 365 days, the minimum cost $f_N(i)$ and optimal policy when entering inventory is i, and for the first few months employ the replenishment policy associated with N = 365. But given the stationarity assumptions, an unbounded horizon model can provide equally excellent answer, and typically requires much less calculation.

Accordingly, one justification for employing a stationary, unbounded horizon model is that in the context of daily operations, the approach is both effective and a relatively easy way to derive optimal decisions for the initial interval of time.

The second type of application pertains to situations in which dynamic optimization models are used to make recurring strategic investment decisions, as in the replacement of expensive equipment. Large pieces of machinery may be replaced as seldom as every 15 or 20 years. Consequently, when the next replacement is necessary, completely new types of machinery are likely to be available. How then, should the current investment decision be made?. One possibility of course, is simply to ignore that fact that the equipment eventually must be replaced. This can be misleading and hazardous, as witnessed by the following illustration, based on an actual case.

A food processing company facing production bottlenecks realized it could alleviate its problem by either prestocking inventory during the slack season for sales during the peak season, or by purchasing new equipment to expand its production capacity. When the ever-recurring cost associated with the increased early season inventory was compared with the initial cost of the added equipment, it seemed preferable to purchase the machinery. However, as soon as the equipment decision was analysed to take account of ever-recurring future replacements, it turned out to be far the less attractive, even with optimistic projection of subsequent replacement costs.

3.2 AN EXAMPLE TIMBER HARVESTING MODEL

We are now ready to see the applications of the various ideas we have learned about optimization in an unbounded horizon. In this section, we will use the ideas in the previous chapter to solve a particular optimization problem.

A Timber Company is planning the forestation of a new area of land. The firm has estimated that a tree fell at the start of the K^{th} period of growth yields a net return $N_k \geq 0$. To keep the discussion simple, assume that all expenditures on planting and maintaining a forest are negligible as compared with the cost of harvesting and transporting trees at the beginning of period K. Further, suppose all the trees are to be cut in the same period. Consequently, N_k represents the revenue received less the costs of cutting all the trees. Assume N_k is available at the start of period k, and k = 1 refers to the current period when the forestation commences.

3.2.1 SINGLE DECISION

The discount factor α^{k-1} is applied to obtain the present value of the return, so the optimization problem is

$$\underset{k=1,2,3,\ldots}{\text{Maximize}}\ \alpha^{k-1} N_k \qquad \text{---------------} \ 3.2.1$$

The formulation (3.2.1) assumes that after the forest is harvested, replanting does not take place. In other words, the problem as stated so far involves a single decision: when to cut down the trees.

Suppose K is a value of k that solves (3.2.1), and assume that the sequence $\alpha^0 N_1, \alpha^1 N_2, \alpha^2 N_3, \ldots$ has the property.

$$\alpha^0 N_1 \leq \alpha^1 N_2 \leq \ldots \leq \alpha^{k-2} N_{k-1} < \alpha^{k-1} N_k \geq \alpha^k N_{k+1} \geq \alpha^{k+1} N_{k+2} \geq \ldots, \quad \text{--------3.2.2}$$

so that the present value is increasing as k goes from 1 to K and then is decreasing for larger k. Then the inequalities

($\alpha^{k-2} N_{k-1} < \alpha^{k-1} N_k$) and ($\alpha^{k-1} N_k \geq \alpha^k N_{k+1}$) simplify to the conditions.

$$\frac{N_{k-1}}{N_k} < \alpha \leq \frac{N_k}{N_{k+1}} \quad \text{------------3.2.3}$$

Therefore the value K can be found by calculating the ratios $\frac{N_k}{N_{k+1}}$ starting with $k = 1$, and terminating as soon as a ratio is at least as large as α.

3.2.2 ILLUSTRATION

Consider the case in which

$N_k = a - b^k$ where $o < b < 1$ and $a \geq b$. -------- 3.2.4

(plot values of N_k for $a = 0.75$, $b = 0.5$, and $k = 1,2,\text{-----}, 8$) then (3.2.3) can be simplified to

$$b^{k-1} > \frac{a(1-\alpha)}{1-\alpha b} > b^k \quad \text{-------- 3.2.5}$$

This case brings up a new difficulty. Let $\alpha = 1$, so that the middle term in (3.2.5) equals 0. Then there is no finite value for K satisfying (3.2.5), since $b^k > 0$ for all k. However, b^k is arbitrarily close to 0 for k sufficiently large. Thus when $\alpha = 1$, it is not meaningful to write, "maximize" in (3.2.1) if N_k is specified by (3.2.4). We can see why a finite maximum does not exist in this circumstance if we refer to the graph of N_k.

To illustrate (3.2.5) by a numerical example, suppose

$$N_k = 0.85 - (0.78)^k \text{ and } \alpha = 0.8 \quad \text{------------ } 3.2.6$$

Then K = 4 satisfies (3.2.5). Let P.V. (k) denote the present value α^{k-1}; then

$$\text{P. V. [1]} = 0.070 \quad \text{P.V. [2]} = 0.193 \quad \text{P.V.[3]} = 0.240$$

$$\text{P.V. [4]} = 0.246 \quad \text{P.V. [5]} = 0.230 \quad \text{----------- } 3.2.7$$

3.2.3 UNBOUNDED HORIZON

Now consider what happens when the forest is replanted in the period following the harvest, and thereafter the process is repeated infinitely often. If, each time, the forest is cut at the beginning of k periods growth, the present discounted value of the return stream over an unbounded horizon is

$$\left.\begin{aligned} F(k) &= \text{P.V. [k]} (1 + \alpha^K + \alpha^{2K} + \ldots) \\ &= \text{P.V. [k]} + \text{P.V.[k]} (\alpha^K + \alpha^{2K} + \ldots.) \\ &= \text{P.V. [k]} + \alpha^K \text{ P.V. [k]} (\alpha^K + \alpha^{2K} + \ldots) \\ &= \text{P.V. [k]} + \alpha^K \ F(k) \end{aligned}\right\} \quad 3.2.8$$

So that

$$F(k) = \frac{\text{P.V.[K]}}{1-\alpha^k} \quad \text{--------------- } 3.2.9$$

An optimal policy over the unbounded horizon is one that maximizes F (k). Let the maximal value of F(k) be denoted by F, which we can according to

$$F = \text{Maximum} \quad \underset{k=1,2,3,\Lambda}{\text{Maximum}} \frac{\text{P.V.[K]}}{1-\alpha^k} \quad \text{------------ } 3.2.10$$

given the formula for F(k) in (3.2.9). An immediate implication of (3.2.10) is

$$F \geq \frac{P.V.[K]}{1-\alpha^k} \text{ --------------- } 3.2.11$$

for every k, or, equivalently,

$$F \geq \alpha^K F + P.V.(k) \text{ ------------- } 3.2.12$$

for very k, where the equality holds in (3.2.11) and (3.2.12) for an optimal policy. This in tern in plies that the value for F must satisfy

$$F = \text{ Maximum } (\alpha^K F + P.V.[k]) \text{ ------------- } 3.2.13$$

Suppose (3.2.10) yields $k = k^1$ as an optimal policy and assume that the sequence $\frac{P.V.[1]}{1-\alpha}, \frac{P.V.[2]}{1-\alpha^2}, \ldots$ has the property

$$\frac{P.V.[1]}{1-\alpha} \leq \text{--------} < \frac{P.V.[K^1]}{1-\alpha^{k1}} \geq \frac{P.V.[K^1+1]}{1-\alpha^{k+1}} \geq \text{ -------} 3.2.14$$

Then the inequalities for k^1-1, k^1, and k^1+1 in (3.2.14) lead to the one analogous to (3.2.3), namely.

$$\frac{\left(1+\alpha+\Lambda+\alpha^{k^1-1}\right)N_{k^1-1}}{\left(1+\alpha+\Lambda+\alpha^{k^1-2}\right)N_{k^1}} < \alpha \leq \frac{\left(1+\alpha+\Lambda+\alpha^{k^1}\right)N_{k^1}}{\left(1+\alpha+\Lambda+\alpha^{k^1-2}\right)N_{k^1+1}} \text{ -------------} 3.2.15$$

The value k^1 can be found by calculating the right–hand side of (3.2.15), starting with k= 1, until the ratio is at least as large as α. For each k, the ratio will be larger than N_k / N_{k+1}, which was calculated to obtain k in (3.2.3). Hence

$$k^1 \leq k, \text{ ----------- } 3.2.16$$

Which means that the forest is usually harvested more often in the unbounded horizon case (and never less often).

When $\alpha = 1$ in (3.2.15), the inequalities can be written as

$$\frac{N_{k^1-1}}{K^1-1} \Big/ \frac{N_{k^1}}{K^1} \quad < \quad 1 \quad \leq \quad \frac{N_{k^1}}{K^1} \Big/ \frac{N_{K^1+1}}{K^1+1} \qquad \text{----------- } 3.2.17$$

implying that k^1 is a policy yielding the maximum average return per period.

For the numerical example in (3.2.6), an optimal policy is $k^1 = 2$ and

$$\left.\begin{array}{lll} F(1) = 0.350 & F(2) = 0.536 & F(3) = 0.492 \\ F(4) = 0.417 & F(5) = 0.342 & \end{array}\right\} \quad 3.2.18$$

where F(k) is the value of (3.2.9). This in the unbounded horizon situation, if we erroneously employed the solution k = 4 from the single decision case, we would receive only $\frac{0.417}{0.536} = 0.78$ of the truly optimal present value.

3.3 INFINITE STAGE REGENERATION MODEL

We have already seen a few applications of optimization with unbounded horizon. One of such example is the timber harvesting and replanting problem in the pervious section. Each time the forest is cut, the problem regenerates itself in the sense that the Timber company must again decide how long to wait until the next harvest period. Another illustration is the problem of equipment replacement discussed in section 3.1. There regeneration occurred each time a machine was replaced. Consequently, the decision variables are really the successive intervals between replacements. Instead of pursuing any particular example in further detail, we will now treat them all in the context of a general model.

Suppose each time the decision process regenerates itself, the decision maker can choose among N alternatives, which are indexed k = 1,2,------, N. Assume that if alternative k is selected at a regeneration period t, then the next regeneration occurs at period t + k, and let

R_k = cost of Alternative k ------- 3.3.1

valued at the start of its regeneration period. Note that (3.3.1) embodies a stationarity assumption R_k does not depend on the particular period when the regeneration occurs. Also observe that since R_k is to be interpreted as a cost, the sense of optimization will be minimization.

3.3.1 FINITE HORIZON

f_n = { present value of an optimal regeneration policy in which an alternative must be chosen when n periods remain unite the end of the planning horizon }

Suppose that we choose Alternative k. Then we immediately incur the cost R_k, and assuming that we act optimally at the next regeneration point, n-k, we subsequently incur the cost $\alpha^k f_{n-k}$, where the factor α^k properly discounts the future cost to the present. Hence, an optimal choice when there are n period remaining until the end of the horizon is a policy that minimizes the sum $\alpha^k f_{n-k} + R_k$, and the corresponding minimum value is f_n. Assuming that $n \geq N$, we can characterize f_n recursively by the relation.

$$f_n = \underset{k=1,2,\ldots,N}{\text{minimum}} \; [\alpha^k f_{n-k} + R_k],\; f_o = 0 \text{ for } 0 \leq \alpha \leq 1 \text{ ----------3.3.3}$$

(if n < N, then the minimum is restricted to k = 1,2,--------, n)

Actually, if the costs of Alternative k occur throughout k periods, then each R_k would also depend on α; but we let this fact remain implicit and use the abbreviated symbol R_k instead of $R_k(\alpha)$.

To see how (3.3.3) works, suppose k=1 is optimal for all horizon lengths n. Then (3.3.3), yields

$$\left.\begin{aligned} f_n &= \alpha f_{n-1} + R_1 = \alpha [\alpha f_{n-2} + R_1] + R_1 \\ &= \alpha [(\alpha f_{n-3} + R_1) + R_1] + R_1 \\ &= \ldots = R_1 + \alpha R_1 + \ldots + \alpha^{n-1} R_1 \end{aligned}\right\} \quad 3.3.4$$

3.3.2 UNBOUNDED HORIZON

Now suppose the planning horizon for the regeneration process is unbounded. Each time regeneration occurs, the decision–maker continues to face an unlimited horizon. It can be proved by rigorous argument that there exists an optimal strategy (or policy) and is stationary: choose the same alternative k at each regeneration point. Then if $\alpha \neq 1$, the appropriate generalization of (3.3.3) is

$$f = \underset{k=1,2,\text{----},N}{\text{Minimum}} [\alpha^k f + R_k] \text{ for } 0 \leq \alpha < 1 \text{ --------- } 3.3.5$$

3.3.3 **REMARKS**

Recall that in (3.2.13) of the previous section, we considered an example of a similar optimization relation, the only important difference being the sense of optimization. There P.V.[k], being a present value, did depend on α.. We also note that here the largest value possible for k is assumed a priori to be N.

The relation (3.3.5) is an example of what is called a **FUNCTIONAL OR EXTREMAL EQUATION**. It is the value of f that is unknown, and (3.3.5) states the optimization relation is that f must satisfy, given that a stationary strategy is used. When dealing with extremal equations, we must always consider the following:

i. Does the equation possess a finite solution?

ii. If so, is the solution unique?

iii. If so, is f the maximal discounted return among all (not necessarily) policies

To see the relevance of these questions, suppose $\alpha = 1$, contrary to the restriction on the right in (3.3.5). If we assume all $R_k > 0$, then no finite value for f satisfies (3.3.5). But if instead, we assume all $R_k = 0$, then any finite value for f will satisfy (3.3.5). Therefore, the functional equation (3.3.5) is not appropriate for $\alpha = 1$.

We can view (3.3.5) as stating that f must satisfy.

$$F \leq \alpha^k f + R_k \text{ or } f \leq \frac{R_k}{1-\alpha^K} \text{ for all k, ------------ 3.3.6}$$

and equality in (3.3.6) must hold for at least one value of k. it follows that a unique finite solution to the extremal equation (3.3.5) does exist and equals

$$f \quad = \quad \text{Minimum} \quad \underset{K=1,2,\Lambda,N}{\text{Minimum}}\left[\frac{R_k}{1-\alpha^K}\right] \quad \text{------------ 3.3.7}$$

An optimal stationary policy corresponds to any Alternative k that yields the optimal value for f.

We can also derive (3.3.7) on the basis of stationarity. Since it is optimal to employ the same Alternative k every time regeneration occurs, the present value of the policy is

$$R_k + \alpha^k R_k + \alpha^{2k} R_k + \alpha^{3k} R_k + \ldots = \frac{R_k}{1-\alpha^K} \quad \text{------------ } 3.3.8$$

Thus an optimal policy is one that minimizes this quantity, as indicated in 3.3.7

So far, the infinite stage problem has been solved assuming $\alpha \neq 1$, and we discovered that 3.3.5 is not appropriate for $\alpha = 1$. However, we can extend the analysis to $\alpha = 1$ by employing the criterion of equivalent average return suggested in chapter two.

CHAPTER FOUR

COMPUTATIONAL RESULTS

4.1 INTRODUCTION

These introductory comments are meant to help us understand that computer programs have been put in place in this project that will assist us easily compute and select optimal strategy discussed in chapter two.

A computer program is also developed to solve a real life problem (Timber Harvesting model) discussed in chapter three. We are now ready to see results generated by computer using solved examples of this project.

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.1	8.7	0
2	.1	12.787	1
2	.1	12.82787	2
2	.1	12.82828	3
2	.1	12.82828	4
2	.1	12.82828	5
2	.1	12.82828	6
2	.1	12.82828	7
2	.1	12.82828	8
2	.1	12.82828	9

Table 4.1.1

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.2	8.7	0
1	.2	10.44	1
1	.2	10.788	2
1	.2	10.8576	3
1	.2	10.87152	4
1	.2	10.8743	5
1	.2	10.87486	6
1	.2	10.87497	7
1	.2	10.87499	8
1	.2	10.875	9
1	.2	10.875	10

Table 4.1.2

1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

	alpha	f	n
	.3	8.7	0
	.3	11.31	1
	.3	12.093	2
	.3	12.3279	3
	.3	12.39837	4
	.3	12.41951	5
	.3	12.42585	6
	.3	12.42776	7
	.3	12.42833	8
	.3	12.4285	9
	.3	12.42855	10
	.3	12.42856	11
	.3	12.42857	12
	.3	12.42857	13

Table 4.1.3

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.4	8.7	0
1	.4	12.18	1
1	.4	12.572	2
1	.4	13.7288	3
1	.4	14.19152	4
1	.4	14.37661	5
1	.4	14.45064	6
1	.4	14.48026	7
1	.4	14.4921	8
1	.4	14.49684	9
1	.4	14.49874	10
1	.4	14.4995	11
1	.4	14.4998	12
1	.4	14.49992	13
1	.4	14.49997	14
1	.4	14.49999	15
1	.4	14.45	16
1	.4	14.48	17
1	.4	14.492	18
1	.4	14.4968	19
1	.4	14.49872	20
1	.4	14.49949	21
1	.4	14.4998	22
1	.4	14.49992	23
1	.4	14.49997	24
1	.4	14.49999	25
1	.4	14.5	26
1	.4	14.5	27

Table 4.1.4

1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

alpha	f	n
.5	8.7	0
.5	13.05	1
.5	15.225	2
.5	16.3125	3
.5	16.73906	4
.5	16.79238	5
.5	16.79905	6
.5	16.79988	7
.5	16.79998	8
.5	16.8	9
.5	16.8	10

Table 4.1.5

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.6	8.7	0
1	.6	13.92	1
1	.6	17.052	2
3	.6	18.38323	3
3	.6	18.67078	4
3	.6	18.73289	5
3	.6	18.7463	6
3	.6	18.7492	7
3	.6	18.74983	8
3	.6	18.74996	9
3	.6	18.74999	10
3	.6	18.75	11
3	.6	18.75	12

Table 4.1.6

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.7	8.7	0
1	.7	14.79	1
1	.7	19.153	2
3	.7	21.26948	3
3	.7	21.99543	4
3	.7	22.24443	5
3	.7	22.32984	6
3	.7	22.35914	7
3	.7	22.36919	8
3	.7	22.37263	9
[illegible]	.7	22.37381	10
[illegible]	.7	22.37422	11
	.7	22.37436	12
	.7	22.37441	13
	.7	22.374412	14
	.7	22.37442	15
	.7	22.37443	16
	.7	22.37443	17

Table 4.1.7

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.8	8.7	0
1	.8	15.66	1
1	.8	21.228	2
3	.8	25.56874	3
3	.8	27.79119	4
3	.8	28.92909	5
3	.8	29.51169	6
3	.8	29.80999	7
3	.8	29.96271	8
3	.8	30.04091	9
3	.8	30.08095	10
3	.8	30.10145	11
3	.8	30.11194	12
3	.8	30.11731	13
3	.8	30.12006	14
3	.8	30.12147	15
3	.8	30.12219	16
3	.8	30.12256	17
3	.8	30.12275	18
3	.8	30.12285	19
3	.8	30.1229	20
3	.8	30.12292	21
3	.8	30.12294	22
3	.8	30.12295	23
3	.8	30.12295	24

Table 4.1.8

R1=8.7 ,R2=12.7 ,R3=14.7 ,R4=19.7 ,R5=28.7 ,R6=30.7 ,R7=35.7 ,R8=38.7 ,R9=40.7 ,R10=42.7 ,

K	alpha	f	n
1	.9	8.7	0
1	.9	16.53	1
1	.9	23.577	2
1	.9	29.9193	3
1	.9	35.62737	4
3	.9	40.67235	5
3	.9	44.35014	6
3	.9	47.03125	7
3	.9	48.98578	8
3	.9	50.41063	9
3	.9	51.44935	10
3	.9	52.20658	11
3	.9	52.7586	12
3	.9	53.16102	13
3	.9	53.45438	14
3	.9	53.66824	15
3	.9	53.82415	16
3	.9	53.93781	17
3	.9	54.02066	18
3	.9	54.08106	19
3	.9	54.12509	20
3	.9	54.15719	21
3	.9	54.18059	22
3	.9	54.19765	23
3	.9	54.21009	24
3	.9	54.21916	25
3	.9	54.22577	26
3	.9	54.23059	27
3	.9	54.2341	28
3	.9	54.23666	29
3	.9	54.23853	30
3	.9	54.23989	31
	.9	54.24088	32
	.9	54.2416	33
	.9	54.24213	34
	.9	54.24251	35
	.9	54.24279	36
	.9	54.24299	37
	.9	54.24314	38
	.9	54.24325	39
	.9	54.24333	40
	.9	54.24339	41
	.9	54.24343	42
	.9	54.24346	43
	.9	54.24348	44
	.9	54.2435	45
	.9	54.24351	46
	.9	54.24352	47
	.9	54.24353	48
	.9	54.24353	49

Table 4.1.9

n	f	K	Alpha
1	15.66	1	.8
2	21.228	1	.8
3	25.56874	3	.8
4	27.6376	3	.8
5	28.85045	3	.8
6	29.47143	3	.8
7	29.78937	3	.8
8	29.95216	3	.8
9	30.03551	3	.8
10	30.07818	3	.8
11	30.10003	3	.8
12	30.11122	3	.8
13	30.11695	3	.8
14	30.11988	3	.8
15	30.12138	3	.8
16	30.12215	3	.8
17	30.12254	3	.8
18	30.12274	3	.8
19	30.12284	3	.8
20	30.12289	3	.8
21	30.12292	3	.8
22	30.12294	3	.8
23	30.12295	3	.8
24	30.12295	3	.8

Tabel 4.2.1

n	f	k	alpha
1	10.26	1	.8
2	13.908	1	.8
3	16.8264	1	.8
4	19.16112	1	.8
5	21.0289	1	.8
6	22.52312	1	.8
7	23.7185	1	.8
8	24.6748	1	.8
9	25.43984	1	.8
10	26.05187	1	.8
11	26.5415	1	.8
12	26.9332	1	.8
13	27.24656	1	.8
14	27.49725	1	.8
15	27.6978	1	.8
16	27.85824	1	.8
17	27.98659	1	.8
18	28.08927	1	.8
19	28.17142	1	.8
20	28.23714	1	.8
21	28.28971	1	.8
22	28.33177	1	.8
23	28.36542	1	.8
24	28.39234	1	.8
25	28.41387	1	.8
26	28.4311	1	.8
27	28.44488	1	.8
28	28.4559	1	.8
29	28.46472	1	.8
30	28.47178	1	.8
31	28.47742	1	.8
32	28.48194	1	.8
33	28.48555	1	.8
34	28.48844	1	.8
35	28.49075	1	.8
36	28.4926	1	.8
37	28.49408	1	.8
38	28.49526	1	.8
39	28.49621	1	.8
40	28.49697	1	.8
41	28.49758	1	.8
42	28.49806	1	.8
43	28.49845	1	.8
44	28.49876	1	.8
45	28.49901	1	.8
46	28.49921	1	.8
47	28.49937	1	.8
48	28.4995	1	.8
49	28.4996	1	.8
50	28.49968	1	.8
51	28.49974	1	.8
52	28.49979	1	.8
53	28.49983	1	.8
54	28.49986	1	.8
55	28.49989	1	.8
56	28.49991	1	.8
57	28.49993	1	.8
58	28.49994	1	.8
59	28.49995	1	.8
60	28.49996	1	.8
61	28.49997	1	.8
62	28.49998	1	.8
63	28.49998	1	.8

Table 4.2.2

4.2 REMARKS

The guiding idea of the proceeding results is to successively approximate the function value f in the extremal equation

$$f^{n+1} = \text{minimum } [\alpha^k f^n + R_k] \text{ for } 0 \leq \alpha < 1 \quad \text{-------} \; 4.1.1$$
$$K = 1, 2, \ldots.., N$$

Accordingly, the process is termed value iteration. In this method, we let f^0 be an initial for f. Then the technique is to compute a sequence of approximations f^1, f^2, f^3, ..., according to the recursion (4.1.1). Where f^n is the trial value for f from iteration n, k is the alternative and R_k is the cost of alternative k and that that is why our (4.1.1) is minimum.

In the proceeding results, we let $f^0=0$ and all $R_k > 0$, so that f^n are monotonically increasing sequence of approximations. We can observe that there is no finite n such that f^n equals f, and further, an alternative may be chosen on the right hand side of (4.1.1) for two or more successive approximations but need not be optimal in an unbounded horizon.

Table 4.1.1 shows that a policy (k=2) can be selected for several successive approximations and may even be optimal solution for an unbounded horizon. Similarly, Tables 4.1.2, 4.1.3, 4.1.4, show that a policy (k=1) can be selected for several successive approximations and may be optimal solution for an unbounded horizon. However, Tables 4.1.5,4.1.6,4.1.7,4.1.8,and 4.1.9 show that a policy (k=1) can be selected for several successive approximations but need not be an optimal solution for the unbounded horizon.

We can easily see that it is possible to select k=1 as an optimal strategy for an arbitrary large number of approximations by reducing R_1 below below 7 as can.

be seen in table 4.2.1. This simply means that the optimality of a strategy depends to some extent on the cost of the strategy. In other words, when a cost of a strategy is small, there is high probability that the strategy may be optimal even over an unbounded horizon.

Tables 4.1.1 to 4.1.8, indicate that there is a fast rate of convergence of f^n to f, but that f^n does not equal f for any finite n. However, in Table 4.1.9, the convergence is not at fast rate.

We equally observe that for $\alpha=1$, the process breaks down. For every n, a k is selected if it produces the minimum R_k. So from the above tables, we can draw the following conclusions:

(i) That the value iteration method given in (4.1.1) actually works for any values of R_k ($R_k > 0$) and initial guess f^0.

(ii) When α is chosen very small (the discount factor), there is a fast rate of convergence of f^n to f

(iii) That a policy (k=1) can be selected for several successive approximations but need not be optimal solution for the unbounded horizon. However, it can be optimal solution for an unbounded horizon if α (discount factor) is chosen close to 0.

(iv) That a policy (k = 1) can be an optimal solution for a finite horizon regardless of the values of α, n and f^n

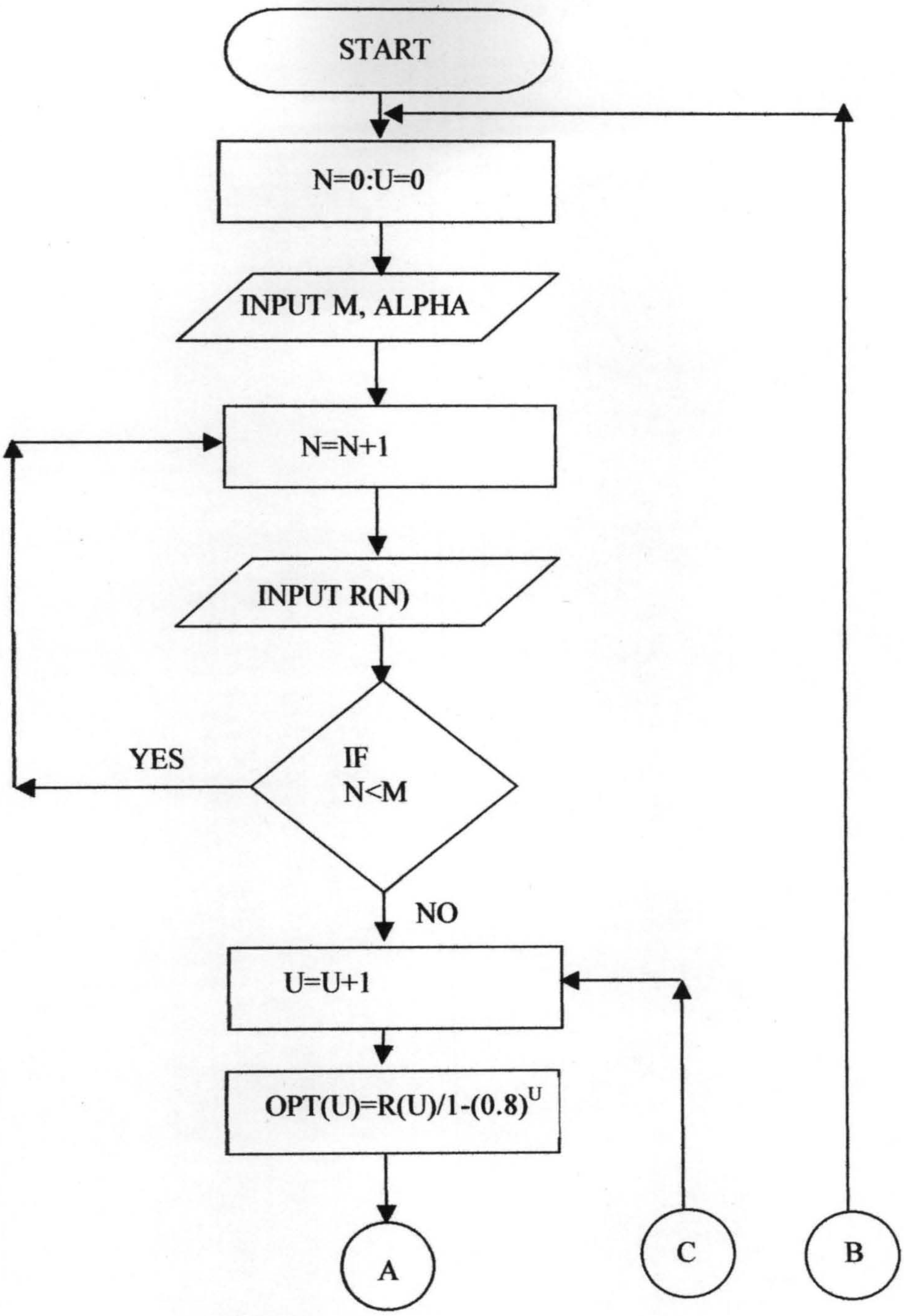
(v) We can easily conclude from the above tables that a policy (k=3) is the best strategy and an optimal solution for an unbounded horizon.

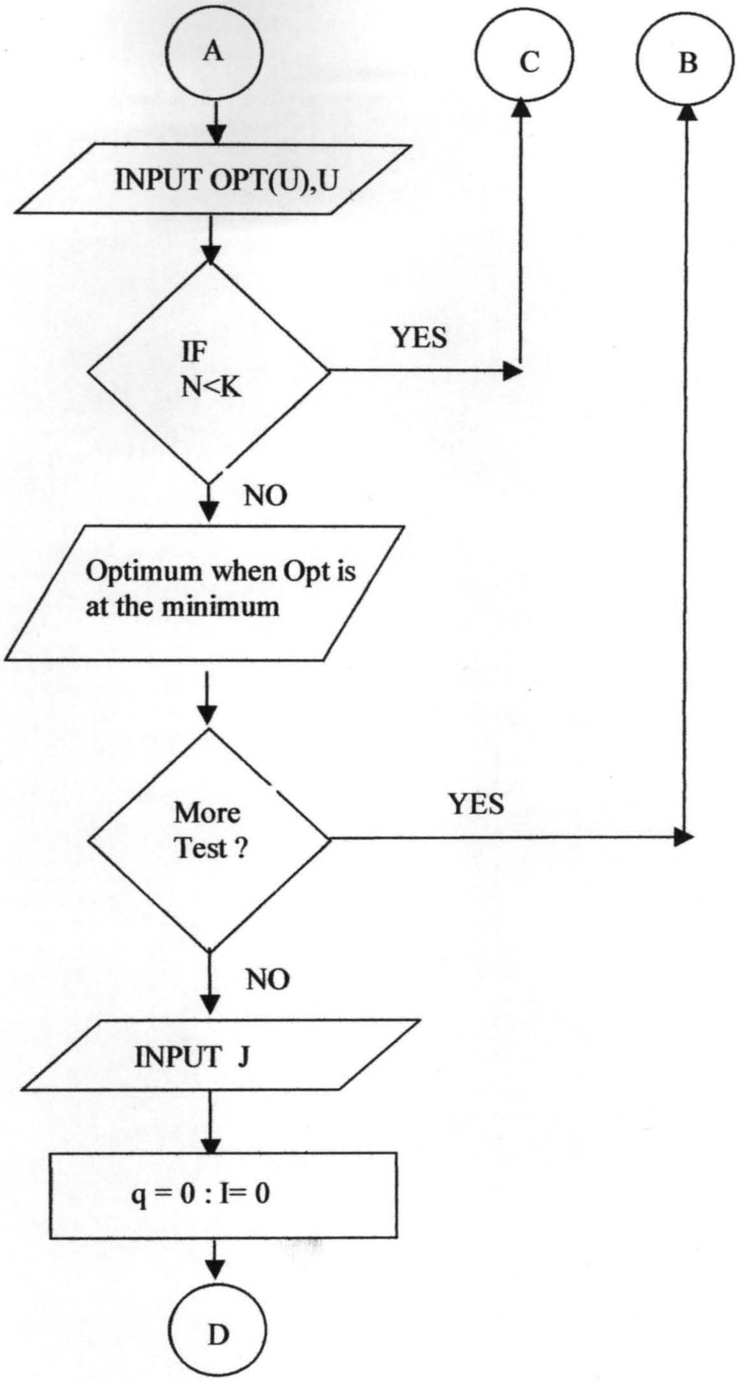
CHAPTER FIVE

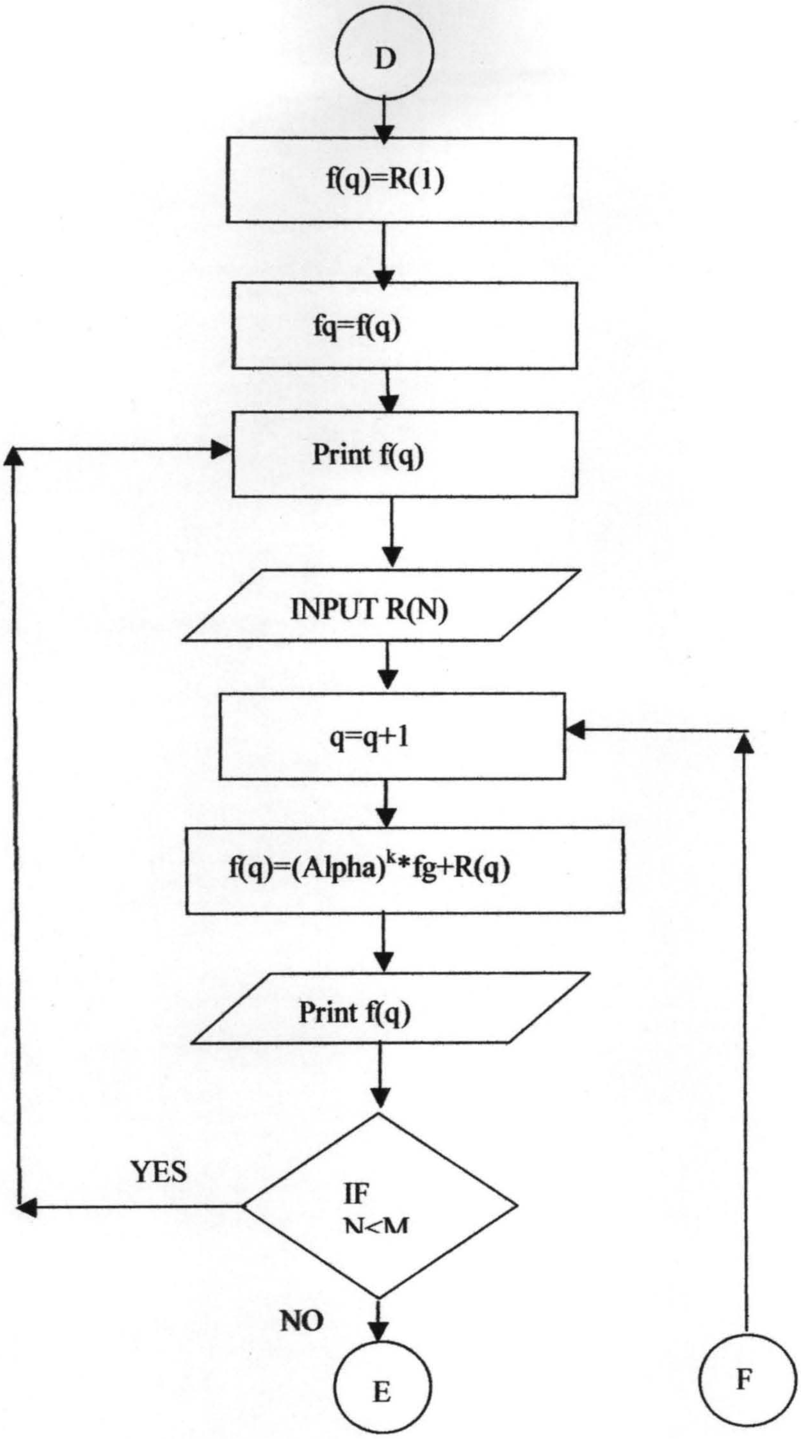
FLOWCHART AND SUMMARY

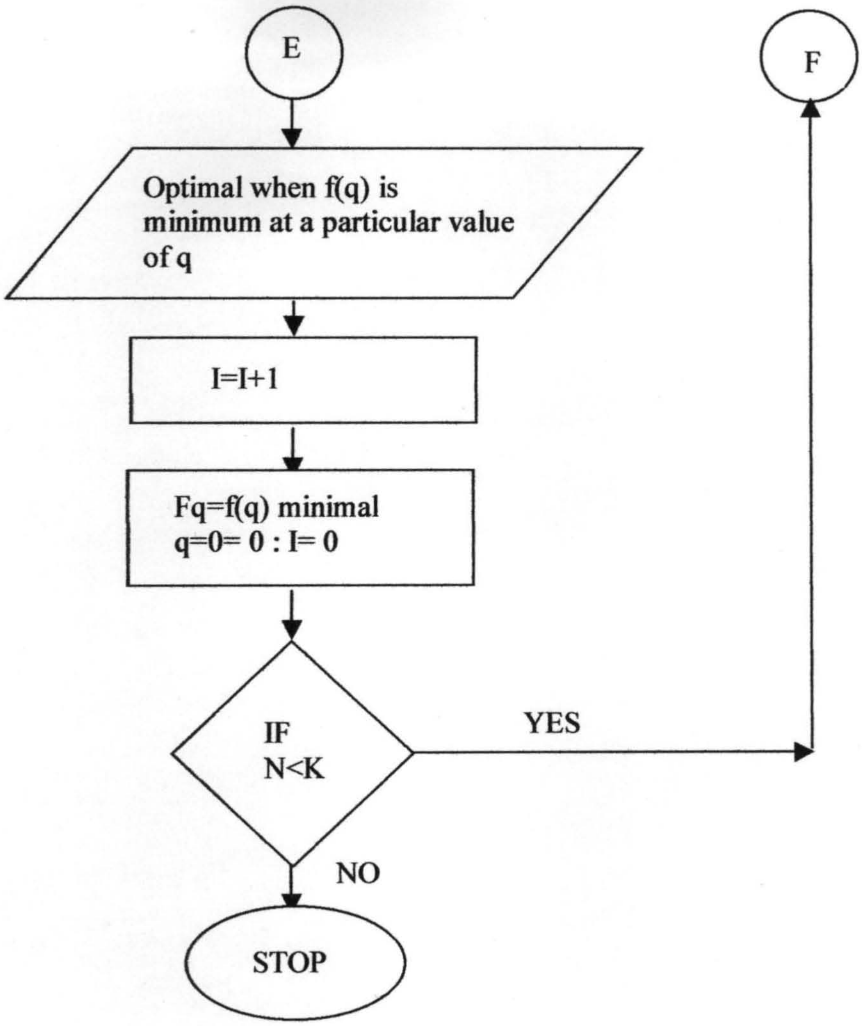
5.1 FLOWCHART

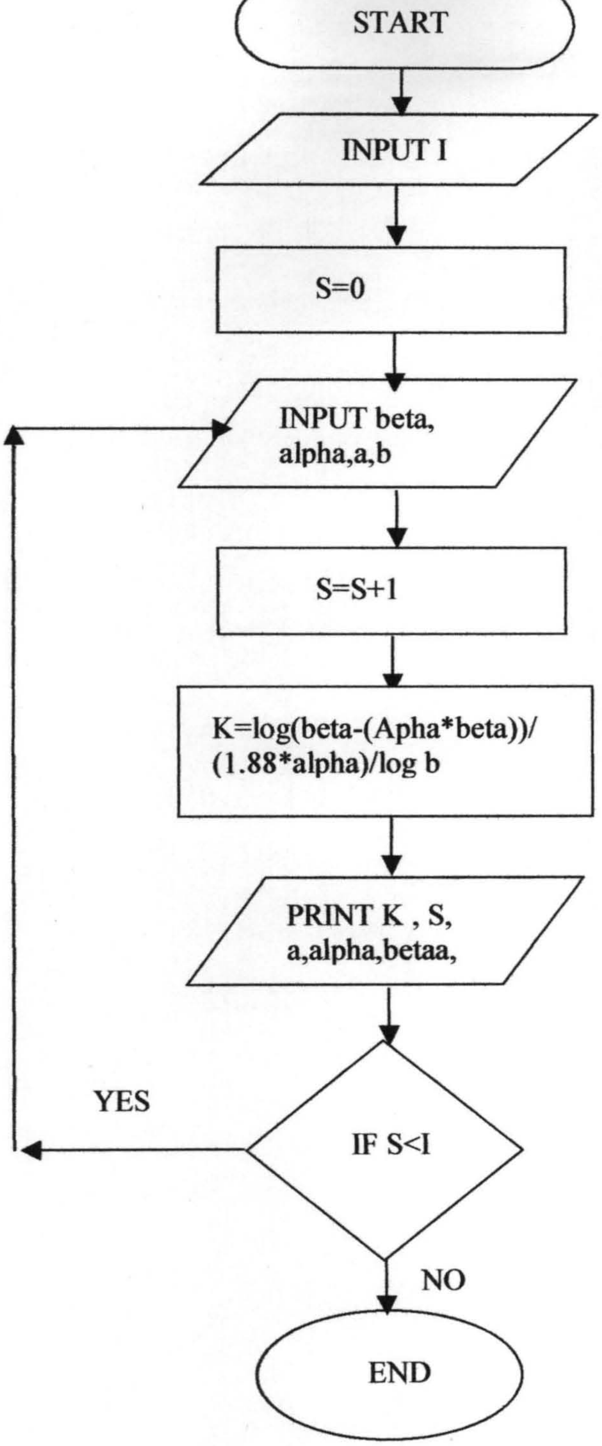
It is said that "**a picture is worth a thousand words**" and so it is with programs. It is much easier to understand the flow of operations from one to the other if we see a diagram than if we hear a description. A flowchart is a diagram of a process. It consists of symbols, indicating operations; and connecting lines, indicating how the process moves from operation to operation. Differently shaped are symbols are used for different types of operations

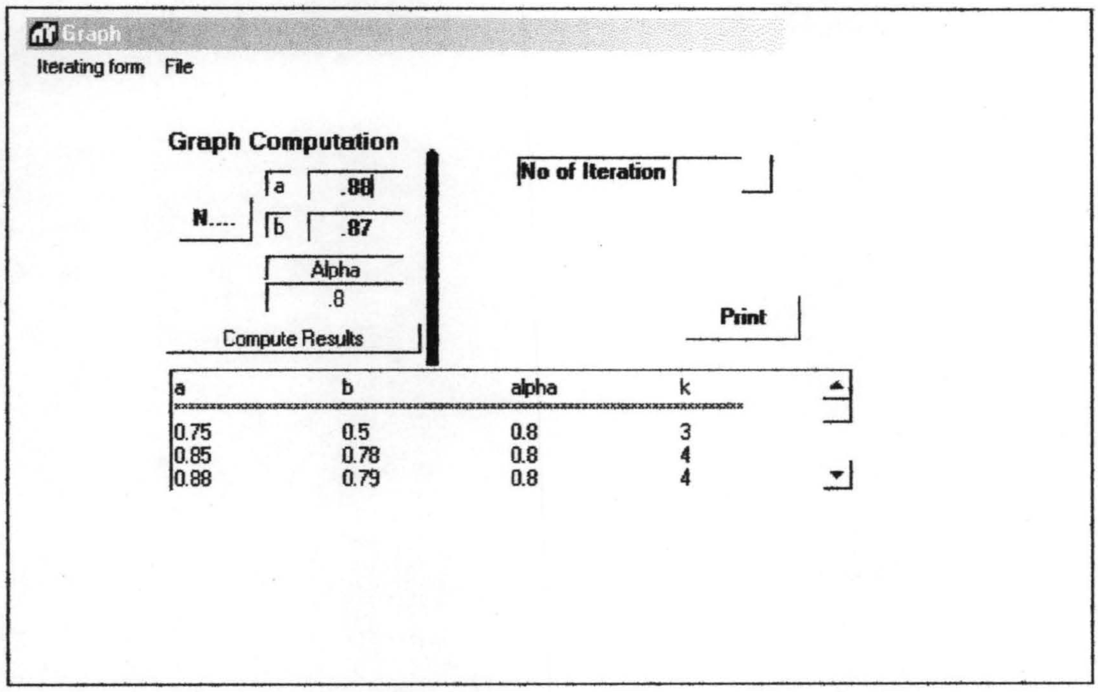

START
N=0:U=0
INPUT M, ALPHA
N=N+1
INPUT R(N)
YES
IF
N<M
NO
U=U+1
OPT(U)=R(U)/1-(0.8)U
A
C
B


A
C
B
INPUT OPT(U),U
IF
N<K
YES
NO
Optimum when Opt is
at the minimum
More
Test ?
YES
NO
INPUT J
q = 0 : I= 0
D

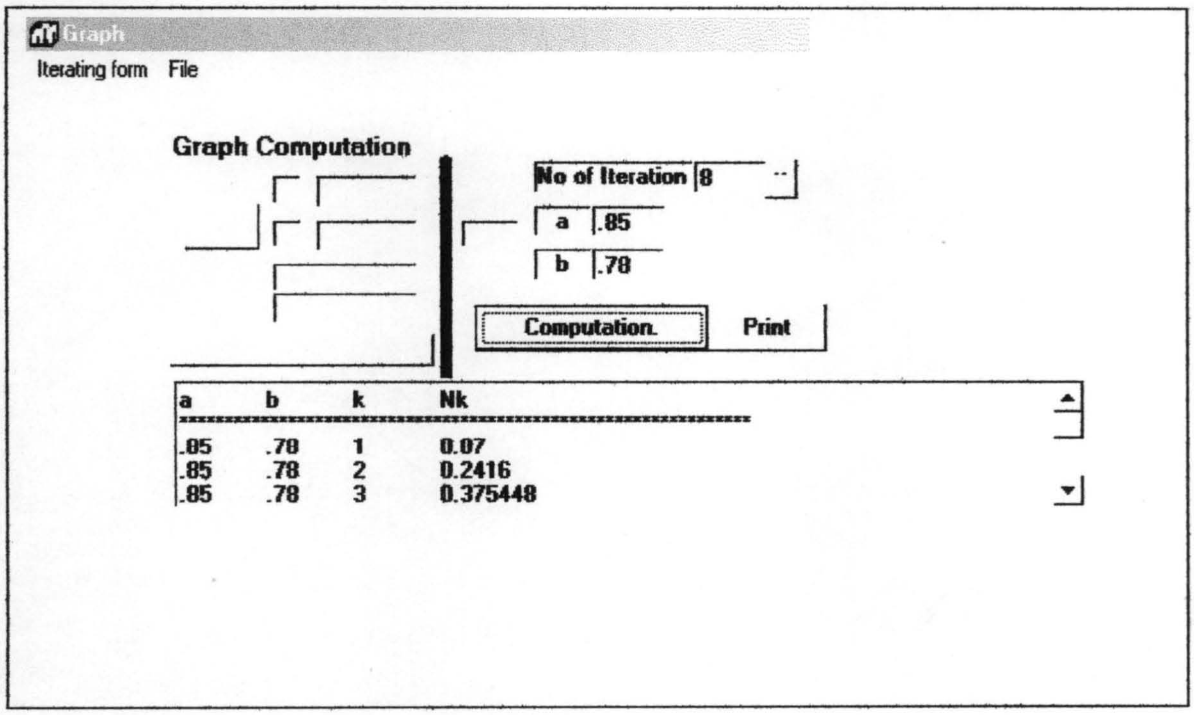

D
f(q)=R(1)
fq=f(q)
Print f(q)
INPUT R(N)
q=q+1
f(q)=(Alpha)^k*fg+R(q)
Print f(q)
YES
IF
N<M
NO
E
F


E
F
Optimal when f(q) is minimum at a particular value of q
I=I+1
Fq=f(q) minimal
q=0= 0 : I= 0
IF
N<K
YES
NO
STOP


START
INPUT I
S=0
INPUT beta,
alpha,a,b
S=S+1
K=log(beta-(Apha*beta))/
(1.88*alpha)/log b
PRINT K , S,
a,alpha,betaa,
YES
IF S<I
NO
END

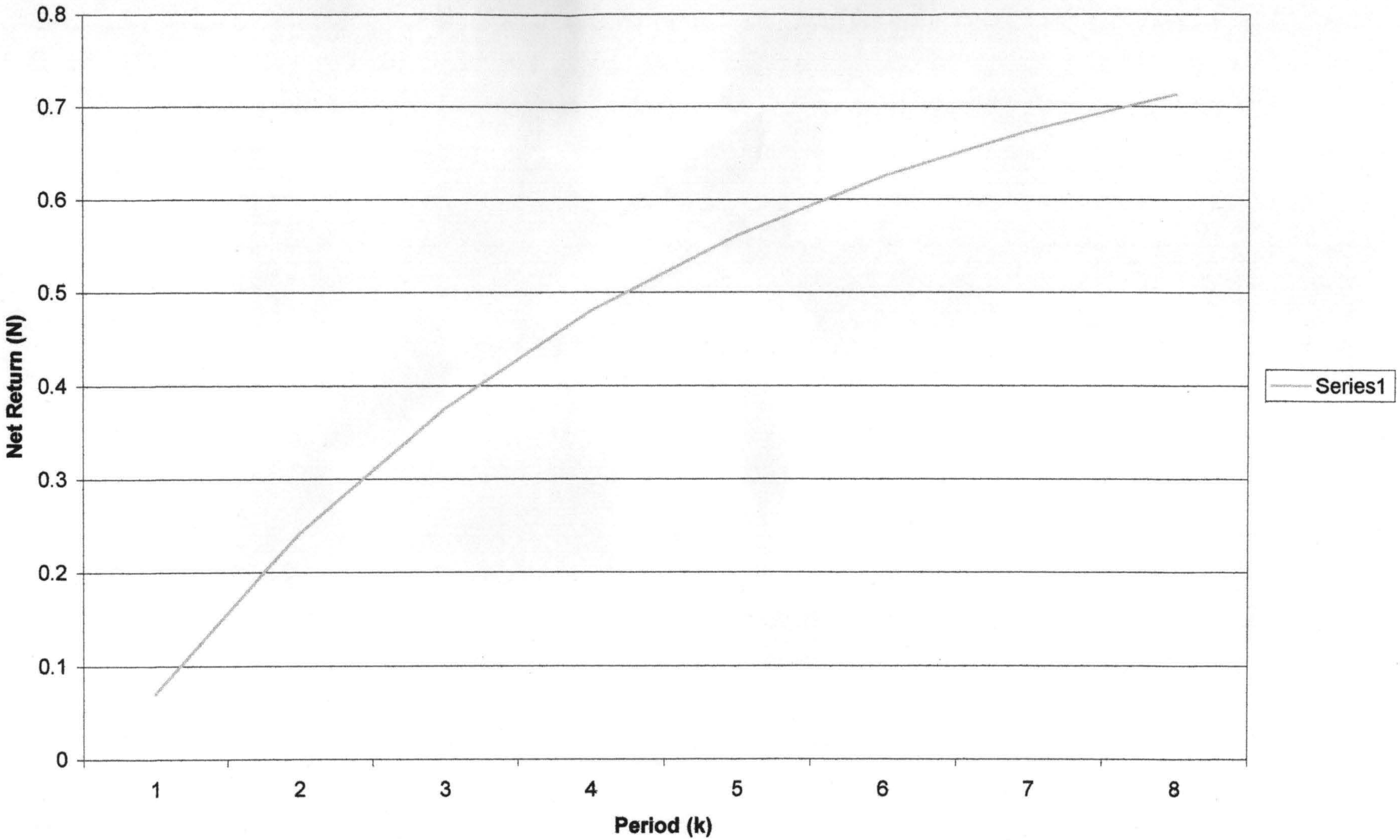

Graph
Iterating form File
Graph Computation
a .88
N.... b .87
Alpha
.8
Compute Results
No of Iteration
Print
a b alpha k
0.75 0.5 0.8 3
0.85 0.78 0.8 4
0.88 0.79 0.8 4

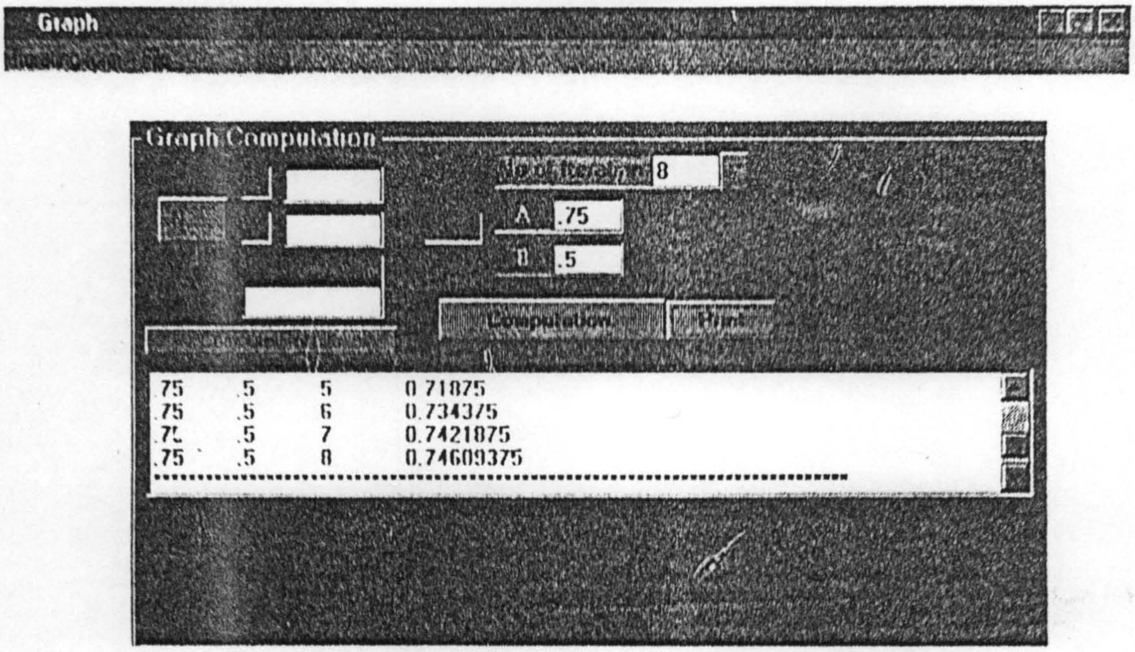
	b	alpha	k
.75	0.5	0.8	2
.85	0.78	0.8	3
.88	0.79	0.8	3
.99	0.89	0.8	3
.66	0.54	0.8	2
.11	0.1	0.8	2
.96	0.92	0.8	4
.98	0.97	0.8	4
.99	0.65	0.8	2
.88	0.87	0.8	4


Graph
Iterating form File
Graph Computation
No of Iteration 8
a .85
b .78
Computation.
Print
a b k Nk
.85 .78 1 0.07
.85 .78 2 0.2416
.85 .78 3 0.375448

	b	k	Nk
85	.78	1	0.07
85	.78	2	0.2416
85	.78	3	0.375448
85	.78	4	0.47984944
85	.78	5	0.5612825632
85	.78	6	0.624800399296
85	.78	7	0.67434431145088
85	.78	8	0.712988562931686

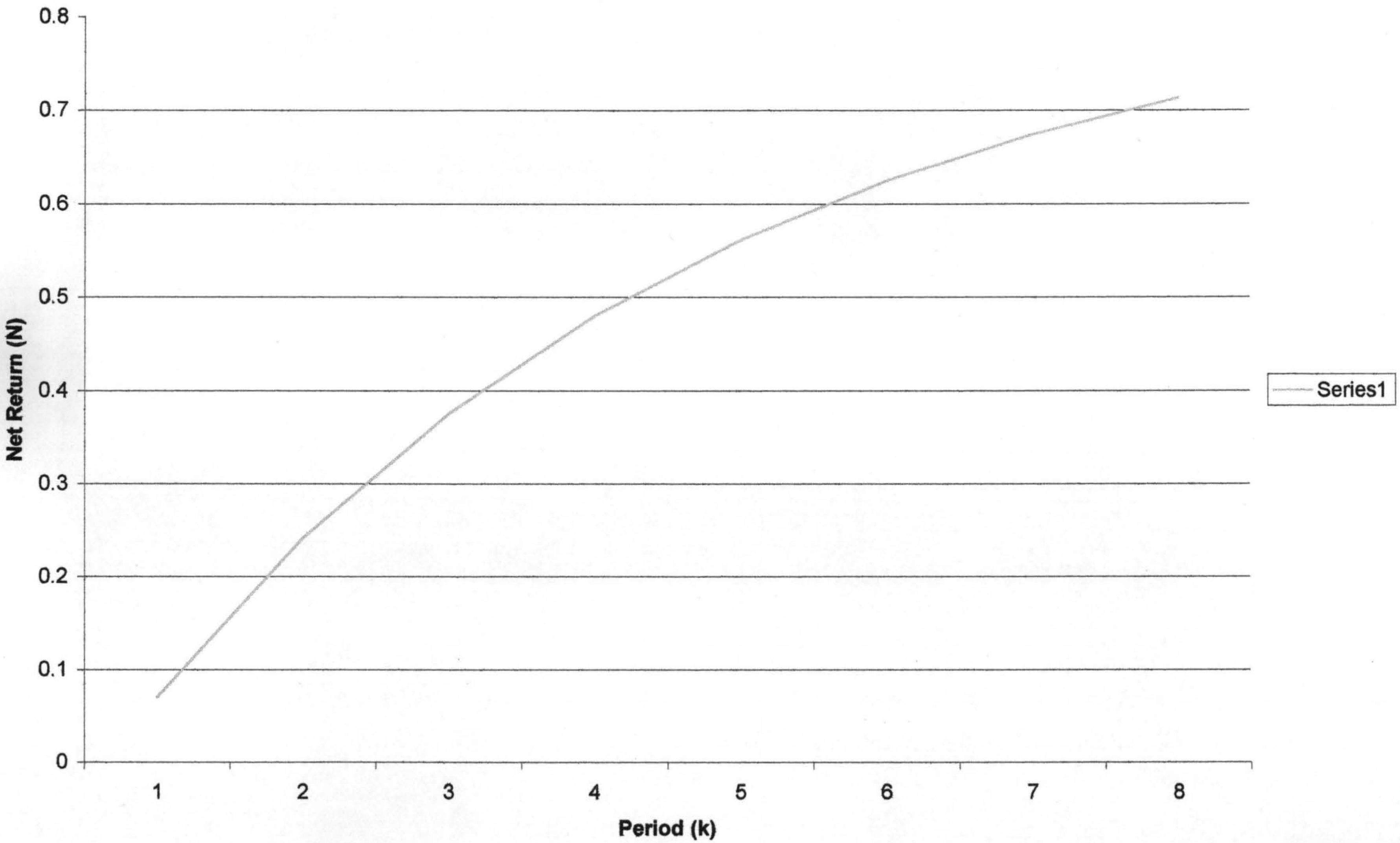
Fig. 5.2.1A Graph Showing The Relationship of Net return with Time




Graph
Graph Computation
8
A .75
B .5
Computation
Print
.75 .5 5 0.71875
.75 .5 6 0.734375
.75 .5 7 0.7421875
.75 .5 8 0.74609375

a	b	k	Nk
.75	.5	1	0.25
.75	.5	2	0.5
.75	.5	3	0.625
.75	.5	4	0.6875
.75	.5	5	0.71875
.75	.5	6	0.734375
.75	.5	7	0.7421875
.75	.5	8	0.74609375

fig. 5.2.2 A Graph Showing he relationship of Net Return with Time



5.3 SUMMARY

The methods of successive approximation in both function space and policy space are applicable to solving the extremal equation for the problem of finding a best route from each node k to the terminal. We should note two points of difference in the application of value iteration to the regeneration model and to the network model. In the regeneration model there was only one unknown, and therefore only one extremal relation, whereas in the network model there are p-1 unknowns and extremal equations. In contrast the horizon in the regeneration model was unbounded and value iteration does not necessarily converge in a finite number of iterations, whereas finite convergence does occur for a shortest route in the network.

However, successive approximation methods are to be applied to extremal equations for dynamic programming models with unbounded horizons that satisfy the assumptions:

(i) The decision outcomes are deterministic

(ii) The state of the system is examined at discrete points in time

(iii) Both the decision and state variables are discrete and have a finite number of possible values

(iv) The system parameters are stationery

However, it is helpful to view certain real situations as if decisions can be made any moment, not only at discrete points in time given in assumption (iii) above. For example, during "rush hours" a supermarket manager may decide to open an additional customer checkout stand whenever he sees the waiting lines getting too long. Obviously, he is not restricted to making this decision only at periodic intervals, such as every five minutes. In these situations the various successive approximations approaches we have studied can be adapted to provide workable optimization techniques.

Assumption (iii), relating to the variables being discrete and finite-valued, is often imposed for either analytical computational convenience. Frequently, however, a real system can be modeled just as well by letting the decision and state variables be continuous, and even unbounded.
The regeneration model in chapter four demonstrated how the numerical solution of a functional equation can be simplified if we have information about the form of an optimal policy.

5.4 REFE RENCES

1. Beale, E.M.L.; (1976), "Optimization in Action", Academic press Inc. New York, USA.
2. Bernard W. Taylor III; (1946) "Introduction to Management Science" Second Edition
3. Dimitri P. Bertsekas; (1982) "Constrained Optimization and Lagrange Multiplier Methods" Academic Press Inc. Harcourt Brace Jovanovich, publishers San Diego New York Berkeley Boston London Sydney Tokyo Toronto.
4. G.R. Walsh (1925) "Methods of Optimization" John Wiley $ Sons
5. London New York Sydney Toronto
6. Harvey M. Wagner (1972) "Principles of Operations Research with Applications" to Managerial Decisions" Prentice-Hall Inc; Engle Wood Cliffs New Jersey.
7. Hillier & Lieberman; (1986), "Introduction to Operations Research" fourth edition Holden-Day Inc.
8. John O. McClaim, L. Joseph Thomas (1929) "Operations Management Production of Good And Services".
8. L.R. Foulds (1984); "Combinatorial Optimization for Under Graduate" Springer-Verlag NewYork.
9. L.R. Foulds (1948) "Optimization Technique an Introduction with 72 illustrations".
10. Magnus R. Hestenes; (1906), "Conjugate Direction Methods in Optimization" Springer-Verlag New York Heidelberg Berlin
11. Rao, S.S. ;(1984), "Optimization Theory and Applications", second edition, Willey Eastern Ltd.

12. Ruth F. Curtain, A.J. Pritchard; (1977); "Functional Analysis in Modern Applied Mathematics" Academic press.

13. Xavier, J.R., et al., (1984), "Mathematical Modelling in Science and Technology"; Fourth International conference, Pergaman Press, Inc. New York, USA.