A DETERMINISTIC MATHEMATICAL MODEL OF THE DYNAMICS OF AVIAN INFLUENZA (BIRD FLU) PANDEMIC INVOLVING HUMAN INTERACTION

BY

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CERTIFICATION

This thesis titled **A DETERMINISTIC MATHEMATICAL MODEL OF THE DYNAMICS OF AVIAN INFLUENZA (BIRD FLU) PANDEMIC INVOLVING HUMAN INTERACTION** by **Abah, Roseline Toyin (M.TEC/SSSE/2005/1394)** meets the regulations governing the award of degree of (M.TECH) of the Federal University of Technology, Minna and is approved for its contribution to scientific knowledge and literary presentation.

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DEDICATION

This project is, specially dedicated to God Almighty for His infinite Mercy.

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ABSTRACT

In this thesis, a deterministic mathematical model of the dynamics of Avian Influenza (bird flu) pandemic with human interaction was proposed as a set of six ordinary differential equations. The model assumes the interaction of three major players of Migratory and Domesticated (Ground) birds' populations with the Human population. The characteristic equation was obtained and the equilibrium state analyzed for stability in order to gain insight into the dynamics of the flu pandemic. The zero equilibrium state was found to be stable while the non – zero equilibrium state is unstable.

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CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

Many real life situations can be transformed into mathematical models. The analysis of the model will then give insight into the dynamics of the real life situation. Problems such as the existence of equilibrium states and their stability are of great interest in the mathematical models of population dynamics as pointed out by Akinwande (1995). In this work, we propose a deterministic mathematical model of the dynamics of avian influenza (birdflu) as a system of six ordinary differential equations. The model assumes the interactions of three major players of Migratory and Domesticated (Ground) birds' populations with the Human population. We obtained the characteristic equation and analyzed the equilibrium states for stability in order to gain insight into the dynamics of the flu. This work has been divided into five chapters. Chapter one is the introduction. Literatures that are relevant to this work are reviewed in chapter two. In chapter three, the model equations are presented as six classes of differential equations with definitions of parameters. The equilibrium states and the corresponding characteristic equation of the model are also obtained. In chapter four, we analyze the equilibrium states for stability. And finally, is the chapter five, which contains the conclusion and recommendations.

1.2 Objectives of the Study

The objectives of this study are to:

i. Propose a mathematical model on the dynamics of Avian Influenza (Bird Flu) pandemic.

ii. Use mathematical tools to analyze the equilibrium states for stability.

iii. Draw conclusion on the nature of the dynamics of the pandemic so as to help policy makers in implementing relevant control measures.

1.3 Significance of the Study

According to the Center for Disease Control (CDC)(2005), "Influenza is a major cause of sickness and death around the world". It is one of the infectious diseases listed as causing years of potential life lost. The spread of infection in birds increases the opportunities for direct infection of humans. If more humans become infected over time, with the outbreaks in poultry and cause death of humans, this could affect the growth of a nation or community with time as the population size of that nation degenerates. John (2008), in his project said experts predict the next avian influenza pandemic could infect 20-50% of the world's total population and may result in 2-50 million deaths; this is obviously alarming. The study of population has been of great relevance to the growth of a nation or community over the ages because of the practical influence it has on human life.

"Population plays a vital role in the economic success of a nation to the extent that she cannot survive without adequate understanding of her population dynamics" Sirajo (2005). Mathematical models are of great contributions in the study of various kinds of dynamics. This research work is of relevance too, since it is a model that will help to understand the nature of the dynamics of the avian influenza pandemic and how to implement relevant control measures.

1.4 Scope and Limitation of the Study

In this work we propose a mathematical model of the dynamics of the flu resulting from the virus infection with the interaction of three populations of Migratory and Domesticated (Ground) birds' populations with the Human population. Each of the population is sub-divided into two classes of the susceptible(S) and the infected (I). In the event of a pandemic, planning and coordinating with public health and emergency management agencies at the local, state, national, and international levels are critical. When pandemic influenza begins, it is likely to spread very rapidly because most people may have little or no immunity to avian influenza pandemic. Information from the Federal Ministry Of Agriculture could not be accessed as a result I relied majorly on the information from the internet

1.5 Avian Influenza Pandemic

1.5.1 Overview of Avian Influenza

Avian influenza is an infection caused by avian (bird) influenza (flu) virus. The virus is very contagious and deadly among birds including domesticated birds such as chickens, ducks, and turkeys among others. Infected birds shed the virus in their saliva, nasal secretions and faeces; other birds get infected when they have contact with these contaminated secretions or excretions or easily picked as they feed together. Records of infections of humans are known to be very deadly as infected humans rarely survive the disease. Humans mostly prone to infections are those keeping poultry either in commercial or domestic quantity. They get infected mainly through contacts with infected birds, CDC W.H.O(2005) (2008). The virus may spread easily, possibly causing serious illness and death. Because so many people are at risk, serious consequences are possible. Historically, pandemic influenza has caused widespread harm and death. Pandemic influenza is different from seasonal influenza (or "the flu"). Seasonal outbreaks of the flu are caused by viruses that are already among people. Pandemic influenza is caused by an influenza virus that is new to people and are likely to affect many more people than seasonal influenza. The timing and consequences of pandemic influenza are difficult to predict. The most serious was the 1918 pandemic which killed tens of millions of people worldwide Pandemic flu (2005).

Infection in domestic poultry causes two main forms of diseases that are distinguished by low and high extremes of virulence. The low pathogenic form may go undetected and usually causes only mild symptoms such as ruffled feathers and a drop in egg production. While the highly pathogenic form spread more rapidly through flocks of poultry, this form may cause disease that affect multiple internal organs and has a mortality rate that can reach 90-100 percent often within 48 hours. The illnesses produced by the different types and strains are similar.

1.5.2 Types of Avian Influenza

There are three types of influenza viruses, known as A, B, and C. Type A, the most dangerous, infects a wide variety of mammals and birds. It causes the most cases of the disease in humans and is the type most likely to become epidemic. Type B infects humans and birds, producing a milder disease that can also cause epidemics. Type C apparently infects only humans. It typically produces either a very mild illness indistinguishable from a common cold or no symptoms at all.

Influenza type A and B viruses continually change. Some changes involve a series of genetic mutations that, over a period of time, cause a gradual evolution of the virus, called antigenic drift, this process accounts for most of the changes in influenza viruses that occur from one year to the next pandemic flu (2005).

1.5.3 Classification of Influenza A Viruses.

There are 15 subtypes of influenza A viruses as shown in figure1.1 Medicalecology (2004).

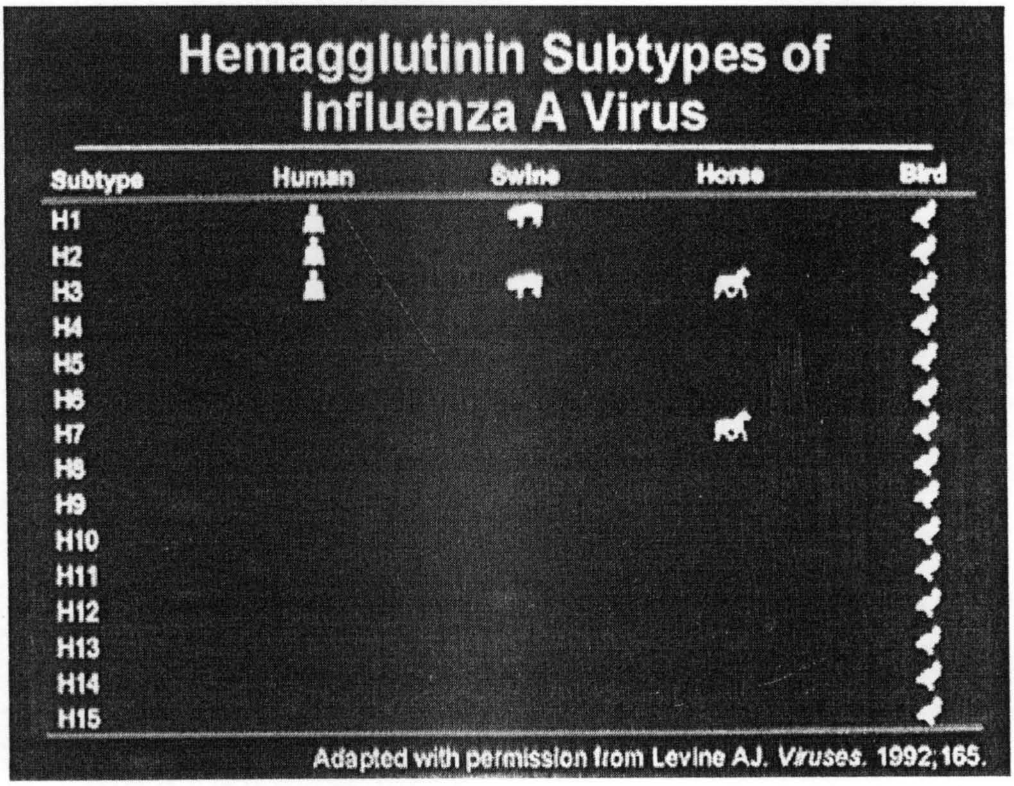

Figure 1.1- (2004) www.Medicalecology.org

1.5.4 Signs and Symptoms of Avian Influenza

Influenza is different from common cold as it affects cells much deeper down in the respiratory tract than its most common complication pneumonia. Although the exact incubation period for bird flu in humans isn't clear, illness seems to develop within one to five days of exposure to the virus. The clinical signs are variable and depend on a range of factors including the virulence of the virus, the species and age of birds infected the presence of concurrent diseases, and the environment. Clinical signs and symptoms of avian influenza include US department of Agic (2008):

- Ruffled feathers.
- Sudden death in several birds.
- Lack of energy and appetite.
- Purple discoloration of the skin.
- Coughing, sneezing, nasal discharge, Sore throat.
- Lack of coordination and diarrhea.
- Unusual neck posture, Swollen head.
- Inability to walk or stand,
- A drop in egg production.
- Fever, Muscle aches

A relatively mild eye infection (conjunctivitis) is sometimes the only indication of the disease. People with bird flu also may develop life-threatening complications, particularly: Acute respiratory distress — the most common cause of bird flu-related deaths, pneumonia and in some cases death.

1.5.5 Structure of Avian Influenza

The neuraminidase antigen exists as a mushroom-shape spike on the surface of the virus. It has a box-like head consisting of four co-planar and roughly spherical subunits, and a hydrophobic region that is embedded within the interior of the viral membrane. Moreover, it is composed of a single polypeptide chain that is oriented in the opposite direction to the haemagglutinin antigen US Department of Agric (2008).

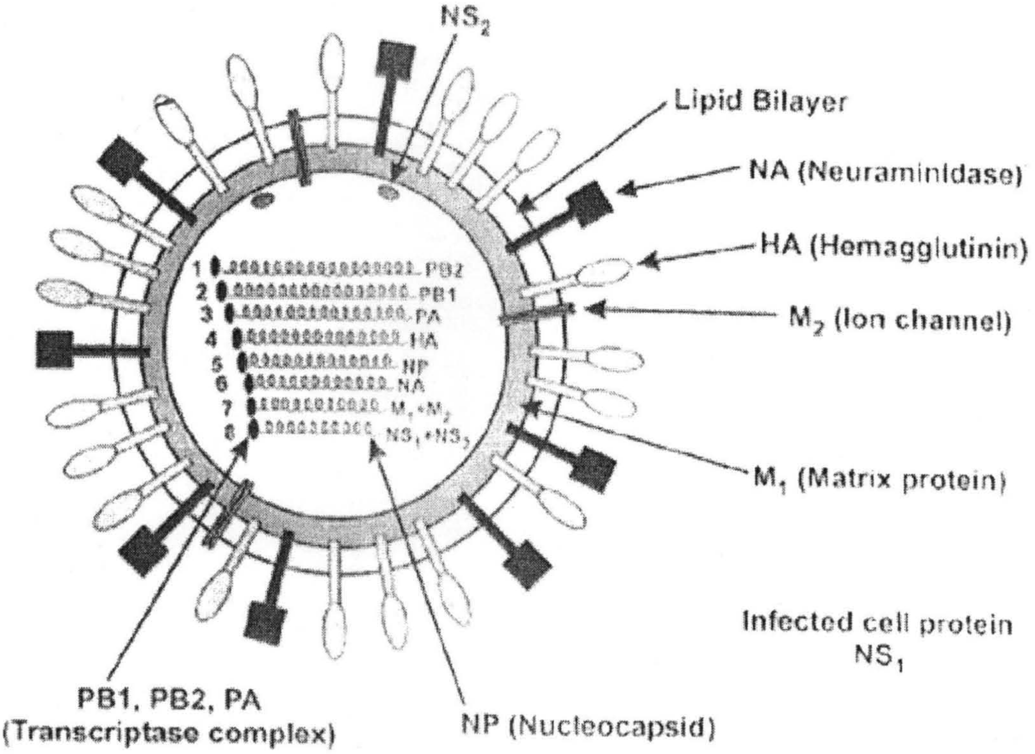

Figure1.2- Source: Medicalecology.org (2004)

Influenza strain diversity is defined by two glycoproteins on the surface of the virion. The first stage of infection with the virus is mediated by the hemagglutinin (HA) membrane glycoprotein that allows for binding and fusion with host cells. The other major glycoprotein, neuraminidase, facilitates viral budding and exit from the host cell. These two glycoproteins exhibit great structural diversity, and at least 15 hemagglutinin and 9 neuraminidase subtypes have been identified to date.

1.6 Mathematical Modeling

1.6.1 Overview of Mathematical Modeling

Mathematical Modeling, which is defined by Benyah (2005) as "the process of creating a mathematical representation of some phenomenon in order to gain a better understanding of that phenomenon", has become an important scientific technique over the last 20years. Mathematical modeling provides an essential tool to capture a set of assumptions and follow them to their precise logical conclusions. They allow us to generate new hypothesis, suggest experiments, and measure crucial parameters. Essentially any real situation in the physical and biological world, whether natural or involving technology and human intervention, is subject to analysis by modeling, if it can be described in quantitative terms. Thus optimization and Control theory may be used to model industrial process, Traffic patterns, sediment transport in streams, and other situations, information and communication theory may be used to model message transmission, linguistic characteristics, and the like and dimension analysis and computer simulation may be used to model atmospheric circulation pattern, stress distribution in engineering structure, the growth and development of land forms and a host of other processes in science and engineering Burghes and Wood (2005)

Serfling developed a mathematical model, Medical ecology (2007) that was used to estimate the magnitude of the 1957 Asian influenza pandemic.

Once a model has been developed and used to answer questions, it should be critically examined and modified to obtain a more accurate reflection of the observed reality of that phenomenon. Mathematical modeling is an evolving process, as new insight is gained the process begins again as additional factors are considered Benyah (2005). Generally, the success of a model depends on how easily it can be used and how accurate are its predictions

1.6.2 The Stages of Mathematical Modeling

Building a mathematical model for a real life situation requires a thorough understanding of the underlying principles of the system to be modeled. During the process of building a Mathematical Model, the modeler will decide what factors are relevant to the problems and what factors can be de-emphasized. Different problems may require very different methods of approach. Benyah (2005) outlined the following steps as a general approach to the formulation of a real- life problem in mathematical terms:

1. Identify the problem
2. Identify the important variables and the parameters

3. Determine how the variables relate to each other, stating the assumptions
4. Develop the equations that express the relationship between the variables and constants.
5. Analyze and solve resulting mathematical problems.

1.6.3 Mathematical Modeling with Differential Equation

Differential equations form very important mathematical tools used in producing models of physical and biological processes. They have been the subject of a great deal of research for more than 100 years. Burghes & Wood (2005) have this to say "..., it could even be claimed that the spread of modern industrial civilization, for better or for worse, is partly a result of man's ability to solve differential equations which govern so many of our industrial processes, be it chemical or engineering".

1.7 EQUILIBRIUM AND STABILITY

1.7.1 Equilibrium

The word equilibrium is quite ancient. The word has the same stem as the name of the constellation "Libra" — the scale. The type of scale in question is the two-pan balance shown in figure 1.3, which has been in use for at least 7000 years. The compound word "equilibrium" translates literally as "equal balance" and means just that: everything in balance, i.e. no unbalanced forces.

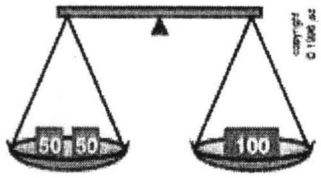

Figure1.3-Source Microsoft corp. (1993 – 2008). Forces in Balance

Equilibrium is a state of a system whose configuration or large-scale properties do not change over time. For example, if a hot penny is dropped into a cup of cold water, the penny will reach equilibrium when both are at the same temperature. At that point, the large-scale properties of the system, namely the temperature of the water and the penny will not change over time. We also see another example in mechanics; a system is at equilibrium if the net force acting on a body is equal to zero. In the case of a stationary body, the large of the position will remain unchanged over time.

If the dynamics of a system is described by a differential equation (or a system of differential equations), then, the equilibra can be estimated by setting a derivative (all derivatives) to zero.

Example: Logistics Model:

$$\frac{dN}{dt} = r_0 N\left(1 - \frac{N}{K}\right)$$

To find equilibra we have to solve the equation: $\frac{dN}{dt} = 0$:

i.e. $r_0\,N\left(1 - \frac{N}{K}\right) = 0$. This equation has two roots: $N = 0$ **and** $N = K$.

1.7.2. Stability

Stability has to do with how the system responds if we move it a little from its equilibrium position. There are three possibilities:

Positive stability means that if the system is displaced a little from its equilibrium position, it will generate a force tending to push it back towards equilibrium. The wheel with the weight positioned at the bottom is an example of positive stability.

Neutral stability (also called zero stability) means that if the system was in equilibrium and you displace it slightly, it remains in equilibrium. No force is generated. The perfectly balanced wheel is an example of this.

Negative stability means that if the system is displaced a little from its equilibrium position, it will generate a force that tends to push it farther from equilibrium. The wheel with the weight at the top is an example of negative stability.The term stable by itself denotes strictly positive stability. The term unstable by itself denotes strictly negative stability. We must beware of the contrast in the following two sentences, both of which are true:

→ A system with zero stability is neither stable nor unstable.

→ A system with zero stability is both neutrally stable and neutrally unstable. A solution f(x) is said to be stable if any other solution of the equation that starts out sufficiently close to it when $x = 0$ remains close to it for succeeding values of x. if the difference approaches zero as it increases, the solution is called asymptotically stable. If a solution does not have either of these properties, it is called unstable.

1.7.3 Illustration Examples

Equilibrium may be stable or unstable. For example, the wheel is more modern than the balance as indicated in figure 1.4, there are three ways to have the wheel be in equilibrium: {1} position the weight at the bottom, {2} remove the weight entirely (or put it at dead center, where the axle is) or {3} position the weight at the top.

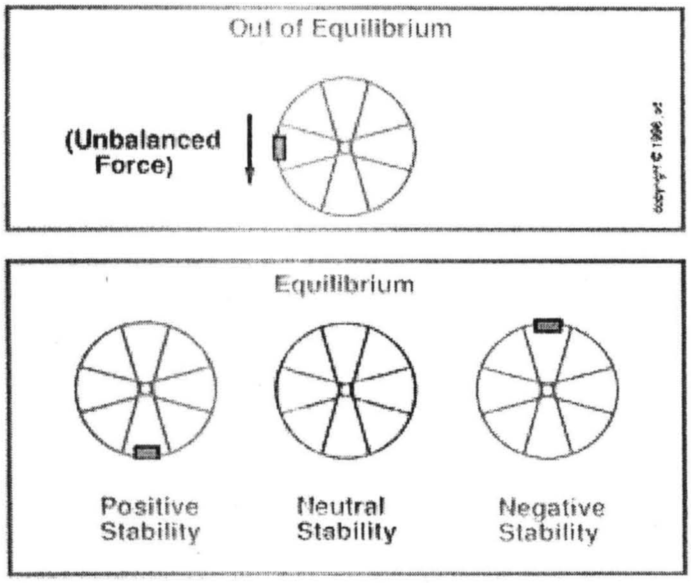

Figure1. 4- Equilibrium and Stability. Source: Microsoft corp. (1993 – 2008)

If we attach the weight to any other point, system will be out of equilibrium. If we then let go, it will immediately start rotating.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The purpose of this chapter is to review some literatures that are relevant to the dynamics of avian influenza pandemic. There are so many mathematical models that have been developed on this topic with different purposes in mind. In order to aid our understanding, this chapter has been divided into the following subsections:

- Avian Influenza and the Immune system
- Avian Influenza and its Transmission/Spread
- Avian Influenza infection in humans
- Avian Influenza in Nigeria
- Prevention and Treatment of Avian Influenza
- Avian influenza connection

2.2 Avian Influenza and the Immune System

Immune System is a group of cells, molecules, and organs that act together to defend the body against foreign invaders that may cause disease, such as bacteria, viruses, and fungi Sirajo (2005). The health of the body is dependent on the immune system's ability to recognize and then repel or destroy these invaders.

Although humans have a degree of immunity to the influenza subtypes that circulate during the winter flu season, the human immune system is unaccustomed to recognizing and fighting off avian influenza. This makes the avian viral strains all the more dangerous.

2.3 Avian Influenza and its Transmission/spread

Avian influenza is most often spread by contact between infected birds and healthy birds CDC (2008). It may also be spread indirectly through contact with contaminated equipment and materials. The avian influenza virus is found in secretions from the nostrils, mouth, and eyes of infected birds. However, the spread of avian influenza between poultry facilities almost always results from the movement of infected birds or contaminated people and equipment (including clothing, boots, and vehicles). Avian influenza virus can also be spread from birds to people as a result of direct contact with infected birds, such as during home slaughter of infected poultry. Public health concerns centre on the potential for the virus to mutate or combine with other influenza viruses to a form that could easily spread from person to person. If that happens, there is a risk that the virus could rapidly spread worldwide and cause large numbers of humans to become ill or die (a pandemic).

2.4 Avian Influenza in Humans

Direct contact with infected poultry, or surfaces and objects contaminated by their droppings is presently considered the main route for infection of humans by the avian H5N1 virus CDC (2005). To date, most human cases have occurred in rural or urban fringe areas where many households keep small poultry flocks, which often roam freely, sometimes entering homes or sharing outdoor areas where children play. As infected birds shed large quantities of virus in their droppings, opportunities for exposure to infected droppings or to environments contaminated by the virus are abundant under such conditions. Moreover, because many households depend on poultry for income and food, families sometimes sell or slaughter and consume birds when signs of illness appear in a flock, rather than disposing of the birds safely, and this practice has proved difficult to change. Exposure is considered most likely during slaughter, plucking and butchering. There is no evidence that properly cooked poultry or eggs can be a source of infection for avian influenza viruses CDC (2008).

2.5 Avian Influenza in Nigeria

Debora in her caption "Nigerian's strain of bird flu bode ill for Africa",

reported that bird flu invaded Nigeria on at least three separate occasions New scientist. Each time, wild birds' were probably responsible, which means that its continued spread across Africa may likely be difficult to halt. After spreading west across Asia, the highly pathogenic virus was identified in Africa the northern Nigerian state of Kaduna. Outbreaks multiplied quickly, reaching Lagos. Muller, et al. took samples from infected birds at two farms less than 50 kilometres apart in Lagos state. The samples were sequenced at the Erasmus Medical Centre in Rotterdam, the Netherlands, and compared with gene sequences of other samples, including the virus from Kaduna.

Nigerian state of Taraba records a new case of bird flu, as the virus breaks out in three local government areas, as reported by the News Agency of Nigeria. The first Bird flu centre (2005)case to appear in the state was reported newscientists.com(2008) Taraba became the 14th of Nigeria's 36 states to report an outbreak . A solution to combat the disease is urgently needed and the government of Nigeria is called upon to aid on the efforts. Experts were sent to the affected areas to control the situation.

The recent outbreak of avian influenza in Nigeria shows that poultry movements can cause the virus to spread across countries and even continents. The Nigerian outbreak further demonstrates that lapses in bio-security are the major reason for avian influenza's continuing spread around the world. Whilst the precise nature of the outbreak is unknown, it seems more than likely that the virus arrived through infected poultry brought into the country in defiance of Nigeria's import controls. Speaking at a press conference, Bello said, "Birds come every day from China, Turkey, into Nigeria, and from Europe and also from Latin America Bird life (2008). So Nigeria is exposed. Illegal importation of poultry by people who have farms, bringing in poultry from places and smuggling them in could also have been a cause."

Globalization has turned the chicken into the world's number one migratory bird species," said Leon Bird life (2008). Movements of chickens around the world take place 365 days a year, unlike the seasonal migrations of wild birds. Millions of people keep chicken in their backyards and poultry is usually transported, sold and handled live, as electricity to run refrigerators is not a common thing.

2.5.1 Effect of Avian influenza (Bird flu) in Nigeria

Avian influenza recorded its first human death in the sub-Saharan region. Test were carried out on a dead woman and proved that she had indeed been infected with the highly pathogenic bird flu virus. The 22 year old woman had previously handled a diseased chicken in her residence in Lagos, Nigeria's capital Newscientist (2008).

2.5.2 Ways Employed to Stopping Further Spread of the Flu in Nigeria

"It is important that strict biosecurity measures are imposed to stop further Nigeria has employed massive culling policy. Nigeria has culled around 700,000 birds in its effort to stop the outbreaks of avian influenza, which was reported in the country, Bird life (2008). The method is an effective way to prevent the spread of the disease. Health workers are expected to cover the whole of Nigeria in an effort of searching for and detecting cases of avian influenza as early as possible. Nigerian authorities have taken standard measures to curb the disease, such as culling poultry, instituting quarantines and banning poultry transports.

2.6 Treatment and Prevention of Influenza Outbreaks

"We really have to get our act together on human and animal surveillance," Osterhaus says Newscientist (2008).

Most countries have the resources to deploy the necessary control measures and provide improved veterinary services and surveillance to support highly pathogenic influenza.

Two new drugs, oseltamivir and zanamivir, can prevent infection with either the influenza virus of type A or type B. These drugs produce minimal side effects. If a secondary bacterial infection develops, antibiotics are added CDC and WHO recommend oseltamivir, a prescription antiviral medication, for treatment and chemoprophylaxis of human infection with avian influenza A viruses WHO (2005). Analyses of available viruses circulating worldwide suggest that most viruses are susceptible to oseltamivir.

Be up-to-date with all the routine vaccinations. Assemble a travel health kit containing basic first aid and medical supplies, include a thermometer and alcohol-based hand gel for hand hygiene. Before you visit a healthcare setting, tell the provider the following: (1) your symptoms, (2) where you traveled, and (3) if you have had direct contact with poultry or close contact with a severely ill person. Avoid all direct contact with poultry, including touching well-appearing, sick, or dead chickens and ducks. Avoid places such as poultry farms and bird markets where live poultry are raised or kept, and avoid handling surfaces contaminated with poultry faeces or secretions. As with other infectious illnesses, one of the most important preventive practices is careful and frequent hand washing. Cleaning your hands often with soap and water removes potentially infectious material from your skin and helps prevent disease transmission. Waterless alcohol-based hand gels may be used when soap is not available and hands are not visibly soiled. Influenza viruses are destroyed by heat; therefore, as a precaution, all foods from poultry, including eggs and poultry blood, should be thoroughly cooked.

2.7 Avian Influenza Connection

There are many different types of influenza viruses circulating in the world at any given time. Some only infect birds, some only infect swine, and others infect only humans. The term "bird flu" or "avian influenza" refers to influenza strains that affect birds. One particular avian strain, known as H5N1, has caused severe illness in birds in recent years. Hundreds of millions of chickens and ducks have died or been killed in an effort to control this strain of influenza, yet the H5N1 virus has still been found in birds in many parts of the world. The virus is probably being spread by wild birds, which may or may not become ill

Although the H5N1 virus primarily affects birds, it can also spread to people. Most people who have gotten the virus work directly with poultry or have had close contact with birds. Public health experts are concerned that the H5N1 virus could change (mutate) into a form that is easily spread from one person to another. We don't know for sure whether that will happen – or when it might happen. But if it does, the result could be a global influenza pandemic. Since few, if any, people would have any immunity to the new influenza virus, it could spread around the world very rapidly, causing serious illness in many people, CDC (2005).

CHAPTER THREE

THE MODEL EQUATIONS AND EQUILIBRUM STATES

3.1 Introduction

The interaction of three populations of Ground, Migratory birds and Human are considered with each partitioned into susceptible and infected classes. The susceptible classes are the birds which are free from the virus but prone to infection as a result of interaction with the infected classes which are carrying the virus. We will be considering the model equations as we assume that the ground birds are susceptible to infections as a result of the infected ground and migratory birds and the infected humans. The migratory birds are susceptible to infection only from their fellow infected migratory birds while the humans are susceptible to infection only from the infected domestic or the ground birds.

3.2 The Model Equations

We propose here a set of six ordinary differential equations for the interactions of these six classes in equations (3.1) to (3.6) as shown below, along with the definitions of parameters. We obtained the equilibrium states and the corresponding characteristic equation. The zero and the non-zero equilibrium states are then analyzed for stability.

The model equations are given as follows:

$$\frac{dG_1}{dt} = (\beta_1 - \mu_1)G_1 - \alpha_1 G_1(G_2 + M_2 + P_2) \tag{3.1}$$

$$\frac{dG_2}{dt} = \alpha_1 G_1(G_2 + M_2 + P_2) + (\beta_1 - \mu_1 - \delta_1)G_2 \tag{3.2}$$

$$\frac{dM_1}{dt} = -\alpha_2 M_1 M_2 + (\beta_2 - \mu_2)M_1 \tag{3.3}$$

$$\frac{dM_2}{dt} = \alpha_2 M_1 M_2 + (\beta_2 - \mu_2 - \delta_2)M_2 \tag{3.4}$$

$$\frac{dP_1}{dt} = -\alpha_3 P_1 G_2 + (\beta_3 - \mu_3)P_1 \tag{3.5}$$

$$\frac{dP_2}{dt} = \alpha_3 P_1 G_2 + (\beta_3 - \mu_3 - \delta_3)P_2 \tag{3.6}$$

Where:

G_1 = the ground birds' population which are susceptible.

G_2 = the ground birds' population which are infected.

M_1 = the migratory birds' population which are susceptible.

M_2 = the migratory birds' population which are infected.

P_1 = the human population which are susceptible.

P_2 = the human population which are infected.

β_1 = natural birth rate for the ground birds' population.

β_2 = natural birth rate for the migratory birds' population.

β_3 = natural birth rate for the human population

μ_1 = natural death rate for the ground birds' population.

μ_2 = natural death rate for migratory birds' population.

μ_3 = natural death rate for the human population.

δ_1 = death rate due to infection for the infected ground birds' populations.

δ_2 = death rate due to the infection for the infected migratory birds' population.

δ_3 = death rate due to the infection for the infected human population.

α_1 = contracting rate for the susceptible ground birds'.

α_2 = contracting rate for the susceptible migratory birds'.

α_3 = contracting rate for the susceptible humans.

t = time.

3.3 Equilibrium states of the model

Here we obtained the equilibrium state of the model. At the equilibrium states the rate of change of the system is zero.

Let $G_1(t), G_2(t), M_1(t), M_2(t), P_1(t), P_2(t) \equiv (u, v, w, x, y, z)$

Equations (3.1) to (3.6), then give

$$(\beta_1 - \mu_1)u - \alpha_1 u(v + x + z) = 0 \tag{3.7}$$

$$\alpha_1 u(v + x + z) + (\beta_1 - \mu_1 - \delta_1)v = 0 \tag{3.8}$$

$$-\alpha_2 wx + (\beta_2 - \mu_2)w = 0 \tag{3.9}$$

$$\alpha_2 wx + (\beta_2 - \mu_2 - \delta_2)x = 0 \tag{3.10}$$

$$-\alpha_3 yv + (\beta_3 - \mu_3)y = 0 \tag{3.11}$$

$$\alpha_3 yv + (\beta_3 - \mu_3 - \delta_3)z = 0 \tag{3.12}$$

The solutions of the non–linear simultaneous equations (3.7) to (3.12) give the equilibrium states of the model equations.

By adding equations (3.7) and (3.8), i.e.

$$(\beta_1 - \mu_1)u - \alpha_1 u(v + x + z) + \alpha_1 u(v + x + z) + (\beta_1 - \mu_1 - \delta_1)v = 0$$

$$(\beta_1 - \mu_1)u + (\beta_1 - \mu_1 - \delta_1)v = 0$$

$$u = -\frac{(\beta_1 - \mu_1 - \delta_1)v}{(\beta_1 - \mu_1)} \tag{3.13}$$

By adding equations (3.9) and (3.10), i.e.

$$-\alpha_2 wx + (\beta_2 - \mu_2)w + \alpha_2 wx + (\beta_2 - \mu_2 - \delta_2)x = 0$$

$$(\beta_2 - \mu_2)w + (\beta_2 - \mu_2 - \delta_2)x = 0$$

$$w = -\frac{(\beta_2 - \mu_2 - \delta_2)x}{(\beta_2 - \mu_2)} \tag{3.14}$$

Adding equations (3.11) and (3.12), we have

$$-\alpha_3 yv + (\beta_3 - \mu_3)y + \alpha_3 yv + (\beta_3 - \mu_3 - \delta_3)z = 0$$

$$(\beta_3 - \mu_3)y + (\beta_3 - \mu_3 - \delta_3)z = 0$$

$$y = -\frac{(\beta_3 - \mu_3 - \delta_3)z}{\beta_3 - \mu_3} \tag{3.15}$$

Then from equation (7), either $u = 0$ *or*

$$(v + x + z) = \frac{\beta_1 - \mu_1}{\alpha_1} \tag{3.16}$$

From (3.8), if $u = 0, then\ v = 0$. Also from (3.9), $w = 0$ or

$$x = \frac{\beta_2 - \mu_2}{\alpha_2} \tag{3.17}$$

From (3.10), if $w = 0,\ then\ x = 0$. Also from (3.11), either $y = 0$ *or*

$$v = \frac{\beta_3 - \mu_3}{\alpha_3} \tag{3.18}$$

From (3.12), if $y = 0, then\ z = 0$. Hence the zero state:

$$(u, v, w, x, y, z) = (0,0,0,0,0,0) \tag{3.19}$$

is an equilibrium state of the model equations.

Now in the case where $u \neq 0$ *and* $w \neq 0$, equations (3.13) and (3.18) give

$$u = -\frac{(\beta_1 - \mu_1 - \delta_1)}{(\beta_1 - \mu_1)} \frac{(\beta_3 - \mu_3)}{\alpha_3} \tag{3.20}$$

Also equations (3.14) and (3.17) give

$$w = -\frac{(\beta_2 - \mu_2 - \delta_2)}{(\beta_2 - \mu_2)}\frac{(\beta_2 - \mu_2)}{\alpha_2} = -\frac{(\beta_2 - \mu_2 - \delta_2)}{\alpha_2} \quad (3.21)$$

By substituting equations (3.17) and (3.18) into (3.16), z gives

$$z = \frac{(\beta_1 - \mu_1)}{\alpha_1} - \frac{(\beta_2 - \mu_2)}{\alpha_2} - \frac{(\beta_3 - \mu_3)}{\alpha_3} \quad (3.22)$$

Equations (3.22) and (3.15) give

$$y = -\frac{(\beta_3 - \mu_3 - \delta_3)}{(\beta_3 - \mu_3)}\left(\frac{(\beta_1 - \mu_1)}{\alpha_1} - \frac{(\beta_2 - \mu_2)}{\alpha_2} - \frac{(\beta_3 - \mu_3)}{\alpha_3}\right) \quad (3.23)$$

The Equilibrium states are

The zero equilibrium state given by

$$(u, v, w, x, y, z) = (0,0,0,0,0,0,0)$$

The non-zero equilibrium state (u, v, w, x, y, z) given by:

$$u = -\frac{(\beta_1 - \mu_1 - \delta_1)}{(\beta_1 - \mu_1)}\frac{(\beta_3 - \mu_3)}{\alpha_3}$$

$$v = \frac{(\beta_3 - \mu_3)}{\alpha_3}$$

$$w = -\frac{(\beta_2 - \mu_2 - \delta_2)}{\alpha_2}$$

$$x = \frac{(\beta_2 \; - \; \mu_2)}{\alpha_2}$$

$$y = -\frac{(\beta_3 - \mu_3 - \delta_3)}{(\beta_3 - \mu_3)}\left(\frac{(\beta_1 - \mu_1)}{\alpha_1} - \frac{(\beta_2 - \mu_2)}{\alpha_2} - \frac{(\beta_3 - \mu_3)}{\alpha_3}\right)$$

$$z = \frac{(\beta_1 - \mu_1)}{\alpha_1} - \frac{(\beta_2 + \mu_2)}{\alpha_2} - \frac{(\beta_3 - \mu_3)}{\alpha_3}$$

3.4 The Characteristics Equation

We perturb the equilibrium states using equations (3.24) to (3.29)

Let :

$$G_1(t) = u + a(t); \; a(t) = \bar{a}e^{\lambda t} \tag{3.24}$$

$$G_2(t) = v + f(t); \; f(t) = \bar{f}e^{\lambda t} \tag{3.25}$$

$$M_1(t) = w + g(t); \; g(t) = \bar{g}e^{\lambda t} \tag{3.26}$$

$$M_2(t) = x + p(t); \; (t) = \bar{p}e^{\lambda t} \tag{3.27}$$

$$P_1(t) = y + q(t); \; q(t) = \bar{q}e^{\lambda t} \tag{3.28}$$

$$P_2(t) = z + r(t);\ r(t) = \bar{r}e^{\lambda t} \tag{3.29}$$

where $\bar{a}, \bar{f}, \bar{g}, \bar{p}, \bar{q}$and $\bar{r}$ are constants.

Substituting (3.24) to (3.29) into the model equations (3.1) to (3.6), i.e.

from (3.1), $\frac{dG_1}{dt} = (\beta_1 - \mu_1)G_1 - \alpha_1 G_1(G_2 + M_2 + P_2)$

$$\frac{d(u + \bar{a}e^{t\lambda})}{dt} = (\beta_1 - \mu_1)(u + \bar{a}e^{t\lambda})$$

$$- \alpha_1(u + \bar{a}e^{t\lambda})\left((v + \bar{f}e^{t\lambda}) + (x + \bar{p}e^{t\lambda}) + (z+\bar{r}e^{t\lambda})\right) \quad \mathbf{(3.30)}$$

$$= \beta_1 u + \beta_1 \bar{a}e^{t\lambda} - \mu_1 u - \mu_1 \bar{a}e^{t\lambda} - \alpha_1(u + \bar{a}e^{t\lambda})(v + \bar{f}e^{t\lambda})$$

$$- \alpha_1(u + \bar{a}e^{t\lambda})(x + \bar{p}e^{t\lambda}) - \alpha_1(u + \bar{a}e^{t\lambda})(z+\bar{r}e^{t\lambda})$$

$$\lambda\bar{a}e^{t\lambda} = (\beta_1 - \mu_1)u + (\beta_1 - \mu_1)\bar{a}e^{t\lambda}$$

$$- \alpha_1(uv + u\bar{f}e^{t\lambda} + v\bar{a}e^{t\lambda} + \bar{a}\bar{f}e^{2t\lambda})$$

$$- \alpha_1(ux + u\bar{p}e^{t\lambda} + x\bar{a}e^{t\lambda} + \bar{a}\,\bar{p}e^{2t\lambda}) - \alpha_1(uz + u\bar{r}e^{t\lambda}$$

$$+ z\bar{a}e^{t\lambda} + \bar{a}\bar{r}e^{2t\lambda})$$

$$= (\beta_1 - \mu_1)u + \beta_1\bar{a}e^{t\lambda} - \mu_1\bar{a}e^{t\lambda} - \alpha_1 uv - \alpha_1 u\bar{f}e^{t\lambda} - \alpha_1 v\bar{a}e^{t\lambda}$$

$$- \alpha_1\bar{a}\bar{f}e^{2t\lambda} - \alpha_1 ux - \alpha_1 u\bar{p}e^{t\lambda} - \alpha_1 x\bar{a}e^{t\lambda} - \alpha_1\bar{a}\,\bar{p}e^{2t\lambda}$$

$$- \alpha_1 uz - \alpha_1 u\bar{r}e^{t\lambda} - \alpha_1 z\bar{a}e^{t\lambda} - \alpha_1\bar{a}\bar{r}e^{2t\lambda}$$

$$= (\beta_1 - \mu_1)u - \alpha_1 u(v + x + z) + \beta_1 \bar{a} e^{t\lambda}$$

$$- \mu_1 \bar{a} e^{t\lambda} - \alpha_1 u \bar{f} e^{t\lambda} - \alpha_1 v \bar{a} e^{t\lambda} - \alpha_1 \bar{a} \bar{f} e^{2t\lambda} - \alpha_1 u \bar{p} e^{t\lambda}$$

$$- \alpha_1 x \bar{a} e^{t\lambda} - \alpha_1 \bar{a} \bar{p} e^{2t\lambda} - \alpha_1 u \bar{r} e^{t\lambda} - \alpha_1 z \bar{a} e^{t\lambda} - \alpha_1 \bar{a} \bar{r} e^{2t\lambda}$$

Neglecting the terms of order 2 in λt and using (3.7) gives

$$\lambda \bar{a} e^{t\lambda} = \beta_1 \bar{a} e^{t\lambda} - \mu_1 \bar{a} e^{t\lambda} - \alpha_1 u \bar{f} e^{t\lambda} - \alpha_1 v \bar{a} e^{t\lambda} - \alpha_1 u \bar{p} e^{t\lambda} - \alpha_1 x \bar{a} e^{t\lambda}$$

$$- \alpha_1 u \bar{r} e^{t\lambda} - \alpha_1 z \bar{a} e^{t\lambda}$$

dividing through by $e^{t\lambda}$gives

$$\lambda \bar{a} = \beta_1 \bar{a} - \mu_1 \bar{a} - \alpha_1 v \bar{a} - \alpha_1 x \bar{a} - \alpha_1 z \bar{a} - \alpha_1 u \bar{f} - \alpha_1 u \bar{p} - \alpha_1 u \bar{r}$$

$$(\beta_1 - \mu_1 - \alpha_1(v + x + z) - \lambda)\bar{a} - \alpha_1 u \bar{f} - \alpha_1 u \bar{p} - \alpha_1 u \bar{r} = 0 \quad (3.31)$$

From (3.2), $\frac{dG_2}{dt} = \alpha_1 G_1(G_2 + M_2 + P_2) + (\beta_1 - \mu_1 - \delta_1)G_2$

$$\frac{d(v+\bar{f}e^{t\lambda})}{dt} = \alpha_1\left(u + \bar{a}e^{t\lambda}\right)\left(\left(v + \bar{f}e^{t\lambda}\right) + \left(x + \bar{p}e^{t\lambda}\right) + \left(z + \bar{r}e^{t\lambda}\right)\right) + (\beta_1 - \mu_1 - \delta_1) \quad (3.32)$$

$$\lambda \bar{f} e^{t\lambda} = \alpha_1\left(u + \bar{a}e^{t\lambda}\right)\left(v + \bar{f}e^{t\lambda}\right) + \alpha_1\left(u + \bar{a}e^{t\lambda}\right)\left(x + \bar{p}e^{t\lambda}\right)$$

$$+ \alpha_1\left(u + \bar{a}e^{t\lambda}\right)\left(z + \bar{r}e^{t\lambda}\right) + (\beta_1 - \mu_1 - \delta_1)\left(v + \bar{f}e^{t\lambda}\right)$$

$$= \alpha_1\left(uv + u\bar{f}e^{t\lambda} + v\bar{a}e^{t\lambda} + \bar{a}\bar{f}e^{2t\lambda}\right)$$
$$+ \alpha_1\left(ux + u\bar{p}e^{t\lambda} + x\bar{a}e^{t\lambda} + \bar{a}\bar{p}e^{2t\lambda}\right)$$
$$+ \alpha_1\left(uz + u\bar{r}e^{t\lambda} + z\bar{a}e^{t\lambda} + \bar{a}\bar{r}e^{2t\lambda}\right) + \beta_1 v + \beta_1\bar{f}e^{t\lambda} - \mu_1 v$$
$$- \mu_1\bar{f}e^{t\lambda} - \delta_1 v - \delta_1\bar{f}e^{t\lambda}$$

$$= \alpha_1 uv + \alpha_1 u\bar{f}e^{t\lambda} + \alpha_1 v\bar{a}e^{t\lambda} + \alpha_1\bar{a}\bar{f}e^{2t\lambda} + \alpha_1 ux + \alpha_1 u\bar{p}e^{t\lambda}$$
$$+ \alpha_1 x\bar{a}e^{t\lambda} + \alpha_1\bar{a}\bar{p}e^{2t\lambda} + \alpha_1 uz + \alpha_1 u\bar{r}e^{t\lambda} + \alpha_1 z\bar{a}e^{t\lambda}$$
$$+ \alpha_1\bar{a}\bar{r}e^{2t\lambda} + \beta_1 v + \beta_1\bar{f}e^{t\lambda} - \mu_1 v - \mu_1\bar{f}e^{t\lambda} - \delta_1 v - \delta_1\bar{f}e^{t\lambda}$$

$$= (\beta_1 - \mu_1 - \delta_1)v + (v + x + z)\alpha_1 u + \alpha_1 v\bar{a}e^{t\lambda} + \alpha_1 x\bar{a}e^{t\lambda} + \alpha_1 z\bar{a}e^{t\lambda}$$
$$+ \alpha_1\bar{a}\bar{f}e^{2t\lambda} + \alpha_1 u\bar{p}e^{t\lambda} + \alpha_1 u\bar{f}e^{t\lambda} + \alpha_1\bar{a}\bar{p}e^{2t\lambda} + \alpha_1 u\bar{r}e^{t\lambda}$$
$$+ \alpha_1\bar{a}\bar{r}e^{2t\lambda} + \beta_1\bar{f}e^{t\lambda} - \mu_1\bar{f}e^{t\lambda} - \delta_1\bar{f}e^{t\lambda}$$

Neglecting the term of other 2 in λt and using equation (3.8) gives

$$\lambda\dot{f}e^{t\lambda} = \alpha_1 v\bar{a}e^{t\lambda} + \alpha_1 x\bar{a}e^{t\lambda} + \alpha_1 z\bar{a}e^{t\lambda} + \alpha_1 u\bar{f}e^{t\lambda} + \beta_1\bar{f}e^{t\lambda} - \delta_1\bar{f}e^{t\lambda}$$
$$- \mu_1\bar{f}e^{t\lambda} + \alpha_1 u\bar{p}e^{t\lambda} + \alpha_1 u\bar{r}e^{t\lambda}$$

$$\lambda\dot{f} = \alpha_1 v\bar{a} + \alpha_1 x\bar{a} + \alpha_1 z\bar{a} + \alpha_1 u\bar{f} + \beta_1\bar{f} - \mu_1\bar{f} - \delta_1\bar{f} + \alpha_1 u\bar{p} + \alpha_1 u\bar{r}$$

$$= (\alpha_1 u + \beta_1 - \delta_1 - \mu_1 + \lambda)\dot{f} + \alpha_1(v + x + z)\bar{a} + \alpha_1 u\bar{p} + \alpha_1 u\bar{r}$$

$$\alpha_1(v+x+z)\bar{a}+(\alpha_1 u+\beta_1-\delta_1-\mu_1-\lambda\bar{f})+\alpha_1 u\bar{p}+\alpha_1 u\bar{r}$$

$$=0 \quad (3.33)$$

From(3.3), $\frac{dM_1}{dt}=-\alpha_2 M_1 M_2+(\beta_2-\mu_2)M_1$

$$\frac{d(w+\bar{g}e^{t\lambda})}{dt}=-\alpha_2(w+\bar{g}e^{t\lambda})(x+\bar{p}e^{t\lambda})+(\beta_2-\mu_2)(w+\bar{g}e^{t\lambda}) \quad (3.34)$$

$$=-\alpha_2(wx+w\bar{p}e^{t\lambda}+x\bar{g}e^{t\lambda}+\bar{g}\bar{p}e^{2t\lambda})$$

$$+(\beta_2 w+\beta_2\bar{g}e^{t\lambda}-\mu_2 w-\mu_2\bar{g}e^{t\lambda})$$

$$\lambda\bar{g}e^{t\lambda}=-\alpha_2 wx-\alpha_2 w\bar{p}e^{t\lambda}-\alpha_2 x\bar{g}e^{t\lambda}-\alpha_2\bar{g}\bar{p}e^{2t\lambda}+(\beta_2-\mu_2)w+(\beta_2-\mu_2)\bar{g}e^{t\lambda}$$

neglecting the term of order 2 in λt and using equation (3.9) gives

$$\lambda\bar{p}e^{t\lambda}=(\beta_2-\mu_2)\bar{g}e^{t\lambda}-\alpha_2 w\bar{p}e^{t\lambda}-\alpha_2 x\bar{g}e^{t\lambda}$$

Dividing through by $e^{t\lambda}$ gives

$$\lambda p=(\beta_2-\mu_2)\bar{g}-\alpha_2 w\bar{p}-\alpha_2 x\bar{g}$$

$$(\beta_2-\mu_2-\alpha_2 x-\lambda)\bar{g}-\alpha_2 w\bar{p}$$

$$=0 \quad (3.35)$$

Also from (3.4),i.e. $\frac{dM_2}{dt} = \alpha_2 M_1 M_2 + (\beta_2 - \mu_2 - \delta_2)$

$$\frac{d(x + \bar{p}e^{t\lambda})}{dt} = \alpha_2(w + \bar{g}e^{t\lambda})(x + \bar{p}e^{t\lambda})$$

$$+ (\beta_2 - \mu_2 - \delta_2)(x + \bar{p}e^{t\lambda}) \quad (3.36)$$

$$= \alpha_2(wx + w\bar{p}e^{t\lambda} + x\bar{g}e^{t\lambda} + \bar{g}\bar{p}e^{2t\lambda})$$

$$+ (\beta_2 x + \beta_2 \bar{p}e^{t\lambda} - \mu_2 x - \mu_2 \bar{p}e^{t\lambda} - \delta_2 x - \delta_2 \bar{p}e^{t\lambda})$$

$\lambda\bar{p}e^{t\lambda} = \alpha_2 wx + \alpha_2 w\bar{p}e^{t\lambda} + \alpha_2 x\bar{g}e^{t\lambda} + \alpha_2 \bar{g}\bar{p}e^{2t\lambda} + \beta_2 x + \beta_2 \bar{p}e^{t\lambda} -$
$\mu_2 x - \mu_2 \bar{p}e^{t\lambda} - \delta_2 x - \delta_2 \bar{p}e^{t\lambda}$

$\lambda\bar{p}e^{t\lambda} = \alpha_2 wx + (\beta_2 - \mu_2 - \delta_2)x + \alpha_2 w\bar{p}e^{t\lambda} + \alpha_2 x\bar{g}e^{t\lambda} + \alpha_2 \bar{g}\bar{p}e^{2t\lambda} +$
$\beta_2 \bar{p}e^{t\lambda} - \mu_2 \bar{p}e^{t\lambda} - \delta_2 \bar{p}e^{t\lambda}$

Neglecting the term of order 2 in $t\lambda$ and using equation (3.10) gives

$$\lambda\bar{p}e^{t\lambda} = \alpha_2 x\bar{g}e^{t\lambda} + \alpha_2 w\bar{p}e^{t\lambda} + \beta_2 \bar{p}e^{t\lambda} - \mu_2 \bar{p}e^{t\lambda} - \delta_2 \bar{p}e^{t\lambda}$$

Dividing through by $e^{t\lambda}$ gives

$$\lambda\bar{p} = \alpha_2 x\bar{g} + (\alpha_2 w + \beta_2 - \mu_2 - \delta_2)$$

$$\bar{p}\alpha_2 x\bar{g} + (\alpha_2 w + \beta_2 - \mu_2 - \delta_2)\bar{p} = 0 \quad (3.37)$$

From (3.5),i.e. $\frac{dP_1}{dt} = -\alpha_3 P_1 G_2 + (\beta_3 - \mu_3)P_1$

$$\frac{d(y + \bar{q}e^{t\lambda})}{dt} = -\alpha_3(y + \bar{q}e^{t\lambda})(v + \bar{f}e^{t\lambda}) + (\beta_3 - \mu_3)(y + \bar{q}e^{t\lambda}) \quad (3.38)$$

$$= -\alpha_3(yv + y\bar{f}e^{t\lambda} + v\bar{q}e^{t\lambda} + \bar{f}\bar{q}e^{2t\lambda})$$

$$+ (\beta_3 y + \beta_3\bar{q}e^{t\lambda} - \mu_3 y - \mu_3\bar{q}e^{t\lambda})$$

$$\lambda\bar{q}e^{t\lambda} = -\alpha_3 yv - \alpha_3 y\bar{f}e^{t\lambda} - \alpha_3 v\bar{q}e^{t\lambda} - \alpha_3\bar{f}\bar{q}e^{2t\lambda} + \beta_3 y + \beta_3\bar{q}e^{t\lambda}$$

$$- \mu_3 y - \mu_3\bar{q}e^{t\lambda}$$

$$\lambda\bar{q}e^{t\lambda} = -\alpha_3 yv + (\beta_3 - \mu_3)y - \alpha_3 y\bar{f}e^{t\lambda} - \alpha_3 v\bar{q}e^{t\lambda} + \beta_3\bar{q}e^{t\lambda} - \mu_3\bar{q}e^{t\lambda}$$

$$- \alpha_3\bar{f}\bar{q}e^{2t\lambda}$$

neglecting the term of order 2 in λt and using equation (3.11) gives,

$$\lambda\bar{q}e^{t\lambda} = -\alpha_3 y\bar{f}e^{t\lambda} - \alpha_3 v\bar{q}e^{t\lambda} + \beta_3\bar{q}e^{t\lambda} - \mu_3\bar{q}e^{t\lambda}$$

$$\lambda\bar{q}e^{t\lambda} = -\alpha_3 y\bar{f}e^{t\lambda} - \alpha_3 v\bar{q}e^{t\lambda} + (\beta_3 - \mu_3)\bar{q}e^{t\lambda}$$

Dividing through by $e^{t\lambda}$ gives

$$\lambda\bar{q} = -\alpha_3 y\bar{f} - \alpha_3 v\bar{q} + (\beta_3 - \mu_3)\bar{q}$$

$$-\alpha_3 y\bar{f} + (\beta_3 - \mu_3 - \alpha_3 v - \lambda)\bar{q} = 0 \quad (3.39)$$

And from (3.6), i.e. $\frac{dP_2}{dt} = \alpha_3 P_1 G_2 + (\beta_3 - \mu_3 - \delta_3)P_2$

$$\frac{d(z + \bar{r}e^{t\lambda})}{dt} = \alpha_3(y + \bar{q}e^{t\lambda})(v + \bar{f}e^{t\lambda})$$

$$+ (\beta_3 - \mu_3 - \delta_3)(z + \bar{r}e^{t\lambda}) \quad (3.40)$$

$$= \alpha_3(yv + y\bar{f}e^{t\lambda} + v\bar{q}e^{t\lambda} + \bar{f}\bar{q}e^{2t\lambda})$$

$$+ (\beta_3 z + \beta_3\bar{r}e^{t\lambda} - \mu_3 z - \mu_3\bar{r}e^{t\lambda} - \delta_3 z - \delta_3\bar{r}e^{t\lambda})$$

$$\lambda\bar{r}e^{t\lambda} = \alpha_3 yv + \alpha_3 y\bar{f}e^{t\lambda} + \alpha_3 v\bar{q}e^{t\lambda} + \alpha_3\bar{f}\bar{q}e^{2t\lambda} + \beta_3 z + \beta_3\bar{r}e^{t\lambda} - \mu_3 z$$

$$- \mu_3\bar{r}e^{t\lambda} - \delta_3 z - \delta_3\bar{r}e^{t\lambda}$$

$$\lambda\bar{r}e^{t\lambda} = \alpha_3 yv + (\beta_3 - \mu_3 - \delta_3)z + \alpha_3 y\bar{f}e^{t\lambda} + \alpha_3 v\bar{q}e^{t\lambda} + \beta_3\bar{r}e^{t\lambda}$$

$$- \mu_3\,\bar{r}e^{t\lambda} - \delta_3\bar{r}e^{t\lambda} + \alpha_3\bar{f}\bar{q}e^{2t\lambda}$$

neglecting the term of order 2 in λt and using equation (3.12) gives,

$$\lambda\bar{r}e^{t\lambda} = \alpha_3 y\bar{f}e^{t\lambda} + \alpha_3 v\bar{q}e^{t\lambda} + \beta_3\bar{r}e^{t\lambda} - \mu_3\bar{r}e^{t\lambda} - \delta_3\bar{r}e^{t\lambda}$$

Dividing through by $e^{t\lambda}$ gives

$$\lambda\bar{r} = \alpha_3 y\bar{f} + \alpha_3 v\bar{q} + (\beta_3 - \mu_3 - \delta_3)\bar{r}$$

$$\alpha_3 y\bar{f} + \alpha_3 v\bar{q} + (\beta_3 - \mu_3 - \delta_3 - \lambda)\bar{r} = 0 \quad (3.41)$$

Where the coefficients $\bar{a}, \bar{f}, \bar{g}, \bar{p}, \bar{q}$ *and* $\bar{r}$ in the equations give the Jacobian matrix for the model equation given by:

$$\begin{vmatrix} \beta_1-\mu_1-\alpha_1(v+x+z)-\lambda & -\alpha_1 u & 0 & -\alpha_1 u & 0 & -\alpha_1 u \\ \alpha_1(v+x+z) & \alpha_1 u+\beta_1-\mu_1-\delta_1-\lambda & 0 & \alpha_1 u & 0 & \alpha_1 u \\ 0 & 0 & -\alpha_2 u+\beta_2-\mu_2-\lambda & -\alpha_2 w & 0 & 0 \\ 0 & 0 & \alpha_2 x & \alpha_2 w+\beta_2-\mu_2-\delta_2-\lambda & 0 & 0 \\ 0 & -\alpha_3 y & 0 & 0 & -\alpha_3 v+\beta_3-\mu_3-\lambda & 0 \\ 0 & \alpha_3 y & 0 & 0 & \alpha_3 v & \beta_3-\mu_3-\delta_3-\lambda \end{vmatrix} = 0 \qquad (3.42)$$

with λ as the eigenvalue. The matrix gives the characteristics

equation. Expanding the determinant (3.42) gives the characteristics equation as a pair of one quadratic equations and one quartic equations given by (3.43) and (3.44).

$$[\beta_2 - \mu_2 - \alpha_2 x - \lambda][\beta_2 - \mu_2 - \delta_2 + \alpha_2 x - \lambda] + \alpha_2^2 xw = 0 \qquad (3.43)$$

$$\{\alpha_1 u(\beta_1 - \mu_1 - \alpha_1(v + x + z) - \lambda) - \alpha_1^2 u(v + x + z)\}$$

$$\{\alpha_3 y(\beta_3 - \mu_3 - \alpha_3 v - \lambda) + \alpha_3^2 vy\} + (\beta_3 - \mu_3 - \alpha_3 v - \lambda)$$

$$(\beta_3 - \mu_3 - \delta_3 - \lambda)\{(\beta_1 - \mu_1 - \alpha_1(v + x + z) - \lambda)\}$$

$$(\beta_1 - \mu_1 - \alpha_1 u - \lambda) + \alpha_1^2 u(v + x + z) = 0 \qquad (3.44)$$

CHAPTER FOUR

STABILITY ANALYSIS OF EQUILIBRIUM STATES

4.1 Stability of the Zero Equilibrium States

At the zero equilibrium state, $(u, v, w, x, y, z) = (0,0,0,0,0,0)$

then the characteristics equations (3.44) and (3.45) give

$$(\beta_2 - \mu_2 - \lambda)(\beta_2 - \mu_2 - \delta_2 - \lambda) = 0 \quad (4.1)$$

$$(\beta_3 - \mu_3 - \lambda)(\beta_3 - \mu_3 - \delta_3 - \lambda)(\beta_1 - \mu_1 - \lambda)(\beta_1 - \mu_1 - \delta_1 - \lambda) = 0 \quad (4.2)$$

From (4.1) and (4.2) we have:

$$(\beta_1 - \mu_1 - \lambda) = 0; \ \lambda_1 = \beta_1 - \mu_1$$

$$(\beta_1 - \mu_1 - \delta_1 - \lambda) = 0; \ \lambda_2 = \beta_1 - \mu_1 - \delta_1$$

$$(\beta_2 - \mu_2 - \lambda) = 0; \ \lambda_3 = \beta_2 - \mu_2$$

$$\beta_2 - \mu_2 - \delta_2 - \lambda = 0; \ \lambda_4 = \beta_2 - \mu_2 - \delta_2$$

$$(\beta_3 - \mu_3 - \lambda) = 0; \ \lambda_5 = \beta_3 - \mu_3$$

$$(\beta_3 - \mu_3 - \delta_3 - \lambda) = 0; \ \lambda_6 = \beta_3 - \mu_3 - \delta_3$$

$$\lambda_1 = \beta_1 - \mu_1; \lambda_2 = \beta_1 - \mu_1 - \delta_1; \ \lambda_3 = \beta_2 - \mu_2; \lambda_4 = \beta_2 - \mu_2 - \delta_2;$$

$$\lambda_5 = \beta_3 - \mu_3; \ \lambda_6 = \beta_3 - \mu_3 - \delta_3 \qquad (4.3)$$

By the theory of stability, the origin is stable only if

$$\beta_1 < \mu_1 \qquad (4.4)$$

$$\beta_1 < \mu_1 + \delta_1 \qquad (4.5)$$

$$\beta_2 < \mu_2 \qquad (4.6)$$

$$\beta_2 < \mu_2 + \delta_2 \qquad (4.7)$$

$$\beta_3 < \mu_3 \qquad (4.8)$$

$$\beta_3 < \beta_3 + \delta_3 \qquad (4.9)$$

We note that for the origin to be stable, it suffices for

$$\beta_j < \mu_j; j = 1,2,3$$

4.2 Stability of the Non-zero Equilibrium State

For the non-zero equilibrium state, it suffices to analyze (3.45), and we shall use the modified Bellman and Cooke theorem (1963). Let the equation (3.45) take the form

$H(\lambda) = 0$ (4.10)

We set $\lambda = iw;\ where\ i = \sqrt{-1}$

$$H(iw) = F(w) + iG(w) \tag{4.11}$$

$F(w)$ and $G(w)$ are respectively given by

$$F(w) = w^4 + \alpha_1\alpha_3 uy\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big)$$

$$(\beta_3 - \mu_3 - \alpha_3 v) + (\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)$$

$$\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big)(\beta_1 - \mu_1 - \delta_1 + \alpha_1 u) - \alpha_1 u$$

$$(\beta_3 - \mu_3 - \delta_3)\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big) - \alpha_1^2\alpha_3 uy$$

$$(v + x + z)(\beta_3 - \mu_3 - \alpha_3 v) - \alpha_1^2 u(v + x + z)$$

$$(\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3) - \alpha_1^2 u(v + x + z)$$

$$w^2 - \{(\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)$$

$$+ \big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big)(\beta_1 - \mu_1 - \delta_1 - \alpha_1 u)\} w^2$$

$$[(\beta_3 - \mu_3 - \alpha_3 v) + (\beta_3 - \mu_3 - \delta_3)]$$

$$\{(\beta_1 - \mu_1 - \alpha_1(v + x + z))$$

$$+ (\beta_1 - \mu_1 - \alpha_1(v + x + z))\}w^2 \qquad (4.12)$$

$$G(w)$$

$$= \{-\alpha_1\alpha_3 uy\,(\beta_1 - \mu_1$$

$$- \alpha_1(v + x + z))$$

$$- \alpha_1\alpha_3 uy(\beta_3 - \mu_3 - \alpha_3 v) + \alpha_1 u\,(\beta_3 - \mu_3 - \delta_3)$$

$$+ \alpha_1^2\alpha_3 uy(v + x + z) - \alpha_1^2 u(v + x + z)(\beta_3 - \mu_3 - \alpha_3 v)$$

$$- \alpha_1^2 u(v + x + z)(\beta_3 - \mu_3 - \delta_3)$$

$$- (\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)\,[(\beta_1 - \mu_1 - \alpha_1(v + x + z))$$

$$+ (\beta_1 - \mu_1 - \delta_1 - \alpha_1 u)] - (\beta_1 - \mu_1 - \alpha_1(v + x + z))(\beta_1 - \mu_1 - \delta_1$$

$$- \alpha_1 u)[(\beta_3 - \mu_3 - \alpha_3 v) + (\beta_3 - \mu_3 - \delta_3)]\}w$$

$$+ \{(\beta_3 - \mu_3 - \alpha_3 v) + (\beta_3 - \mu_3 - \delta_3) + (\beta_1 - \mu_1 - \alpha_1(v + x + z))$$

$$+ (\beta_1 - \mu_1 - \delta_1$$

$$- \alpha_1 u)\}w^3 \qquad (4.13)$$

From (4.12) and (4.13) we note that

$$F'(0) = 0\,; G(0) = 0 \qquad (4.14)$$

and

$$F(0) = \alpha_1\alpha_3 uv\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big)(\beta_3 - \mu_3 - \alpha_3 v)$$

$$+(\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big)(\beta_1 - \mu_1$$
$$- \delta_1 - \alpha_1 u)$$
$$- \alpha_1 u\,(\beta_3 - \mu_3 - \delta_3)\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big)$$

$$-\alpha_1^2\alpha_3 uy(v + x + z)(\beta_3 - \mu_3 - \alpha_3 v)$$
$$+ \alpha_1^2 u(v + x + z)(\beta_3 - \mu_3 - \alpha_3 v)$$

$$(\beta_3 - \mu_3 - \delta_3) \qquad (4.15)$$

While

$$G'(0) = -\alpha_1\alpha_3 uv\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big) - \alpha_3 uy(\beta_3 - \mu_3 - \alpha_3 v) + \alpha_1 u\,(\beta_3 - \mu_3 - \delta_3) + \alpha_1^2\alpha_3 uy(v + x + z) - \alpha_1^2 u(v + x + z)(\beta_3 - \mu_3 - \alpha_3 v) - \alpha_1^2 u(v + x + z)(\beta_3 - \mu_3 - \delta_3) - (\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)\big[\big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big) + (\beta_1 - \mu_1 - \delta_1 - \alpha_1 u)\big] - \big(\beta_1 - \mu_1 - \alpha_1(v + x + z)\big) + (\beta_1 - \mu_1 - \delta_1 - \alpha_1 u)[(\beta_3 - \mu_3 - \alpha_3 v) + (\beta_3 - \mu_3 - \delta_3)] \qquad (4.16)$$

For the local stability of the non-zero equilibrium state the inequality

$$F(0)G'(0) > 0 \quad (4.17)$$

needs to hold, using the condition of the Bellman and Cooke (1963). A

necessary and sufficient condition for (4.17) to hold is for

$$sgnF(0) = sgnG'(0) \quad (4.18)$$

A close look at the expression for F(0) and G′(0) shows that u is

predominant, thus we investigate the inequality (4.17) in the neighbourhood of $u = 0$, this gives

$$F(0)|_{u=0} = (\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)\left[\left(\beta_1 - \mu_1 - \alpha_1(v + x + z)\right) + (\beta_1 - \mu_1 - \delta_1)\right] \quad (4.19)$$

$$G'(0)|_{u=0} = -(\beta_3 - \mu_3 - \alpha_3 v)(\beta_3 - \mu_3 - \delta_3)\left[\left(\beta_1 - \mu_1 - \alpha_1(v + x + z)\right) + (\beta_1 - \mu_1 - \delta_1)\right] - \left(\beta_1 - \mu_1 - \alpha_1(v + x + z)\right)(\beta_1 - \mu_1 - \delta_1)\,[(\beta_3 - \mu_3 - \alpha_3 v) + (\beta_3 - \mu_3 - \delta_3)] \quad (4.20)$$

Indicating that

$$\operatorname{sgn} F(0) \neq \operatorname{sgn} G'(0) \quad (4.21)$$

In some neighbourhood of $u = 0$, hence the non-zero equilibrium state is not locally stable.

CHARPTER FIVE

CONCLUSSION AND RECOMENDATION

5.1 Conclusion

The analysis reveals that once the epidemics is introduced into the populations involved in the dynamics, the tendency for the wiping out of the populations is imminent; since the zero state is stable while the non – zero state is unstable.

The result lays credence to the current control measure whereby infected humans are quarantined while infected birds are wiped out and the environs cordoned and heavily fumigated.

5.2 Recommendations

1. From the Analysis the flu pandemic scenarios is life threatening, hence the need to adequately prepare against the occurrence .

2. Continued surveillance efforts both in humans and animal (avian) species must be stepped up and reinforced.

3. Other efforts could include the stockpiling of antiviral drugs and a greater commitment by governments to an early warning system.

The tendency of influenza viruses to undergo frequent and permanent antigenic changes necessitates constant monitoring of the global influenza situation and annual adjustments in the composition of influenza vaccines.

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APPENDIX I

THEOREM (BELLMAN AND COOKE,(1963)

Theorem: Let $\Delta(z) = p\ (z,e^{z})$ Where p(z,w) is a polynomial with principal term. Suppose Δ (iy), y∈|, is separated into its real and imaginary parts,

Δ (iy) = F(y) + iG(y). If all zeros of Δ (z) have negatives real parts, then the zeros of F(y) and G(y) are real, simple, alternate and

$$F(0)G'(0) - F'(0)G(0) > 0 \qquad (A)$$

For y∈|, conversely, all zeros of $\Delta(z)$ will be in the left half-plane provided that either of the following conditions is satisfied:

(i) All the zeros of F(y) and G(y) are real simple, and alternate and inequality (A) is satisfied for at least one y.

(ii) All the zeros of F(y) are real and, for each zero, Relation (A) is satisfied.

(iii) All the zeros of G(y) are real and, for each zero, Relation (A) is satisfied.

(iv) All the zeros of G(y) are real and, for each zero, Relation (A) is satisfied.