APPLICATION OF OPTIMAL CONTROL MODEL TO RIVER POLLUTION PROBLEM

BY

PAUL GANA

SSSE/MTECH/2003/1020

EPARTMENT OF MATHEMATIC/COMPUTER SCIENCE,

FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA

NOVEMBER 2007

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A THESIS SUBMITTED IN THE
DEPARTMENT OF MATHEMATICS/COMPUTER SCIENCE
TO THE SCHOOL OF POSTGRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR
THE AWARD OF MASTER OF TECHNOLOGY,
OF FEDERAL UNIVERSITY OF TECHNOLOGY,
MINNA, NIGERIA

NOVEMBER 2007

CERTIFICATION

This thesis titled **APPLICATION OF OPTIMAL CONTROL MODEL TO RIVER POLLUTION PROBLEM** by **PAUL GANA** matric no **SSSE/MTECH/2003/1020** of the Department of Mathematics and Computer Science, School of Science Education, Federal University of Technology, meets the regulation governing the award of Masters Degree in Applied Mathematics, and is approved for its contribution to knowledge and literary presentation.

Prof K.R. ADEBOYE 29/1/2008 Supervisor	Date
Dr. V.O WAZIRI Co-supervisor	8/1/2008 Date
Dr. N. I. AKINWANDE (Head of Department)	4-2-08 Date
Prof. M. GALADIMA (Dean, School of Science & Science Education)	16/5/08 Date
Prof. S. L. LAMAI (Dean, Post-graduate School)	27/8/08 Date
External Examiner	Date

DEDICATION

This Thesis is dedicated to Almighty God; the Creator of Heaven and Earth, and to my Heart Throb wife who stands by me in all pleasantness and unpleasantness.

ACKNOWLEDGEMENT

I give many Thanks to the Almighty God who ensured the completion of this M.Tech Thesis at this august period. Without God's intervention, I would not have been able to meet this accomplishment. I am most grateful for ever more.

I wish to express my profound gratitude to my God gifted supervisor Prof. K. R. Adeboye whose supervisory expertise strengthened the structure of this Thesis. All his impeccable and well-defined pieces of advice during the writing of this project were excellently followed. I am more grateful to Prof. K. R. Adeboye for seeing me through my undergraduate study and the M. Tech course work.

Dr. V. O. Waziri is my project co-supervisor. He gave me clearance in the intricacies of the manipulation of the optimal control system and he also taught me Matlab graphical plotting. I am most grateful to him.

I wish to express my gratitude to the Head of Department, Dr. N. I. Akinwanwande, a gentleman to the core. As the postgraduate coordinator, he pleaded my cause with post-graduate school when my time for the Thesis writing was ebbing. The accomplishment of this Thesis was accelerated due to his encouragement.

I pay my tribute to the entire members of staff in the department; worthy of note is Dr. Ayeisimi Yomi who has been my long time instructor since my undergraduate days. My honest respect goes to Dr. N. L. Ezeako as the former H.O.D and as my instructor at the undergraduate level.

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ABBREVIATION OF TERMS

DO: dissolved oxygen

BOD: Biochemical Oxygen Demand

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ABSTRACT

In this research work, we worked on an optimal control model for a polluted river with consideration of multiple effluents into some categorized River lengths termed Reach. The Hamiltonian forms and the maximum principles are employed to solve the model equations using both Matlab and Mathcad computing packages. Besides, graphical interpretations are plotted from the optimal solutions using Mathlab. This assisted us in interpreting the exponential solution outputs so obtained. We discover that the Biochemical Oxygen Demand (BOD) declines in magnitude and then reaches a minimum peak at some time equivalent to 0.15 of a second

CHAPTER ONE

INTRODUCTION

1.1 River Pollution

River pollution involves the release into lakes, streams, rivers, and oceans of substances that are dissolved or suspended in the water or deposited upon the bottom and accumulate to the extent that they interfere with the functioning of aquatic ecosystems. It may also include the release of energy in the form of radioactivity or heat as in the case of thermal pollution. A body of water has the capacity to absorb, break down, or recycle introduced materials. Under normal circumstances, inorganic substances are widely dispersed and have little or no effect on life within the bodies of water into which they are released; organic materials are broken down by bacteria or other organisms and converted into a form in which they are useful to aquatic life. But, if the capacity of a body of water to dissolve, disperse, or recycle is exceeded all additional substances or forms of energy become pollutants. Thus, thermal pollution which is usually caused by the discharge of water that has been used as a coolant in fossil-fueled or nuclear-power plants, can favour a diversity of aquatic life in waters that would otherwise be too cold. In a warmer body of water, however, the addition of heat changes its characteristics and may make it less suited to species that are considered desirable. Pollution may begin as water moves through the air, if the air is polluted. Soil erosion adds silt as a pollutant. The use of chemical fertilizers, pesticides, or other materials on watershed lands is an additional factor contributing to water pollution. The runoff from septic tanks and the outflow of manures from livestock feedlots along the watershed are sources of organic pollutants. Industries located along waterways downstream contribute a number of chemical pollutants, some of which are toxic if present in any concentration. Finally, cities and towns contribute their loads of sewage and other urban wastes. Thus, a community far upstream in a watershed may receive relatively clean water, whereas one farther downstream receives a partly diluted mixture of urban, industrial, and rural wastes. The cost of cleaning and purifying this water for community use may be high, and the process may be only partially effective. To add to the problem, the cities and towns in the lower, or downstream regions of the river basin contribute additional wastes that flow into estuaries, creating new pollution problems.

The output of industries, agriculture and urban communities generally exceeds the biological capacities of aquatic systems, causing water to become choked with excess organic substances and organisms to be poisoned by toxic materials. When organic matter exceeds the capacity of those microorganisms in water that breaks it down and recycle it, the excess of nutrients in such matter encourages rapid growth of algae. When they die, the remains of the dead algae add further to the organic wastes already in the water; eventually, the water becomes deficient in oxygen. Anaerobic organisms (those that do not require oxygen to live) then attack the organic wastes, releasing gases such as methane and hydrogen sulfide which are harmful to the oxygen-requiring (aerobic) forms of life. The result is a foul-smelling, waste-filled body of water and this is a growing problem in freshwater lakes. The process by which a lake or any other body of water changes from a clean, clear condition - with a relatively low concentration of dissolved nutrients and a balanced aquatic community-to a nutrient-rich, algae-filled body and hence to an oxygen-deficient, waste-filled condition is known as accelerated eutrophication.

1.2 Statement of the Problem

Consider the model stated by Singh et al. (1978) for a single reach with an effluent:

Find the optimal control trajectory Δm_1 for the river system:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}\begin{bmatrix} z_1 \\ q_1 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}\Delta m_1 + \begin{bmatrix} 0.9z_0 & 5.35 \\ 0.9q_0 & 1.9 \end{bmatrix}$$

with $z_0 = 10$, $q_0 = 0$ and the cost function to be minimized is

$$J = \int_0^s \{(z_1 - 4.06)^2 + 2(q-8)^2 + \Delta m_1^2\}dt$$

1.3 Aim and Objectives

Aim: The aim of this project is to compute the optimal control value for a river polluted by an industry.

Objectives:

The objectives of this research are:

1) To compute the optimal Biochemical Oxygen Demand (BOD) of reach of a river from an industrial effluent.
2) To compute the optimal dissolved Oxygen (BO) in a reach with effluents of river
3) To obtain the co-states of the adjoined objective functional with the continuous constraint equations

4) To simulate graphically and interpret the numerical outputs obtained in (2) through (4)

1. 4 Methodology

The methods we shall use to fulfill the aim and objectives listed in section (1.3) are:

a) The Hamiltonian form

b) The maximum principle

1.5 Scope of Study

River pollution control can be studied for both continuous and discrete objective functional. The scope of this study is limited to the optimization of problems involving continuous functionals. We will exam the river pollution control model developed by Beck (1974) using the method of computational analysis.

1.6 Significance of Study

This study has some importance in many ways. These include:

(1) Determining the minimum control for a polluted river

(2) Preservation of aquatic lives in rivers

(3) Improvement of water quality in urban and industrialized communities

CHAPTER TWO

LITERATURE REVIEW

2.1 Models Of Pollution Control

Water pollution has been a crucial problem in many countries of the world as well as a subject of growing public concern which has attracted researchers attention from all over the world.

Thousands of mathematical models have been developed on this subject with different purposes in mind. Mathematical modeling has become an important instrument for the solution of water management problems. The application of mathematical models for this purpose dates back to the initial studies of oxygen depletion due to organic waste pollution. Since then, models have been constantly refined and updated to meet new and emerging problems of surface, acute and chronic toxicity, etc.

Mathematical models are used extensively in research on the transport of pollution self purification of river water as well as in the design and assessment of water quality management measures. Water contamination as investigated by (Katsuhia, 1972) is caused by pollutant discharges from point sources into rivers or streams (e. g factories and sewage-treatment plants) and surface runoff into streams or leaching into groundwater from non-point sources (e. g. cropland and urban storm-water). Sources are generally easier to identify, monitor and regulate than non-point sources. Water contamination may occur in surface waters or groundwater or both, depending on solubility of the pollutant, soil permeability, etc.

Stock pollutants include heavy metals which may be bio-accumulate in the food chain. Surface water can assimilate various pollutants which ground water cannot.

According to (Katsuhia, 1972), technology standard specifies the method of pollution control used to reduce pollutant loadings. Water pollution policy is almost entirely based on "command-and-control" standards. There are three basic types of standard. An ambient standard specifies minimum dissolved oxygen (DO) levels or maximum pollutant concentrations at receptor points.

River pollution control as a large scale problem, is not that much a problem in Nigeria rural areas as it is in the urban areas such as Lagos where there are industries and lagoons all over the territory. Hence this study has the urban pollution as the target.

River water quality models seek to describe the spatial and temporal changes of constituents of concern. Over the past seven decades components or state variables have been gradually incorporate into models following the evolution of water quality models characters among others oxygen household, nutrients and so on.

The complexity covers a broad range from the simple Streeter –Phelps model (Streeter Phelps. 1925).

Water quality changes in rivers due to physical transport and exchanges processes (such as advection and diffusion/dispersion, the description of which requires one way or another application of a hydraulic models as an input and biological, chemical, biochemical and physical conversion processes. The above process is governed by a set of well known extended transport equations. (Somlyody and Van Straten 1986)

$$\frac{\partial c}{\partial t} = u\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y} - w\frac{\partial c}{\partial z} + \frac{\partial}{\partial x}\left[\varepsilon_x \frac{\partial c}{\partial x}\right] + \frac{\partial}{\partial y}\left[\varepsilon_y \frac{\partial x}{\partial y}\right] + \frac{\partial}{\partial z}\left[\varepsilon_z \frac{\partial c}{dz}\right] + r(c, p).$$

where C – dimensional mars concentration vector for the n state variables: t-time, x, y, z – spatial Co-ordinates; u, v and w – corresponding velocity components; ε_x, ε_y, ε_z - turbulent diffusion coefficient for the direction x, y, and z respectively: r – n-dimensional vector of rates of change of state variables due to biological, chemical, and other conversion processes as a function of concentrations, c, and model parameters, p.

The above equation is a well know partial differential equation which offers not only the basic governing equation of water quantity models, but also specifies a useful framework and the main model elements. These include:

- ✓ The hydrodynamic model for deriving velocity components u, v, and w, and turbulent diffusion co-efficient ε_x, ε_y and ε_z
- ✓ The transport (or advection – diffusion) equation (describing the behaviour of so-called conservation substances) and if solution.

- ✓ The conversion of sub-model r(c, p). it has much less solid theoretical ground than hydrodynamics and, thus, for a development an adequate combination of theoretical and empirical knowledge is needed.
- ✓ For the latter purpose, methodologies such as calibration, validation, identification, sensitivity and uncertainty analysis are required. (Back, 1987).
- ✓ The model designed on the basis of the above steps and elements should be implemented on a computer, which raises a number of software and hardware issues.

2.2 HYDRO DYNAMICS AND HYDRAULICS

Here flow of water in a river is described by the continuity and momentum equations. The latter is known as Navier-Stokes or Reynoids equation. The actual form of a hydrodynamic model depends on assumption made on characterizing turbulence. Methods vary from the use of eddy viscosity as known parameters to the application of the so-called K- ε theory (Bedford et al: 1988 or Rodi, 1993)

The hydrodynamic equations are generally solved by efficient finite difference methods. (Mahmood and Yevjevich, 1975).

2.3 WATER QUALITY MANAGEMENT MODELS

A number of water quality management models have been developed in the past for the allocation of assimilative capacity of a river system. Model results help in setting the amount of waste that can be disposed into the river from various points and non-point sources without violating the water quality standards, the intended purpose of these models is to provide economic and technologically feasible solutions acceptable to both the pollution control agency and the dischargers. Water quality management problems have been addressed as multiple objective optimization problems by many researchers such as Cohon 1978; Loucks et al. 1981; Loucks 1983; Burn and Mebean 1985. General methods of solution include the weighting method and the constraint method. Although these methods provide acceptable solutions, they are characterized by the difficulty of assigning unknown relative weights and setting upper bounds in the problem formulation, Loucks (1983). This results in an improper accounting of the aspirations of the various groups such as the pollution control agency and the dischargers.

Water quality management problems are characterized by various types of uncertainties at different stages of the decision-making process to arrive at the optimal allocation of the assimilative capacity of the river system. The type of uncertainty that has received much attention is that due to randomness associated with various components of water quality system. Two major components considered for randomness are river flow and effluent flow (Lohani and Thanh 1978, 1979; Bum and McBean 1985, 1986; Fugiwara et al. 1986, 1987; 1988; Ellis 1987; Cardwell and Ellis 1993). Another type of uncertainty prominent in the management of water quality systems is the uncertainty due to vagueness associated with describing the goals related to water quality and pollutant abatement. Desirable and permissible water quality criteria and minimal pollutant treatment levels are set up depending on the environmental objectives.

The model uses the concepts of fuzzy set theory (Zadeh 1965). Fuzzy decision making (Bellman and Zadeh 1970), fuzzy mathematical programming (Zimmermann 1978, 1985), and fuzzy resource allocation (Kindler 1992).

Figure 2.1 is a schematic diagram of a part of a stream (as can be deduced from Tamura (1974)) into which sources (industries and municipalities) discharge polluting effluents. The pollutants consist of various materials, but for simplicity of exposition we assume that their impact on the quality of the river is measured in terms of a single quantity, namely the biochemical oxygen demand (BOD) which they place on the dissolved oxygen (DO) in the stream. Since the DO in the stream is used to breakdown chemically the pollutants into harmless substances, the quality of the stream improves with the amount of (DO) and decreases with increasing BOD. It is a well-advertised fact that if the DO drops below a certain concentration, then life in the river is seriously threatened; Therefore, it is important to treat the effluents before they are discharged into the stream in order to reduce the BOD to concentration levels which can be safely absorbed by the DO in the stream. In this example, we are concerned with ending the optimal balance between costs of waste treatment and costs of high BOD in the stream.

We at once derive the equations which govern the evolution in terms of BOD and DO in the areas of the river as outlined by (Katsuhisa, 1972).

The fluctuations of BOD and DO will be cyclical with a period of 24 hours. Hence, it is enough to study the problem over a 24-hour period.

We divide this period into T intervals, t = 1;........ ; T . During interval t and in an area i, let

z_i (t) = concentration of BOD measured in mg/liter,
q_i (t) = concentration of DO measured in mg/liter,
s_i (t) = concentration of BOD of effluent discharge in mg/liter, and
m_i (t) = amount of effluent discharge in liters.

The principle of concentration of mass gives us equations (2.1) and (2.2) as in (Beck, 1974) which are derived from figure 2.1 as shown immediately below:

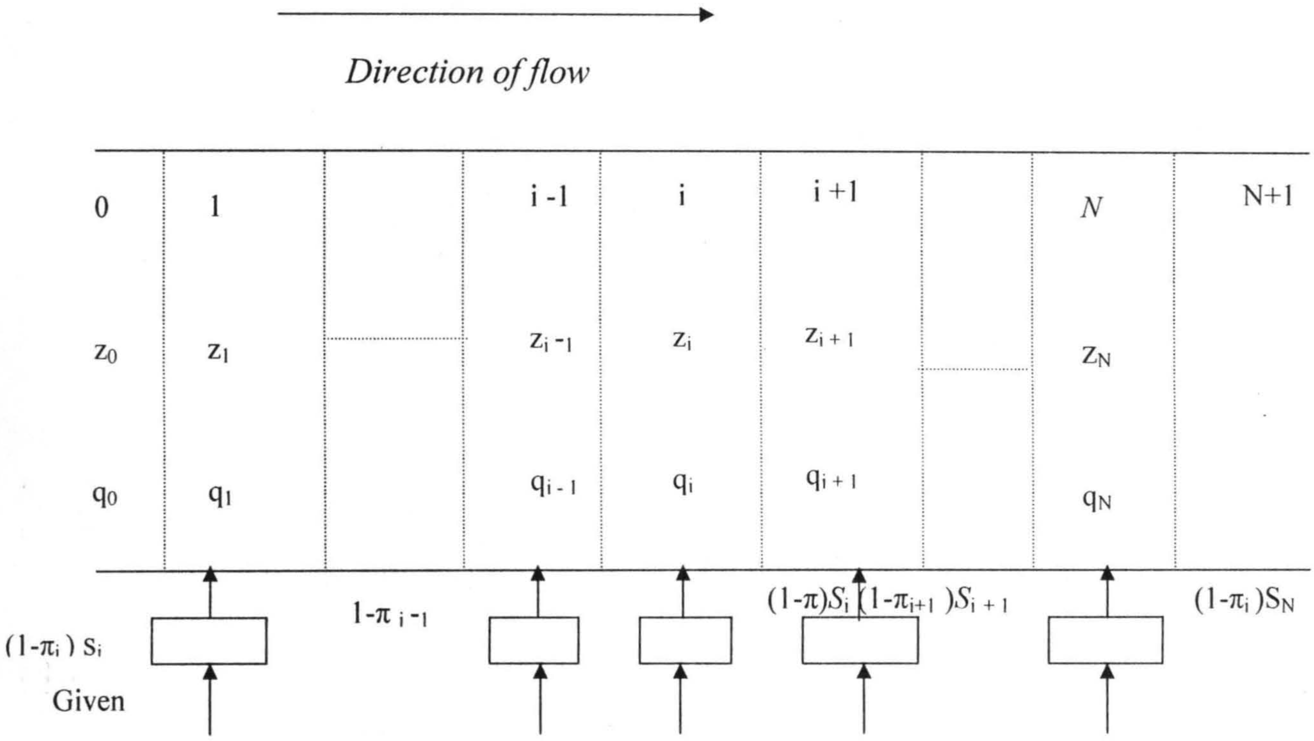

Figure 2.1 Schematic of river with affluent discharge

$$z_i(t+1) = z_i(t) - \alpha_i z_i(t) + \frac{\psi_i(t) z_{i-1}(t)}{v_i} - \frac{\psi_i z_i(t)}{v_i} + \frac{s_i(t) m_i(t)}{v_i} \quad 2.1$$

$$q_i(t+1) = q_i(t) + \beta_i(q^2_{z_i}(t) - q_i(t)) + \frac{q_{i-1}(t) z_{i-1}(t)}{v_i} - \frac{\psi_i q_i(t)}{v_i} + \alpha_i z_i(t) - \eta_i v_i \quad 2.2$$

$$\forall\ t = 1,\dots\dots\dots, T \quad and \quad i = 1,\dots\dots\dots, N$$

Here v_i is the volume of water in i in liters, ψ_i is the volume of water which flows from area i to area $i+1$ in each period in liters, α_i is the rate of decay of BOD per interval. This decay occurs by combination of BOD per interval.

2.4 The Optimal Control Problem

In the previous section, we surveyed the control of water pollution in a river. In this section as expunged by (Singh, et al. 1978, Waziri and Adeboye 2007), we review some computational methods that would be applicable in obtaining the optimal control of a polluted river in the next chapter.

CHAPTER THREE

RIVER POLLUTION CONTROL MODEL

3.1 Single Reach Pollution Control Model

Considering a reach of a river (figure 3.1) as a stretched portion of a river of some convenient length which receives one major controlled effluent discharge from a sewage treatment facility. Beck (1974) developed a second order state space equation, which describes the BOD and DO relationship at some average point in the reach as shown hereunder. The figure depicts a single reach and an effluent from a sewage station. Each reach is thought of as an ideal stirred tank reactor, as shown in figure 3.2 hence the parameters and variables are uniform throughout the stretch of the reach. Then, from the mass balance considerations, we have the following equations:

$$BOD: \ \dot{z}_i = -k_{1i}z_i + \frac{Q_{i-1}}{V_i}z_{i-1} - \frac{Q_i + Q_E}{z_i}z_i + \frac{\eta_i Q_E}{V_i} \qquad 3.1$$

$$DO: \quad \dot{q}_i = k_{2i}(q_i^s - q_i) - \frac{q_{i-1}}{V_i}q_{i-1} - \frac{Q_i + Q_E}{V_i}q_i - k_{1i}z_i - \frac{\eta_i}{V_i} \qquad 3.2$$

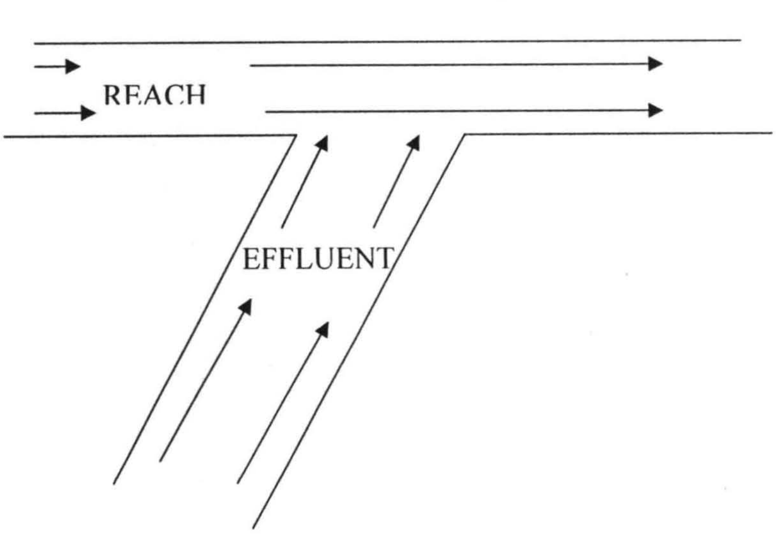

Fig3.1: An Effluent into a reach of a river

V_1 is the volume of water in reach i in million gallons

Q_E is the flow rate of effluent in reach i in million gallons/day

z_i, z_{i-1} are the concentration of BOD in reaches i and $i-1$ in mg/litre

q_i, q_{i-1} are the concentration of DO in reaches i and $i-1$ in mg/litre

k_{1i} is the BOD decay rate (day)$^{-1}$ in reach i

k_{2i} is the DO reaction (day)$^{-1}$ in reach i.

Q_1, Q_{i-1} are the stream flow rates in reaches i and $i-1$ in million gallons/day

q_i is the DO saturation level for the ith reach (mg/litre)

$\frac{\eta_i}{V_i}$ is the removal of DO due to bottom sludge requirements (mg/litre(day)$^{-1}$)

m_i is the concentration of BOD in the effluent in mg/litre

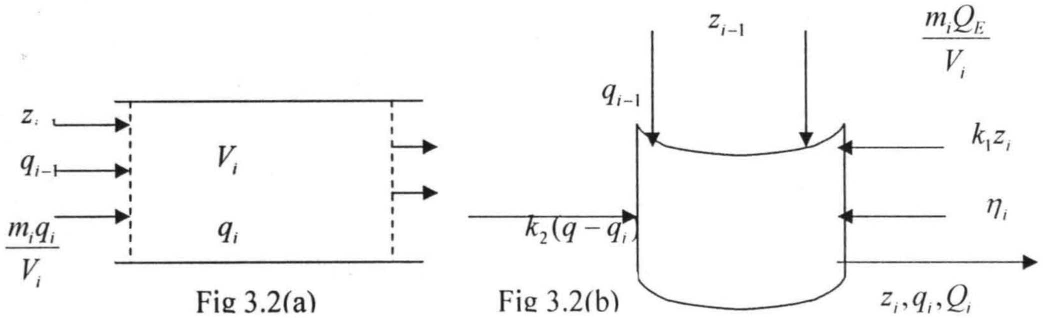

Figure 3.2: An ideal stirring tank reactor model for a reach of a river

(Beck 1974) found that for a section of the river CAM, the following values for the coefficients in the set of equations (3.1) and (3.2), are appropriate:

$$k_{1i} = 0.32 day^{-1},\ k_{21i} = 0.2 day^{-1},\ \frac{\eta_i}{V_i} = 0.1 mg/litre\, day^{-1},\ q_i^s = 10 mg/litre$$

$$\frac{Q_E}{V} = 0.1 \ and\ \frac{Q}{V} = 0.9$$

Thus, for the i^{th} reach, equations (3.1) and (3.2) can be rewritten as:

$$\frac{d}{dt}\begin{bmatrix} z_i \\ q_i \end{bmatrix} = \begin{bmatrix} -1.32 & 0 \\ -0.31 & -1.2 \end{bmatrix}\begin{bmatrix} z_i \\ q_i \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} m_i + \begin{bmatrix} 0.9z_{i-1} \\ 0.9q_{i-1} + 1.9 \end{bmatrix} \qquad 3.3$$

With stipulated initial conditions, this single reach problem is solvable using the Riccati equation.

3.2 Multiple Effluent Inputs to a System of Reaches Model

The model in section (3.1) is based on a single reach of a river which has only one effluent input. Here in this section, we consider the one with multiple effluents inputs. Here it is assumed that there is a system of reaches each of which has the properties of the one in section (3.2). This model is credited to Tamura (1974) who assumed that each reach was separated from the next by a distributed delay. This model is able to account for the dispersion of pollutants which actually occur in rivers. In this model, for $j = 1, 2, ..., s,$ a fraction α_j of BOD and DO in the $i-1^{th}$ reach at time $(t-\theta_j)$ arrives in the i^{th} reach at time t, that is, the transport delays are distributed in time between θ_1 and θ_s. Thus z_{i-1}, q_{i-1} are given by:

$$z_{i-1}(t) = \sum_{j=1}^{\infty} \alpha_j z_{i-1}(t-\theta_j) \qquad 3.4$$

$$q_{i-1}(t) = \sum_{j=1}^{\infty} \alpha_j q_{i-1}(t-\theta_j) \qquad 3.5$$

$$\sum_{j=1}^{s} \alpha_j = 1; \text{ mean of } \alpha_j = \theta_0;\ \theta_1 < \theta_2 < .... < \theta_s$$

The state equations for a three reach system with distributed delays could be written as:

$$\dot{z}_1 = -1.32s_1 + 0.1m_1 + 0.9z_0 + 5.35 \qquad 3.6$$

$$\dot{z}_1 = -1.32x_1 + 1.2q_1 + 0.9q_0 + 1.9 \qquad 3.7$$

$$\dot{z}_2 = 0.9\sum_{j=1}^{s}\alpha_j z_1(t-\tau_j) - 1.32z_2 + 0.1m_2 + 4.19 \qquad 3.8$$

$$\dot{q}_2 = 0.9\sum_{j=1}^{s}\alpha_j q_1(t-\tau_j) - 1.32z_2 - 1.29q_2 + 1.9 \qquad 3.9$$

$$\dot{z}_3 = 0.9\sum_{j=1}^{s}\alpha_j z_2(t-\tau_j) - 1.32z_3 + 0.1m_3 + 4.19 \qquad 3.10$$

$$\dot{q}_3 = 0.9\sum_{j=1}^{s}\alpha_j q_2(t-\tau_j) - 1.32z_3 - 1.29q_3 + 1.9 \qquad 3.11$$

As in (Beck 1974) model, (Tamura 1974) gives the following values for s, τ, α, etc for each of the distributed delays:

$$s = 3,\ \tau_1 = 0;\ \tau_2 = \frac{1}{2}day;\ \tau_3 = 1day,\ z_0 = 0;\ q_0 = 10;\ \alpha_1 = 0.15,\ \alpha_2 = 0.7\ and\ \alpha_3 = 0.15$$

The described by the system (3.6)-(3.11) is normally of infinite dimension in the state space. By expanding the delay terms in a Taylor series and taking the first two terms, we can obtain a good finite dimensional approximation. Take for instance equation (3.8) can be rewritten as:

$$\dot{z}_2 = 0.9(0.15z_1(t) + 0.7z_1(t-0.5) + 0.15z_1(t-1) - 1.32z_2 + 0.1m_2 + 4.19$$

Here, we have 2 delays so it is necessary to introduce four additional states. Let these be given by z_4, z_5, z_6, z_7. Let $z_4(t) = z_1(t-0.5)$; then $z_4(s) = z_1(s)e^{-0.5s}$. Now

$$z_1(s)e^{-0.5s} = z_1(s)[1 + 0.5s + \frac{0.25}{2}s^2 + ...]^{-1}$$

Taking only the first three terms:

$$z_1(t) = z_4(t) + 0.5\dot{z}_4(t) + 0.125\ddot{z}_4(t)$$

Let $\dot{z}_4 = z_5$ 3.12

then $\dot{z}_5 = 8z_1 - 8z_4 - 4z_5$ 3.13

Similarly for the other delay, we let:

$$z_1(t-1) = z_6 \qquad 3.14$$

then
$$\dot{z}_6 = z_7 \qquad 3.15$$

$$\dot{z}_7 = z_7 - 2z_6 - 2z_7 \qquad 3.16$$

Thus, we can rewrite equations 3.8 as

$$\dot{z}_2 = 0.135z_1 + 0.63z_4 + 0.135z_6 - 1.32z_2 + 0.1m_2 + 4.19 \qquad 3.17$$

In the same way, we can introduce 4 more additional variables for each of the delays in equations (3.9)-(3.11).

This makes the overall system of order 22. Solving this large problem of order 22 is not feasible using the current single PC with little processor. Even using Pentium 4 is practically impossible. In view of this difficulty, we consider the simple problems for our data analysis involving one and two reaches.

CHAPTER FOUR

NUMERICAL COMPUTATION OF THE RIVER POLLUTION CONTROL

4. 1 Statement of Problem

We consider the model for the one reach with one effluent problem posed but not solved in Singh et al (1978):

Find the optimal control trajectory Δm_1 for the river system:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}\begin{bmatrix} z_1 \\ q_1 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}\Delta m_1 + \begin{bmatrix} 0.9z_0 & 5.35 \\ 0.9q_0 & 1.9 \end{bmatrix} \qquad 4.1$$

with $z_0 = 10, q_0 = 0$ and the cost function to be minimized is

$$J = \int_0^s \left\{(z_1 - 4.06)^2 + 2(q-8)^2 + \Delta m_1^2\right\} dt \qquad 4.2$$

with initial conditions

$$z_1(0) = 10mg/l,\ q_1(0) = 7mg/l \qquad 4.3$$

$z_1(0)$ *and* $q_1(0)$ which are the concentrations immediately upstream from area 1 are assumed known.

This problem is solvable using the Maximum principle:

a) The Hamiltonian form is:

$$H = \|z_1 - 4.06\|^2 + 2\|q-8\|^2 + \Delta m_1^2 + \lambda_1[-1.32z_1 + 0.1\Delta m_1 + 0.9z_0 + 5.35] + \lambda_2[-3.2z_1 - 1.2q + 0.9q_0 + 1.9] \qquad 4.4$$

b) The maximum principle conditions are;

i. $$\frac{\partial H}{\partial(\Delta m_1)} = 2\Delta m_1 + 0.1\lambda_1 = 0,\ \textit{this implies that}\ \ \Delta m_1 = 0.05\lambda_1 \qquad 4.5$$

ii. $$\frac{\partial H}{\partial z_1} = -\dot{\lambda}_1 = -2(z_1 - 4.06) + 1.32\lambda_1 + 1.32\lambda_2 \qquad 4.6$$

$$\frac{\partial H}{\partial q} = \dot{\lambda}_2 = -4(q-8) + 1.2\lambda_2 \qquad 4.7$$

$$\frac{\partial H}{\partial \lambda_1} = \dot{z}_1 = -1.32z_1 + 0.1\Delta m_1 + 0.9z_0 + 5.35 \qquad 4.8$$

$$\frac{\partial H}{\partial \lambda_2} = \dot{q}_2 = -3.2z_1 - 1.2q + 0.9q_0 + 1.9 \qquad 4.9$$

c) Rearranging group b(ii) and with consideration of equation (4.5), we have:

$$\dot{z}_1 = -1.32z_1 + 0.005\lambda_1 + 0.9z_0 + 5.35 \qquad 4.10$$

$$\dot{q}_1 = -3.2z_1 - 1.2q_1 + 0.9q_0 + 1.9 \qquad 4.11$$

$$\dot{\lambda}_1 = -2z_1 + 1.32\lambda_1 + 1.32\lambda_2 + 8.12 \qquad 4.12$$

$$\dot{\lambda}_2 = -4q + 1.2\lambda_2 + 32 \qquad 4.13$$

Solving this system of first order differential equations by rewriting the equations in canonical form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{q}_1 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -1.32 & 0 & 0.005 & 0 \\ -3.2 & -1.2 & 0 & 0 \\ -2 & 0 & 1.32 & 1.32 \\ -3.2 & -1.2 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ q_1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0.9z_0 + 5.35 \\ 0.9q_0 + 1.9 \\ 8.12 \\ 32 \end{bmatrix} \qquad 4.14$$

Solving the canonical system using

$$x = \begin{pmatrix} 64.4021 & 62.4642 & -20.3986 & -72.2514 \\ 62.4642 & 67.9971 & -14.3726 & -75.3770 \\ -20.3986 & -14.3726 & 11.1645 & 18.6559 \\ -72.2514 & -75.3770 & 18.6559 & 85.1208 \end{pmatrix}$$

$$\begin{pmatrix} -20.2686 \\ \\ \\ \end{pmatrix}$$

L = -3.0182

-0.0699

-1.6739

G = (10.1988 4.3257 -5.5018 -7.0888)

The values for x matrix are the eigenvectors, L denote the roots of the canonical equation 4.14 The Gain is that gain of the canonical form which represents $-R^{-1}B^{T}P(t;P_f,t_f)$ of the optimal control as derived in equation (2.70). We are now to compute the BOD, the DO and the co-states of our problem statement:

$$\begin{bmatrix} z_1 \\ q_1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = k_1 \begin{bmatrix} 64.4021 \\ 62.4642 \\ -20.3986 \\ -72.2514 \end{bmatrix} e^{-20.2686t} + k_2 \begin{bmatrix} 62.4642 \\ 67.9971 \\ -14.3726 \\ -75.3770 \end{bmatrix} e^{-3.0182t} + k_3 \begin{bmatrix} -20.3986 \\ -14.3726 \\ 11.1645 \\ 18.6559 \end{bmatrix} e^{-0.0699.t} + k_4 \begin{bmatrix} -72.2514 \\ -75.3770 \\ 18.6559 \\ 85.1208 \end{bmatrix} e^{-1.6739t}$$

(4.15)

The optimal control Δm_1 is defined as:

$$\Delta m_1 = -R^{-1}B^{T}P(t;P_f,t_f)x$$

which implies that:

$$\Delta m_1 = Gx \qquad 4.16$$

Solving the above problem in (4.15) for Δm_1 is not feasible. We therefore employ the use of mathcad to compute it. This yields:

$$G := (10.1988\ \ 4.3257\ \ -5.5018\ \ -7.0888)$$

$$X := \begin{pmatrix} 64.4021 & 62.4642 & -20.3986 & -72.2514 \\ 62.4642 & 67.9971 & -14.3726 & -75.3770 \\ -20.3986 & -14.3726 & 11.1645 & 18.6559 \\ -72.2514 & -75.3770 & 18.6559 & 85.1208 \end{pmatrix}$$

$$\Delta M_1 := G \cdot X$$

$$\Delta M_1 = \left(1.551 \times 10^3\ \ 1.545 \times 10^3\ \ -463.886\ \ -1.769 \times 10^3\right) \qquad 4.18$$

The numerical values to equation explicitly define the optimal control to our statement of the problem. Now, invoking the initial conditions from equation (4.3), and with $\lambda_1 = 5.94 mg/l \ and \ \lambda_2 = 6 mg/l$ when can compute the BOD, DO and co-states in this order:

$$z_1(0) = 64.4021k_1 + 62.4642k_2 - 20.3986k_3 - 72.2514k_4 = 10 \qquad 4.19$$

$$q_1(0) = 62.4642k_1 + 67.9971k_2 - 14.3726k_3 - 75.3770k_4 = 7 \qquad 4.20$$

$$\lambda_1(0) = -20.3986k_1 - 14.3726k_2 + 11.1645k_3 + 18.6559k_4 = 5.94 \qquad 4.21$$

$$\lambda_2(0) = -72.2514k_1 - 75.3770k_2 + 18.6559k_3 + 85.1208k_4 = 6 \qquad 4.22$$

Solving equations (4.18)-(4.22) simultaneously using software package yields:

$$M := \begin{pmatrix} 64.402 & 62.4642 & -20.3986 & -72.2514 \\ 62.4642 & 67.9971 & -14.372 & -75.3770 \\ -20.3986 & -14.372 & 11.1645 & 18.6559 \\ -72.2514 & -75.3770 & 18.6559 & 85.1208 \end{pmatrix}$$

$$v := \begin{pmatrix} 10 \\ 7 \\ 5.94 \\ 6 \end{pmatrix}$$

$$soln := lsolve(M, v)$$

$$soln = \begin{pmatrix} 199.722 \\ 62.191 \\ 110.576 \\ 200.433 \end{pmatrix}$$

From the computational algorithm,

$$k_1 = 199.722,\ k_2 = 62.191,\ k_3 = 110.576\ and\ k_4 = 200.433 \qquad 4.23$$

From the set of equations (4.23) and considering the system of equation (4.15), we have the BOD, DO and the co-states as follows:

$$z_1(t) = 12860e^{-20.2686t} + 12480e^{-3.0182t} - 4074e^{-0.0699t} - 14430e^{-1.6739t} \qquad 4.24$$

$$q_1(t) = 3885e^{-20.2686t} + 4229e^{-3.0182t} - 893.809e^{-0.0699t} - 4088e^{-1.6739t} \qquad 4.25$$

$$\lambda_1(t) = -2256e^{-20.2686t} - 1589e^{-3.0182t} + 1235e^{-0.699t} + 2063e^{-1.6739t} \qquad 4.26$$

$$\lambda_2(t) = -14480e^{-20.2686t} - 15110e^{-3.0182t} + 3739e^{-0.699t} + 17060e^{-1.6739t} \qquad 4.27$$

The next step is to have the visual representation graphically of equations (4.24)-(4.27); these demonstrated in the figures below:

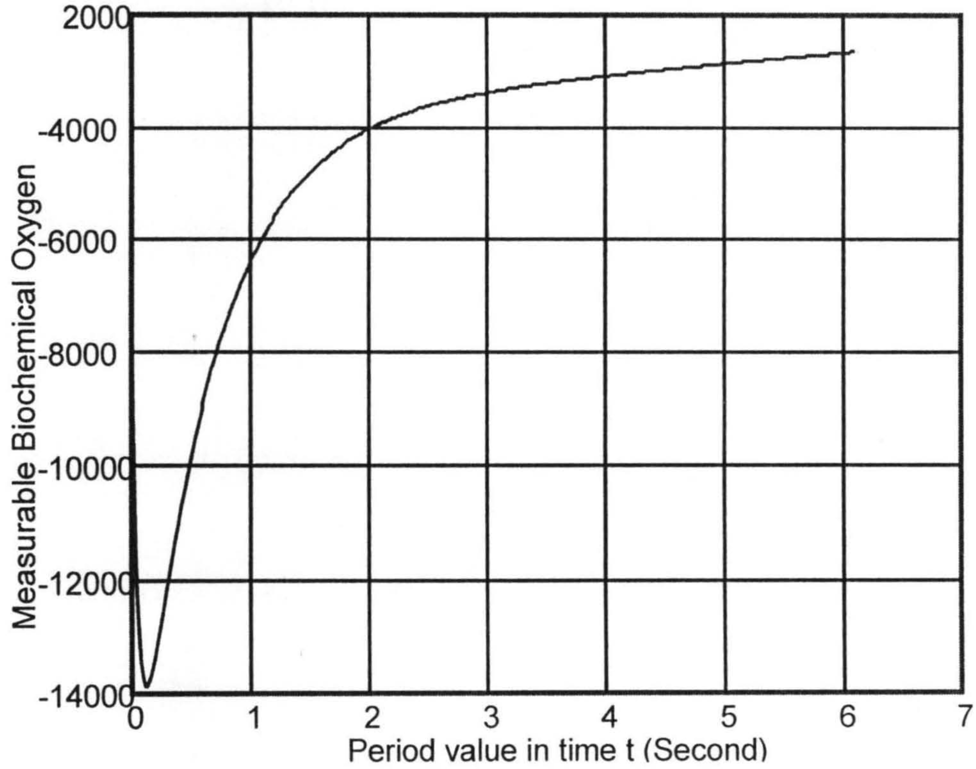

Figure 4.1: Plot of BOD versus Time

Figure 4.1 depicts the graphical representation of the Biochemical Oxygen Dissolved in a

River with one reach and one effluent. It is curious to note that the BOD activities at first declines in magnitude and then reaches a minimum peak at some time equivalent to 0.15. It then rises shapely in a monotonic increasing fashion within time 2; thereafter asymptotic increment is achieved. The explanation may be fetched in this manner; as the effluent reaches a reach, the decay activities falls in a negative direction for within a short period of time, after the attainment of this fall, the BOD activities increase significantly to a given peak. Upon reaching some peak (as identified at time t=2), the decay of the activities in the water maintain some stability in a positive monotonic increment.

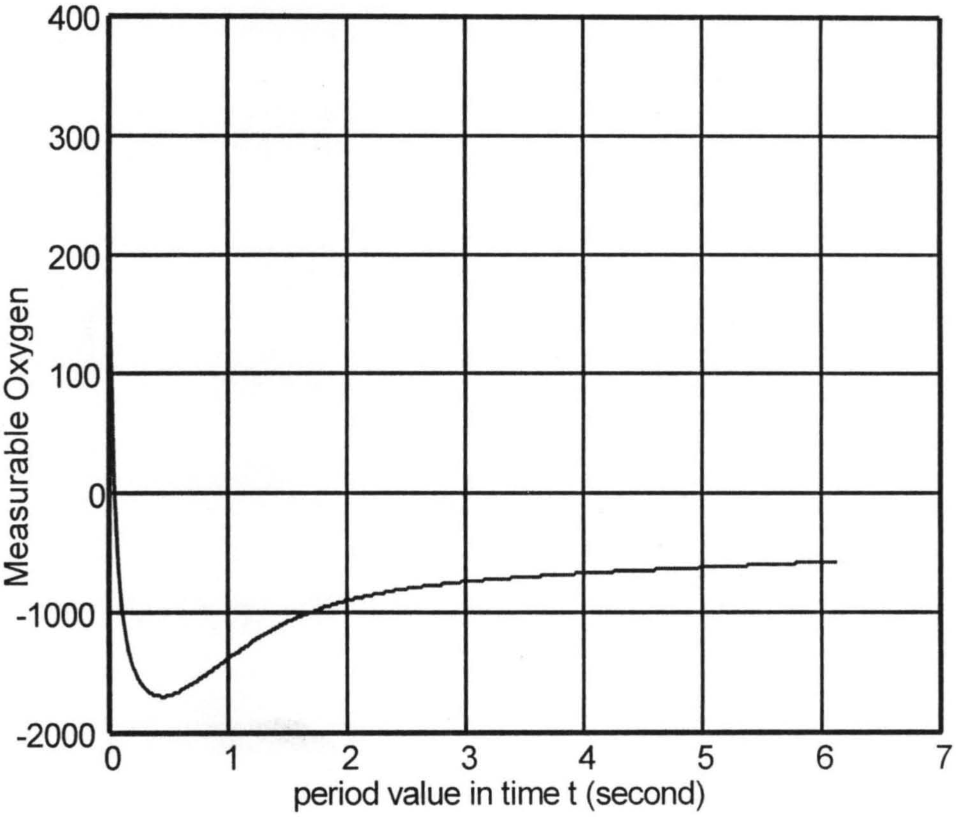

Figure 4.2: Plot of OD versus Time

Figure 4.2 is the graphical representation of rate of rearation of the Dissolved Oxygen (DO) in the water. Like in BOD above a decline in the rearation is first observed and then increase within the negative belt of observations.

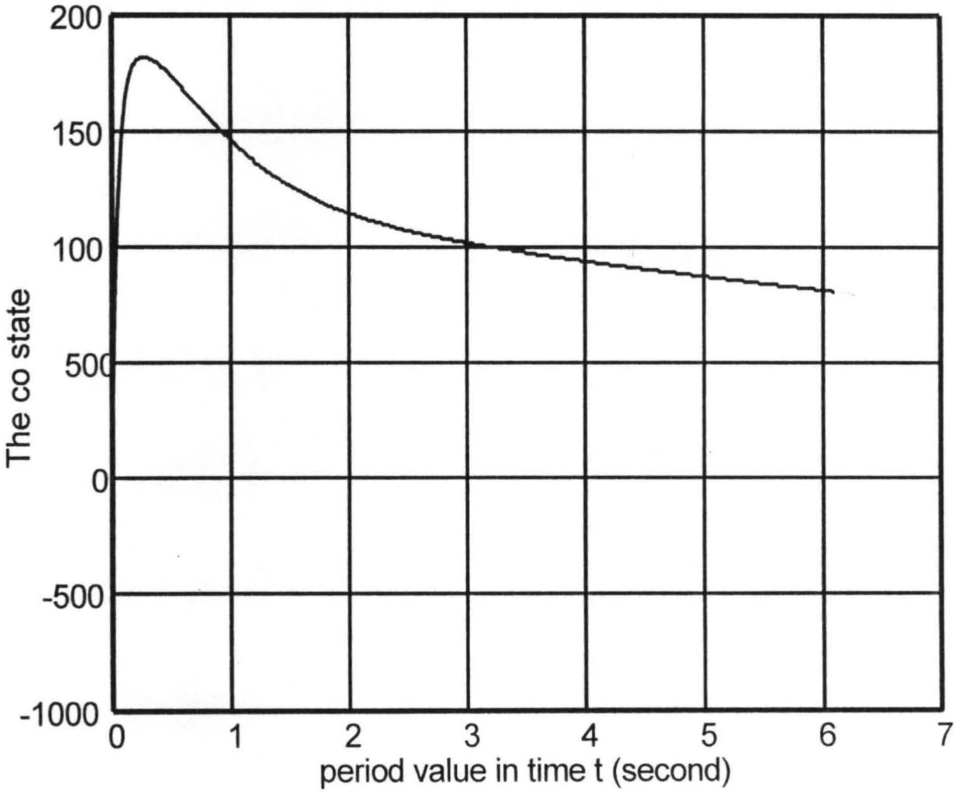

Figure 4.3: Plot of co-state 1 versus

Figure 4.3 depicts the co-existence of the BOD and DO with the co-states. This cohabitation results to a raise and reaching some significant optimum; and then decline negatively in an asymptotic order.

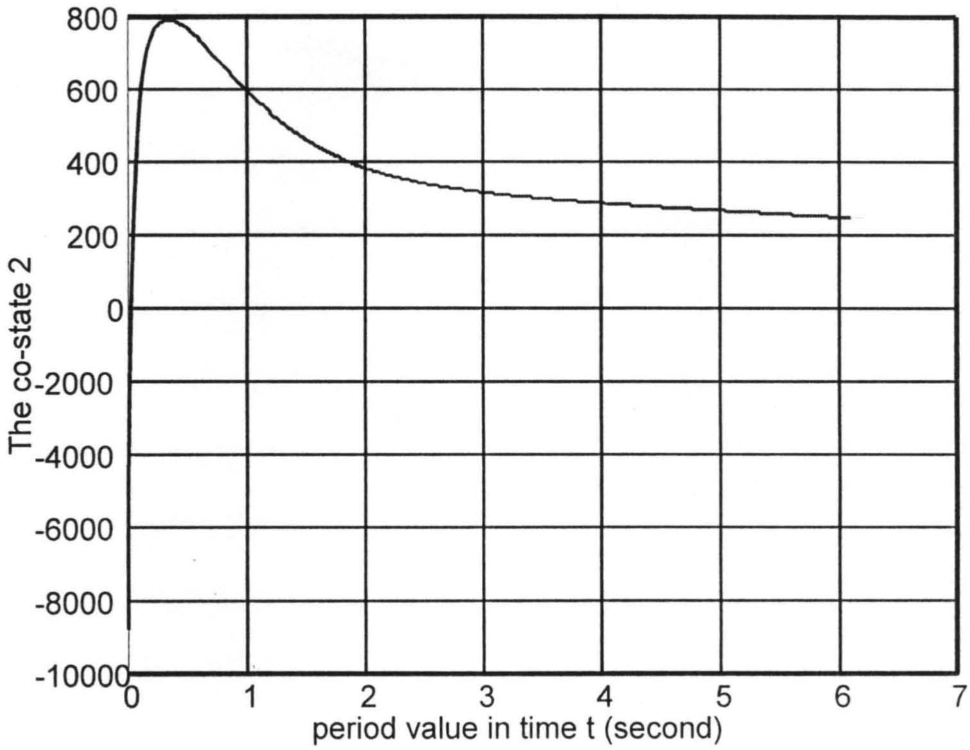

Figure 4.4: Plot of co state 2 versus

Figure 4.4 depicts the co-existence of the BOD and DO with the co-states. This co-habitation results to a raise and reaching some significant optimum; and then decline negatively in an asymptotic order.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATION

5.1 CONCLUSION

This research is on the optimal river pollution control. The research visualized the reaches with many effluents of which Beck (1974) and Tamura (1974) river pollution models were studied. Beck's model was on a stretch of reach with an effluent from an industry flowing into the river. The computational processes were observed to be relatively simpler than that of Tamura's model which consists of many influents into many specified reaches. For instance, Tamura's model of three reaches and three effluents resulted in the formulation of a 22 by 22 matrix. The computation of this model resulted into an overflow; meaning that the computer space is inadequate.

In view of the computational difficulty encountered, a one reach and one effluent modeled by Beck (1974) were chosen as the standard computational model. Hence, a problem posed but not solved by Singh and Titli (1978) was taken as our computational example and this itself is a major achievement. The outcome of the result for the optimal control is given by the numerical value 4.18. The Biological Oxygen Demand (BOD), the Dissolved Oxygen (DO) and co-states vectors were obtained using the Maximum principle. Tracking revelations were obtained and are systematically discussed under the graphical simulations as exhibited in figures (4.1)-(4.4).

Research for the river pollution control should not be limited to the treatment of the effluents in the industrialized environment only. Government could tax other agencies that contribute to the pollution of the environment which further has adverse effects on the water pollution. Such taxes would help in checking reckless utility of those substances that could contribute to the environmental and water pollution. Such taxes could also further enhance government revenue.

5.2 RECOMMENDATION

The study has been primarily on continuous computational river control pollution. The research can be extended to the discrete variables consideration. If this is done, the researcher suggests the following discrete computational river control pollution problem for further investigation:

$$J = \frac{1}{2}\begin{bmatrix} x_1(6) \\ x_2(6) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(6) \\ x_2(6) \end{bmatrix} + \sum_{k=0}^{5} \begin{bmatrix} x_1(6) \\ x_2(6) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(6) \\ x_2(6) \end{bmatrix} + \begin{bmatrix} u_1(6) \\ u_2(6) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1(6) \\ u_2(6) \end{bmatrix} \quad 5.1$$

subject to the constraints

$$\begin{aligned} x_1(k+1) &= 2x_1(k) + x_2^2(k) \\ x_2(k+1) &= x_1(k) + x_2(k) + u_1(k) + u_2(k) \end{aligned} \quad 5.2$$

Apart from computing the optimal controls of the polluted reaches, the cost of controlling the rate of decay of BOD and DO could be considered for study in some extended area of the reaches per liter per interval. This decay occurs through the combination of BOD and DO. The increase in DO is due to various natural oxygen-producing biochemical reactions in the stream and the increase is proportional to (q_s / q_i) where q_s is the saturation level of DO in the stream. Finally, q_i is the DO requirement in the bottom sludge.

References

Beck, M. B. (1974): The application of control and systems theory to problems of river pollution. Cambridge University PH. D. Thesis (Unpublished).

Burn, D, H, and McBean, E, A, (1985): "Optimization modeling of water quality in an uncertain environment." Water Resource 21(7), 934-940.

Burrton, T. D. (1994): Introduction to dynamic systems analysis. Mcgraw-Hill series in mechanical engineering.

Edward, T. D. (1972): Introduction to Mathematical economics (Third Edition). McGraw-Hill International Edition. 494-503

James A., and Elliot, D. t. (1993): Models of Water quality in rivers." An introduction to water quality modeling, (2nd Ed.), A. James ed., John Wiley & Sons Inc. New York, N.Y. 141 – 181.

Katsuhisa, F., Akira, S. and Atherton, D. (1988): State Variable Methods in Automatic Control. John Wiley and Sons; New York-Singapore-Toronto-Brisbane - Chichester.

Loucks, D.P. (1983): "Models for management applications." Mathematical modeling for water quality: Streams lakes and reservoirs, G. T. orlob, ed., John Wiley & Sons, Inc New York N.Y. 468 – 509.

Rodi W. (1993): Turbulence models and their application in hydraulic, 3rd ed., IAHR/AIRH monograph, Balkema, Rotterdam

Singh, M. G. and Titli A. (1978): Systems; decomposition, optimization and control. Pergamon International Library of Science, Technology, Engineering and Social Sciences.

Tamura, H. (1974): A discrete dynamical model with distributed transport delays and its hierarchical optimization to preserve stream quality. I.E.E.E. Trans. SMC 4, 424-429.

Waziri V. O. and Adeboye K.R. (2007): The optimal control of second order linear Equipontential fluids. Leonardo Electronic Journal of Practices and Technologies (LEJPT) Issue 10, (109 – 122).