AN OPTIMAL 6–STEP IMPLICIT LINEAR MULTISTEP METHOD FOR INITIAL VALUE PROBLEMS

BY

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CERTIFICATION

This thesis titled "AN OPTIMAL SIX-STEP IMPLICIT LINEAR MULTISTEP METHOD FOR INITIAL VALUE PROBLEMS" by Abdulrahman Ndanusa, meets the regulations governing the award of the degree of Masters of Technology in Mathematics, Federal University of Technology Minna and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

TO

UMMU

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ABSTRACT

One of the major problems of numerical analysis is that of solving differential equations. A great many variety of methods have been developed which enable us to provide solutions to many differential equations, even to those that have defied solution analytically. In this research work, we examine existing processes, how they are derived, how they are proved mathematically and their limitations. Based on such analysis, we derive, through Taylor series expansion, a new 6-step implicit linear multistep method of order eight for solving initial value problems. By assigning suitable values to the free parameters involved, we develop three different schemes. For acceptability, the schemes so derived are tested for consistency and zero stability. Hence, their convergence is established. Also provided are examples of initial value problems solved with the new schemes.

CHAPTER ONE

INTRODUCTION

1.1 PREAMBLE

The use of simple arithmetic operations to find approximate solutions to complex problems constitutes the main goal of numerical analysis. And one of the major problems of numerical analysis is that of solving differential equations, which are just relationships involving an independent variables x, a dependent variable y, and one or more differential coefficients of y with respect to x. An example of differential equation will be

$$y'' + 2y' + xy = 0$$

Differential equations represent dynamic relationships; i.e. quantities that change, and are thus frequently occurring in scientific, engineering, as well as social problems. The solution of a differential equation thus provides solution to the physical problem it represents.

Traditionally, solutions to differential equations were derived using analytical, or exact methods. These solutions are often useful as they provide excellent insight into the behaviour of some systems. However, analytical solutions can be derived for only a limited class of problems. These include those that can be approximated with linear models and those that have simple geometry and low dimensionality. Consequently, analytical solutions are of limited practical value because most real life problems are nonlinear and involve complex shapes and processes.

In such cases, where a differential equation defies solution analytically, an approximate solution is often obtainable by the application of numerical methods. Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations. This means that the relevant particular solution is obtained as a set of function values for the range of values of the independent variable. This set of points is an approximation of exact solution at these points.

The goal of finding numerical solutions to differential equations is to get a method that will give an answer that will be (if possible) the same as the exact solution.

A broad variety of methods or schemes have been derived for solving differential equations. These methods can be classified into two thus: One-step and multistep methods.

One-step methods permit the calculation of y_{i+1}, given the differential equation and y_i. They utilize information at a single point x_i to predict a value of the dependent variable y_{i+1} at a future point x_{i+1}. There are two families of one-step methods: methods of Taylor and methods of Runge-Kutta. The methods of Taylor are further classified into Euler method and methods of Taylor of greater order. (Chapra and Canale ,1998).

Multistep methods require additional values of y other than at i. Multistep methods are based on the insight that, once the computation has begun, valuable information from previous points is at our disposal. The curvature of the lines connecting these previous values provides information regarding the trajectory of the solution. The multistep methods exploit this information to solve ordinary differential equations (Chapra and Canale, 1998).

Some famous sub-classes of multistep methods are, Adams-Moulton, Adams-Bashforth and Predictor-Corrector methods.

Although there are many kinds of numerical methods; they have one common characteristic: they invariably involve large numbers of tedious arithmetic calculations. In fact, it is a business of solving hard problems by doing lots of easy steps (Chapra and Canale,1998)

Various reasons determine the choice of one method over another, two obvious criteria being speed and accuracy. However, the advent of fast and efficient digital computers has increased dramatically the role of numerical methods in solving scientific, engineering as well as social problems. (Scheid, 1998).

The application of computer algorithm to implement a new 6-step implicit linear multistep method for initial value problems forms the basis for this project.

1.2 LITERATURE REVIEW

The family of multistep methods for solving initial value problems offers a wide range of methods which are further grouped into sub classes. A great many methods have been developed in this direction, and yet others are still being developed. Many have undergone changes either to improve their accuracies, or their error control strategies, or to shed more light on their behaviours in general.

DAHLQUIST (1956), made the first investigation that brings strict mathematical analysis to the problem of the convergence of numerical solutions to initial value problems, and ushered in a new era in the subject. In seeking the highest possible order thet can be achieved by a linear k-step method, the consistency condition is automatically achieved, but we come across what is known as the 'first Dahlquist barrier'; which arises in attempting to satisfy the 'root condition' (Lambert,1973). The highest order that can be attained by a linear k-step method is 2k if the method is implicit, and 2k-1 if it is explicit. Linear k-step methods achieving such orders are called MAXIMAL. However, maximal methods, in general, fail to satisfy the 'root condition' and are thus Zero-Unstable. The following theorem thus encapsulates the first Dahlquist barrier.

Theorem: No zero-stable linear k-step method can have order exceeding k+1 when k is odd and k+2 when k is even(Lambert, 1973).

HULL and CREAMER (1963), pointed out that for the convergence of predictor-corrector methods using a constant step size, an optimal step size should be small enough to ensure convergence within two iterations of the corrector. In addition, it must be small enough to yield a sufficiently small truncation error. At the same, the step size should be large as possible to minimize runtime cost and round-off error. As with other methods for ODES, the only practical way to assess the magnitude of the global error is to compare the results for the same problem but with a halved step size.

GEAR (1971), developed a special series of implicit schemes that have very large stability limits based on backward difference formulas. Extensive efforts have been made to develop software to efficiently implement Gear's methods. As a result, this is probably the most widely used method to solve stiff systems.

LAMBERT (1973), showed that the highest order we can expect from a linear k-step method is *2k* if the method is implicit, and *2k-1* if it is explicit.

ONUMANYI et al. (1981), developed a software for a method of finite approximations for the numerical solution of differential equations, which was based on the Tau method. According to them problems with complex initial boundary or mixed conditions involving combinations of function and derivative values, can be dealt with by means of their program. Consequently, encouraging results have been obtained in the solutions with regions of rapid variation, oscillatory behaviour and stiffness.

FATUNLA (1987), derived some new predictor-corrector formulas using an arbitrary step number *k*. According to him a matrix representation in the spirit of Gear was incorporated so as to facilitate variable step, variable order modes. Accordingly for even step numbers, the proposed algorithms are nearly symmetric and hence perform better than the Adams- Bashforth- Moulton predictor - corrector formulas on oscillatory initial value problems.

AWOYEMI (1994), derived a two-step method for the continuous solution of initial value problems for the second order differential equations without first derivative explicitly present. The method is based on collocation at the grid points and at one off-grid point x_{n+v} , (x_n, x_{n+t}). In addition, the Numerov's method of order four is recovered for any v , $(0,2)$ $v \neq 1$ *at* $x = x_{n+2}$.

SIRISENA and ONUMANYI (1994), developed a continuous formulation of a modified self-starting Numerov method for the second order differential equations. According to them, the uniformly accurate discrete schemes obtained by evaluating the continuous formula at certain points are solved simultaneously as block methods for a uniform treatment of initial and boundary value problems.

SIRISENA et al. (1996), attempted to seek other alternatives to the class of Adams-predictor-corrector method, two new families of linear multistep methods are developed, one explicit and the other implicit. They are based on collocation at arbitrarily selected mesh points and two point interpolation. The predictor schemes of step numbers $k= 2, 4, 6,\ldots$ are symmetric which is also a desirable property for extrapolation process. They have smaller error constants than the corresponding Adams-Bashforth methods, implying greater accuracy, though their intervals of absolute stability are located at the origin, due to symmetry. The corrector schemes on the other hand are competitive with Adams-Moulton methods for the higher order methods. The order of the proposed methods is $p=t+m-1$ where $t=2$ is the number of distinct interpolation points (x_n *and* x_{n+k-1}) used. The number of distinct collocation used is $m=k-1$ with collocation at $x = x_{n+1}, \ldots \ldots, x_{n+k-1}$ for the predictor while $m=k$ with collocation at $x= x_{n+1,\ldots\ldots\ldots} x_{n+k}$ for the correctors.

ONUMANYI et al. (1997), discussed the improvements in the multistep methods for solving ordinary differential equations (ODES). They stated that of recent, discrete multistep methods have been extended to continuous forms based on multistep collocation. According to them following the extension, continuous ones have more ability to solve the ODES than the discrete ones, that is, initial value problem (IVP) is solved without looking for any other method to start the integration process. They claimed that even the problem of overlap of solution models usually associated with multistep finite difference methods is overcome, and on the same fixed meshes the higher order methods can be applied successively by choosing different values of the step number.

CHAPRA and CANALE (1998), showed that if the predictor and the corrector of a multistep method are of the same order, the local truncation error, *Ec*, may be estimated during the course of a computation by

$$E_c = -\frac{y_{i+1}^{0} - y_{i+1}^{m}}{5} \tag{1.2}$$

As an example, using the corrector equation for the Hewn method,

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})h}{2} \tag{1.3}$$

Its local truncation error is analysed as

$$E_c = -\tfrac{1}{2}h^3 y^{(3)}(\xi_c) = -\tfrac{1}{2}h^3 f''(\xi_c) \tag{1.4}$$

Where the subscript *c* designates that this is the error of the corrector.

Also, the predictor for the non-self-starting Heun method is

$$y_{i+1} = y_{i-1} + 2hf(x_i, y_i) \tag{1.5}$$

Its local truncation error is taken as

$$E_p = \tfrac{1}{3}h^3 y^{(3)}(\xi_p) = \tfrac{1}{3}h^3 f''(\xi_p) \tag{1.6}$$

Where the subscript *p* designates that this is the error of the predictor.

Equation (1.6) above can be combined with the estimate of y_{i+1} from the predictor step (eqn.(1.5)) to yield:

$$True\ value = approximation + error$$

i.e.

$$True\ \ value = y_{i+1}^{0} + \tfrac{1}{3}h^{3}y^{(3)}(\xi_{p}) \qquad (1.7)$$

Using a similar approach, the error estimate for the corrector (eqn(1.4)) can be combined with the corrector result y_{i+1} to give

$$True\ value = approximation + error$$

i.e. ,

$$True\ \ value = y_{i+1}^{m} - \tfrac{1}{12}h^{3}y^{(3)}(\xi_{c}) \qquad (1.8)$$

Eqn (1.7) can be subtracted from eqn (1.8) to yield

$$0 = y_{i+1}^{m} - y_{i+1}^{0} - \tfrac{5}{12}h^{3}y^{(3)}(\xi) \qquad (1.9)$$

Where ξ is now between x_{i-1} and x_{i+1}. Now, dividing eqn (1.9) by 5 and rearranging the result gives

$$\frac{y_{i+1}^{0} - y_{i+1}^{m}}{5} = -\frac{1}{12}h^{3}y^{(3)}(\xi) \qquad (1.10)$$

The Right hand side of equations (1.4) and (1.10) are identical, with the exception of the argument of the third derivative. If the third derivative does not vary appreciably over the interval in question, we can assume that the right hand sides are equal, and therefore, the left hand sides should also be equivalent, as in

$$E_{c} = -\frac{y_{i+1}^{0} - y_{i+1}^{m}}{5}$$

CHAPRA and CANALE (1998), outlined two criteria that are typically used to decide whether a change in step size is warranted.

First, if eqn (1.2) is greater than some prespecified criterium, the step size is decreased. Second, the step size is chosen so that the convergence criterion of the corrector is satisfied in two iterations. This criterion is intended to account for the trade-off between the rate of convergence and the total number of steps in the calculation. For smaller values of *h*, convergence will be more rapid but more steps are required. For larger *h*, convergence is slower but few steps result. Experience (Hull and Creamer, 1963) suggests that the total steps will be minimized if *h* is chosen so that the corrector converges within two iterations. Therefore, if over two iterations are required, the step size is decreased, and if less than two iterations are required, the step size is increased.

CHAPRA AND CANALE (1998), outlined some software libraries and packages that have great capabilities for solving ODES and determining eigenvalues. According to them, the following are some of the ways the packages can be applied for this purpose.

EXCEL

Excel's direct capabilities for solving eigenvalue problems and ODES is limited. However, if some programming is done (e.g. macros), they can be combined with Excel's visualization and optimization tools to implement some interesting applications.

MATHCAD

Mathcad has a number of different functions that determine eigenvalues and eigenvectors and solve differential equations.

MATLAB

As might be expected the standard MATLAB package has excellent capabilities for determining eigenvalues and eigenvectors. However, it also has built in functions for solving ODE. The standard ODE solver5 include two functions to implement the adaptive stepsize Runge-Kutta Fehlberg method. These are ODE23, which uses second-and-third-order formula to attain medium accuracy,and ODE45, which uses fourth-and fifth-order formula to attain higher accuracy.

IMSL

IMSL has a variety of routines for solving ODES and determining eigenvalues. The IVPRK routine, for example, integrates a system of ODES using the Runge-kutta method.

1.3 NOMENCLATURE AND DEFINITIONS

1.3.1 INITIAL VALUE PROBLEM

A first order differential equation, $y' = f(t, y)$, together with an initial condition, $y(t_o)=y_o$, constitutes an initial value problem:

$$y' = f(t,y), \qquad y(t_0) = y_0, \qquad t > t_0 \qquad (1.11)$$

The following theorem, whose proof may be found in Henrici (1962), states condition on $f(t,y)$ which guarantee the existence of a unique solution of the IVP (1.11)

Theorem:

Let $f(t,y)$ be defined and continuous for all points (t,y) in the region D defined by $t_0 \le t \le t_N$, $-\infty < y < \infty$, t_0 and t_N finite, and let there exist a constant L such that, for every t, y, y^* such that (t, y) and (t, y^*) are both in D,

$$|f(t,y) - f(t,y^*)| \le L|y - y^*| \qquad (1.12)$$

Then, if y_o is any given number, there exists a unique solution $y(t)$ of the initial value problem (1.11), where $y(t)$ is continuous and differentiable for all (t,y) in D.

The requirement (1.12) is known as a Lipschitz condition, and the constant L as a Lipschitz constant .(Lambert,1973)

1.3.2 NUMERICAL SOLUTION

We wish to solve the standard initial value problem given by equation (1.11) above. Since analytical or exact solutions are not always possible to find, it is essential to work with techniques which work without them. One approach is the numerical analysis, which tries to find good algorithms to approximate solutions. This simply means finding procedures by which computers can do the solving for us. We seek a solution on the interval $[t_o, t_N]$ of t where t_o and t_N are finite. We assume that equation (1.11) has a unique solution. The t-axes is discretized over a finite interval $[t_o, t_N]$; that is, the continuous interval $[t_o, t_N]$ of t is replaced by the descrete point set, $\{t_n\}$, defined by $t_n=t_o+nh$, $n=0,1,2,\ldots\ldots,N=(b-a)/h$. The subdivision points t_n are often equally spaced, that is $t_n=t_o+nh$ where the parameter h called the step length or step size is defined by

$$h=\frac{t_N-t_0}{N}>0\,.$$

We let $y(t_n)$ denote the exact solution $y(t)$ of equation (1.11) at the point t_n and y_n to denote the numerical slution at t_n.

We seek to find a way of producing a sequence of values $\{y_n\}$ that approximates the solution of (1.11) on the discrete point set $\{t_n\}$; such a sequence constitutes a numerical solution of the IVP (1.11) (Lambert, 1991)

1.3.3 NUMERICAL METHOD

A numerical method can be defined as a difference equation that involves a number of consecutive approximations y_{n+j}, $j=0,1,\ldots,k$, from which if will be possible to compute sequentially the sequence $\{y_n/n=0,1,2,\ldots,N\}$. (Lambert,1991). Although numerical methods for IVPs can take many forms, all of them can be written in the general form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\phi_f\left(y_{n+k}, y_{n+k-1}, \cdots y_n, t_n; h\right) \qquad (1.13)$$

$$y_i = \mu_i(h), \qquad i = 0,1,\ldots,k-1 \qquad (1.14)$$

Where subscript f indicates that the dependence of ϕ on y_{n+k}, $y_{n+k-1}\ldots,y_n$; t_n is through the function $f(t,y)$, and $\{\mu_i\,(h)\}_{i=0,1,\ldots,k-1}$ are the initial points (Patrizia, 2001)

1.3.4 ORDER OF ACCURACY

To enable us quantify the order of accuracy of a numerical approximation we consider the Taylor series expansion of $y(t) \in C^r$ around t_n , i.e.

$$y_{n+1}^T = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \ldots + \frac{h^r}{r!}y^{(r)}(\xi) \qquad (1.15)$$

Where $\xi \in [t_n, t_{n+1}]$ and subtitute this into the numerical method under consideration . Let $f \in C^r$, i.e. of the same order of the solution . Then we say that a numerical method is r-th order accurate if the term in h^r in the Taylor expansion of the unknown is correctly reproduced .This is indicated by $O(h^r)$ .The oder r allows us to tell by how much the results are improved when the step is reduced . In general, one prefers methods with a large r , since a reduction of h promises a large gain in accuracy .

We note that the order r depends only on the method and not on the differential equation, provided f satisfies the assumption. An obvious property to require of any numerical method is that the approximation solution $\{y_n\}$, defined by equation (1.13), gives the exact solution over interval $[t_o, t_N]$. (Partrizia,2001)

1.3.5 CONVERGENCE

A numerical method is convergence if

$$\lim_{h \to 0} y_n = y(t)$$

for all t over the finite interval $[t_o, t_N]$, i.e. if the sequence of improved values converges to the true value of y. A method is not convergent is said to be DIVERGENT (Patrizia,2001).

1.3.3 LOCAL TRUNCATION ERROR

The leading order deviation is called LOCAL TRUNCATION ERROR at the *nth-step*

$$E_{n+k} = \sum_{j=0}^{k} \alpha_j y(t_{n+j}) - h\phi_f(y(t_{n+k}), y(t_{n+k-1}), \ldots, y(t_n), t_n; h) \qquad (1.16)$$

and the LOCAL UNIT TRUNCATION ERROR is given by

$$\theta_{n+k} = \frac{E_{n+k}}{h}, \qquad (1.17)$$

Starting at the exact value at t_n, i.e. if it is calculated with single- step of integration at the position (t_{n+kh}).

These errors are local (that is, error per step). The total error usually builds up as one moves away from the initial value y_o, and it is composed of the accumulated local truncation error and of machine error. This last one is due to finite numerical accuracy of the calculation, which depends on the computing and its method of rounding decimal (round-off error).

1.3.4 GLOBAL OR TOTAL TRUNCATION ERROR

The total truncation error is the difference between the solution $y(x_{n+1})$ and y_{n+1} (the solution calculated after $n+1$ steps):

$$e_{n+1} = \|y(x_{n+1}) - y_{n+1}\| \qquad (1.18)$$

The definition of convergence can be formulated using the total error: a numerical method is convergent if

$$\lim_{h \to 0} \max_{n=0,1,\ldots,N} \|e_n\| = 0 \qquad (1.19)$$

It is natural to expect that the error will accumulate steadily as integration proceeds, but that is not so. The stability of the problem can cause errors to decrease as well as to increase.

A first thought on the appropriate level of accuracy, that might be needed for convergence, is that $E_{n+1} \to 0$ as $h \to 0$. In-depth remarks show that this is not going to be enough. The appropriate level is to demand that $\theta_{n+k} \to 0$ as $h \to 0$.

1.3.5 CONSISTENCY

A numerical method is called CONSISTENT if the local truncation error satisfies:

$$\lim_{h \to 0} \theta_{n+k} = 0 \qquad (1.20)$$

The necessary and sufficient conditions, which must be satisfied by a numerical method (eqn 1.13) to be consistent are:

$$\sum_{j=0}^{k} \alpha_j = 0$$

and

$$\frac{\phi_f\left(y(t_{n+k}), y(t_{n+k-1}), \ldots, y(t_n), t_n; 0\right)}{\sum_{j=0}^{k} j\alpha_j} = f(y(t_n)) \qquad (1.22)$$

which using the first characteristic polynomial $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \quad \xi \in \mathbb{C}$, it is possible to rewrite the two equalities in the usual form

$$\rho(1) = 0 \qquad (1.23)$$

$$\frac{\phi_f\left(y(t_{n+k}), y(t_{n+k-1}), \ldots, y(t_n), t_n; 0\right)}{\rho'(1)} = f(y(t_n)) \qquad (1.24)$$

(Patrizia,2001)

Specific condition for single class of methods are being found.

For linear multistep methods, consistency demands that

(i) $\rho(1) = 0$

(ii) $\rho'(1) = \sigma(1)$; *where* $\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$

For general Range-Kutta methods, consistency demands that

$$\sum_{i=1}^{s} b_i = 1$$

(Lamert,1991)

We apply the above conditions to the following examples of numerical methods:

1. $y_{n+2} + y_{n+1} - 2y_n = \frac{h}{4}[f(t_{n+2}, y_{n+2}) + 8f(t_{n+1}, y_{n+1}) + 3f(t_n, y_n)]$
2. $y_{n+2} - y_{n+1} = \frac{h}{3}[3f(t_{n+1}, y_{n+1}) - 2f(t_n, y_n)]$
3. $y_{n+1} - y_n = \frac{h}{4}(k_1 + 3k_3)$

where

$k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1)$

$k_3 = f(t_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_2)$

4. $y_{n+1} - y_n = \frac{h}{2}(k_1 + k_2)$

where

$k_1 = f(t_n, y_n)$

$k_2 = f(t_n + h, y_n + \frac{1}{2}hk_1 + \frac{1}{2}hk_2)$

It is easily seen that condition (1.23) is satisfied for all of them. It is straightforward to see that (1.24) holds for example (1). For example (2), we have,

$$\rho(\xi) = \xi^2 - \xi, \qquad \textit{where} \quad \rho'(1) = 1 \quad \textit{and}$$

$$\frac{\phi_f(y(t_n), y(t_n), \ldots, y(t_n), t_n; 0)}{\rho'(1)} = \frac{1}{3} f(t_n, y(t_n))$$

and the method is inconsistent. If it is applied to the initial value problem (1.11) it will attempt to solve instead the problem $y' = \frac{1}{3}f(t, y), \ y(t_0) = y_0$. For examples (3) and (4), it is clear that when $h=0$ and y_n is replaced by $y(t_n)$, each of the k_i reduces to $f(t_n, y(t_n))$, and (1.24) is satisfied. Thus all of the examples except example (2) are consistent. (Lambert,1991)

1.3.9 STABILITY

A numerical method is said to be STABLE if a small deviation from the true solution does not tend to grow as the solution is iterated.

We consider a form of stability, which is concerned with the stability of the difference system in the limit as h tends to zero.

Let $\{\delta_n, n=0,1,\ldots,N\}$ and $\{\delta_n^*, n=0,1,\ldots,N\}$ be any two perturbations of the discretized Problem (i.e. difference equation generated by the method) and let $\{y_n, n=0,1,..,N\}$ and $\{y_n^*, n=0,1,\ldots N\}$ be the resulting perturbed solutions. Then if there exist positive constants S and h_o such that, for all $h \in (0, h_o]$

$$\|y_n - y_n^*\| \le S_\varepsilon \quad whenever \quad \|\delta_n - \delta_n^*\| \le \varepsilon, \quad 0 \le n \le N \qquad (1.25)$$

then the method is said to be ZERO-STABLE. (Lambert,1991)

It is a characteristic of non-zero stable methods, that decreasing the step-size actually makes matters worse, i.e. the error grows at an increasing pace. Any error due to discretization and round-ff could be interpreted as being equivalent to perturbing the problem. The zero-stability is thus a requirement that the difference system be likewise insensitive to perturbations. If the inequality (1.25) is not satisfied, then no acceptable solution will be producd.

An algebraic alternative definition to the zero-stability is given using the ROOT CONDITION.

A numerical method is said to satisfy the ROOT CONDITION if all of the roots of the first characteristic polynomial have modulol ess than or equal to unity, and those modulo unity are simple. (Lambert,1991).

Thus the necessary and sufficient condition for a numerical method to be zero-stable is that it satisfies the root condition. (Lambert,1991)

The following linear multistep method is used to illustrate zero-stability:

$$y_{n+2}-(1+a)y_{n+1}+ay_n=\tfrac{1}{2}h[(3-a)f_{n+1}-(1+a)f_n]$$

The first characteristic polynomial for this method is:

$$\rho(\xi)=\xi^2-(1+a)\xi+a=(\xi-1)(\xi-a)$$

Thus, when $a=0$ the method is zero-stable and when $a=-5$ it is zero-unstable (Lambert,1991)

From the above facts, we conclude that the necessary and sufficient condition for a numerical method to be convergent are that it be consistent and zero stable. That is, it must satisfy the following conditions:

(i) $\rho(1)=0$

(ii) $\dfrac{\phi_f(y(t_{n+k}),y(t_{n+k-1}),\ldots,y(t_n),t_n;0)}{\rho'(1)}=f(y(t_n))$

(iii) No root of the equation: $\rho(\xi)=0$, has modulo greater than 1, and every root with modulo is simple.

here $\rho(\xi)=\sum_{j=0}^{k}\alpha_j\xi^j$

elow are some examples of convergent and non convergent schemes:

. $y_{n+2}+y_{n+1}-2y_n=\frac{h}{4}\left[f(t_{n+2},y_{n+2})+8f(t_{n+1},y_{n+1})+3f(t_n,y_n)\right]$

or this method $\rho(1)=0$; $\rho^{\prime}(1)=\Phi(1)=3$. Thus satisfying conditions (*i*) and (*ii*). The roots of

$\rho(>)$ are $> = 1, -2$; which does not satisfy condition (*iii*) and therefore Zero unstable. Hence,

he method is divergent.

2. $y_{n+2}-y_{n+1}=\frac{h}{3}\left[3f(t_{n+1},y_{n+1})-2f(t_n,y_n)\right]$

Here, $\rho(1)=0$; $\rho^{\prime}(1)=1$, $\Phi(1)=1/3$. Hence, inconsistent (not satisfying condition (*ii*)). The

oots of $\rho(>)$ satisfy condition (*iii*). Thereby making it zero-stable. Therefore, it is divergent.

$y_{n+1}-y_n=\frac{h}{4}(k_1+3k_3)$

here

$$k_1=f(t_n,y_n)$$
$$k_2=f(t_n+\tfrac{1}{3}h,\ y_n+\tfrac{1}{3}hk_1)$$
$$k_3=f(t_n+\tfrac{2}{3}h,\ y_n+\tfrac{2}{3}hk_2)$$

or this method, $\sum_{i=1}^{s}b_i=\frac{1}{4}(1+3)=\frac{4}{4}=1$. Thereby satisfying conditions (*i*) and (*ii*). The root of

$\rho(>)=1$; also satisfying condition (*iii*). This makes the method both consistent and zero-stable.

herefore it is convergent.

.4 AIM AND OBJECTIVES

he aim of this study is to derive an optimal 6-step implicit LMM for the solution of IVPs.

he objectives of the study include the following:

To derive a convergent 6-step implicit linear multistep method, which is optimal.

i. To verify the accuracy of the method by making comparison with the exact solutions and

nown methods of similar steps.

ii. To use the method to solve some differential equations.

CHAPTER TWO

NUMERICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

2.1 NUMERICAL METHODS FOR INITIAL VALUE PROBLEMS (IVPS)

In the preceding chapter, we made an introduction into what numerical methods for solving IVPs are all about. A great many of such methods have been developed, and yet many more are still being produced. Although all the methods have certain fundamental properties common to them all of them are further classified into different sub-classes, with specific characteristics peculiar to each class. It is this classification of numerical methods we shall discuss in this chapter.

2.2 ONE – STEP METHODS

One – step methods are numerical methods that determine the solution at the support times through the recursive formula

$$y_{n+1} = y_n + h\phi(t_{n,} y_n ; h) , \qquad n \in N \qquad (2.1)$$

i.e. $k = 1$ *in the formula*

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\phi_f(y_{n+k}, y_{n+k-1}, \dots, y_n, t_n ; h)$$

There are two families of one-step methods:

(i) Methods of Taylor

(ii) Methods of Runge – Kutta

The methods of Taylor type are further classified into Euler method and methods of Taylor of greater order.

2.2.1 EULER METHOD

If we take the first two terms of the Taylor series, which describes the exact solution at t_{n+1},

$$y(t_{n+1}) = \sum_{r=0}^{\infty} \frac{h}{r!} y^{(r)}(t_n)$$

to compute

$$y(t_1) \approx y(t_0) + hy'(t_0) = y_0 + hf_0 = y_1$$

After n steps it yields

$$y_{n+1} = y_n + hf_n \qquad (2.2)$$

Equation (2.2) above is called the Euler's formula or the Euler method; the simplest of the numerical methods for solving first – order differential equations.

Graphical Interpretation of Euler Method

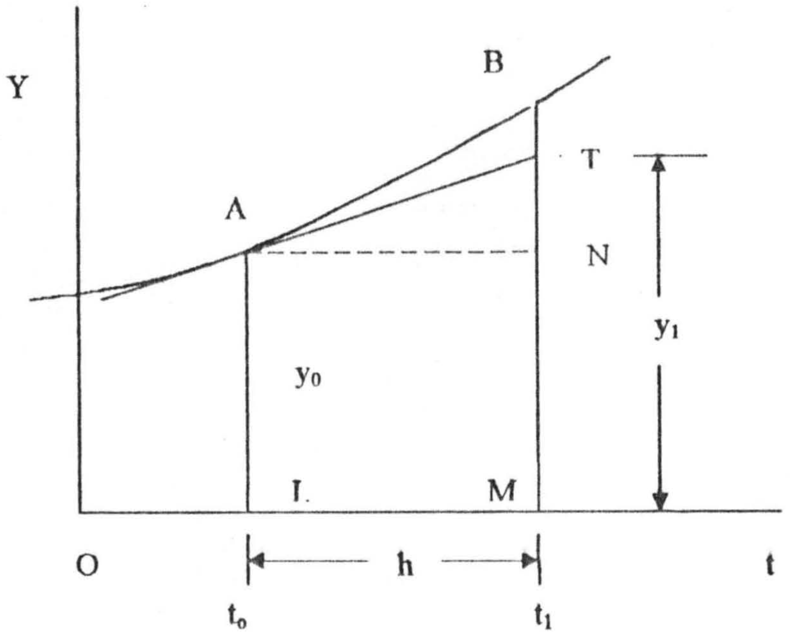

FIG 2.1

If AT is the tangent to the curve at A, then

$$\frac{NT}{AN} = \left.\frac{dy}{dt}\right|_{t=t_0} = y_0'$$

$$\frac{NT}{h} = y_0'$$

$$\because NT = y_0'$$

$$\because At \;\; t = t_1, \;\; MT = y_0 + hy_0'$$

By Euler's relationship, $y_1 = y_0 + hy_0'$ *i.e.* MT.

The difference between the calculated value of y, i.e. MT, and the actual value of the function y, i.e. MB, at $t = t_1$, is indicated by TB. This error can be considerable depending on the curvature of the graph and the size of the interval h. It is inherent to the method and corresponds to the truncation of the Taylor's series after the second term.

Although Euler method is simple in procedure it is lacking in accuracy, especially away from the starter values of the initial condition. And it is of use only for very small values of the interval h (Stroud, 1996).

In spite of its practical limitations, it is the foundation of several more sophisticated methods.

2.2.2 METHODS OF TAYLOR OF GREATER ORDER

In order to obtain a numerical method with greater order of accuracy than the Euler Method, we could just as well take more terms of the Taylor's series expansion. A method of second order looks like.

$$y_{n+1} y_n + hy_n' + \frac{h^2}{2} y_n''.$$

$$Since \quad y_n' = f(x_n, y_n) = f_n \qquad then$$

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f_n'$$

This implies the truncation error

$$E_2 = y(t_1) - y_1 = \frac{h^3}{6} y'''(\xi) \approx O(h^3)$$

More generally, a K-order numerical method is:

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} y_n'' + \ldots + \frac{h^k}{k!} y_n^k$$

With a truncation error

$$E_k = y(t_1) - y_1 = \frac{h^{k+1}}{(k+1)!} y^{k+1}(\xi), \qquad \xi \in [t_0, t_1]$$

About the convergence, the following is valid .

LEMMA: All the consistent One-step methods satisfy the root condition (Patrizia, 2001).

PROPOSITION: Given a numerical method of Taylor type, if $\phi(t_n, y_n)$ satisfies the Lipschitz condition and the method is consistent then, from the above lemma, the method is convergent (Patrizia, 2001).

2.2.3 RUNGE – KUTTA METHODS

The idea of extending the Euler method, by allowing for a multiplicity of evaluations of the function f within each step, was originally proposed by Runge (1895), further contributions were made by Heun (1900) and by kutta (1901).

Given y_n as an approximation to $y(t_n)$, where y satisfies the differential equation system,

$$y'(t) = f(t, y), \qquad y(t_0) = y_0, \qquad f: R \; X \; R^m \to R^m,$$

the approximation y_{n+1} to $y(t_{n+1})$ is computed by evaluating

$$\left.\begin{aligned} &y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i \\ &where \\ &k_i = f\left(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij} k_i\right), \qquad i = 1,2,\ldots,s \end{aligned}\right\} \quad (2.3)$$

(Lambert, 1991)

An alternative form of the above, is,

$$\left.\begin{aligned} &y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(t_n + c_i h, Y_i) \\ &where \\ &Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j), \qquad i = 1,2,\ldots,s \end{aligned}\right\} \quad (2.4)$$

The two forms are equivalent by making the interpretation

$$k_i = f(t_n + c_i h, Y_i), \qquad i = 1,2,\ldots,s$$

(Lambert, 1991)

The integer s is the number of stages of the method and measures its complexity, since the number of evaluations of f per step equals s. The set $\{a_{ij}, b_i\}_{i=1,\ldots,s}$ of constants characterizes a particular method of this type.

The quantities Y_i are approximations to solution values $y(t)$ for t ranging through various values near t_n. Also $f(Y_i)$ are approximations to $y^1(t)$ at the same values t. (Patrizia, 2001)

Runga-kutta methods are often represented using the Butcher array as follows:

$$\begin{array}{c|ccccc} c_1 & a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2s} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ c_s & a_{s1} & a_{s2} & \cdot & \cdot & \cdot & a_{ss} \\ \hline & b_1 & b_2 & \cdot & \cdot & \cdot & b_s \end{array}$$

An s – stage Runge-kutta method is completely specified by its butcher array as

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array},$$

$$C = [c_1, c_2, \ldots, c_s]^T, b = [b_1, b_2, \ldots, b_s]^T, A = (a_{ij})$$

The components of C are the row sums of A

(Lambert, 1991)

From the definition, a Runga-kutta is consistent when

$$\sum_{i=1}^{s} b_i = 1$$

(Lambert 1991)

And when y_{n+1} depends only upon the evaluations of the previous points $f(y_i)_{i=0,1,\ldots,n}$, if $a_{ij} = 0$ for all $1 \le i \le j \le s$, it is called EXPLICIT. Otherwise it is said to be IMPLICIT.

We present below some explicit Runge-kutta methods:

ONE STAGE

The general s-stage Runge-kutta method (2.3) becomes 1-stage if we set $b_2 = b_3 = 0$.

Then

$$y_{n+1} = y(t_n) + hb_1 f$$

From the Taylor expansion follows that the best one can do is set $b_1 = 1$, hence $E_{n+1} = 0(h^2)$.

Thus there exists only one explicit one-stage Runge-kutta method of order 1, namely Euler's Rule. (Lambert, 1991)

TWO STAGE

If we set $b_3 = 0$, the method becomes two-stage

$$y_{n+1} = y(x_n) + h(b_1 + b_2)f + h^2 b_2 c_2 F + \tfrac{1}{2} h^3 b_2 c_2^2 G + 0(h^4)$$

where

$$F := f_x + ff_y, \quad G := f_{xx} + 2ff_{xy} + f^2 f_{yy},$$

(Lambert, 1991)

On comparing with the expansion for $y\ (x_{n+1})$,

$$y(x_{n+1}) = y(x_n) + hf + \tfrac{1}{2} h^2 F + \tfrac{1}{6} h^3 (Ff_y + G) + 0(h^4),$$

We see that order 2 can be achieved by choosing

$$b_1 + b_2 = 1, \qquad b_2 c_2 = \tfrac{1}{2}$$

There exists an infinite family of explicit two-stage Runge-kutta methods of order 2. Two solutions yield well-known methods:

(*i*) The Modified Euler (or Improved Polygon) method

$$b_1 = 0, \; b_2 = 1, \; c_2 = ½ .$$

Its Butcher array is

0		
½	½	
	0	1

(*ii*) The Improved Euler (or Heun) method

$$b_1 = b_2 = \tfrac{1}{2},\ c_2 = 1$$

Its Butcher array is

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \tfrac{1}{2} & \tfrac{1}{2} \end{array}$$

THREE STAGE

By satisfying the following coefficient conditions one can achieve order 3.

$$\begin{aligned} b_1 + b_2 + b_3 &= 1 \\ b_2c_2 + b_3c_3 &= \tfrac{1}{2} \\ b_2c_2^2 + b_3c_3^2 &= \tfrac{1}{3} \\ b_3c_2a_{32} &= \tfrac{1}{6} \end{aligned}$$

Two particular solutions lead to well-known methods:

(*i*) Heun's third order formula.

Its Butcher array is

$$\begin{array}{c|ccc} 0 & & & \\ \tfrac{1}{3} & \tfrac{1}{3} & & \\ \tfrac{2}{3} & 0 & \tfrac{2}{3} & \\ \hline & \tfrac{1}{4} & 0 & \tfrac{3}{4} \end{array}$$

(*ii*) Kutta's third order formula

It has the Butcher array

0			
$\frac{1}{2}$	$\frac{1}{2}$		
1	-1	2	
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

FOUR STAGE

The most popular Runge-kutta scheme is the classical Runge-kutta method of order four (4). so popular is this method that when one sees a reference to a problem having been solved by "the Runge-kutta method", it is almost certainly the classical Runge-kutta method that has been used.

It has the following Butcher array:

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

The classical Runge-kutta scheme is as follows:

$$y_{n+1} = y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(x_{n,}y_n) \qquad k_3 = f(x_n + h/2, y_n + h/2\,k_2)$$

$$k_2 = f(x_n + h/2, y_n + h/2\,k_1) \qquad k_4 = f(x_n + h, y_n + hk_3)$$

The absence of k_1 in the evaluation of k_3, and absence of k_1, k_2 in the evaluation of k_4 may have played a role in making this method popular. However, Lambert (1991) suggests another reason for the popularity of the method: "In the pre-computer days, computations were performed on purely mechanical devices. Multiplication or division was a tiresome business on such machines. Since the main computation is in the evaluation of the functions to produce the k_i' s. That the c_i' s and a_i' s are always either 1 or ½ increased the chances of any division in the evaluation of f terminating quickly"

2.3 MULTISTEP METHODS

As stated in the preceding chapter, we can write a numerical method for solving IVPs in the general form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\phi_f(y_{n+k}, y_{n+k-1}, \ldots, y_n, t_n; h)$$
$$y_i = \mu_i(h) \qquad for \qquad i = 0, \ldots, k-1$$

If $k > 1$ in the above formula then the numerical method is called multistep, because it determines the solution at the support times using k values. (Patrizia, 2001)

2.3.1 LINEAR MULTISTEP METHODS (LMMs)

Let y_n be an approximation to the theoretical solution at t_n, that is, to $y(t_n)$, and let $f_n = f(t_n, y_n)$. Then, we say a linear multistep method of step number k, or a linear k-step method is a computational method for determining, the sequence $\{y_n\}$ that takes the form of a linear relationship between y_{n+j}, f_{n+j}, $j = 0, 1, \ldots, k$.

Thus the general lmm may be written.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2.5)$$

Where α_j and β_j are constants; we assume $\alpha_k = 1$ and that not both α_0 and β_0 are zero. We say that the method is explicit if $\beta_k = 0$, and implicit if $\beta_k \neq 0$. (Lambert, 1973)

The above definition may be formulated in a more compact alternative notation. In chapter one, we introduced the first characteristic polynomial $\rho(\xi)$ associated with the general form of a numerical method. In the case of linear multistep methods we define a similar polynomial $\sigma(\xi)$, which is said to be the second characteristic polynomial of (2.5); i.e.

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \qquad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$$

Where $\xi \in C$ is a dummy variable.

Thus the lmm (2.5) can now be written in the form

$$\rho(E)y_n = h\sigma(E)f_n \qquad (2.6)$$

Where E is the forward shift operator

$$E(y_n) = y_{n+1} \qquad \textit{for all} \quad n \in N$$

It is easily established that the consistency condition is guaranteed if

$$\rho(1) = 0 \qquad and \qquad \rho'(1) = \sigma(1) \qquad (2.7)$$

(Lambert, 1973)

And the zero-stability should be verified through the root condition.

Within the general class (2.5) of linear multistep methods, there are several well known sub-classes. The sub-class of methods of ADAMS TYPE are characterized by

$$\rho(\xi) = \xi^k - \xi^{k-1}.$$

Since the spurious roots of ρ are all situated at the origin of the complex plane, methods of Adams type are clearly zero-stable for all values of k. Methods of Adams type which have the maximum possible accuracy are known as ADAMS METHODS; if they are explicit they are known as ADAMS-BASHFORTH METHODS, and if implicit as ADAMS-MOULTON METHODS.

Then 1–step Adams-Bashforth method is Euler's Rule,

$$y_{n+1} = y_n + hf_n,$$

While the 1–step Adams-Moulton method is the Trapezoidal Rule,

$$y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n)$$

Other sub-classes are characterized by

$$\rho(\xi) = \xi^k - \xi^{k-2}$$

are clearly also zero-stable for all k. Explicit members of this sub-class are known as NYSTROM METHODS, and implicit members as GENERALIZED MILNE-SIMPSON METHODS. A well known example of a Nystrom method is the mid-point Rule

$$y_{n+2} - y_n = 2hf_{n+1},$$

and of a Generalized Milne-Simpson method is Simpson's Rule.

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+2} + 4f_{n+1} + f_n)$$

A sub-class that is important in dealing with stiffness consists of the BACKWARD DIFFERENTIATON FORMULAE or BDF, which are implicit methods with $\sigma(\xi) = \beta_k \xi^k$. (Lambert, 1991)

The following

$$\sum_{j=1}^{k} \frac{1}{j} \Delta y_{n+k} = hf(y_{n+k}),$$

Where $\Delta y_n = y_n - y_{n-1}$ and recursively $\Delta^i y_n = \Delta(\Delta^{i-1} y_n)$ describes a general BDF method. It can be proved that the method is convergent for only $k = 1, \ldots, 6$. (Patrizia, 2001)

2.4 PREDICTOR – CORRECTOR METHODS

At each step of an implicit method we must solve for y_{n+k},

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(t_{n+k}, y_{n+k}) + h\sum_{j=0}^{k-1} \beta_j f_{n+j}$$

In general this equation will be nonlinear in y_{n+k}. One way of solving this equation is Newton's method (Mackenzie, 2000). A much simpler way of obtaining y_{n+k} is by using the iterative algorithm.

$$y_{n+k}^{(s+1)} = h\beta_k f\left(t_{n+k}, y_{n+k}^{(s)}\right) - \sum_{j=0}^{k-1} \left(\alpha_j y_{n+j} - \beta_j f_{n+j}\right), \qquad (2.8)$$

$$s = 0, 1, \ldots$$

Where $y_{n+k}^{(s)}$ denotes the approximation of y_{n+k} after s iterations. The solution y_{n+k} is therefore the fixed point of the function.

$$g(y_{n+k}) = h\beta_k f(t_{n+k}, y_{n+k}) - \sum_{j=0}^{k-1} \left(\alpha_j y_{n+j} - \beta_j f_{n+j}\right)$$

The fixed point iteration (2.8) will converge to the unique solution in the sense that

$$\lim_{s \to \infty} \left| y_{n+k}^{(s+1)} - y_{n+k}^{(s)} \right| = 0$$

for any arbitarary guess of $y_{n+k}^{(0)}$ if the condition $|g'| < 1$, where differentiation is with respect to y_{n+k}. If $f\ (t,\ y)$ is continuously differentiable and $\left|\partial f / \partial y\right| < L$ then we are guaranteed to have convergence so long as

$$h < \frac{1}{L|\beta_k|} \qquad (2.9)$$

This condition on the maximum value of h is only likely to be severe for problems with large Lipshitz constants i.e. stiff problems. For non-stiff problem the step size will be determined by accuracy considerations leading to step- sizes that will be smaller than (2.9).

In practice, a pre-assigned tolerance ε would be chosen and the iteration performed until

$$\left|y_{n+k}^{(s+1)} - y_{n+k}^{(s)}\right| < \varepsilon \qquad (2.10)$$

To speed up the rate of convergence we would like a cheap way of providing a good initial guess for y_{n+k}. This could be done by using an explicit method to predict the value of $y_{n+k}^{(0)}$. The fixed point iteration could then be used to correct the solution. There are two ways that the correction stage could be carried out. First we could just iterate until (2.10) is satisfied. This approach is called CORRECTING TO CONVERGENCE. The overall method would therefore just be the implicit method as the effect of the explicit predictor would be lost once convergence was reached. On the other hand we could decide to only compute a fixed number of iterations of the correction stage and then pass into the next step. The advantage of this approach is that we can know in advance how much effort is going to be expended for each step. One other advantage is that this approach is computationally less expensive than correcting to convergence. However, since we are not guaranteed to have converged using a fixed number of iteration it is not clear exactly what method we have used now being a mixture of the predictor and corrector. If we let P denote an application of the predictor, C a single application of the corrector, and E an evaluation of f, then if we compute $y_{n+k}^{(0)}$ from the predictor, evaluate $f_{n+k}^{(0)} = f\left(t_{n+k}, y_{n+k}^{(0)}\right)$ and then apply the corrector once to get $y_{n+k}^{'}$, the calculation is denoted symbolically by PEC. If we decide to carry out another correction step we need to first evaluate $f_{n+k}^{(1)} = f\left(t_{n+k}, y_{n+k}^{(1)}\right)$ and then correct to get $y_{n+k}^{(2)}$. The total algorithm at this stage is denoted by PECEC or $P(EC)^2$. If we apply the corrector m times the algorithm is denoted by $(P(EC)^m$. After we have completed m correction steps we have the option of evaluating $f_{n+k}^{(m)} = f\left(t_{n+k}, y_{n+k}^{(m)}\right)$ which will be used as part of the prediction stage to calculate y_{n+k+1}. This mode of operation is denoted by $P(EC)^mE$. This will give us a slightly different algorithm to the $P(EC)^m$ mode where f_{n+k} is not updated using $y_{n+k}^{(m)}$.

To summarize the above let the k-step explicit predictor method denoted by

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j^{*} y_{n+k} = h \sum_{j=0}^{k-1} \beta_j^{*} f_{n+j}$$

and the implicit corrector method by

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f_{n+j} + h \sum_{j=0}^{k-1} \beta_j f_{n+j}$$

The $P(EC)^mE$ mode is defined as follows

$$y_{n+k}^{(0)} + \sum_{j=0}^{k-1} \alpha_j^* y_{n+j}^{(m)} = h \sum_{j=0}^{k-1} \beta_j^* f_{n+j}^{(m)} \qquad \textit{predict}$$

$for \quad s = 0, 1, \ldots, m-1$

$$f_{n+k}^{(s)} = f\left(t_{n+k}, y_{n+k}^{(s)}\right), \qquad \textit{Evaluate}$$

$$y_{n+k}^{(s+1)} + \sum_{j=0}^{k-1} \alpha_j y_{n+j}^{(m)} = h\beta_k f_{n+k}^{(s)} + h \sum_{j=0}^{k-1} \beta_j f_{n+j}^{(m)} \qquad \textit{Correct}$$

$$f_{n+k}^{(m)} = f\left(t_{n+k}, y_{n+k}^{(m)}\right), \qquad \textit{Evaluate}$$

The $P(EC)^m$ algorithm takes the form

$$y_{n+k}^{(0)} + \sum_{j=0}^{k-1} \alpha_j^* y_{n+j}^{(m)} = h \sum_{j=0}^{k-1} \beta_j^* f_{n+j}^{(m-1)} \qquad \textit{predict}$$

$for \quad s = 0, 1, \ldots, m-1$

$$f_{n+k}^{(s)} = f\left(t_{n+k}, y_{n+k}^{(s)}\right), \qquad \textit{Evaluate}$$

$$y_{n+k}^{(s+1)} + \sum_{j=0}^{k-1} \alpha_j y_{n+j}^{(m)} = h\beta_k f_{n+k}^{(s)} + h \sum_{j=0}^{k-1} \beta_j f_{n+j}^{(m-1)} \qquad \textit{Correct}$$

$$f_{n+k}^{(m)} = f\left(t_{n+k}, y_{n+k}^{(m)}\right), \qquad \textit{Evaluate}$$

We would like to derive an expression for the leading term in the truncation error of a predictor – corrector method. We will concentrate on the analysis for the $P(EC)^m$ mode of operation as the $P(EC)^mE$ approach is done similarly. Let assume that the predictor is of order p^* and has error constant $C^*_{p^*+1}$ i.e.

$$T^*_{n+k} = C^*_{p^*+1} h^{p^*+1} y^{(p^*+1)}(t_n) + 0\left(h^{p^*+2}\right)$$

and that the corrector is of order p and has error constant C_{p+1} i.e.

$$T_{n+k} = C_{p+1} h^{p+1} y^{(p+1)}(t_n) + 0\left(h^{p+2}\right)$$

Mackenzie (2000), showed that we can deduce the following about the truncation error of the predictor-corrector method.

1. If the order of the predictor is greater than or equal to that of the corrector, i.e. $p^* \geq p$ *and* $m \geq 1$ *then* $m + p^* + 1 \geq p + 2$ and the leading term in the truncation error is that of the corrector.

2. If the order of the predictor is less than that of the corrector i.e. $p^* = p - q < p$ *and* $0 < q \leq p - 1$, then the leading term of the local truncation error is

   (a) that of the corrector alone when $m \geq q + 1$ *as* $m + p - q \geq p + 2$.

   (b) of the same order as that of the corrector but not identical if $m = q$ *as* $m+p-q+1 = p+1$.

   (c) Of the form $0\,(h^{(p-q+m+l)})$ when $m \geq q + 1$ *as* $m + p - q + 1 < p + 1$.

From the above it is not clear what the ideal combination of predictor, corrector and the number of correction steps should be. However, if we use the same order of predictor and corrector then for free we can get an estimate of the local truncation error which could be used to adaptively change the step size.

One example of a popular predictor – corrector algorithm is the following fourth – order Adams – Bashforth – Moulton pair:

$$
\begin{aligned}
&\textit{predictor}: \quad y_{n+4} - y_{n+3} = \tfrac{h}{24}\left(55 f_{n+3} - 59 f_{n+2} + 37 f_{n+1} - 9 f_n\right) \\
&\textit{corrector}: \quad y_{n+4} - y_{n+3} = \tfrac{h}{24}\left(9 f_{n+4} + 19 f_{n+3} - 5 f_{n+2} + f_{n+1}\right) \\
&\textit{Error estimate}: \; C_5 h^5 y^{(5)}(t_n) \approx -\tfrac{19}{270}\left(y_{n+4}^{(1)} - y_{n+4}^{(0)}\right) \\
&\textit{Interval of absolute stability in PECE}\ \text{mode}: \; (-1.25, 0)
\end{aligned}
$$

CHAPTER THREE

DERIVATION OF A SIX-STEP IMPLICIT LINEAR MULTISTEP METHOD (LMM)

3.1 THE SIX-STEP IMPLICIT LMM OF ORDER EIGHT

Although a number of ways for deriving a lmm exist, one of the best of such methods is through Taylor series expansion. Lambert (1973) described the process thus:

Let □ be the linear difference operator defined by

$$\square[y(t)\,;h)] = \sum_{j=0}^{k}[\alpha_j y(t+jh) - h\beta_j y^{'}(t+jh)] \qquad (2.1)$$

where *y(t)* is an arbitrary function, continuously differentiable on [a , b]. if we expand *y(t+jh)* and its derivative *y'(t+jh)* as Taylor series about t, and collecting like terms we have

$$\square[y(t);h] = c_0 y(t) + c_1 h y^{'}(t) + c_2 h^2 y^{''}(t) + \ldots + c_q h^q y^q(t) + \ldots \qquad (2.2)$$

where c_q are constants.

The constants c_q are expressed in terms of the coefficients $\forall_j$, $\exists_j$ thus:

$$\left.\begin{aligned}
c_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \ldots \alpha_k \\
c_1 &= \alpha_1 + 2\alpha_2 \ldots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \ldots \beta_k) \\
c_2 &= \tfrac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + \ldots k^2\alpha_k) - (\beta_1 + 2\beta_2 + \ldots k\beta_k) \\
&. \\
&. \\
&. \\
c_q &= \tfrac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \ldots k^q\alpha_k) - \tfrac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + 3^{q-1}\beta_3 \ldots k^{q-1}\beta_k) \\
q &= 2, 3, \ldots
\end{aligned}\right\} \qquad (2.3)$$

we can use the above formulae to derive a linear multistep method of given structure and maximal order.

Suppose we choose to expand $y(t+jh)$ and $y'(t+jh)$ about $t+rh$; where r need not necessarily be an integer. We obtain

$$\mathcal{L}[y(t);h] = D_0 y(t+rh) + D_1 h y'(t+rh) + D_2 h^2 y''(t+rh) + \ldots + D_q h^q y^q(t+rh) \qquad (2.4)$$

If we employ the Taylor expansions

$$y^{(q)}(t+rh) = y^{(q)}(t) + rhy^{(q+1)}(t) + \ldots + \tfrac{r^s h^s}{s!} y^{(q+s)}(t) + \ldots$$
$$q = 0, 1, 2, \ldots$$

where $y^{(0)}(t) = y(t)$; and substitute in (2.4), we obtain on equating term by term with (2.2)

$$\left.\begin{aligned}
c_0 &= D_0 \\
c_1 &= D_1 + rD_0 \\
c_2 &= D_2 + rD_1 + \tfrac{r^2}{2!} D_0 \\
&\;\;\vdots \\
c_p &= D_p + rD_{p-1} + \ldots + \tfrac{r^p}{p!} D_0 \\
c_{p+1} &= D_{p+1} + rD_p + \ldots + \tfrac{r^{p+1}}{(p+1)!} D_0
\end{aligned}\right\} \qquad (2.5)$$

It follows that $c_o = c_1 = \ldots c_p = 0$ iff $D_0 = D_1 = \ldots D_p = 0$

The formulae for the constants D_q expressed in terms of α_j, β_j are

$$
\left.\begin{aligned}
D_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \dots \alpha_k \\
D_1 &= -r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + \dots + (k-r)\alpha_k \\
&\quad -(\beta_0 + \beta_1 + \beta_2 + \dots \beta_k) \\
&\;\;\cdot \\
&\;\;\cdot \\
&\;\;\cdot \\
D_q &= \tfrac{1}{q!}\left[(-r)^q \alpha_0 + (1-r)^q \alpha_1 + (2-r)^q \alpha_2 + \dots (k-r)^q \alpha_k\right] \\
&\quad - \tfrac{1}{(q-1)!}\left[(-r)^{q-1} \beta_0 + (1-r)^{q-1} \beta_1 + (2-r)^{q-1} \beta_2 + \dots + (k-r)^{q-1} \beta_k\right], \\
&\quad q = 2, 3, \dots
\end{aligned}\right\} \qquad (2.6)
$$

where $r = 0$ (2.6) reverts to (2.3). A judicious choice for r can sometimes reduce the labour in deriving linear multistep methods.

In this research work, we wish to derive an optimal 6-step method. Therefore, all the roots of the first characteristic polynomial $\rho(\xi)$ must be on the unit circle. We know that $\rho(\xi)$ is a polynomial of degree 6. Hence, by consistency, it has one real root at +1 and another real root at -1. The four remaining roots must be complex.

Hence we have

$\xi_1 = +1, \qquad \xi_2 = -1, \qquad \xi_3 = e^{i\theta_1}, \qquad \xi_4 = e^{-i\theta_1} \qquad \xi_5 = e^{i\theta_2} \qquad \xi_6 = e^{-i\theta_2}$

Hence

$$
\begin{aligned}
\rho(\xi) &= (\xi - 1)(\xi + 1)(\xi - e^{i\theta_1})(\xi - e^{-i\theta_1})(\xi - e^{i\theta_2})(\xi - e^{-i\theta_2}) \\
&= (\xi^4 - 2\cos\theta_1 \xi^3 + 2\cos\theta_1 \xi - 1)(\xi^2 - 2\cos\theta_2 + 1) \\
&= \xi^6 - 2\cos\theta_2 \xi^5 + \xi^4 - 2\cos\theta_1 \xi^5 + 4\cos\theta_1 \cos\theta_2 \xi^4 - 2\cos\theta_1 \xi^3
\end{aligned}
$$

$$+2\cos\theta_1\xi^3 - 4\cos\theta_1\cos\theta_2\xi^2 + 2\cos\theta_1\xi - \xi^2 + 2\cos\theta_2\xi - 1$$
$$= \xi^6 - (2\cos\theta_2 + 2\cos\theta_1)\xi^5 + (4\cos\theta_1\cos\theta_2 + 1)\xi^4 - (4\cos\theta_1\cos\theta_2 + 1)\xi^2$$
$$+ 2(\cos\theta_1 + 2\cos\theta_2)\xi - 1$$
$$= \xi^6 + 2(\cos\theta_1 + \cos\theta_2)\xi^5 + (4\cos\theta_1\cos\theta_2 + 1)\xi^4 - (4\cos\theta_1\cos\theta_2 + 1)\xi^2$$
$$+ 2(\cos\theta_1 + 2\cos\theta_2)\xi - 1$$

Set $cos\theta_1 = a,\ cos\theta_2 = b$

$\Rightarrow$

$$\rho(\xi) = \xi^6 - 2(a+b)\xi^5 + (4ab+1)\xi^4 - (4ab+1)\xi^2 + 2(a+b)\xi - 1$$

$\Rightarrow$

$$\alpha_6 = +1,\quad \alpha_5 = -2(a+b),\quad \alpha_4 = (4ab+1),\quad \alpha_3 = 0,$$
$$\alpha_2 = -(4ab+1),\quad \alpha_1 = 2(a+b),\quad \alpha_0 = -1$$

We require the method to have order $k+2$. We now state the order requirement in terms of the coefficients D_q rather than in terms of the C_q.

From (2.6) we have the following:

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$$
$$D_1 = -r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + (3-r)\alpha_3 + (4-r)\alpha_4 + (5-r)\alpha_5 + (6-r)\alpha_6$$
$$-(\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)$$
$$D_2 = \tfrac{1}{2!}\left[(-r)^2\alpha_0 + (1-r)^2\alpha_1 + (2-r)^2\alpha_2 + (3-r)^2\alpha_3 + (4-r)^2\alpha_4 + (5-r)^2\alpha_5 + (6-r)^2\alpha_6\right]$$
$$-\left[-r\beta_0 + (1-r)\beta_1 + (2-r)\beta_2 + (3-r)\beta_3 + (4-r)\beta_4 + (5-r)\beta_5 + (6-r)\beta_6\right]$$
$$D_3 = \tfrac{1}{3!}\left[(-r)^3\alpha_0 + (1-r)^3\alpha_1 + (2-r)^3\alpha_2 + (3-r)^3\alpha_3 + (4-r)^3\alpha_4 + (5-r)^3\alpha_5 + (6-r)^3\alpha_6\right]$$
$$-\tfrac{1}{2!}\left[(-r)^2\beta_0 + (1-r)^2\beta_1 + (2-r)^2\beta_2 + (3-r)^2\beta_3 + (4-r)^2\beta_4 + (5-r)^2\beta_5 + (6-r)^2\beta_6\right]$$
$$D_4 = \tfrac{1}{4!}\left[(-r)^4\alpha_0 + (1-r)^4\alpha_1 + (2-r)^4\alpha_2 + (3-r)^4\alpha_3 + (4-r)^4\alpha_4 + (5-r)^4\alpha_5 + (6-r)^4\alpha_6\right]$$
$$-\tfrac{1}{3!}\left[(-r)^3\beta_0 + (1-r)^3\beta_1 + (2-r)^3\beta_2 + (3-r)^3\beta_3 + (4-r)^3\beta_4 + (5-r)^3\beta_5 + (6-r)^3\beta_6\right]$$
$$D_5 = \tfrac{1}{5!}\left[(-r)^5\alpha_0 + (1-r)^5\alpha_1 + (2-r)^5\alpha_2 + (3-r)^5\alpha_3 + (4-r)^5\alpha_4 + (5-r)^5\alpha_5 + (6-r)^5\alpha_6\right]$$
$$-\tfrac{1}{4!}\left[(-r)^4\beta_0 + (1-r)^4\beta_1 + (2-r)^4\beta_2 + (3-r)^4\beta_3 + (4-r)^4\beta_4 + (5-r)^4\beta_5 + (6-r)^4\beta_6\right]$$
$$D_6 = \tfrac{1}{6!}\left[(-r)^6\alpha_0 + (1-r)^6\alpha_1 + (2-r)^6\alpha_2 + (3-r)^6\alpha_3 + (4-r)^6\alpha_4 + (5-r)^6\alpha_5 + (6-r)^6\alpha_6\right]$$
$$-\tfrac{1}{5!}\left[(-r)^5\beta_0 + (1-r)^5\beta_1 + (2-r)^5\beta_2 + (3-r)^5\beta_3 + (4-r)^5\beta_4 + (5-r)^5\beta_5 + (6-r)^5\beta_6\right]$$
$$D_7 = \tfrac{1}{7!}\left[(-r)^7\alpha_0 + (1-r)^7\alpha_1 + (2-r)^7\alpha_2 + (3-r)^7\alpha_3 + (4-r)^7\alpha_4 + (5-r)^7\alpha_5 + (6-r)^7\alpha_6\right]$$
$$-\tfrac{1}{6!}\left[(-r)^6\beta_0 + (1-r)^6\beta_1 + (2-r)^6\beta_2 + (3-r)^6\beta_3 + (4-r)^6\beta_4 + (5-r)^6\beta_5 + (6-r)^6\beta_6\right]$$

$$D_8 = \tfrac{1}{8!}\left[(-r)^8\alpha_0 + (1-r)^8\alpha_1 + (2-r)^8\alpha_2 + (3-r)^8\alpha_3 + (4-r)^8\alpha_4 + (5-r)^8\alpha_5 + (6-r)^8\alpha_6\right]$$
$$-\tfrac{1}{7!}\left[(-r)^7\beta_0 + (1-r)^7\beta_1 + (2-r)^7\beta_2 + (3-r)^7\beta_3 + (4-r)^7\beta_4 + (5-r)^7\beta_5 + (6-r)^7\beta_6\right]$$
$$D_9 = \tfrac{1}{9!}\left[(-r)^9\alpha_0 + (1-r)^9\alpha_1 + (2-r)^9\alpha_2 + (3-r)^9\alpha_3 + (4-r)^9\alpha_4 + (5-r)^9\alpha_5 + (6-r)^9\alpha_6\right]$$
$$-\tfrac{1}{8!}\left[(-r)^8\beta_0 + (1-r)^8\beta_1 + (2-r)^8\beta_2 + (3-r)^8\beta_3 + (4-r)^8\beta_4 + (5-r)^8\beta_5 + (6-r)^8\beta_6\right]$$

Setting $r = 3$ and $D_q = 0,\ q = 2, 3, 4, 5, 6, 7, 8$ we have,

$$D_2 = \tfrac{1}{2!}\left[3^2\alpha_0 + 2^2\alpha_1 + \alpha_2 + \alpha_4 + 2^2\alpha_5 + 3^2\alpha_6\right] - \left[-3\beta_0 - 2\beta_1 - \beta_2 + \beta_4 + 2\beta_5 + 3\beta_6\right] = 0$$
$$D_3 = \tfrac{1}{3!}\left[-3^3\alpha_0 - 2^3\alpha_1 - \alpha_2 + \alpha_4 + 2^3\alpha_5 + 3^3\alpha_6\right] - \tfrac{1}{2!}\left[3^2\beta_0 + 2^2\beta_1 + \beta_2 + \beta_4 + 2^2\beta_5 + 3^2\beta_6\right] = 0$$
$$D_4 = \tfrac{1}{4!}\left[3^4\alpha_0 + 2^4\alpha_1 + \alpha_2 + 2^4\alpha_5 + 3^4\alpha_6\right] - \tfrac{1}{3!}\left[3^3\beta_0 + 2^3\beta_1 - \beta_2 + \beta_4 + 2^3\beta_5 + 3^3\beta_6\right] = 0$$
$$D_5 = \tfrac{1}{5!}\left[-3^5\alpha_0 - 2^5\alpha_1 - \alpha_2 + \alpha_4 + 2^5\alpha_5 + 3^5\alpha_6\right] - \tfrac{1}{4!}\left[3^4\beta_0 + 2^4\beta_1 + \beta_2 + \beta_4 + 2^4\beta_5 + 3^4\beta_6\right] = 0$$
$$D_6 = \tfrac{1}{6!}\left[3^6\alpha_0 + 2^6\alpha_1 + \alpha_2 + \alpha_4 + 2^6\alpha_5 + 3^6\alpha_6\right] - \tfrac{1}{5!}\left[-3^5\beta_0 - 2^5\beta_1 - \beta_2 + \beta_4 + 2^5\beta_5 + 3^5\beta_6\right] = 0$$
$$D_7 = \tfrac{1}{7!}\left[-3^7\alpha_0 - 2^7\alpha_1 - \alpha_2 + \alpha_4 + 2^7\alpha_5 + 3^7\alpha_6\right] - \tfrac{1}{6!}\left[3^6\beta_0 + 2^6\beta_1 + \beta_2 + \beta_4 + 2^6\beta_5 + 3^6\beta_6\right] = 0$$
$$D_8 = \tfrac{1}{8!}\left[3^8\alpha_0 + 2^8\alpha_1 + \alpha_2 + \alpha_4 + 2^8\alpha_5 + 3^8\alpha_6\right] - \tfrac{1}{7!}\left[-3^7\beta_0 - 2^7\beta_1 - \beta_2 + \beta_4 + 2^7\beta_5 + 3^7\beta_6\right] = 0$$

However, on inserting the values we have obtained for the α_j into these equations we have

$$\left.\begin{aligned}
&-3\beta_0 - 2\beta_1 - \beta_2 + \beta_4 + 2\beta_5 + 3\beta_6 = 0\\
&3^2\beta_0 + 2^2\beta_1 + \beta_2 + \beta_4 + 2^2\beta_5 + 3^2\beta_6 = \tfrac{2}{3}\left[28 + 4ab - 16(a+b)\right]\\
&-3^3\beta_0 + 2^3\beta_1 - \beta_2 + \beta_4 + 2^3\beta_5 + 3^3\beta_6 = 0\\
&3^4\beta_0 + 2^4\beta_1 + \beta_2 + \beta_4 + 2^4\beta_5 + 3^4\beta_6 = \tfrac{2}{5}\left[244 + 4ab - 64(a+b)\right]\\
&-3^5\beta_0 - 2^5\beta_1 - \beta_2 + \beta_4 + 2^5\beta_5 + 3^5\beta_6 = 0\\
&3^6\beta_0 + 2^6\beta_1 + \beta_2 + \beta_4 + 2^6\beta_5 + 3^6\beta_6 = \tfrac{2}{7}\left[2188 + 4ab - 256(a+b)\right]\\
&-3^7\beta_0 - 2^7\beta_1 - \beta_2 + \beta_4 + 2^7\beta_5 + 3^7\beta_6 = 0
\end{aligned}\right\} \quad (2.7)$$

We can satisfy the first, third, fifth and seventh of these equations if we choose

$\beta_2 = \beta_4,\ \beta_1 = \beta_5,\ \beta_0 = \beta_6$

The remaining three equations give

$$3^2\beta_0 + 2^2\beta_1 + \beta_2 = \tfrac{1}{3}[28 + 4ab - 16(a+b)] \quad (2.8)$$
$$3^4\beta_0 + 2^4\beta_1 + \beta_2 = \tfrac{1}{5}[244 + 4ab - 64(a+b)] \quad (2.9)$$
$$3^6\beta_0 + 2^6\beta_1 + \beta_2 = \tfrac{1}{7}[2188 + 4ab - 256(a+b)] \quad (2.10)$$

From (2.8) we have

$$\beta_2 = \tfrac{1}{3}[28 + 4ab - 16((a+b)] - 9\beta_0 - 4\beta_1 \quad (2.11)$$

Substituting (2.11) into (2.9)

$$81\beta_0 + 16\beta_1 + \tfrac{1}{3}[28 + 4ab - 16(a+b)] - 9\beta_0 - 4\beta_1 = \tfrac{1}{5}[244 + 4ab - 64(a+b)]$$
$$72\beta_0 + 12\beta_1 = \tfrac{1}{5}[244 - 4ab - 64(a+b)] - \tfrac{1}{3}[28 + 4ab - 16(a+b)]$$
$$72\beta_0 + 12\beta_1 = \tfrac{1}{15}[592 - 8ab - 112(a+b)] \quad (2.12)$$

Substituting (2.11) into (2.10)

$$729\beta_0 + 64\beta_1 + \tfrac{1}{3}[28 + 4ab - 16(a+b)] - 9\beta_0 - 4\beta_1 = \tfrac{1}{7}[2188 + 4ab - 256(a+b)]$$
$$720\beta_0 + 60\beta_1 = \tfrac{1}{7}[2188 + 4ab - 256(a+b)] - \tfrac{1}{3}[28 + 4ab - 16(a+b)]$$
$$720\beta_0 + 60\beta_1 = \tfrac{1}{21}[6368 - 16ab - 656(a+b)] \quad (2.13)$$

From (2.12) and (2.13), solving simultaneously,

$$72\beta_0 + 12\beta_1 = \tfrac{1}{15}[592 - 8ab - 112(a+b)] \qquad \text{X } 60$$
$$720\beta_0 + 60\beta_1 = \tfrac{1}{21}[6368 - 16ab - 656(a+b)] \qquad \text{X } 12$$

$$4320\beta_0 + 720\beta_1 = 4[592 - 8ab - 112(a+b)]$$
$$\underline{8640\beta_0 + 720\beta_1 = \tfrac{4}{7}[6368 - 16ab - 656(a+b)]}$$

$$-4320\beta_0 = \tfrac{-8896}{7} - \tfrac{160}{7}ab - \tfrac{512}{7}(a+b)$$
$$\Rightarrow$$
$$\beta_0 = \tfrac{1}{945}[278 + 5ab + 16(a+b)] = \beta_6 \quad (2.14)$$

Substituting (2.14) into (2.12)

$$72\beta_0 + 12\beta_1 = \tfrac{1}{15}[592 - 8ab - 112(a+b)]$$
$$12\beta_1 = \tfrac{1}{15}[592 - 8ab - 112(a+b)] - \tfrac{72}{30240}[8896 + 160ab + 512(a+b)]$$
$$\Rightarrow$$
$$\beta_1 = \tfrac{1}{105}[160 - 8ab - 76(a+b)] = \beta_5 \qquad (2.15)$$

Substituting (2.14) and (2.15) into (2.11)

$$\beta_2 = \tfrac{1}{3}[28 + 4ab - 16(a+b)] - 9\beta_0 - 4\beta_1$$
$$\beta_2 = \tfrac{1}{3}[28 + 4ab - 16(a+b)] - \tfrac{9}{30240}[8896 + 160ab + 512(a+b)]$$
$$- \tfrac{4}{420}[640 - 32ab - 304(a+b)]$$
$$\Rightarrow$$
$$\beta_0 = \tfrac{1}{105}[62 + 167ab - 272(a+b)] = \beta_4$$

Finally, solving $D_l = 0$ gives

$$-3\alpha_0 - 2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5 + 3\alpha_6 - 3\beta_3 - (\beta_0 + \beta_1 + \beta_2 + \beta_4 + \beta_5 + \beta_6) = 0$$
$$\beta_3 = -(\beta_0 + \beta_1 + \beta_2 + \beta_4 + \beta_5 + \beta_6) - 3\alpha_0 - 2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5 + 3\alpha_6$$
$$\beta_3 = -2\{\beta_0 + \beta_1 + \beta_2\} - 3(-1) - 2[2(a+b)] + (4ab+1) + (4ab+1) - 2[2(a+b)] + 3(+1)$$
$$= 2\begin{Bmatrix} \tfrac{1}{945}(278 + 5ab + 16(a+b)) + \tfrac{1}{105}(160 - 8ab - 76(a+b)) \\ + \tfrac{1}{105}(62 + 167ab - 272(a+b)) \end{Bmatrix} + 8(1 + ab - (a+b))$$
$$\Rightarrow \beta_3 = \tfrac{1}{945}(3008 + 5688ab - 1328(a+b)) \qquad (2.17)$$

We solve for the error constant, D_9

$$
\begin{aligned}
D_9 &= \tfrac{1}{91}\left[-3^9\alpha_0 - 2^9\alpha_1 - \alpha_2 + \alpha_4 + 2^9\alpha_5 + 3^9\alpha_6\right] - \tfrac{1}{81}\begin{bmatrix} 3^8\beta_0 + 2^8\beta_1 + \beta_2 + \beta_4 + 2^8\beta_5 \\ + 3^8\beta_6 \end{bmatrix} \\
&= \tfrac{2}{91}\left[-3^9\alpha_0 - 2^9\alpha_1 - \alpha_2\right] - \tfrac{2}{81}\left[3^8\beta_0 + 2^8\beta_1 + \beta_2\right] \\
&= \tfrac{2}{91}\left[-3^9(-1) - 2^9(2[a+b]) + (4ab+1)\right] - \tfrac{2}{81}\left[3^8\left\{\tfrac{1}{945}(278) + 5ab + 16(a+b)\right\}\right. \\
&\quad \left. + 2^8\left\{\tfrac{1}{105}(160 - 8ab - 76(a+b))\right\} + \left\{\tfrac{1}{105}(62 + 167ab - 272(a+b))\right\}\right] \\
&= \tfrac{2}{91}\left[19684 + 4ab + 1024(a+b)\right] - \tfrac{2}{81}\left[\tfrac{2193156}{945} + \tfrac{84}{5}ab - \tfrac{384}{5}(a+b)\right] \\
&= \tfrac{2}{81}\left[-\tfrac{6016}{45} - \tfrac{736}{45}ab + \tfrac{8576}{45}(a+b)\right]
\end{aligned}
$$

$$
D_9 = -\tfrac{1}{907200}\left[6016 + 736ab - 8576(a+b)\right] \qquad 2.18
$$

Since $a = \cos\theta_1,\ b = \cos\theta_2,\ 0 < \theta_1 < \pi,\ 0 < \theta_2 < \pi$ a and b are restricted to the range $-1 < a < 1$ and $-1 < b < 1$. So there is no allowable value for a and b which will cause D_9 to vanish, that is cause the order of the method to exceed 8; we also observe that there is no allowable value for a and b which will cause β_6 to vanish, thus making the method explicit.

Our choice of values for a and b is guided by the fact that we like to minimize the error constant as well as the need to develop a method that makes computation easier by reducing te number of operations involved.

Although it is possible to get so many schemes out of the method we have developed, by simply changing the variables a and b, we limit our work to three (3) schemes as follows:

SCHEME 1

The following values are therefore, assigned to the variables:

$a = \frac{3}{4}$

$b = -\frac{1}{3}$

Since this causes two coefficients α_2 and α_4 to vanish.

Hence the following values are obtained for the coefficients α_i, β_i

$$\begin{aligned}
\alpha_6 &= +1 & \beta_0 &= \tfrac{3401}{11340} = \beta_6 \\
\alpha_5 &= -\tfrac{5}{6} & \beta_1 &= \tfrac{391}{315} = \beta_5 \\
\alpha_4 &= 0 & \beta_2 &= \tfrac{-1117}{1260} = \beta_4 \\
\alpha_3 &= 0 & \beta_3 &= \tfrac{3848}{2835} \\
\alpha_2 &= 0 \\
\alpha_1 &= \tfrac{5}{6} \\
\alpha_0 &= -1
\end{aligned}$$

Substituting the values of a and b into (2.18), the Error constant is: -0.002489711924

And finally, we have the scheme:

$$y_{n+6} - \tfrac{5}{6} y_{n+5} + \tfrac{5}{6} y_{n+1} - y_n = h\begin{bmatrix} \frac{3401}{11340} f_{n+6} + \frac{391}{315} f_{n+5} - \frac{1117}{1260} f_{n+4} + \frac{3848}{2835} f_{n+3} \\ - \frac{1117}{1260} f_{n+2} + \frac{391}{315} f_{n+1} + \frac{3401}{11340} f_n \end{bmatrix} \qquad (2.19)$$

SCHEME 2

Similarly, we generate another set of values for a and b in order to get a second scheme.

The following values are, therefore assigned to the variables:

$a = \frac{7}{8}$
$b = \frac{-7}{8}$

This causes two coefficients α_5 and α_1 to vanish. Hence the following values are obtained for the coefficients α, β_i.

$\alpha_6 = +1$ $\alpha_2 = {}^{33}\!/_{16}$ $\beta_0 = \frac{17547}{60480} = \beta_6$

$\alpha_5 = 0$ $\alpha_1 = 0$ $\beta_1 = \frac{443}{280} = \beta_5$

$\alpha_4 = {}^{-33}\!/_{16}$ $\alpha_0 = -1$ $\beta_2 = \frac{-281}{448} = \beta_4$

$\alpha_3 = 0$ $\beta_3 = \frac{-155}{252}$

When the values of a and b are substituted into (2.18) , we obtain the Error constant as: -0.006010251323

And we have a second scheme:

$$y_{n+6} - {}^{33}\!/_{16}\, y_{n+4} + {}^{33}\!/_{16}\, y_{n+2} - y_n = h\begin{bmatrix} \frac{17547}{60480} f_{n+6} + \frac{443}{280} f_{n+5} - \frac{281}{448} f_{n+4} - \frac{155}{252} f_{n+3} \\ - \frac{281}{448} f_{n+2} + \frac{443}{280} f_{n+1} + \frac{17547}{60480} f_n \end{bmatrix} \qquad (2.20)$$

SCHEME 3

In the same way, we obtain a third scheme by assigning the following values to a and b.

$a = \frac{1}{2}$

$b = -\frac{1}{2}$

This causes four coefficients α_1, α_2, α_4, and α_5 to vanish. Thus we have the following values for the coefficients α_i, β_i

$\alpha_6 = +1$ $\beta_0 = \frac{41}{140} = \beta_6$

$\alpha_5 = 0$ $\beta_1 = \frac{162}{105} = \beta_5$

$\alpha_4 = 0$ $\beta_2 = \frac{27}{140} = \beta_4$

$\alpha_3 = 0$ $\beta_3 = \frac{68}{35}$

$\alpha_2 = 0$

$\alpha_1 = 0$

$\alpha_0 = -1$

Substituting the values of a and b into (2.18), the Error constant is: -0.006428571429

Hence our third scheme is as follows:

$$y_{n+6} - y_n = h\left[\tfrac{41}{140} f_{n+6} + \tfrac{162}{105} f_{n+5} + \tfrac{27}{140} f_{n+4} + \tfrac{68}{35} f_{n+3} + \tfrac{27}{140} f_{n+2} + \tfrac{162}{105} f_{n+1} + \tfrac{41}{140} f_n\right] \qquad (2.21)$$

3.2 TEST FOR CONVERGENCE

SCHEME 1 (a = 3/4, b = -1/3)

To prove that scheme 1, given by (2.19) converges, it is sufficient for us to show that it is consistent as well as zero-satble.

CONSISTENCY

From (2.19), the first characteristic polynomial $\rho(\xi)$ is given by

$$\rho(\xi) = \sum_{j=0}^{6} \alpha_j \xi^j \qquad (2.22)$$

$\Rightarrow$

$$\rho'(\xi) = \sum_{j=0}^{6} j\alpha_j \xi^{j-1}$$

$\Rightarrow$

$$\rho(1) = \sum_{j=0}^{6} \alpha_j = 1 - 5/6 + 5/6 - 1 = 0 \qquad (2.23)$$

$$\rho'(1) = \sum_{j=0}^{6} j\alpha_j = 6(1) - 5(5/6) + 1(5/6) - 0(0)$$

$$= 8/3 = 2.666666667 \qquad (2.24)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi) = \sum_{j=0}^{6} \beta_j \xi^j \qquad (2.25)$$

$$\sigma(1) = \sum_{j=0}^{6} \beta_j$$

$$= \frac{3401}{11340} + \frac{391}{315} - \frac{1117}{1260} + \frac{3848}{2835} - \frac{1117}{1260} + \frac{391}{315} + \frac{3401}{11340}$$

$$=2.666666667 \qquad (2.26)$$

From (2.23), (2.24) and (2.26) we have

$$\left.\begin{array}{l}(i)\ \rho(1) = 0 \\ (ii)\ \rho'(1) = \sigma(1)\end{array}\right\} \qquad (2.27)$$

Hence the scheme (2.19) (a = 3/4, b = -1/3) is consistent.

ZERO-STABILITY

$$\rho(\xi) = \xi^6 - {}^5\!/_6\,\xi^5 + {}^5\!/_6\,\xi - 1 \qquad (2.28)$$

Eqn (2.28) represents the first characteristic polynomial of scheme 1 (2.19). We are expected to show that no root of (2.28) has modulus greater than 1 and that every root with modulus 1 is simple, to establish zero-stability.

Already we know that (2.28) has two real roots at +1 and -1, i.e., (ξ^2 -1). To obtain the remaining four complex roots we carry out long division to obtain

$$\xi^4 - {}^5\!/_6\,\xi^3 + \xi^2 - {}^5\!/_6\,\xi + 1 = 0$$

we divide through by ξ^2 to have

$$\xi^2 - \tfrac{5}{6}\xi + 1 - \tfrac{5}{6} * \tfrac{1}{\xi} + \tfrac{1}{\xi^2} = 0$$

$$\xi^2 + \tfrac{1}{\xi^2} - \tfrac{5}{6}\left(\xi + \tfrac{1}{\xi}\right) + 1 = 0 \qquad (2.29)$$

$$let \;\; y = \xi + \tfrac{1}{\xi} \qquad (2.30)$$

from (2.30)

$$y^2 = \xi^2 + \tfrac{1}{\xi^2} + 2$$

$$\left(y^2 - 2\right) = \xi^2 + \tfrac{1}{\xi^2} \qquad (2.31)$$

substituting (2.31) int *o* (2.29)

$$y^2 - 2 - \tfrac{5}{6}y + 1 = 0$$

$$y^2 - \tfrac{5}{6}y - 1 = 0$$

$\Rightarrow$

$$y = \frac{\tfrac{5}{6} \pm \sqrt{\tfrac{25}{36} + 4}}{2} = \frac{\tfrac{5}{6} \pm \sqrt{\tfrac{169}{36}}}{2}$$

$$= \frac{\tfrac{5}{6} \pm \tfrac{13}{6}}{2}$$

$\Rightarrow$

$$y = \tfrac{3}{2} \qquad (2.32)$$

$$y = -\tfrac{2}{3} \qquad (2.33)$$

substituting (2.32) into (2.30)

$$\xi + \tfrac{1}{\xi} = \tfrac{3}{2}$$

$$2\xi^2 + 2 = 3\xi$$

$$2\xi^2 - 3\xi + 2 = 0$$

$\Rightarrow$

$$\xi = \frac{3 \pm \sqrt{9 - 16}}{4} = \frac{3 \pm \sqrt{-7}}{4}$$

$\Rightarrow$

$$\xi = \frac{3 + \sqrt{7}}{4} i \qquad or \qquad \xi = \frac{3 - \sqrt{7}}{4} i$$

substituting (2.33) into (2.30)

$$\xi + \tfrac{1}{\xi} = -\tfrac{2}{3}$$

$$3\xi^2 + 3 = -2\xi$$

$$3\xi^2 + 2\xi + 3 = 0$$

$\Rightarrow$

$$\xi = \frac{-2 \pm \sqrt{4 - 36}}{6} = \frac{-2 \pm \sqrt{-32}}{6}$$

$\Rightarrow$

$$\xi = \frac{-1 + 2\sqrt{2}}{3} i \qquad or \qquad \xi = \frac{-1 - 2\sqrt{2}}{3} i$$

Therefore, the six roots of $\rho\,(\xi)$ are,

$$\left.\begin{aligned}
\xi_1 &= +1 \\
\xi_2 &= -1 \\
\xi_3 &= \frac{3+\sqrt{7}}{4}i \\
\xi_4 &= \frac{3-\sqrt{7}}{4}i \\
\xi_5 &= \frac{-1+2\sqrt{2}}{3}i \\
\xi_6 &= \frac{-1-2\sqrt{2}}{3}i
\end{aligned}\right\} \qquad (2.34)$$

Next, we show that $|\xi_i| \leq 1$, $i = 3,\ 4,\ 5,\ 6$. It is obvious that $|\xi_i| = 1$, $i = 1,\ 2$.

$$|\xi_3| = \left|\frac{3+\sqrt{7}}{4}i\right| = \left|\frac{3}{4} + \frac{\sqrt{7}}{4}i\right| = \left[\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right]^{1/2}$$

$$= \left(\frac{9}{16} + \frac{7}{16}\right)^{1/2} = \left(\frac{16}{16}\right)^{1/2} = (1)^{1/2} \ 1 \qquad (2.35)$$

from (2.35), *it follows that* $|\xi_4| = \left|\frac{3-\sqrt{7}}{4}i\right| = 1$

$$|\xi_5 = \left|\frac{-1+2\sqrt{2}}{3}i\right| = \left|\frac{-1}{3} + \frac{2\sqrt{2}}{3}i\right| = \left[\left(\frac{-1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2\right]^{1/2}$$

$$= \left[\frac{1}{9} + \frac{8}{9}\right]^{1/2} = \left(\frac{9}{9}\right) = (1)^{1/2} = 1 \qquad (2.36)$$

Similarly, from (2.36)

$\Rightarrow$

$$|\xi_6| = \left| \frac{-1 - 2\sqrt{2}}{3} i \right| \quad = \quad 1$$

Thus ξ_i, $i = 1, 2, 3, 4, 5, 6$ satisfy the zero-stability condition.

Hence we conclude that scheme 1 (a = 3/4, b = -1/3) is convergent.

SCHEME 2 (a = 7/8, b = -7/8)

CONSISTENCY

From (2.20), the first characteristic polynomial $\rho(\xi)$ is given

$$\rho(\xi) = \sum_{j=0}^{6} \alpha_j \xi^j \qquad (2.37)$$

$$= \xi^6 - \tfrac{33}{16}\xi^4 + \tfrac{33}{16}\xi^2 - 1 \qquad (2.38)$$

$\Rightarrow$

$$\rho(1) = 1 - \tfrac{33}{16} + \tfrac{33}{16} - 1 = 0 \qquad (2.39)$$

And

$$\rho'(\xi) = \sum_{j=0}^{6} j\alpha_j \xi^{j-1}$$

$$= 6\xi^5 - \tfrac{33}{4}\xi^3 + \tfrac{33}{8}$$

$\Rightarrow$

$$\rho'(1) = 6 - \tfrac{33}{4} + \tfrac{33}{8} \quad \tfrac{15}{8} \quad = 1.875 \qquad (2.40)$$

The second characteristic polynomial $\sigma(\xi)$ is given by

$$\sigma(\xi)=\sum_{j=0}^{6}\beta_j\xi^j$$

$$=\tfrac{17547}{60480}\xi^6+\tfrac{443}{280}\xi^5-\tfrac{281}{448}\xi^4-\tfrac{155}{252}\xi^3-\tfrac{281}{448}\xi^2+\tfrac{443}{280}\xi+\tfrac{17547}{60480}$$

$$\sigma(1)=1.875 \qquad (2.41)$$

And from (2.39), (2.40) *and* (2.41) *we observe that*

$(i)\ \rho(1) = 0$

$(ii)\ \rho'(1) = \sigma(1)$

Hence scheme 2 (a = 7/8, b = -7/8) is consistent.

ZERO-STABILITY

We look for the roots of $\rho(\xi)$,

$$\rho(\xi)=\xi^6-\tfrac{33}{16}\xi^4+\tfrac{33}{16}\xi^2-1$$

We know that $\rho(\xi)$ has two real roots, $\xi_1 = +1$ and $\xi_2 = -1$, which implies that (ξ^2-1) is a factor. We carry out long division and obtain,

$$\xi^4 - \frac{17}{16}\xi^2 + 1 = 0$$

Dividing through by ξ^2

$$\xi^2 - \frac{17^2}{16} + \frac{1}{\xi^2} = 0$$

$$\xi^2 + \frac{1}{\xi^2} - \frac{17}{16} = 0 \qquad (2.42)$$

$$let \;\; y = \xi + \frac{1}{\xi} \qquad (2.43)$$

$\Rightarrow$

$$y^2 = \xi^2 + \frac{1}{\xi^2} + 2$$

$$\left(y^2 - 2\right) = \xi^2 + \frac{1}{\xi^2}$$

Substituting (2.44) into (2.42)

$$4\xi^2 - 7\xi + 4 = 0$$

$\Rightarrow$

$$\xi = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(4)}}{2(4)}$$

$$= \frac{7 \pm \sqrt{15}}{8}i$$

$\Rightarrow$

$$\xi_3 = \frac{7 \pm \sqrt{15}}{8}i$$

$$\xi_4 = \frac{7 - \sqrt{15}}{8}i$$

Similarly, substituting y = -7/4 into (2.43)

$$4\xi^2 + 7\xi + 4 = 0$$

This gives

$$\xi_5 = \frac{-7 + \sqrt{15}}{8}i$$

$$\xi_6 = \frac{-7-\sqrt{15}}{8}i$$

Therefore, the six roots of $\rho(\xi)$ (2.38) are

$$\left.\begin{aligned} \xi_1 &= +1 \\ \xi_2 &= -1 \\ \xi_3 &= \frac{7+\sqrt{15}}{8}i \\ \xi_4 &= \frac{7-\sqrt{15}}{8}i \\ \xi_5 &= \frac{-7+\sqrt{15}}{8}i \\ \xi_6 &= \frac{-7-\sqrt{15}}{8}i \end{aligned}\right\} \qquad (2.45)$$

And lastly, we show that $|\xi_i| \leq 1,\ i = 3,\ldots,6$

$$|\xi_3| = \left|\frac{7+\sqrt{15}}{8}i\right| = \left|\frac{7}{8} + \frac{\sqrt{15}}{8}i\right|$$

$$= \sqrt{\frac{7^2}{8} + \frac{15}{8^2}} = \sqrt{\frac{49}{64} + \frac{15}{64}}$$

$$= \sqrt{\left(\frac{64}{64}\right)} = 1$$

In view of the symmetry of the roots, it implies $|\xi_i| \leq 1,\ i = 3,\ldots,6$. Thereby establishing zero-stability.

And Hence, scheme 2(a = 7/8, b = -7/8) is convergent.

SCHEME 3 (a = 1/2, b = -1/2)

CONSISTENCY

From (2.1) above, the first characteristic polynomial is given as

$$\rho(\xi)=\sum_{j=0}^{6}\alpha_j\xi^j \qquad (2.46)$$
$$=\xi^6-1$$

$\Rightarrow$

$$\rho(1)=1-1=0 \qquad (2.47)$$

$$\rho(\xi)=\sum_{j=0}^{6}j\alpha_j\xi^j$$
$$=6\xi^5$$

$\Rightarrow$

$$\rho'(1)=6(1)=6 \qquad (2.48)$$

The second characteristic polynomial $\sigma(\xi)$ is,

$$\sigma(\xi)=\sum_{j=0}^{6}\beta_j\xi^j$$
$$=\frac{41}{140}\xi^6+\frac{162}{105}\xi^5+\frac{27}{140}\xi^4+\frac{68}{35}\xi^3+\frac{27}{140}\xi^2+\frac{162}{105}\xi+\frac{41}{140}$$
$$\sigma(1)=\frac{41}{140}+\frac{162}{105}+\frac{27}{140}+\frac{68}{35}+\frac{27}{140}+\frac{162}{105}+\frac{41}{140}$$
$$=6 \qquad (2.49)$$

from (2.47), (2.48), *and* (2.49) *we observe the following*

(i) $\rho(1)=0$

(ii) $\rho'(1)=\sigma(1)$

We therefore conclude that scheme 3 (a = 1/2, b = -1/2) is consistent.

ZERO-STABILITY

We find the roots of $\rho(\xi)$

$$\rho(\xi)=\xi^6-1=0$$

And we have the following real roots

$$\xi_1 = +1, \quad \xi_2 = -1$$

$$\Rightarrow$$

$$(\xi^2 - 1) \ \textit{is a factor of} \ \rho(\xi)$$

The other factor of $\sigma(\xi)$ is : $\xi^4+\xi^2+1$. To find the other roots, we put

$$\xi^4 + \xi^2 + 1 = 0$$

Dividing through by ξ^2

$$\xi^2 + \frac{1}{\xi^2} + 1 = 0 \qquad (2.50)$$

$$let \quad y = \xi + \frac{1}{\xi} \qquad (2.51)$$

$$\Rightarrow$$

$$y^2 = \xi^2 + \frac{1}{\xi^2} + 2$$

$$y^2 - 2 = \xi^2 + \frac{1}{\xi^2} \qquad (2.52)$$

substituting (2.52) *into* (2.50)

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y = \pm\sqrt{1}$$

$$= \pm 1$$

subtituting $y = 1$ *into* (2.51)

$$\xi + \frac{1}{\xi} = 1$$

$$\frac{\xi^2 + 1}{\xi} = 1$$

$$\xi^2 + 1 = \xi$$

$$\xi^2 - \xi + 1 = 0$$

$$\xi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}}{2} i$$

$$\Rightarrow$$

$$\xi_3 = \frac{1 + \sqrt{3}}{2} i,$$

$$\xi_4 = \frac{1 - \sqrt{3}}{2} i$$

Substituting $y = -1$ into (2.51)

$$\xi + \frac{1}{\xi} = -1$$

$$\xi^2 + 1 = -\xi$$

$$\xi^2 + \xi + 1 = 0$$

$$\xi = \frac{-1 \pm \sqrt{-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}}{2} i$$

$$\xi_5 = \frac{-1 + \sqrt{3}}{2} i$$

$$\xi_6 = \frac{-1 - \sqrt{3}}{2} i$$

Thus the six roots of $\rho(\xi)$ *are*

$$\xi_1 = +1$$

$$\xi_2 = -1$$

$$\xi_3 = \frac{1 + \sqrt{3}}{2} i$$

$$\xi_4 = \frac{1 - \sqrt{3}}{2} i$$

$$\xi_5 = \frac{-1 + \sqrt{3}}{2} i$$

$$\xi_6 = \frac{-1 - \sqrt{3}}{2} i$$

Lastly, we show that $|\xi_i| \leq 1,\ i = 3, 4, 5, 6.$

$$|\xi_3| = \left|\frac{1 + \sqrt{3}}{2} i\right| = \left[\left(\frac{1}{2}\right)^2 + \frac{3}{4}\right]^{½} = \left[\frac{1}{4} + \frac{3}{4}\right]^{½} = \left(\frac{4}{4}\right)^{½}$$

$$= (1)^{½} = 1$$

$$= |\xi_4| = |\xi_5| = |\xi_6|$$

This implies that the scheme is zero-stable. Therefore, the scheme (a = 1/2 , b = -1/2) is convergent.

3.3 TEST FOR ABSOLUTE STABILITY

The linear multistep methods we have developed are optimal methods (i.e. of order k+2. In this case, order 8). And as with all optimal methods they have no interval of absolute stability (Lambert, 1973).

CHAPTER FOUR

APPLICATION AND COMPARISON OF RESULTS

4.1 APPLICATIONS

We use the 3 schemes to solve various differential equations. To start with, we solve the following differential equation

$$y' = x + y\,;\;\; y(0) = 1\,,\;\; h = 0.1 \qquad (4.1)$$

STARTING VALUES

As with all k-step methods(k>1) we face the problem of generating additional starting values. Also, we demand that these starting values should be calculated to an accuracy at least as high as the local accuracy of the main method. This means that any method we use to calculate the starting values must itself require no starting values other than y_0.

In this work, we decide to use the Fourth Order Runge-Kutta method to evaluate the starting values y_n , n=0, 1, 2,…,5, since the Runge-Kutta methods constitute the most efficient method for generating starting values for linear multistep methods.

The Fourth order Runge-Kutta method is given below:

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1) \qquad (4.2) \\ k_3 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2) \\ k_4 &= f(x_n + h, y_n + hk_3) \end{aligned}$$

To use the above for generating $y_0, y_1, \ldots, y_5$ for problem (4.1) we have

n = 0

$x_0 = 0$

$y_0 = 1$

$$k_1 = f(x_0, y_0) = f(0,) = 0 + 1$$

$$k_2 = f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}hk_1) = f(0 + 0.5*0.1, 1 + 0.5*0.1*1)$$

$$= f(0.05, 1.05) = 0.05 + 1.05$$

$$= 1.1$$

$$k3 = f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}hk_2) = f(0 + 0.5*0.1, 1 + 0.5*0.1*1.1)$$

$$= f(0.05, 1.055) = 0.05 + 1.055$$

$$= 1.105$$

$$k_4 = f(x_0 + h,\ y_0 + hk_3) = f(0 + 0.1, 1 + 0.1*1.105)$$

$$= f(0.1, 1.1105) = 0.1 + 1.1105$$

$$= 1.2105$$

substituting y_0, h, $k_1, \ldots, k_4$, into (4.2) gives

$$y_1 = y_0 + \tfrac{h}{6}(1 + 2(1.1) + 2(1.105) + 1.2105)$$

$$= 1 + \tfrac{0.1}{6}(6.6205) = 1 + 0.110341667$$

$$y_1 = 1.110341667$$

$$n = 1$$

$$x_1 = 0.1$$

$$y_1 = 1.110341667$$

$$y_2 = y_1 + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (4.3)$$

$$k_1 = f(x_1, y_1) = f(0.1, 1.110341667) = 0.1 + 1.110341667$$

$$= 1.210341667$$

$$k_2 = f(x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}hk_1)$$

$$= f(0.1 + 0.5*0.1, 1.110341667 + 0.5*0.1*1.210341667)$$

$$= f(0.15, 1.17085875) = 0.15 + 1.17085875$$

$$= 1.32085875$$

$$k_3 = f(x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}hk_2)$$

$$= f(0.1 + 0.5*0.1, 1.110341667 + 0.5*0.1*1.32085875)$$

$$= f(0.15, 1.176384605)$$

$$= 1.326384605$$

$$k_4 = f(x_1 + h,\ y_1 + hk_3)$$

$$= f(0.1 + 0.1, 1.110341667 + 0.1*1.326384605)$$

$$= f(0.2, 1.242980128) = 0.2 + 1.242980128$$

$$= 1.442980158$$

On substituting y_1, h, $k_1, \ldots, k_4$ into (4.3)

$y_2 = 1.110341667 + {}^{0.1}\!/_6(1.210341667 + 2(1.32085875) + 2(1.326384605) + 1.442980128)$

$= 1.110341667 + 0.13246475$

$y_2 = 1.242805142$

$n = 2$

$x_2 = 0.2$

$y_2 = 1.242805142$

$$y_3 = y_2 + {}^{h}\!/_6(k_1 + 2k_2 + 2k_3 + k_4) \qquad (4.4)$$

$k_1 = f(x_2, y_2) = f(0.2, 1.242805142) = 0.2 + 1.242805142$

$= 1.442805142$

$k_2 = f(x_2 + \tfrac{1}{2}h,\ y_2 + \tfrac{1}{2}hk_1)$

$= f(0.2 + 0.5 * 0.1, 1.242805142 + 0.5 * 0.1 * 1.442805142)$

$= f(0.25, 1.314945399)$

$= 0.25 + 1.314945399$

$= 1.564945399$

$k_3 = f(x_2 + \tfrac{1}{2}h,\ y_2 + \tfrac{1}{2}hk_2)$

$= f(0.2 + 0.5 * 0.1, 1.242805142 + 0.5 * 0.1 * 1.564945399)$

$= f(0.25, 1.321052412)$

$= 0.25 + 1.321052412$

$= 1.571052412$

$k_4 = f(x_2 + h,\ y_2 + hk_3)$

$= f(0.2 + 0.1, 1.242805142 + 0.1 * 1.571052412)$

$= f(0.3, 1.399910383)$

$= 1.699910383$

And (4.4) becomes

$y_3 = 1.242805142 + {}^{0.1}\!/_6(1.442805142 + 2(1.564945399) + 2(1.571052412) + 1.699910383)$

$= 1.242805142 + 0.156911852$

$y_3 = 1.399716994$

$n = 3$

$x_3 = 0.3$

$y_3 = 1.399716994$

$$y_4 = y_3 + {}^{h}\!/_6(k_1 + 2k_2 + 2k_3 + k_4) \qquad (4.5)$$

$k_1 = f(x_3, y_3) = f(0.3, 1.399716994)$

$=1.699716994$

$k_2 = f(x_3 + \frac{1}{2}h, y_3 + \frac{1}{2}hk_1)$

$= f(0.3 + 0.5 * 0.1, 1.399716994 + 0.5 * 0.1 * 1.699716994)$

$= f(0.35, 1.484702844)$

$= 0.35 + 1.484702844$

$= 1.834702844$

$k_3 = f(x_3 + \frac{1}{2}h, y_3 + \frac{1}{2}hk_2)$

$= f(0.3 + 0.5 * 0.1, 1.399716994 + 0.5 * 0.1 * 1.834702844)$

$= f(0.35, 1.491452136)$

$= 0.35 + 1.491452136$

$= 1.841452136$

$k_4 = f(x_3 + h, y_3 + hk_3)$

$= f(0.3 + 0.1, 1.399716994 + 0.1 * 1.841452136)$

$= f(0.4, 1.583862208)$

$= 0.4 + 1.583862208$

$= 1.983862208$

Therefore (4.5) becomes

$y_4 = 1.399716994 + 0.183931486$

$y_4 = 1.583648480$

$n = 4$

$x_4 = 0.4$

$y_4 = 1.583648480$

$$y_5 = y_4 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (4.6)$$

$k_1 = f(x_4, y_4)$

$= f(0.4, 1.583648480)$

$= 0.4 + 1.583648480$

$= 1.98364848$

$k_2 = f(x_4 + \frac{1}{2}h, y_4 + \frac{1}{2}hk_1)$

$= f(0.4 + 0.5 * 0.1, 1.583648480 + 0.5 * 0.1 * 1.98364848)$

$= f(0.45, 1.682830904)$

$= 0.45 + 1.652830904$

$= 2.132830904$

$k_3 = f(x_4 + \frac{1}{2}h, y_4 + \frac{1}{2}hk_2)$

$= f(0.4 + 0.5 * 0.1, 1.583648480 + 0.5 * 0.1 * 2.132830904)$

$= f(0.45, 1.690290025)$

$= 0.45 + 1.690290025$

$= 2.140290025$

$k_4 = f(x_4 + h, y_4 + hk_3)$

$= f(0.4 + 0.1, 1.583648480 + 0.1 * 2.140290025)$

$= f(0.5, 1.797677483)$

$= 0.5 + 1.797677483$

$= 2.297677483$

And (4.6) becomes

$y_5 = 1.583648480 + {}^{0.1}\!/_6(1.98364848 + 2(1.32830904) + 2(2.140290025) + 20297677483$

$= 1.583648480 + 0.213792797$

$y_5 = 1.797441277$

THE PREDICTOR

We choose as our predictor, the fourth order Adams-Bashforth method:

$$y_{n+4} = y_{n+3} + \tfrac{h}{24}(55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n)$$

in order to solve the given problem $(y' = x + y)$

when n = 2

$$y_6^P = y_5 + \tfrac{h}{24}(55f_5 - 59f_4 + 37f_3 - 9f_2)$$

$$\begin{aligned} f_2 &= f(x_2, y_2) \\ &= f(0.2, 1.242805142) \\ &= 0.2 + 1.242805142 \\ &= 1.442805142 \end{aligned}$$

$$\begin{aligned} f_3 &= f(x_3, y_3) \\ &= f(0.3, 1.399716994) \\ &= 0.3 + 1.399716994 \\ &= 1.699716994 \end{aligned}$$

$$\begin{aligned} f_4 &= f(x_4, y_4) \\ &= f(0.4, 1.583648480) \\ &= 0.4 + 1.583648480 \\ &= 1.98364848 \end{aligned}$$

$$\begin{aligned} f_5 &= f(x_5, y_5) \\ &= f(0.5, 1.797441277) \\ &= 0.5 + 1.797441277 \\ &= 2.297441277 \end{aligned}$$

$$\begin{aligned} y_6^P &= 1.797441277 + \tfrac{0.1}{24}[55(2.297441277) - 59(1.98364848) + 37(1.699716994) - 9(1.442805142)] \\ &= 1.797441277 + 0.246784551 \end{aligned}$$

$$y_6^P = 2.044225829$$

HE CORRECTORS

r 6-step implicit linear multistep method contains two free parameters(a and b). We ve produced three(3) schemes by assigning values to these parameters. We now solve given problem by each of these schemes and compare their accuracies thus:

SCHEME 1 (a=3/4, b=-1/3)

$$y^c_{n+6} = \tfrac{5}{6} y_{n+5} - \tfrac{5}{6} y_{n+1} + y_n + h\left\{\tfrac{3401}{11340} f^p_{n+6} + \tfrac{391}{315} f_{n+5} - \tfrac{1117}{1260} f_{n+4} + \tfrac{3848}{2835} f_{n+3} - \tfrac{1117}{1260} f_{n+2} + \tfrac{391}{315} f_{n+1} + \tfrac{3401}{11340} f_n\right\}$$

When n = 0

$$y^c_6 = \tfrac{5}{6} y_5 - \tfrac{5}{6} y_1 + y_0 + h\left\{\tfrac{3401}{11340} f^p_6 + \tfrac{391}{315} f_5 - \tfrac{1117}{1260} f_4 + \tfrac{3848}{2835} f_3 - \tfrac{1117}{1260} f_2 + \tfrac{391}{315} f_1 + \tfrac{3401}{11340} f_0\right\}$$

values for y_5, y_1, and y_0 are known from the calculated starting values. So also, $f_2, \ldots,$ f_5.

$$\begin{aligned} f_0 &= f(x_0, y_0) \\ &= f(0,1) \\ &= 0+1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f_1 &= f(x_1, y_1) \\ &= f(0.1, 1.110341667) \\ &= 0.1 + 1.110341667 \\ &= 1.210341667 \end{aligned}$$

$$\begin{aligned} f_6 &= f(x_6, y^p_6) \\ &= f(0.6, 2.044225829) \\ &= 0.6 + 2.044225829 \\ &= 2.644225829 \end{aligned}$$

$$y^c_6 = \tfrac{5}{6}(1.797441277) - \tfrac{5}{6}(1.110341667) + 1 + 0.1\left\{\tfrac{3401}{11340}(2.644225829) + \tfrac{391}{315}(2.297441277)\right.$$

$$\left. - \tfrac{1117}{1260}(1.98364848) + \tfrac{3848}{2835}(1.699716994) - \tfrac{1117}{1260}(1.442805142) + \tfrac{391}{315}(1.210341667) + \tfrac{3401}{11340}(1)\right\}$$

$=1.572583008 + 0.471653178$

$y^c_6 = 2.044236187$

SCHEME 2 (a=7/8, b=-7/8)

$$y^c_{n+6} = \tfrac{33}{16} y_{n+4} - \tfrac{33}{16} y_{n+2} + y_n + h\left\{\tfrac{17547}{60480} f^p_{n+6} + \tfrac{443}{280} f_{n+5} - \tfrac{281}{448} f_{n+4} - \tfrac{155}{252} f_{n+3} - \tfrac{281}{448} f_{n+2} + \tfrac{443}{280} f_{n+1} + \tfrac{17547}{60480} f_n\right\}$$

When n = 0

$$y_6^c = {}^{33}\!/_{16}y_4 - {}^{33}\!/_{16}y_2 + y_0 + h\left\{\tfrac{17547}{60480}f_6^P + \tfrac{443}{280}f_5 - \tfrac{281}{448}f_4 - \tfrac{155}{252}f_3 - \tfrac{281}{448}f_2 + \tfrac{443}{280}f_1 + \tfrac{17547}{60480}f_0\right\}$$

$$y_6^c = {}^{33}\!/_{16}(1.583648480) - {}^{33}\!/_{16}(1.242805142) + 1 + 0.1\{\tfrac{17547}{60480}(2.644225829) + \tfrac{443}{280}(2.297441277)$$

$$-\tfrac{281}{448}(1.98364848) - \tfrac{155}{252}(1.699716994) - \tfrac{281}{448}(1.442805142) + \tfrac{443}{280}(1.210341667) + \tfrac{17547}{60480}(1)\}$$

$$=1.702989385 + 0.341246651$$

$$y_6^c = 2.044236036$$

SCHEME 3 (a=1/2, b=-1/2)

$$y_{n+6}^c = y_n + h\left\{\tfrac{41}{140}f_{n+6}^P + \tfrac{162}{105}f_{n+5} - \tfrac{27}{140}f_{n+4} - \tfrac{68}{35}f_{n+3} + \tfrac{27}{140}f_{n+2} + \tfrac{162}{105}f_{n+1} + \tfrac{41}{140}f_n\right\}$$

When n = 0

$$y_6^c = y_0 + h\left\{\tfrac{41}{140}f_6^P + \tfrac{162}{105}f_5 - \tfrac{27}{140}f_4 - \tfrac{68}{35}f_3 + \tfrac{27}{140}f_2 + \tfrac{162}{105}f_1 + \tfrac{41}{140}f_0\right\}$$

$$y_{n+6}^c = 1 + 0.1\left\{\begin{matrix}\tfrac{41}{140}(2.644225829) + \tfrac{162}{105}(2.297441277) + \tfrac{27}{140}(1.98364848) + \tfrac{68}{35}(1.699716994) \\ + \tfrac{27}{140}(1.442805142) + \tfrac{162}{105}(1.210341667) + \tfrac{41}{140}(1)\end{matrix}\right\}$$

$$=1+1.044236889$$

$$y_6^c = 2.044236889$$

4.2 COMPARISON OF RESULTS

In the same way we solve the following problems, using the three schemes. Their results are obtained and compared for accuracy in the following pages. The problems are implemented on computer using Microsoft Excel software package.

1. $y' = x + y$; $y(0) = 1$, $h = 0.1$
2. $y' = {}^{(1+y)}\!/_{(2+x)}$; $y(0) = 1$; $h = 0.1$
3. $y' = 4xy^{1/2}$; $y(0) = 1$; $h = 0.1$
4. $y' = 3x^2 - 6x + 5$; $y(0) = 1$, $h = 0.1$
5. $y' = x^5 + 2x^4 + 3x^3$; $y(0) = 1$, $h = 0.1$

TABLE 4.1

PROBLEM: F=X+Y ; Y(0)=1 ; h=0.1

EXACT SOLUTION: Y(X)=2*EXP(X)-X-1

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.1103418362	1.1103416667	1.6948462878E-07	1.1103416667	1.6948462878E-07	1.1103416667	1.6948462878E-07
0.2	1.2428055163	1.2428051417	3.7461895075E-07	1.2428051417	3.7461895075E-07	1.2428051417	3.7461895075E-07
0.3	1.3997176152	1.3997169941	6.2102693121E-07	1.3997169941	6.2102693121E-07	1.3997169941	6.2102693121E-07
0.4	1.5836493953	1.5836484802	9.1512116951E-07	1.5836484802	9.1512116951E-07	1.5836484802	9.1512116951E-07
0.5	1.7974425414	1.7974412772	1.2642065803E-06	1.7974412772	1.2642065803E-06	1.7974412772	1.2642065803E-06
0.6	2.0442376008	2.0442361876	1.4132161703E-06	2.0442360367	1.5640399451E-06	2.0442368893	7.1145714031E-07
0.7	2.3275054149	2.3275025204	2.8945628872E-06	2.3275017844	3.6305474218E-06	2.3275027970	2.6179097690E-06
0.8	2.6510818570	2.6510783589	3.4980860688E-06	2.6510783329	3.5240506189E-06	2.6510785490	3.3079643345E-06
0.9	3.0192062223	3.0192004614	5.7608878645E-06	3.0191990040	7.2183356430E-06	3.0191999381	6.2842580402E-06
1.0	3.4365636569	3.4365566462	7.0107547572E-06	3.4365575986	6.0583367980E-06	3.4365562683	7.3885805181E-06
1.1	3.9083320479	3.9083216889	1.0358962205E-05	3.9083199922	1.2055702603E-05	3.9083210361	1.1011780565E-05
1.2	4.4402338455	4.4402212279	1.2617575122E-05	4.4402236466	1.0198845668E-05	4.4402219958	1.1849689056E-05
1.3	5.0385933352	5.0385763860	1.6949279956E-05	5.0385745510	1.8784218881E-05	5.0385768406	1.6494681973E-05
1.4	5.7103999337	5.7103801780	1.9755707239E-05	5.7103833676	1.6566062193E-05	5.7103811441	1.8789605289E-05
1.5	6.4633781407	6.4633529509	2.5189778107E-05	6.4633497956	2.8345026974E-05	6.4633527945	2.5346198219E-05
1.6	7.3060648488	7.3060359464	2.8902397775E-05	7.3060396496	2.5199228915E-05	7.3060362346	2.8614217014E-05
1.7	8.2478947835	8.2478589594	3.5824020866E-05	8.2478531024	4.1681091048E-05	8.2478579347	3.6848744299E-05
1.8	9.2992949288	9.2992536541	4.1274708437E-05	9.2992589103	3.6018491270E-05	9.2992543020	4.0626866555E-05
1.9	10.4717888846	10.4717380840	5.0800516968E-05	10.4717292061	5.9678501644E-05	10.4717377097	5.1174813139E-05
2.0	11.7781121979	11.7780537315	5.8466353456E-05	11.7780627858	4.9412037782E-05	11.7780548023	5.7395610558E-05
2.1	13.2323398251	13.2322688646	7.0960547497E-05	13.2322562777	8.3547452116E-05	13.2322682828	7.1542355320E-05
2.2	14.8500269989	14.8499456795	8.1319408944E-05	14.8499606151	6.6383748036E-05	14.8499470060	7.9992830319E-05
2.3	16.6483649096	16.6482678243	9.7085343281E-05	16.6482495609	1.1534875772E-04	16.6482669767	9.7932908339E-05
2.4	18.6463527613	18.6462423132	1.1044809521E-04	18.6462645674	8.8193876170E-05	18.6462441964	1.0856490735E-04
2.5	20.8649879214	20.8648572068	1.3071457665E-04	20.8648296964	1.5822501940E-04	20.8648564410	1.3148036366E-04

2.6	23.3274760700	23.3273275599	1.4851010909E-04	23.3273601907	1.1587931595E-04	23.3273293895	1.4668045978E-04
2.7	26.0594634497	26.0592887044	1.7474534302E-04	26.0592471700	2.1627975152E-04	26.0592868419	1.7660784666E-04
2.8	29.0892935422	29.0890948527	1.9868948952E-04	29.0891434972	1.5004498071E-04	29.0890967908	1.9675134766E-04
2.9	32.4482907389	32.4480579595	2.3277937285E-04	32.4479962035	2.9453539831E-04	32.4480559810	2.3475790909E-04
3.0	36.1710738464	36.1708097603	2.6408608579E-04	36.1708828835	1.9096282971E-04	36.1708132391	2.6060725960E-04

TABLE 4.2

PROBLEM: F=(1+Y)/(2+X) ; Y(0)=1 ; h=0.1

EXACT SOLUTION: Y(X)=2+X-1

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.1000000000	1.1000000000	0.0000000000E+00	1.1000000000	0.0000000000E+00	1.1000000000	0.0000000000E+00
0.2	1.2000000000	1.2000000000	0.0000000000E+00	1.2000000000	0.0000000000E+00	1.2000000000	0.0000000000E+00
0.3	1.3000000000	1.3000000000	0.0000000000E+00	1.3000000000	0.0000000000E+00	1.3000000000	0.0000000000E+00
0.4	1.4000000000	1.4000000000	0.0000000000E+00	1.4000000000	0.0000000000E+00	1.4000000000	0.0000000000E+00
0.5	1.5000000000	1.5000000000	0.0000000000E+00	1.5000000000	0.0000000000E+00	1.5000000000	0.0000000000E+00
0.6	1.6000000000	1.6000000000	0.0000000000E+00	1.6000000000	0.0000000000E+00	1.6000000000	0.0000000000E+00
0.7	1.7000000000	1.7000000000	0.0000000000E+00	1.7000000000	0.0000000000E+00	1.7000000000	0.0000000000E+00
0.8	1.8000000000	1.8000000000	0.0000000000E+00	1.8000000000	0.0000000000E+00	1.8000000000	0.0000000000E+00
0.9	1.9000000000	1.9000000000	0.0000000000E+00	1.9000000000	0.0000000000E+00	1.9000000000	0.0000000000E+00
1.0	2.0000000000	2.0000000000	0.0000000000E+00	2.0000000000	0.0000000000E+00	2.0000000000	0.0000000000E+00
1.1	2.1000000000	2.1000000000	0.0000000000E+00	2.1000000000	0.0000000000E+00	2.1000000000	0.0000000000E+00
1.2	2.2000000000	2.2000000000	0.0000000000E+00	2.2000000000	0.0000000000E+00	2.2000000000	0.0000000000E+00
1.3	2.3000000000	2.3000000000	0.0000000000E+00	2.3000000000	0.0000000000E+00	2.3000000000	0.0000000000E+00
1.4	2.4000000000	2.4000000000	0.0000000000E+00	2.4000000000	0.0000000000E+00	2.4000000000	0.0000000000E+00
1.5	2.5000000000	2.5000000000	0.0000000000E+00	2.5000000000	0.0000000000E+00	2.5000000000	0.0000000000E+00
1.6	2.6000000000	2.6000000000	0.0000000000E+00	2.6000000000	0.0000000000E+00	2.6000000000	0.0000000000E+00
1.7	2.7000000000	2.7000000000	0.0000000000E+00	2.7000000000	0.0000000000E+00	2.7000000000	0.0000000000E+00
1.8	2.8000000000	2.8000000000	0.0000000000E+00	2.8000000000	0.0000000000E+00	2.8000000000	0.0000000000E+00
1.9	2.9000000000	2.9000000000	0.0000000000E+00	2.9000000000	0.0000000000E+00	2.9000000000	0.0000000000E+00
2.0	3.0000000000	3.0000000000	0.0000000000E+00	3.0000000000	0.0000000000E+00	3.0000000000	0.0000000000E+00
2.1	3.1000000000	3.1000000000	0.0000000000E+00	3.1000000000	0.0000000000E+00	3.1000000000	0.0000000000E+00
2.2	3.2000000000	3.2000000000	0.0000000000E+00	3.2000000000	0.0000000000E+00	3.2000000000	0.0000000000E+00
2.3	3.3000000000	3.3000000000	0.0000000000E+00	3.3000000000	0.0000000000E+00	3.3000000000	0.0000000000E+00
2.4	3.4000000000	3.4000000000	0.0000000000E+00	3.4000000000	0.0000000000E+00	3.4000000000	0.0000000000E+00
2.5	3.5000000000	3.5000000000	0.0000000000E+00	3.5000000000	0.0000000000E+00	3.5000000000	0.0000000000E+00

2.6	3.6000000000	3.6000000000	0.0000000000E+00	3.6000000000	0.0000000000E+00	3.6000000000	0.0000000000E+00
2.7	3.7000000000	3.7000000000	0.0000000000E+00	3.7000000000	0.0000000000E+00	3.7000000000	0.0000000000E+00
2.8	3.8000000000	3.8000000000	0.0000000000E+00	3.8000000000	0.0000000000E+00	3.8000000000	0.0000000000E+00
2.9	3.9000000000	3.9000000000	0.0000000000E+00	3.9000000000	0.0000000000E+00	3.9000000000	0.0000000000E+00
3.0	4.0000000000	4.0000000000	0.0000000000E+00	4.0000000000	0.0000000000E+00	4.0000000000	0.0000000000E+00

TABLE 4.3

PROBLEM: $F=4XY^{1/2}$; $Y(0)=1$; $h=0.1$

EXACT SOLUTION: $Y(X)=(1+X^2)^2$

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.0201000000	1.0200999163	8.3740235413E-08	1.0200999163	8.3740235413E-08	1.0200999163	8.3740235413E-08
0.2	1.0816000000	1.0815996645	3.3550094458E-07	1.0815996645	3.3550094458E-07	1.0815996645	3.3550094458E-07
0.3	1.1881000000	1.1880992364	7.6362208823E-07	1.1880992364	7.6362208823E-07	1.1880992364	7.6362208823E-07
0.4	1.3456000000	1.3455986013	1.3986813359E-06	1.3455986013	1.3986813359E-06	1.3455986013	1.3986813359E-06
0.5	1.5625000000	1.5624976976	2.3024470428E-06	1.5624976976	2.3024470428E-06	1.5624976976	2.3024470428E-06
0.6	1.8496000000	1.8495978929	2.1071298080E-06	1.8495975424	2.4576180864E-06	1.8495995447	4.5533008630E-07
0.7	2.2201000000	2.2200981516	1.8483568627E-06	2.2200964953	3.5046751483E-06	2.2200993118	6.8822349020E-07
0.8	2.6896000000	2.6895984215	1.5784759917E-06	2.6895972208	2.7791784922E-06	2.6895991149	8.8509088902E-07
0.9	3.2761000000	3.2760986987	1.3012728237E-06	3.2760963052	3.6947909670E-06	3.2760984239	1.5761347201E-06
1.0	4.0000000000	3.9999990162	9.8383706826E-07	3.9999975811	2.4189371315E-06	3.9999979918	2.0081629217E-06
1.1	4.8841000000	4.8840981581	1.8418632006E-06	4.8840967202	3.2797895031E-06	4.8840967455	3.2544755930E-06
1.2	5.9536000000	5.9535974469	2.5530644727E-06	5.9535980322	1.9677553009E-06	5.9535987717	1.2282875765E-06
1.3	7.2361000000	7.2360967391	3.2608515799E-06	7.2360965978	3.4022265156E-06	7.2360981558	1.8442473371E-06
1.4	8.7616000000	8.7615962507	3.7492648612E-06	8.7615978500	2.1500092213E-06	8.7615982078	1.7921617435E-06
1.5	10.5625000000	10.5624956727	4.3272978019E-06	10.5624951201	4.8799002830E-06	10.5624969526	3.0473593657E-06
1.6	12.6736000000	12.6735962988	3.7012086089E-06	12.6735968589	3.1411350534E-06	12.6735969586	3.0414207082E-06
1:7	15.1321000000	15.1320963823	3.6176896288E-06	15.1320925581	7.4419399390E-06	15.1320950636	4.9364353139E-06
1.8	17.9776000000	17.9775963939	3.6061437712E-06	17.9775956647	4.3352589891E-06	17.9775975260	2.4740253792E-06
1.9	21.2521000000	21.2520958930	4.1069518950E-06	21.2520899631	1.0036875629E-05	21.2520962936	3.7064239535E-06
2.0	25.0000000000	24.9999955725	4.4274541793E-06	24.9999951624	4.8376380164E-06	24.9999968078	3.1922454156E-06
2.1	29.2681000000	29.2680939912	6.0088282900E-06	29.2680881025	1.1897538226E-05	29.2680947794	5.2205605847E-06
2.2	34.1056000000	34.1055933300	6.6700044954E-06	34.1055957348	4.2652129011E-06	34.1055954115	4.5885001470E-06
2.3	39.5641000000	39.5640926544	7.3455959537E-06	39.5640866823	1.3317726157E-05	39.5640926835	7.3165207866E-06
2.4	45.6976000000	45.6975926406	7.3594409287E-06	45.6975969072	3.0928104309E-06	45.6975957723	4.2277215755E-06

2.5	52.5625000000	52.5624921577	7.8423257932E-06	52.5624845958	1.5404220747E-05	52.5624937303	6.2697053380E-06
2.6	60.2176000000	60.2175928744	7.1256417371E-06	60.2175978211	2.1788532223E-06	60.2175948951	5.1048865330E-06
2.7	68.7241000000	68.7240922089	7.7911376621E-06	68.7240809352	1.9064828791E-05	68.7240919232	8.0767626969E-06
2.8	78.1456000000	78.1455916661	8.3339246970E-06	78.1455981052	1.8947844893E-06	78.1455933373	6.6627318063E-06
2.9	88.5481000000	88.5480903074	9.6926423794E-06	88.5480757926	2.4207356077E-05	88.5480896261	1.0373880329E-05
3.0	100.0000000000	99.9999897082	1.0291754307E-05	99.9999983084	1.6915813177E-06	99.9999934915	6.5085275764E-06

TABLE 4.4

PROBLEM: $F=3X^2-6X+5$; $Y(0)=1$; $h=0.1$

EXACT SOLUTION: $Y(X)=X^3-3X^2+5X+1$

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.4710000000	1.4710000000	0.0000000000E+00	1.4710000000	0.0000000000E+00	1.4710000000	0.0000000000E+00
0.2	1.8880000000	1.8880000000	0.0000000000E+00	1.8880000000	0.0000000000E+00	1.8880000000	0.0000000000E+00
0.3	2.2570000000	2.2570000000	0.0000000000E+00	2.2570000000	0.0000000000E+00	2.2570000000	0.0000000000E+00
0.4	2.5840000000	2.5840000000	0.0000000000E+00	2.5840000000	0.0000000000E+00	2.5840000000	0.0000000000E+00
0.5	2.8750000000	2.8750000000	0.0000000000E+00	2.8750000000	0.0000000000E+00	2.8750000000	0.0000000000E+00
0.6	3.1360000000	3.1360000000	0.0000000000E+00	3.1360000000	0.0000000000E+00	3.1360000000	0.0000000000E+00
0.7	3.3730000000	3.3730000000	0.0000000000E+00	3.3730000000	0.0000000000E+00	3.3730000000	0.0000000000E+00
0.8	3.5920000000	3.5920000000	0.0000000000E+00	3.5920000000	0.0000000000E+00	3.5920000000	0.0000000000E+00
0.9	3.7990000000	3.7990000000	0.0000000000E+00	3.7990000000	0.0000000000E+00	3.7990000000	0.0000000000E+00
1.0	4.0000000000	4.0000000000	0.0000000000E+00	4.0000000000	0.0000000000E+00	4.0000000000	0.0000000000E+00
1.1	4.2010000000	4.2010000000	0.0000000000E+00	4.2010000000	0.0000000000E+00	4.2010000000	0.0000000000E+00
1.2	4.4080000000	4.4080000000	0.0000000000E+00	4.4080000000	0.0000000000E+00	4.4080000000	0.0000000000E+00
1.3	4.6270000000	4.6270000000	0.0000000000E+00	4.6270000000	0.0000000000E+00	4.6270000000	0.0000000000E+00
1.4	4.8640000000	4.8640000000	0.0000000000E+00	4.8640000000	0.0000000000E+00	4.8640000000	0.0000000000E+00
1.5	5.1250000000	5.1250000000	0.0000000000E+00	5.1250000000	0.0000000000E+00	5.1250000000	0.0000000000E+00
1.6	5.4160000000	5.4160000000	0.0000000000E+00	5.4160000000	0.0000000000E+00	5.4160000000	0.0000000000E+00
1.7	5.7430000000	5.7430000000	0.0000000000E+00	5.7430000000	0.0000000000E+00	5.7430000000	0.0000000000E+00
1.8	6.1120000000	6.1120000000	0.0000000000E+00	6.1120000000	0.0000000000E+00	6.1120000000	0.0000000000E+00
1.9	6.5290000000	6.5290000000	0.0000000000E+00	6.5290000000	0.0000000000E+00	6.5290000000	0.0000000000E+00
2.0	7.0000000000	7.0000000000	0.0000000000E+00	7.0000000000	0.0000000000E+00	7.0000000000	0.0000000000E+00
2.1	7.5310000000	7.5310000000	0.0000000000E+00	7.5310000000	0.0000000000E+00	7.5310000000	0.0000000000E+00
2.2	8.1280000000	8.1280000000	0.0000000000E+00	8.1280000000	0.0000000000E+00	8.1280000000	0.0000000000E+00
2.3	8.7970000000	8.7970000000	0.0000000000E+00	8.7970000000	0.0000000000E+00	8.7970000000	0.0000000000E+00
2.4	9.5440000000	9.5440000000	0.0000000000E+00	9.5440000000	0.0000000000E+00	9.5440000000	0.0000000000E+00

2.5	10.3750000000	10.3750000000	0.0000000000E+00	10.3750000000	0.0000000000E+00	10.3750000000	0.0000000000E+00
2.6	11.2960000000	11.2960000000	0.0000000000E+00	11.2960000000	0.0000000000E+00	11.2960000000	0.0000000000E+00
2.7	12.3130000000	12.3130000000	0.0000000000E+00	12.3130000000	0.0000000000E+00	12.3130000000	0.0000000000E+00
2.8	13.4320000000	13.4320000000	0.0000000000E+00	13.4320000000	0.0000000000E+00	13.4320000000	0.0000000000E+00
2.9	14.6590000000	14.6590000000	0.0000000000E+00	14.6590000000	0.0000000000E+00	14.6590000000	0.0000000000E+00
3.0	16.0000000000	16.0000000000	0.0000000000E+00	16.0000000000	0.0000000000E+00	16.0000000000	0.0000000000E+00

TABLE 4.5

PROBLEM: $F=X^5+2X^4+3X^3$; $Y(0)=1$; $h=0.1$

EXACT SOLUTION: $Y(X)=(X^6/6)+(2X^5/5)+(3X^4/4)+1$

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.0000791667	1.0000793542	1.8749999997E-07	1.0000793542	1.8749999997E-07	1.0000793542	1.8749999997E-07
0.2	1.0013386667	1.0013390833	4.1666666672E-07	1.0013390833	4.1666666672E-07	1.0013390833	4.1666666672E-07
0.3	1.0071685000	1.0071691875	6.8750000004E-07	1.0071691875	6.8750000004E-07	1.0071691875	6.8750000004E-07
0.4	1.0239786667	1.0239796667	9.9999999992E-07	1.0239796667	9.9999999992E-07	1.0239796667	9.9999999992E-07
0.5	1.0619791667	1.0619805208	1.3541666666E-06	1.0619805208	1.3541666666E-06	1.0619805208	1.3541666666E-06
0.6	1.1360800000	1.1360800000	0.0000000000E+00	1.1360800000	0.0000000000E+00	1.1360800000	0.0000000000E+00
0.7	1.2669111667	1.2669111667	0.0000000000E+00	1.2669111667	0.0000000000E+00	1.2669111667	0.0000000000E+00
0.8	1.4819626667	1.4819626667	0.0000000000E+00	1.4819626667	0.0000000000E+00	1.4819626667	0.0000000000E+00
0.9	1.8168445000	1.8168445000	0.0000000000E+00	1.8168445000	0.0000000000E+00	1.8168445000	0.0000000000E+00
1.0	2.3166666667	2.3166666667	0.0000000000E+00	2.3166666667	0.0000000000E+00	2.3166666667	0.0000000000E+00
1.1	3.0375391667	3.0375391667	0.0000000000E+00	3.0375391667	0.0000000000E+00	3.0375391667	0.0000000000E+00
1.2	4.0481920000	4.0481920000	0.0000000000E+00	4.0481920000	0.0000000000E+00	4.0481920000	0.0000000000E+00
1.3	5.4317151667	5.4317151667	0.0000000000E+00	5.4317151667	0.0000000000E+00	5.4317151667	0.0000000000E+00
1.4	7.2874186667	7.2874186667	0.0000000000E+00	7.2874186667	0.0000000000E+00	7.2874186667	0.0000000000E+00
1.5	9.7328125000	9.7328125000	0.0000000000E+00	9.7328125000	0.0000000000E+00	9.7328125000	0.0000000000E+00
1.6	12.9057066667	12.9057066667	0.0000000000E+00	12.9057066667	0.0000000000E+00	12.9057066667	0.0000000000E+00
1.7	16.9664311667	16.9664311667	0.0000000000E+00	16.9664311667	0.0000000000E+00	16.9664311667	0.0000000000E+00
1.8	22.1001760000	22.1001760000	-7.1054273576E-15	22.1001760000	0.0000000000E+00	22.1001760000	0.0000000000E+00
1.9	28.5194511667	28.519451167	0.0000000000E+00	28.519451167	0.0000000000E+00	28.519451167	0.0000000000E+00
2.0	36.4666666667	36.4666666667	0.0000000000E+00	36.4666666667	0.0000000000E+00	36.4666666667	0.0000000000E+00
2.1	46.2168325000	46.2168325000	0.0000000000E+00	46.2168325000	0.0000000000E+00	46.2168325000	0.0000000000E+00
2.2	58.0803786667	58.0803786667	0.0000000000E+00	58.0803786667	0.0000000000E+00	58.0803786667	0.0000000000E+00
2.3	72.4060951667	72.4060951667	0.0000000000E+00	72.4060951667	0.0000000000E+00	72.4060951667	0.0000000000E+00
2.4	89.5841920000	89.5841920000	0.0000000000E+00	89.5841920000	0.0000000000E+00	89.5841920000	0.0000000000E+00

2.5	110.0494791667	110.0494791667	0.0000000000E+00	110.0494791667	0.0000000000E+00	110.0494791667	0.0000000000E+00
2.6	134.2846666667	134.2846666667	0.0000000000E+00	134.2846666667	0.0000000000E+00	134.2846666667	0.0000000000E+00
2.7	162.8237845000	162.8237845000	0.0000000000E+00	162.8237845000	0.0000000000E+00	162.8237845000	0.0000000000E+00
2.8	196.2557226667	196.2557226667	0.0000000000E+00	196.2557226667	0.0000000000E+00	196.2557226667	0.0000000000E+00
2.9	235.2278911667	235.2278911667	0.0000000000E+00	235.2278911667	0.0000000000E+00	235.2278911667	0.0000000000E+00
3.0	280.4500000000	280.4500000000	0.0000000000E+00	280.4500000000	0.0000000000E+00	280.4500000000	0.0000000000E+00

4.3 ANALYSIS OF RESULTS

In Table 4.1 the three schemes produced errors. For schemes 1 and 3 there is a steady growth in the errors as the step increases. For the first few steps (i.e. $y(0) - y(0.8)$) scheme 3 is slightly more accurate. This trend is reversed between $y(0.9)$ and $y(1.2)$. Again scheme 3 is better for $y(1.2) - y(1.4)$. Thereafter, accuracy alternates between the two schemes. Scheme 2 on the other hand does not show a steady rise in its error as the step increases. Rather, there is a fluctuation at each step. However, right from $y(0.9)$, whenever it decreases, it shows more accuracy than the other two.

In Table 4.2, we observe that all the three schemes do not exhibit any error (up to ten decimal places).

In Table 4.3, the three schemes exhibit errors. The behaviour here is similar to that obtainedin Table 4.1. Notably, scheme 3 maintains a lead in accuracy between $y(1.2)$ and $y(2.1)$. For other step values, accuracy cuts across the three schemes.

The three schemes are very accurate as shown in Table 4.4; there is no error at all. This is understandable since the solution of the differential equation is a polynomial of degree three.

And lastly, all the three schemes produce accurate results in Table 4.5. Also, this is according to expectation since the solution of the differential equation is a polynomial of degree six and the schemes are each of order six.

CHAPTER FIVE

ERROR ESTIMATION

5.1 VARIABLE STEP SIZE

When solving an initial value problem we can achieve better results by varying the step size. Mathews (1992), stated that one way to guarantee accuracy of an initial value problem is to solve the problem twice using step sizes *h* and *1/2h* and compare answers at the mesh-points corresponding to the larger step sizes.

However, changing the step size is not without its difficulties. Lambert(1991) staed that predictor-corrector methods possess many advantages, notably the facility for monitoring the local truncation error cheaply and efficiently. However, there is a balancing disadvantage, shared by all multistep methods, the difficulties encountered in implementing a change in step size.

Besides the difficulties involved in changing the step size. We need to know when to change the step size. Chapra and Canale (1998) out lined two criteria that are typically used to decide whether a change in step size is warranted. First, if the local truncation error estimated (see eqn (1.2)) is greater than some specified criterion, the step size is decreased. Second, the step size is chosen so that convergence criterion of the corrector is satisfied in two iterations. This criterion is intended to account for the trade-off between the rate of convergence and the total number of steps in the calculation. For smaller values of h, convergence will be more rapid but more steps are required. For larger h, convergence is slower but few steps result.

We illustrate this by solving the differential equations

$(i)\ y' = x + y; \qquad y(0) = 1$

$(ii)\ y' = 4xy^{1/2}\ ; \quad y(0) = 1$

at different step sizes .

The differential equations are solved for the following step sizes: 0.025, 0.05 and 0.1. The results obtained are as follows:

Table 5.1

PROBLEM: F=X+Y ; Y(0)=1 ; h=0.025

EXACT SOLUTION: Y(X)=2*EXP(X)-X-1

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.000	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.025	1.0256302410	1.0256302409	1.6344103848E-10	1.0256302409	1.6344103848E-10	1.0256302409	1.6344103848E-10
0.050	1.0525421928	1.0525421924	3.3515723530E-10	1.0525421924	3.3515723530E-10	1.0525421924	3.3515723530E-10
0.075	1.0807683018	1.0807683013	5.1546211743E-10	1.0807683013	5.1546211743E-10	1.0807683013	5.1546211743E-10
0.100	1.1103418362	1.1103418354	7.0468186841E-10	1.1103418354	7.0468186841E-10	1.1103418354	7.0468186841E-10
0.125	1.1412969061	1.1412969052	9.0315110945E-10	1.1412969052	9.0315110945E-10	1.1412969052	9.0315110945E-10
0.150	1.1736684855	1.1736684847	7.0727868007E-10	1.1736684846	8.4134588185E-10	1.1736684853	1.3301093560E-10
0.175	1.2074924332	1.2074924324	8.0280049275E-10	1.2074924319	1.3404957144E-09	1.2074924326	6.0152571812E-10
0.200	1.2428055163	1.2428055156	7.4779227255E-10	1.2428055154	8.8858076452E-10	1.2428055155	8.0948359127E-10
0.225	1.2796454324	1.2796454314	9.9448893565E-10	1.2796454308	1.6036167949E-09	1.2796454310	1.3939003285E-09
0.250	1.3180508334	1.3180508323	1.0640872627E-09	1.3180508325	8.6085516493E-10	1.3180508318	1.6214580789E-09
0.275	1.3580613497	1.3580613480	1.7443648748E-09	1.3580613479	1.8313204286E-09	1.3580613476	2.1823367558E-09
0.300	1.3997176152	1.3997176131	2.0937105383E-09	1.3997176140	1.1907346220E-09	1.3997176137	1.4902723500E-09
0.325	1.4430612920	1.4430612894	2.5764828049E-09	1.4430612897	2.2941610833E-09	1.4430612899	2.0546602197E-09
0.350	1.4881350972	1.4881350945	2.7247257783E-09	1.4881350952	2.0081936114E-09	1.4881350949	2.2958019930E-09
0.375	1.5349828292	1.5349828261	3.0922526761E-09	1.5349828261	3.0906810444E-09	1.5349828262	2.9913076460E-09
0.400	1.5836493953	1.5836493924	2.9105067245E-09	1.5836493923	3.0296334330E-09	1.5836493920	3.2464624322E-09
0.425	1.6341808393	1.6341808361	3.2041238551E-09	1.6341808353	4.0510317323E-09	1.6341808354	3.9258201134E-09
0.450	1.6866243710	1.6866243676	3.4072462629E-09	1.6866243671	3.8324965423E-09	1.6866243677	3.2737779154E-09
0.475	1.7410283950	1.7410283910	3.9970164956E-09	1.7410283901	4.8974888589E-09	1.7410283910	3.9548710973E-09
0.500	1.7974425414	1.7974425371	4.3418475482E-09	1.7974425371	4.2533885303E-09	1.7974425372	4.2407823919E-09
0.525	1.8559176968	1.8559176915	5.2119892846E-09	1.8559176912	5.5134681265E-09	1.8559176917	5.0689934472E-09
0.550	1.9165060357	1.9165060302	5.5264810506E-09	1.9165060312	4.5380679214E-09	1.9165060304	5.3643831599E-09
0.575	1.9792610538	1.9792610478	5.9796116947E-09	1.9792610478	6.0640519273E-09	1.9792610476	6.1849814159E-09
0.600	2.0442376008	2.0442375947	6.0848579508E-09	2.0442375957	5.1093445030E-09	2.0442375952	5.5866129500E-09
0.625	2.1114919149	2.1114919083	6.5475447286E-09	2.1114919080	6.8483516635E-09	2.1114919085	6.4087339879E-09

0.650	2.1810816580	2.1810816515	6.5717373765E-09	2.1810816519	6.1755551783E-09	2.1810816513	6.7539187576E-09
0.675	2.2530659519	2.2530659447	7.2741750401E-09	2.2530659439	8.0199571428E-09	2.2530659442	7.7406663301E-09
0.700	2.3275054149	2.3275054071	7.8188149288E-09	2.3275054074	7.5454757997E-09	2.3275054068	8.0920763423E-09
0.725	2.4044621999	2.4044621912	8.7183877895E-09	2.4044621905	9.4417611507E-09	2.4044621909	9.0811247411E-09
0.750	2.4840000332	2.4840000240	9.2106202665E-09	2.4840000244	8.8172193990E-09	2.4840000247	8.5534326288E-09
0.775	2.5661842544	2.5661842442	1.0147866103E-08	2.5661842435	1.0817498897E-08	2.5661842448	9.5456194060E-09
0.800	2.6510818570	2.6510818466	1.0395917016E-08	2.6510818472	9.7591952297E-09	2.6510818470	9.9681605192E-09
0.825	2.7387615307	2.7387615197	1.0939306794E-08	2.7387615187	1.1978946279E-08	2.7387615195	1.1144269951E-08
0.850	2.8292937039	2.8292936926	1.1216884310E-08	2.8292936933	1.0526848282E-08	2.8292936923	1.1571305247E-08
0.875	2.9227505879	2.9227505759	1.2033085195E-08	2.9227505749	1.3055946546E-08	2.9227505752	1.2761312007E-08
0.900	3.0192062223	3.0192062099	1.2457582521E-08	3.0192062108	1.1517409249E-08	3.0192062100	1.2325368726E-08
0.925	3.1187365207	3.1187365071	1.3640498064E-08	3.1187365064	1.4364123224E-08	3.1187365072	1.3522273079E-08
0.950	3.2214193186	3.2214193042	1.4459711650E-08	3.2214193056	1.3003420118E-08	3.2214193046	1.4044601038E-08
0.975	3.3273344220	3.3273344064	1.5560150057E-08	3.3273344059	1.6113308021E-08	3.3273344065	1.5446845580E-08
1.000	3.4365636569	3.4365636408	1.6114338752E-08	3.4365636420	1.4888128952E-08	3.4365636409	1.5973748102E-08

Table 5.2

PROBLEM: F=X+Y ; Y(0)=1 ; h=0.05

EXACT SOLUTION: Y(X)=2*EXP(X)-X-1

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.000	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.050	1.0525421928	1.0525421875	5.2520483518E-09	1.0525421875	5.2520483518E-09	1.0525421875	5.2520483518E-09
0.100	1.1103418362	1.1103418251	1.1042652792E-08	1.1103418251	1.1042652792E-08	1.1103418251	1.1042652792E-08
0.150	1.1736684855	1.1736684680	1.7413232412E-08	1.1736684680	1.7413232412E-08	1.1736684680	1.7413232412E-08
0.200	1.2428055163	1.2428054919	2.4408037147E-08	1.2428054919	2.4408037147E-08	1.2428054919	2.4408037147E-08
0.250	1.3180508334	1.3180508013	3.2074329859E-08	1.3180508013	3.2074329859E-08	1.3180508013	3.2074329859E-08
0.300	1.3997176152	1.3997175864	2.8768430971E-08	1.3997175819	3.3214656359E-08	1.3997176059	9.2968024390E-09
0.350	1.4881350972	1.4881350537	4.3462413890E-08	1.4881350348	6.2373089138E-08	1.4881350609	3.6328872577E-08
0.400	1.5836493953	1.5836493492	4.6089604577E-08	1.5836493456	4.9657913559E-08	1.5836493494	4.5835412577E-08
0.450	1.6866243710	1.6866243021	6.8930033681E-08	1.6866242752	9.5778061260E-08	1.6866242882	8.2740877261E-08
0.500	1.7974425414	1.7974424632	7.8151945138E-08	1.7974424760	6.5411750239E-08	1.7974424467	9.4701267273E-08
0.550	1.9165060357	1.9165059188	1.1696528812E-07	1.9165059030	1.3276968325E-07	1.9165059030	1.3268584009E-07
0.600	2.0442376008	2.0442374622	1.3856508430E-07	2.0442375024	9.8418945882E-08	2.0442374820	1.1875140782E-07
0.650	2.1810816580	2.1810814814	1.7664024377E-07	2.1810814766	1.8144730074E-07	2.1810814980	1.6007333992E-07
0.700	2.3275054149	2.3275052212	1.9374384719E-07	2.3275052593	1.5564801092E-07	2.3275052385	1.7639426364E-07
0.750	2.4840000332	2.4839998014	2.3179548325E-07	2.4839997856	2.4764942053E-07	2.4839998031	2.3010148897E-07
0.800	2.6510818570	2.6510816142	2.4279325972E-07	2.6510816288	2.2818528977E-07	2.6510816073	2.4965090217E-07
0.850	2.8292937039	2.8292934215	2.8232080629E-07	2.8292933739	3.2996250354E-07	2.8292933971	3.0672107387E-07
0.900	3.0192062223	3.0192059141	3.0816896945E-07	3.0192059210	3.0132977358E-07	3.0192059200	3.0233813630E-07
0.950	3.2214193186	3.2214189543	3.6437282702E-07	3.2214188966	4.2202707728E-07	3.2214189539	3.6475590681E-07
1.000	3.4365636569	3.4365632566	4.0028160386E-07	3.4365632873	3.6961057681E-07	3.4365632645	3.9239484639E-07

Table 5.3

PROBLEM: F=X+Y ; Y(0)=1 ; h=0.1

EXACT SOLUTION: Y(X)=2*EXP(X)-X-1

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.000	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.100	1.1103418362	1.1103416667	1.6948462878E-07	1.1103416667	1.6948462878E-07	1.1103416667	1.6948462878E-07
0.200	1.2428055163	1.2428051417	3.7461895075E-07	1.2428051417	3.7461895075E-07	1.2428051417	3.7461895075E-07
0.300	1.3997176152	1.3997169941	6.2102693121E-07	1.3997169941	6.2102693121E-07	1.3997169941	6.2102693121E-07
0.400	1.5836493953	1.5836484802	9.1512116951E-07	1.5836484802	9.1512116951E-07	1.5836484802	9.1512116951E-07
0.500	1.7974425414	1.7974412772	1.2642065803E-06	1.7974412772	1.2642065803E-06	1.7974412772	1.2642065803E-06
0.600	2.0442376008	2.0442361876	1.4132161703E-06	2.0442360367	1.5640399451E-06	2.0442368893	7.1145714031E-07
0.700	2.3275054149	2.3275025204	2.8945628872E-06	2.3275017844	3.6305474218E-06	2.3275027970	2.6179097690E-06
0.800	2.6510818570	2.6510783589	3.4980860688E-06	2.6510783329	3.5240506189E-06	2.6510785490	3.3079643345E-06
0.900	3.0192062223	3.0192004614	5.7608878645E-06	3.0191990040	7.2183356430E-06	3.0191999381	6.2842580402E-06
1.000	3.4365636569	3.4365566462	7.0107547572E-06	3.4365575986	6.0583367980E-06	3.4365562683	7.3885805181E-06

Table 5.4

PROBLEM: $F=4XY^{1/2}$; $Y(0)=1$; $h=0.025$

EXACT SOLUTION: $Y(X)=(1+X^2)^2$

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.000	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.025	1.0012503906	1.0012503906	2.0351498264E-11	1.0012503906	2.0351498264E-11	1.0012503906	2.0351498264E-11
0.050	1.0050062500	1.0050062499	8.1405993058E-11	1.0050062499	8.1405993058E-11	1.0050062499	8.1405993058E-11
0.075	1.0112816406	1.0112816404	1.8317392048E-10	1.0112816404	1.8317392048E-10	1.0112816404	1.8317392048E-10
0.100	1.0201000000	1.0200999997	3.2569902331E-10	1.0200999997	3.2569902331E-10	1.0200999997	3.2569902331E-10
0.125	1.0314941406	1.0314941401	5.0910187177E-10	1.0314941401	5.0910187177E-10	1.0314941401	5.0910187177E-10
0.150	1.0455062500	1.0455062496	4.1165915121E-10	1.0455062495	5.0838444565E-10	1.0455062500	7.6525452641E-12
0.175	1.0621878906	1.0621878903	3.0035240961E-10	1.0621878899	6.9673733449E-10	1.0621878906	3.0355051805E-11
0.200	1.0816000000	1.0815999998	1.8467649632E-10	1.0815999995	4.6287262911E-10	1.0815999999	8.9772411727E-11
0.225	1.1038128906	1.1038128906	7.1298300597E-11	1.1038128900	5.7808624554E-10	1.1038128904	1.9458634704E-10
0.250	1.1289062500	1.1289062500	3.3200775462E-11	1.1289062498	2.3782487091E-10	1.1289062497	3.3400215926E-10
0.275	1.1569691406	1.1569691405	1.4399126336E-10	1.1569691404	2.7307534012E-10	1.1569691401	5.2152904217E-10
0.300	1.1881000000	1.1880999997	2.8629054682E-10	1.1881000000	4.6387560459E-11	1.1881000000	1.9996448941E-11
0.325	1.2224066406	1.2224066402	3.9329806079E-10	1.2224066405	7.6712858288E-11	1.2224066406	4.7714499019E-11
0.350	1.2600062500	1.2600062495	4.6034820400E-10	1.2600062499	7.0558003884E-11	1.2600062499	1.0211409496E-10
0.375	1.3010253906	1.3010253901	4.9368109600E-10	1.3010253904	1.8320833739E-10	1.3010253904	2.1536283867E-10
0.400	1.3456000000	1.3455999997	2.6849344970E-10	1.3455999997	2.9356828080E-10	1.3455999997	3.4692226869E-10
0.425	1.3938753906	1.3938753905	1.4164180939E-10	1.3938753901	5.0455617462E-10	1.3938753901	5.4387561121E-10
0.450	1.4460062500	1.4460062499	8.6533002985E-11	1.4460062495	5.1620197006E-10	1.4460062500	3.9394043583E-11
0.475	1.5021566406	1.5021566405	9.5354391050E-11	1.5021566399	7.5383987941E-10	1.5021566406	7.4282580087E-11
0.500	1.5625000000	1.5624999999	1.3898637796E-10	1.5624999995	5.4103344027E-10	1.5624999999	1.2135048522E-10
0.525	1.6272191406	1.6272191402	4.0003356183E-10	1.6272191399	7.1493921894E-10	1.6272191404	2.4673441068E-10
0.550	1.6965062500	1.6965062495	4.9682546965E-10	1.6965062496	3.5438008084E-10	1.6965062496	3.6656611080E-10
0.575	1.7705628906	1.7705628901	5.0105875005E-10	1.7705628902	4.4523473797E-10	1.7705628900	5.7648064100E-10
0.600	1.8496000000	1.8495999996	4.3895220792E-10	1.8495999999	1.3850431912E-10	1.8495999999	6.5727867593E-11
0.625	1.9338378906	1.9338378903	3.6491254463E-10	1.9338378904	2.2090063112E-10	1.9338378905	1.1013190360E-10

0.650	2.0235062500	2.0235062499	1.2600631649E-10	2.0235062499	1.0314726850E-10	2.0235062499	1.4754686362E-10
0.675	2.1188441406	2.1188441405	1.1092593510E-10	2.1188441403	2.7660584934E-10	2.1188441403	2.8845859035E-10
0.700	2.2201000000	2.2200999998	1.8834489524E-10	2.2200999997	2.9177638083E-10	2.2200999996	3.9314995703E-10
0.725	2.3275316406	2.3275316403	3.1444491455E-10	2.3275316400	5.8494453725E-10	2.3275316400	6.1923088879E-10
0.750	2.4414062500	2.4414062496	4.1466385881E-10	2.4414062495	5.4127857751E-10	2.4414062499	9.9004804355E-11
0.775	2.5620003906	2.5620003900	6.2894534025E-10	2.5620003897	8.8318286018E-10	2.5620003905	1.5535484010E-10
0.800	2.6896000000	2.6895999994	5.7819526944E-10	2.6895999994	6.3251182070E-10	2.6895999998	1.8078916142E-10
0.825	2.8245003906	2.8245003902	4.6143089349E-10	2.8245003897	9.2154639475E-10	2.8245003903	3.4027491935E-10
0.850	2.9670062500	2.9670062497	3.3175773240E-10	2.9670062495	4.9299053728E-10	2.9670062496	4.2686654211E-10
0.875	3.1174316406	3.1174316404	2.7299185135E-10	3.1174316399	6.9852079676E-10	3.1174316400	6.7194871889E-10
0.900	3.2761000000	3.2760999999	1.4078205268E-10	3.2760999997	2.6432500633E-10	3.2760999999	1.3930279152E-10
0.925	3.4433441406	3.4433441403	2.9252422706E-10	3.4433441402	4.6148018740E-10	3.4433441404	2.1002266593E-10
0.950	3.6195062500	3.6195062495	4.6155701483E-10	3.6195062498	1.7144019537E-10	3.6195062498	2.2117463416E-10
0.975	3.8049378906	3.8049378900	6.0038907179E-10	3.8049378902	4.7277781690E-10	3.8049378902	4.0194381157E-10
1.000	4.0000000000	3.9999999994	6.2954885749E-10	3.9999999997	3.1400571032E-10	3.9999999995	4.6787995700E-10

Table 5.5

PROBLEM: $F=4XY^{1/2}$; $Y(0)=1$; $h=0.05$

EXACT SOLUTION: $Y(X)=(1+X^2)^2$

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.000	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.050	1.0050062500	1.0050062487	1.3037011470E-09	1.0050062487	1.3037011470E-09	1.0050062487	1.3037011470E-09
0.100	1.0201000000	1.0200999948	5.2153668051E-09	1.0200999948	5.2153668051E-09	1.0200999948	5.2153668051E-09
0.150	1.0455062500	1.0455062383	1.1744571093E-08	1.0455062383	1.1744571093E-08	1.0455062383	1.1744571093E-08
0.200	1.0816000000	1.0815999791	2.0932845457E-08	1.0815999791	2.0932845457E-08	1.0815999791	2.0932845457E-08
0.250	1.1289062500	1.1289062171	3.2886015910E-08	1.1289062171	3.2886015910E-08	1.1289062171	3.2886015910E-08
0.300	1.1881000000	1.1880999726	2.7398315172E-08	1.1880999665	3.3531799337E-08	1.1880999981	1.8995824913E-09
0.350	1.2600062500	1.2600062290	2.0981690607E-08	1.2600062041	4.5937669890E-08	1.2600062462	3.7961904731E-09
0.400	1.3456000000	1.3455999857	1.4307481422E-08	1.3455999677	3.2346600509E-08	1.3455999927	7.3400165945E-09
0.450	1.4460062500	1.4460062423	7.6804342886E-09	1.4460062094	4.0591534578E-08	1.4460062353	1.4729287567E-08
0.500	1.5625000000	1.5624999985	1.4723042785E-09	1.5624999800	1.9974369936E-08	1.5624999769	2.3108848168E-08
0.550	1.6965062500	1.6965062372	1.2838119900E-08	1.6965062259	2.4078415262E-08	1.6965062138	3.6202221887E-08
0.600	1.8496000000	1.8495999780	2.2035101299E-08	1.8495999913	8.7307061580E-09	1.8495999950	4.9689079518E-09
0.650	2.0235062500	2.0235062204	2.9585494055E-08	2.0235062356	1.4415782790E-08	2.0235062418	8.1638242833E-09
0.700	2.2201000000	2.2200999655	3.4454226139E-08	2.2200999900	1.0031971254E-08	2.2200999894	1.0601280298E-08
0.750	2.4414062500	2.4414062121	3.7923221896E-08	2.4414062264	2.3643248248E-08	2.4414062298	2.0174736992E-08
0.800	2.6896000000	2.6895999751	2.4851178004E-08	2.6895999759	2.4123242959E-08	2.6895999734	2.6642562823E-08
0.850	2.9670062500	2.9670062317	1.8281956393E-08	2.9670062034	4.6556068778E-08	2.9670062078	4.2215482221E-08
0.900	3.2761000000	3.2760999847	1.5270871234E-08	3.2760999602	3.9776271610E-08	3.2760999903	9.7339403204E-09
0.950	3.6195062500	3.6195062328	1.7174432987E-08	3.6195061843	6.5707651320E-08	3.6195062351	1.4947320270E-08
1.000	4.0000000000	3.9999999799	2.0117174149E-08	3.9999999556	4.4362432838E-08	3.9999999843	1.5671610232E-08

Table 5.6

PROBLEM: $F=4XY^{1/2}$; $Y(0)=1$; $h=0.1$

EXACT SOLUTION: $Y(X)=(1+X^2)^2$

X	Y(X)						
		a=3/4, b=-1/3		a=7/8, b=-7/8		a=1/2, b=-1/2	
	EXACT	Y(X)	ERROR	Y(X)	ERROR	Y(X)	ERROR
0.000	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.100	1.0201000000	1.0200999163	8.3740235413E-08	1.0200999163	8.3740235413E-08	1.0200999163	8.3740235413E-08
0.200	1.0816000000	1.0815996645	3.3550094458E-07	1.0815996645	3.3550094458E-07	1.0815996645	3.3550094458E-07
0.300	1.1881000000	1.1880992364	7.6362208823E-07	1.1880992364	7.6362208823E-07	1.1880992364	7.6362208823E-07
0.400	1.3456000000	1.3455986013	1.3986813359E-06	1.3455986013	1.3986813359E-06	1.3455986013	1.3986813359E-06
0.500	1.5625000000	1.5624976976	2.3024470428E-06	1.5624976976	2.3024470428E-06	1.5624976976	2.3024470428E-06
0.600	1.8496000000	1.8495978929	2.1071298080E-06	1.8495975424	2.4576180864E-06	1.8495995447	4.5533008630E-07
0.700	2.2201000000	2.2200981516	1.8483568627E-06	2.2200964953	3.5046751483E-06	2.2200993118	6.8822349020E-07
0.800	2.6896000000	2.6895984215	1.5784759917E-06	2.6895972208	2.7791784922E-06	2.6895991149	8.8509088902E-07
0.900	3.2761000000	3.2760986987	1.3012728237E-06	3.2760963052	3.6947909670E-06	3.2760984239	1.5761347201E-06
1.000	4.0000000000	3.9999990162	9.8383706826E-07	3.9999975811	2.4189371315E-06	3.9999979918	2.0081629217E-06

For the differential equation $y' = x + y$; $y(0) = 1$, using scheme1 $(a = 3/4, b = -1/3)$ as an example, the error for $h = 0.1$ at x = 1 (Table 5.3) is $7.0107547572*10^{-6}$. This error is reduced to $4.0028160386*10^{-7}$ when $h = 0.05$ (Table 5.2); and this is further reduced to $1.6114338752*10^{-8}$ when $h = 0.025$ (Table 5.1).

The same trend is noticed for the other two schemes as well as in the solution for $y' = 4xy^{1/2}$, $y(0) = 1$.

We show the errors at various step lengths.in the following tables. They help to confirm the observation earlier on made on the rate of convergence as h decreases.

Table 5.7
PROBLEM: F=X+Y ; Y(0)=1
SCHEME 1 a=3/4, b=-1/3

	ERROR		
X	h=0.1	h=0.05	h=0.025
0.05		5.2520483518E-09	3.3515723530E-10
0.10	1.6948462878E-07	1.1042652792E-08	7.0468186841E-10
0.15		1.7413232412E-08	7.0727868007E-10
0.20	3.7461895075E-07	2.4408037147E-08	7.4779227255E-10
0.25		3.2074329859E-08	1.0640872627E-09
0.30	6.2102693121E-07	2.8768430971E-08	2.0937105383E-09
0.35		4.3462413890E-08	2.7247257783E-09
0.40	9.1512116951E-07	4.6089604577E-08	2.9105067245E-09
0.45		6.8930033681E-08	3.4072462629E-09
0.50	1.2642065803E-06	7.8151945138E-08	4.3418475482E-09

Table 5.8
PROBLEM: F=X+Y ; Y(0)=1
SCHEME 2 a=7/8, b=-7/8

	ERROR		
X	h=0.1	h=0.05	h=0.025
0.05		5.2520483518E-09	3.3515723530E-10
0.10	1.6948462878E-07	1.1042652792E-08	7.0468186841E-10
0.15		1.7413232412E-08	8.4134588185E-10
0.20	3.7461895075E-07	2.4408037147E-08	8.8858076452E-10
0.25		3.2074329859E-08	8.6085516493E-10
0.30	6.2102693121E-07	3.3214656359E-08	1.1907346220E-09
0.35		6.2373089138E-08	2.0081936114E-09
0.40	9.1512116951E-07	4.9657913559E-08	3.0296334330E-09
0.45		9.5778061260E-08	3.8324965423E-09
0.50	1.2642065803E-06	6.5411750239E-08	4.2533885303E-09

Table 5.9
PROBLEM: F=X+Y ; Y(0)=1
SCHEME 3 a=1/2, b=-1/2

	ERROR		
X	h=0.1	h=0.05	h=0.025
0.05		5.2520483518E-09	3.3515723530E-10
0.10	1.6948462878E-07	1.1042652792E-08	7.0468186841E-10
0.15		1.7413232412E-08	1.3301093560E-10
0.20	3.7461895075E-07	2.4408037147E-08	8.0948359127E-10
0.25		3.2074329859E-08	1.6214580789E-09
0.30	6.2102693121E-07	9.2968024390E-09	1.4902723500E-09
0.35		3.6328872577E-08	2.2958019930E-09
0.40	9.1512116951E-07	4.5835412577E-08	3.2464624322E-09
0.45		8.2740877261E-08	3.2737779154E-09
0.50	1.2642065803E-06	9.4701267273E-08	4.2407823919E-09

Table 5.10

PROBLEM: $F=4XY^{1/2}$; Y(0)=1

SCHEME 1 a=3/4, b=-1/3

	ERROR		
X	h=0.1	h=0.05	h=0.025
0.05		1.3037011470E-09	8.1405993058E-11
0.10	8.3740235413E-08	5.2153668051E-09	3.2569902331E-10
0.15		1.1744571093E-08	4.1165915121E-10
0.20	3.3550094458E-07	2.0932845457E-08	1.8467649632E-10
0.25		3.2886015910E-08	3.3200775462E-11
0.30	7.6362208823E-07	2.7398315172E-08	2.8629054682E-10
0.35		2.0981690607E-08	4.6034820400E-10
0.40	1.3986813359E-06	1.4307481422E-08	2.6849344970E-10
0.45		7.6804342886E-09	8.6533002985E-11
0.50	2.3024470428E-06	1.4723042785E-09	1.3898637796E-10

Table 5.11

PROBLEM: $F=4XY^{1/2}$; Y(0)=1

SCHEME 2 a=7/8, b=-7/8

	ERROR		
X	h=0.1	h=0.05	h=0.025
0.05		1.3037011470E-09	8.1405993058E-11
0.10	8.3740235413E-08	5.2153668051E-09	3.2569902331E-10
0.15		1.1744571093E-08	5.0838444565E-10
0.20	3.3550094458E-07	2.0932845457E-08	4.6287262911E-10
0.25		3.2886015910E-08	2.3782487091E-10
0.30	7.6362208823E-07	3.3531799337E-08	4.6387560459E-11
0.35		4.5937669890E-08	7.0558003884E-11
0.40	1.3986813359E-06	3.2346600509E-08	2.9356828080E-10
0.45		4.0591534578E-08	5.1620197006E-10
0.50	2.3024470428E-06	1.9974369936E-08	5.4103344027E-10

Table 5.12

PROBLEM: $F=4XY^{1/2}$; Y(0)=1

SCHEME 3 a=1/2, b=-1/2

	ERROR		
X	h=0.1	h=0.05	h=0.025
0.05		1.3037011470E-09	8.1405993058E-11
0.10	8.3740235413E-08	5.2153668051E-09	3.2569902331E-10
0.15		1.1744571093E-08	7.6525452841E-12
0.20	3.3550094458E-07	2.0932845457E-08	8.9772411727E-11
0.25		3.2886015910E-08	3.3400215926E-10
0.30	7.6362208823E-07	1.8995824913E-09	1.9996448941E-11
0.35		3.7961904731E-09	1.0211409496E-10
0.40	1.3986813359E-06	7.3400165945E-09	3.4692226869E-10
0.45		1.4729287567E-08	3.9394043583E-11
0.50	2.3024470428E-06	2.3108848168E-08	1.2135048522E-10

5.2 RECOMMENDATIONS

The main business of numerical analysis is to provide us with computational methods for the study and solution of mathematical problems. However, most numerical methods give answers that are only approximations to the desired true solution. Consequently , a numerical result is seldom free of error. It is recommended that further work be done in producing schemes with higher accuracies.

Furthermore, suitable free parameters can be chosen so as to reduce the functional evaluation at each step of computation.

5.3 SUMMARY

In this work we have been able to drive a 6-step implicit linear multistep method of order eight. And by assigning suitable values for the free parameters we obtain the following three schemes:

Scheme $1\left(a = \frac{3}{4},\ b = -\frac{1}{3}\right)$:

$$y_{n+6} - \frac{5}{6}y_{n+5} + \frac{5}{6}y_{n+1} - y_n = h\left[\frac{3401}{11340}f_{n+6} + \frac{391}{315}f_{n+5} - \frac{1117}{1260}f_{n+4} + \frac{3848}{2835}f_{n+3} - \frac{1117}{1260}f_{n+2} + \frac{391}{315}f_{n+1} + \frac{3401}{11340}f_n\right]$$

Scheme $2\left(a = \frac{7}{8},\ b = -\frac{7}{8}\right)$:

$$y_{n+6} - \frac{33}{16}y_{n+4} + \frac{33}{16}y_{n+2} - y_n = h\left[\frac{17547}{60480}f_{n+6} + \frac{443}{280}f_{n+5} - \frac{281}{448}f_{n+4} - \frac{155}{252}f_{n+3} - \frac{281}{448}f_{n+2} + \frac{443}{280}f_{n+1} + \frac{17547}{60480}f_n\right]$$

Scheme $3\left(a = \frac{1}{2},\ b = -\frac{1}{2}\right)$:

$$y_{n+6} - y_n = h\left[\frac{41}{140}f_{n+6} + \frac{162}{105}f_{n+5} + \frac{27}{140}f_{n+4} + \frac{68}{35}f_{n+3} + \frac{27}{140}f_{n+2} + \frac{162}{105}f_{n+1} + \frac{41}{140}f_n\right]$$

We have also solved various differential equations using the three schemes. To assist us in solving differential equations, a computer implementation program using Microsoft Excel software package was used.

5.4 CONCLUSION

Since we have used the methods to solve various differential equations we conclude that our 6-step implicit linear multistep methods are accurate as they produce results which are comparable with those produced by other similar methods.

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