BLOCK ERROR ANALYSIS OF SOME ADAMS BASHFORTH AND ADAMS MOULTON SCHEMES

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF TECHNOLOGY IN APPLIED MATHEMATICS

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ABSTRACT

In the past, the analysis of order and error constants of block linear multistep methods are usually determined on the individual members of the block. In this project, we proposed the analysis of the schemes in block form as they appeared for implementation. Specifically, cases k= 2, 3, 4 and 5 for both Adams Moulton (implicit) and Adams Bashforth (explicit) were reformulated as continuous schemes to generate a number of sufficient schemes necessary to make the methods self-starting. The derivation was done through the continuous collocation technique using power series as basis function and the property of order and error constants is examined on the entire block of each step number considered. Numerical experiment was also conducted on the methods considered and it was observed that the accuracy of the methods increases as the step number increases. Furthermore, the Adams Moulton methods produced more accurate results than the corresponding Adams Bashforth methods.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

There are a number of differential equations which are studied in calculus to get close form solution. However, not all differential equations possess closed form or finite solution. Numerical methods for ordinary differential equations (ODEs) are methods used to produce numerical approximations to the solution of ordinary differential equations in discrete form. Many differential equations cannot be solved using analytical method, therefore, for practical purposes such as in engineering, a numeric approximation to the solution is often sufficient (Akinfenwa *et al.,* 2011).

Numerical methods for solving first-order initial value problems (IVPs) are often categorized into linear multistep methods (LMM), or Runge-Kutta methods (Akinfenwa *et al.*, 2011). A further division can be realized by dividing the methods into explicit and implicit methods. Explicit linear multistep methods include Adams – Bashforth methods, and any Runge-Kutta method with a lower diagonal Butcher tableau is explicit. While implicit linear multistep methods include Adams-Moulton methods, and Backward Differentiation Formula (BDF). Linear Multistep Methods require less evaluation of the derivative function f than one step methods in the range of integral $\left[x_0, b\right]$. For this reason, they have been very popular and important for solving ordinary differential equations numerically (Muhammed *et al.*, 2014). However, these methods have certain limitations such as the overlap of solution models and the requirement of a starting value. Other limitations include yielding the discrete solution values $y_n, ..., y_N$ hence uneconomical for producing large output. A continuous formulation is desirable in this respect. The collocation method is probably the most important numerical procedure for the construction of continuous methods – (Lie and

Norsett, 1989; Onumanyi *et al.*, 1994; 1999; 2001). The continuous method preserves the Runge-kutta traditional advantage as it allows generation of necessary and sufficient number of schemes which makes the method self-starting and is more accurate since it is implemented as a block method. Block methods were first introduced by Milne (1953) for the purpose of obtaining starting values for predictor-corrector algorithms (Sarafyan, 1965). However, Rosser (1967), developed Milne's idea into algorithms for general use. Block methods have also been considered by Shampine and Watts (1969), Musa *et al.* (2012), Jator and Li, (2012), Akinfenwa, *et al.*, 2013; Mohammed and Adeniyi, (2014), Badmus *et al.* (2015), Omar and Adeyeye (2016), Akinfenwa *et al.*, (2017). Furthermore, error analysis of numerical methods is crucial; an acceptable linear multistep method (LMM) must be convergent. Consistency and zero stability are the necessary and sufficient conditions for convergence of a LMM. According to Musa *et al.* (2012), consistency controls the magnitude of the local truncation error while zero stability controls the manner in which the error is propagated at each step of the computation. A method which is not both consistent and zero stable is rejected outright and has no practical interest. In the past times, analysis of these properties are being carried out on the individual member of a block linear multistep methods - (Ibrahim *et al*., 2011; Muhammad *et al.*, 2014), whose results may not be assumed for the entire block method. However, in this project work, we carry out the analysis of the derived block methods on the entire block.

1.2 Statement of the Research Problem

Linear multistep methods implemented by the predictor-corrector method have been found to be very expensive to implement in terms of developing predictors. Furthermore, this predictor developed is usually of lower order to the corrector, thus it has great consequence on the accuracy of the corrector results. The block multistep methods are one of the numerical methods which have been suggested by scholars to cater for the shortcoming of predictor-corrector method, since block method are self-starting which eliminates the use of predictors and provides solutions at each grid (and off-grid) within the interval of integration without overlapping thereby eradicating the idea of predictor. However, prior to the implementation of numerical methods, convergence analysis is a very key instrument to determine the reliability of such numerical methods. Before now, the determination of the order of accuracy and error constants of block methods, their convergence and plotting their absolute stability regions have been done for the individual members of the block and whose result may not be assumed for the entire block. In this research, the properties (particularly, order and error constants) of the entire block of some Adams Bashforth and Adams Moulton methods are considered.

1.3 Aim and Objectives of the Study

The aim of this study is to carry out block error analysis of some Adams Bashforth and Adams Moulton schemes. The specific objectives are to:

1. derive the continuous formulation of both Adams Bashforth and Adams Moulton methods when $k=2, k=3, k=4$ and $k=5$;
2. obtain from the continuous method, the schemes necessary and sufficient to make the methods self-starting;
3. investigate the properties of the entire members of the block methods (in terms of convergence and stability) as a single entity.
4. carryout numerical experiments using the derived block methods.

1.4 Significance of the Study

Error in numerical analysis is key, prior to this work, authors determine the error analysis of block linear multistep methods using the individual scheme that constitute the block members of the method. Consequently, in this research work, rather than using the conventional approach of obtaining order and error constants of individual member in the block method, an efficient approach that yields the order and error constants of block members at once is analysed thereby saving computing time.

1.5 Scope and Limitations of the Study

This research work considers the block error analysis of some selected step numbers of the Adams Bashforth and Adams Moulton schemes for the numerical solution of first order ordinary differential equation.

1.6 Definition of some Terms

1.6.1 Initial value problem (Akinfenwa *et al.,* 2011)

This is any differential equation in which the initial condition of the problem is given, it is given in the form

$$y^n(x) = f(x, y, y' \ldots y^{n-1}), \;\; y(x_0) = y_0, \;\; y'(x_0) = y'_0 \ldots y^{n-1}(x_0) = y_0^{n-1}$$

$a \le x \le b,$ given $a = x_0 \prec x_1 \ldots \prec N = b$

1.6.2 Linear multistep method (LMM)

Unlike the one step method that utilizes one previous value of the numerical solution to approximate the subsequent value, a k step multistep method utilizes $k-1$ previous value.

A general k-step LMM is given in the form

$$y_n = \sum_{j=1}^{k} \alpha_j y_{n+j} + h^{\mu} \sum_{j=0}^{k} \beta_j f_{n+j}$$

where μ is the order of differential equation in consideration. (Yahaya, 2004)

1.6.3 Block method

A block method is formulated in linear multistep method form which preserves the advantage of one step method of being self-starting and permitting the change of step length. Its general form is given as:

$$Y_m = \sum_{j=0}^{k} A_j y_{m-j} + h\sum_{j=o}^{k} B_j F_{m-j}$$

where $Y_m = (y_n, y_{n+1}, \ldots y_{n+r-1})^T$, $F_m = (f_n, f_{n+1}, \ldots f_{n+r-1})^T$ $A_j\ 's$ and $B_j\ 's$ are properly chosen $r \times r$ matrix coefficients and $m = 0,1,2\ldots$represents the number $n = mr$ is the first step number of the nth block and r is the proposed block size (Yahaya, 2004).

1.6.4 Consistency

Linear multistep methods are said to be consistent if it has order $p \geq 1$ (Muhammad *et al.*, 2014).

1.6.5 Zero stability

A block method is said to be zero stable if the roots $\lambda_i, i = 1,2,3\ldots s$ of the first characteristics polynomial $\rho(\lambda)$ defined by $\rho(\lambda) = \left|\sum_{i=0}^{s} A^i \lambda^{s-1}\right| = 0$ satisfies $|\lambda_1| \leq 1$ and for the roots with $|\lambda_1| = 1$ the multiplicity must not exceed the order of the differential equation in consideration; (Muhammad *et al.*, 2014).

1.6.6 Convergence

If a linear multistep method is stable and of order r , then it is convergent of order r . Convergence of order r means that for sufficiently accurate starting approximations $y_0, \ldots, y_{k-1}$ the global error satisfies $y_n, \ldots, y(t_n) = O(h^r)$ on compact intervals $[0, nh]$, where $nh \leq T$. The constant symbolized by the $O(h^r)$ notation is bounded by $exp(TL)$, where L is a Lipschitz constant of the vector field $f(t, y)$.

1.6.7 Truncation error

This is of two kind: The local truncation error at a time step $n+1$ is given as

$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}$. It is the difference between the exact solution and the approximation applied to the exact solution at time t. The global truncation amount to error that occurs in the use of a numerical approximation to solve a problem.

1.6.8 Matrix

A matrix is a set of real or complex numbers (or elements) arranged in rows and columns to form a rectangular or square array. Example, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a 2×2 matrix

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Numerical Methods

Numerical analysis is concerned with all aspects of the numerical solution of a problem, from the theoretical development and understanding of numerical methods to their practical implementation as reliable and efficient computer programs. Numerical analysis is the study of algorithms that use numerical approximation for the issues of mathematical analysis of real world problems arising from all fields of engineering, physical sciences, life sciences, social sciences, medicine and business. In most cases, a number of these problems are dynamical in nature with relation to time, space and other physical quantities which might be transformed into ordinary differential equations (ODEs). However, most numerical analysts concentrate on small subfields, but they share some common concerns, perspectives, and mathematical methods of research. When presented with a problem that cannot be solved directly, they exchange it with a “nearby problem” which will be solved more easily. Examples are the utilization of interpolation in developing numerical integration methods and root-finding methods – (Onumanyi *et al.*, 2001; Yahaya, 2004, Mohammed and Yahaya, 2010). There is widespread use of the language and results of algebra, real analysis, and functional analysis (with its simplifying notation of norms, vector spaces, and operators). There is a fundamental concern with error, its size, and its analytic form. When approximating a problem, it is prudent to know the characteristics of the error within the computed solution. Moreover, understanding the shape of the error allows creation of extrapolation processes to boost the convergence behaviour of the numerical method. Numerical analysts are concerned with stability, a concept referring to the sensitivity of the solution of a problem to small changes in the data or the parameters of the problem. Numerical analysts are very curious about the consequences of using finite precision computer arithmetic, this can be especially important in numerical algebra, as large problems contain many rounding errors. Numerical analysts are generally interested about measuring the efficiency (or "cost") of an algorithm, for an example, the employment of Gaussian elimination to solve a linear system $Ax = B$ containing n equations would require approximately $\frac{2n^3}{3}$ arithmetic operations. Numerical analysts would want to understand how this method compares with other methods for solving the problem.

However, a number of the ODEs don't have analytical solution, therefore one amongst the possible ways to tackle this problem is to think about a discrete domain instead of a continuous one. Hence for practical purposes like engineering, a numerical approximation is commonly sufficient. Numerical methods for ODEs are methods accustomed to find numerical approximations to the solutions of ODEs. Conceptually, a numerical method starts from an initial point then takes a brief step forward in time to search out the subsequent solution point. The overall numerical methods for approximating initial value problems (IVPs) of ODEs for the value of $y(x)$ at discrete times t_i can be written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\varphi_f\left(y_{n+k}, y_{n+k-1}, ..., y_n, x_n, h\right) \tag{2.1}$$

$y_n \approx y(x_n)$ where $x_n = x_0 + nh$ for

where h is the time step and n is an integer.

The common numerical methods used to solve ODEs are categorized as one-step $(k = 1)$ methods and multistep ($k > 1$) methods (Akinfenwa *et al.*, 2011).

2.2 Linear multistep method (LMM)

Linear multistep methods are used for the numerical solution of ordinary differential equations. Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution. Single-step methods (such as Euler's method) refer to only one previous point and its derivative to determine the current value. Methods such as Runge–Kutta take some intermediate steps (for example, a half-step) to obtain a higher order method, but then discard all previous information before taking a second step (Butcher, 2008). Multistep methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, multistep methods refer to several previous points and derivative values. In the case of linear multistep methods, a linear combination of the previous points and derivative values is used.

The general k-step linear multistep method is given as

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \ldots + \alpha_1 y_{n+1} + \alpha_0 y_n = h\left(\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \ldots + \beta_1 f_{n+1} + \beta_0 f_n\right) \qquad (2.2)$$

or equivalently (Ndanusa, 2007).

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2.3)$$

It is always the case that $\alpha_k = 1$, also at least one of the α_0 and β_0 will be non-zero.

A linear multistep method is defined by a choice of quantities α_j's and β_j's. If $\beta_k = 0$, the method is called explicit (because the method can directly compute y_{n+k}). If $\beta_k \neq 0$, the method is called implicit (since the value of y_{n+k} depends on the value of $f\left(x_{n+k}, y_{n+k}\right)$, and the equation must be solved for y_{n+k}. Sometimes an explicit multistep method is used to “predict” the value of

y_{n+k}. That value is then used in an implicit formula to "correct" the value. The result is a predictor – corrector method. However, the predictor-corrector method has a great setback of starting values which means more time and human efforts are needed to derive another method(s) to allow the main method to be implemented. Moreover, the predictors developed are of lower order to the corrector; therefore, corrupting the expected results generated by the corrector. Linear multistep techniques (for various families as expressed above) have been found to create generally higher order of exactness to differential equations by numerous scientists – Ndanusa (2007), Ndanusa and Adeboye (2008), Mohammed and Yahaya (2010), Chollom *et al.* (2014), Akinfenwa *et al.* (2014) and Akinfenwa *et al.* (2017).

2.2.1 Families of linear multistep methods

There are three families of linear multistep methods that are commonly used:

(i) The Adams–Bashforth methods allow us explicitly to compute the approximate solution at an instant time from the solutions in previous instants. The Adams – Bashforth methods are explicit methods and the coefficients are $\alpha_0 = \alpha_1 = ... = \alpha_{k-2} = 0$ and $\alpha_{k-1} = -1$, while $\beta_j 's$ are chosen such that the methods have an order k. The Adams – Bashforth methods with k=1,2,3,4,5 are: (Hairer *et al.*, 1993).

$$\left.\begin{aligned}
&y_{n+1} = y_n + hf\left(x_n, y_n\right) \\
&y_{n+2} = y_{n+1} + h\left(-\frac{1}{2} f\left(x_n, y_n\right) + \frac{3}{2} f\left(x_{n+1}, y_{n+1}\right)\right) \\
&y_{n+3} = y_{n+2} + \frac{h}{12}\left(5 f\left(x_n, y_n\right) - 16 f\left(x_{n+1}, y_{n+1}\right) + 23 f\left(x_{n+2}, y_{n+2}\right)\right) \\
&y_{n+4} = y_{n+3} + \frac{h}{24}\left(-9 f\left(x_n, y_n\right) + 37 f\left(x_{n+1}, y_{n+1}\right) - 59 f\left(x_{n+2}, y_{n+2}\right) + 55 f\left(x_{n+3}, y_{n+3}\right)\right) \\
&y_{n+5} = y_{n+4} + \frac{h}{720}\begin{pmatrix} 251 f\left(x_n, y_n\right) - 1274 f\left(x_{n+1}, y_{n+1}\right) + 2616 f\left(x_{n+2}, y_{n+2}\right) - \\ 2774 f\left(x_{n+3}, y_{n+3}\right) + 1901 f\left(x_{n+4}, y_{n+4}\right) \end{pmatrix}
\end{aligned}\right\} \quad (2.4)$$

(ii) The Adam-Moulton Methods – are implicit and the coefficients are $\alpha_0 = \alpha_1 = ... = \alpha_{k-2} = 0$ and $\alpha_{k-1} = -1$, while β_j's are chosen such that the methods have a highest order possible and $\beta_k \neq 0$. By removing the restriction that $\beta_k = 0$ a k-step Adams Moulton method can reach order k+1, while the Adams – Bashforth methods has only order k. In each step of Adams–Moulton methods an algebraic matrix Riccati equation (AMRE) is obtained, which is solved by means of Newton's method. The Adams – Moulton methods with k=1,2,3,4 are: (Hairer *et al.*, 1993).

$$\left.\begin{aligned}
y_{n+1} &= y_n + hf\left(x_{n+1}, y_{n+1}\right) \\
y_{n+1} &= y_n + \frac{h}{2}\left(f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right)\right) \\
y_{n+2} &= y_{n+1} + \frac{h}{12}\left(-f\left(x_n, y_n\right) + 8f\left(x_{n+1}, y_{n+1}\right) + 5f\left(x_{n+2}, y_{n+2}\right)\right) \\
y_{n+3} &= y_{n+2} + \frac{h}{24}\left(f\left(x_n, y_n\right) - 5f\left(x_{n+1}, y_{n+1}\right) + 19f\left(x_{n+2}, y_{n+2}\right) + 9f\left(x_{n+3}, y_{n+3}\right)\right) \\
y_{n+4} &= y_{n+3} + \frac{h}{720}\begin{pmatrix} -19f\left(x_n, y_n\right) + 106f\left(x_{n+1}, y_{n+1}\right) - 264f\left(x_{n+2}, y_{n+2}\right) + 646f\left(x_{n+3}, y_{n+3}\right) + \\ 251f\left(x_{n+4}, y_{n+4}\right) \end{pmatrix}
\end{aligned}\right\} \quad (2.5)$$

(iii) The backward differentiation formula (BDF) – is a family of implicit methods in which for a given function and time, approximate the derivative of a function using information from already computed times, thereby increasing the accuracy of the approximation. These methods are especially used for the solution of stiff differential equations (Curtiss and Hirschflder, 1952). The general formula for a BDF can be written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta f\left(x_{n+k}, y_{n+k}\right) \quad (2.6)$$

The coefficients α_j's and β are chosen so that the method achieves order k, which is the maximum possible.

2.3 Block Method

According to Yahaya and Sagir (2013), a block method can be defined as follows;

Let Y_m and F_m be defined by $Y_m = (y_n, y_{n+1}, ..., y_{n+r-1}), F_m = (f_n, f_{n+1}, ..., f_{n+r-1})$. Then a general k-block, r-point block method is a matrix of finite difference equation of the form;

$$Y_m = \sum_{j=1}^{k} A_i y_{m-1} + h \sum_{i=0}^{k} \beta_i f_{m-1} \qquad (2.7)$$

where all the A'_i and $\beta'_i s$ are properly chosen *r* x *r* matrix coefficients and $m = 0,1,2,....$ represents the block number, $n = m$ is the first step number of mth block and r is the proposed block size. The block method was first proposed by Milne (1953) who advocated their utilization just as a method for obtaining starting values for predictor – corrector schemes (Sarafyan, 1965) and later continuing in blocks for general use was created by Rosser (1967). There are impressive writing on the method of solution of ordinary differential equations (ODEs) by predictor-corrector techniques as reported in (Lambert, 1973; 1991; Onumanyi *et al*., 1994; Fatunla 1994; Awoyemi and Idowu, 2005; Areo and Adeniyi, 2013; Ndanusa and Tafida, 2016). The predictor-corrector method has a great setback of starting values which implies additional time and human efforts are expected to develop another method(s) to permit the main scheme to be implemented. Also, the predictor derived are usually of lower order to the corrector; consequently, defiling the normal outcomes produced by the corrector. Nonetheless, to beat the test in creating separate predictors, and different deficiencies in the predictor-corrector technique, the development of the block method was proposed (Milne, 1953). This contains main and additional methods generated from the same continuous scheme, which are normally consolidated to at the same time produce simultaneously discrete solutions for IVPs at non-overlapping points $(x_{n+1}, x_{n+2}, ..., x_{n+N})$; thus, it is self-starting. Implementation of linear multistep methods in block form has been found to give better estimation as found in the studies by (Fatunla, 1994, Yahaya, 2004; Badmus and Yahaya, 2009; Jator and Li, 2012; Mohammed, 2011; Yahaya and Sagir, 2013; Akinfenwa, *et al*., 2013; Mohammed and Adeniyi, 2014; Badmus *et al*., 2015; Omar and Adeyeye, 2016; Akinfenwa *et al*., 2017).

Furthermore, the central concepts in the analysis of linear multistep methods, and indeed any numerical method for differential equations, are convergence, and stability. However, in past time, the determination of the order and error constants for the block methods, has been accomplished for the single individual members (particularly the main schemes - Yahaya, 2004; Badmus and Yahaya, 2009; Badmus, *et al*., 2015; Akinfenwa *et al*., 2017) of the block and whose outcome may not be true for other members of the block. Consequently, in this research work, the determination of the order and error constants of block linear multistep method are established using the entire block members concurrently.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Derivation of Block Adams Methods

For the purpose of the aim of this project, the block forms of Adams methods for two-step, three-step, four-step and five-step are derived in this subsection. The continuous collocation technique is employed in the derivation process of these schemes.

A power series of a single variable x in the form (3.1):

$$p(x) = \sum_{j=0}^{\infty} a_j x^j \tag{3.1}$$

is used as the basis or trial function, to produce the approximate solution given as

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{3.2}$$

where $a_j \in \mathbb{R}$ are unknown coefficients to be determined, r and s are the numbers of interpolation and collocation points respectively.

Differentiating (3.2),

$$y'(x) = f\left(x, y\right) = \sum_{j=1}^{r+s-1} j a_j x^{j-1} \tag{3.3}$$

Interpolating (3.2) and collocating (3.3) at specified points lead to a system of nonlinear equations of the form

$$AX = B \tag{3.4}$$

Solving the equation (3.4) via matrix inversion technique, the values of a_j's are obtained and substituted back into (3.2) to yield the continuous scheme of the Adams method in the form

$$y\left(x\right) = \alpha_{k-1}\left(x\right) y_{n+k-1} + h\sum_{j=0}^{k} \beta_j\left(x\right) f_{n+j} \tag{3.5}$$

In order to obtain the sufficient number of equations required to solve an ODE, the continuous scheme is evaluated at the non-interpolating points.

3.2 Derivation of Adams Bashforth Block Methods

The Adams Bashforth method is an explicit method; meaning $\beta_k = 0$ in equation (3.5).

3.2.1 Two-step Block Adams Bashforth Method

For this method one interpolation point and two collocation points are considered, $r = 1$ and $s = 2$ and a polynomial of degree 2 is obtained as follows;

$$y(x) = \sum_{j=0}^{2} a_j x^j \tag{3.6}$$

The derivative is given as

$$y'(x) = \sum_{j=1}^{2} j a_j x^{j-1} \tag{3.7}$$

Interpolating (3.6) at x_{n+1} and collocating (3.7) at x_n and x_{n+1} and representing in matrix form:

$$\begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 \\ 0 & 1 & 2x_n \\ 0 & 1 & 2x_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_n \\ f_{n+1} \end{pmatrix} \tag{3.8}$$

Using matrix inversion the values of a_j's are given as

$$\left.\begin{aligned} a_0 &= y_{n+1} - \tfrac{1}{2} hf_n - \tfrac{1}{2} hf_{n+1} \\ a_1 &= f_n \\ a_2 &= -\tfrac{1}{2h} f_n + \tfrac{1}{2h} f_{n+1} \end{aligned}\right\} \tag{3.9}$$

Substituting (3.9) into (3.6) and collecting like terms gives the continuous scheme

$$y(x) = y_{n+1} + (-\tfrac{1}{2} h + x - \tfrac{x^2}{2h}) f_n + (-\tfrac{1}{2} h + \tfrac{x}{2h}) f_{n+1} \tag{3.10}$$

Evaluating (3.10) at $x = x_n$ and x_{n+2} gives the block members of the 2 step Adams Bashforth

$$\left.\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{2}h\left(f_n + f_{n+1}\right) \\ y_{n+2} &= y_{n+1} - \tfrac{1}{2}h\left(f_n - 3f_{n+1}\right) \end{aligned}\right\} \tag{3.11}$$

3.2.2 Three-step block Adams Bashforth method

For this method, one interpolation point and three collocation points are considered, $r = 1$ and $s = 3$ and a polynomial of degree 3 is obtained as follows;

$$y(x) = \sum_{j=0}^{3} a_j x^j \tag{3.12}$$

The derivative is given as

$$y'(x) = \sum_{j=1}^{3} j a_j x^{j-1} \tag{3.13}$$

Interpolating (3.12) at x_{n+2} and collocating (3.13) at x_n, x_{n+1} and x_{n+2} and representing in matrix form:

$$\begin{pmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2(x_{n+1}) & 3(x_{n+1})^2 \\ 0 & 1 & 2(x_{n+2}) & 3(x_{n+2})^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{3.14}$$

Using matrix inversion the values of a_j's are given as

$$\left.\begin{aligned} a_0 &= y_{n+2} - \tfrac{1}{3}hf_n - \tfrac{4}{3}hf_{n+1} - \tfrac{1}{3}hf_{n+2} \\ a_1 &= f_n \\ a_2 &= -\tfrac{3}{4}\tfrac{f_n}{h} + \tfrac{f_{n+1}}{h} + \tfrac{1}{6}\tfrac{f_{n+2}}{h} \\ a_3 &= \tfrac{1}{6}\tfrac{f_n}{h^2} - \tfrac{1}{3}\tfrac{f_{n+1}}{h^2} + \tfrac{1}{6}\tfrac{f_{n+2}}{h^2} \end{aligned}\right\} \tag{3.15}$$

Substituting (3.15) into (3.12) and collecting like terms gives the continuous scheme

$$y(x) = y_{n+2} + \left(\tfrac{1}{6}\tfrac{x^3}{h^2} - \tfrac{3}{4}\tfrac{x^2}{h} + x - \tfrac{1}{3}h\right)f_n + \left(-\tfrac{1}{3}\tfrac{x^3}{h^2} + \tfrac{x^2}{h} - \tfrac{4}{3}h\right)f_{n+1} + \left(\tfrac{1}{6}\tfrac{x^3}{h^2} - \tfrac{1}{4}\tfrac{x^2}{h} - \tfrac{1}{3}h\right)f_{n+2} \tag{3.16}$$

Evaluating (3.16) at $x = x_n, x_{n+1}$ and x_{n+3} gives the block members of the 3 step Adams Bashforth method

$$\left.\begin{aligned} y_{n+1} &= y_{n+2} + \tfrac{1}{12}hf_n - \tfrac{2}{3}hf_{n+1} - \tfrac{5}{12}hf_{n+2} \\ y_{n+2} &= y_n + \tfrac{1}{3}hf_n + \tfrac{4}{3}hf_{n+1} + \tfrac{1}{3}hf_{n+2} \\ y_{n+3} &= y_{n+2} + \tfrac{5}{12}hf_n - \tfrac{4}{3}hf_{n+1} + \tfrac{23}{12}hf_{n+2} \end{aligned}\right\} \tag{3.17}$$

3.2.3 Four-step block Adams Bashforth method

Following similar procedure in the above two methods derived, the four-step block Adams Bashforth method is presented as follows:

$$\left.\begin{aligned} y_{n+1} &= y_{n+3} - \tfrac{1}{3}hf_{n+1} - \tfrac{4}{3}hf_{n+2} - \tfrac{1}{3}hf_{n+3} \\ y_{n+2} &= y_{n+3} - \tfrac{1}{24}hf_n + \tfrac{5}{24}hf_{n+1} - \tfrac{19}{24}hf_{n+2} - \tfrac{3}{8}hf_{n+3} \\ y_{n+3} &= y_n + \tfrac{3}{8}hf_n + \tfrac{9}{8}hf_{n+1} + \tfrac{9}{8}hf_{n+2} + \tfrac{3}{8}hf_{n+3} \\ y_{n+4} &= y_{n+3} - \tfrac{3}{8}hf_n + \tfrac{37}{24}hf_{n+1} - \tfrac{59}{24}hf_{n+2} + \tfrac{55}{24}hf_{n+3} \end{aligned}\right\} \tag{3.18}$$

3.2.4 Five-step block Adams Bashforth method

Similarly, the five-step block Adams Bashforth method is presented as follows:

$$\left.\begin{aligned} y_{n+1} &= y_{n+4} + \frac{3}{80}f_n - \frac{21}{40}hf_{n+1} - \frac{4}{3}hf_{n+2} - \frac{9}{10}hf_{n+3} - \frac{27}{80}hf_{n+4} \\ y_{n+2} &= y_{n+4} + \frac{1}{90}hf_n - \frac{2}{45}hf_{n+1} - \frac{19}{24}hf_{n+2} - \frac{4}{15}hf_{n+3} - \frac{29}{90}hf_{n+4} \\ y_{n+3} &= y_{n+4} + \frac{19}{720}hf_n - \frac{53}{360}hf_{n+1} + \frac{11}{30}hf_{n+3} - \frac{251}{720}hf_{n+4} \\ y_{n+4} &= y_n + \frac{14}{45}hf_n + \frac{64}{45}hf_{n+1} + \frac{9}{8}hf_{n+2} + \frac{8}{15}hf_{n+3} + \frac{14}{45}hf_{n+4} \\ y_{n+5} &= y_{n+4} + \frac{251}{720}hf_n - \frac{637}{360}hf_{n+1} - \frac{44}{3}hf_{n+2} + \frac{109}{30}hf_{n+3} + \frac{1901}{720}hf_{n+4} \end{aligned}\right\} \tag{3.19}$$

3.3 Derivation of Adams Moulton Block Methods

The Adams Moulton method is an implicit method; meaning $\beta_k \neq 0$ in equation (3.5).

3.3.1 Two-step Block Adams Moulton Method

For this method one interpolation point and three collocation points are considered, $r=1$ and $s=3$ and a polynomial of degree 3 is obtained as follows;

$$y(x)=\sum_{j=0}^{3} a_j x^j \tag{3.20}$$

The derivative is given as

$$y'(x)=\sum_{j=1}^{3} j a_j x^{j-1} \tag{3.21}$$

Interpolating (3.20) at x_{n+1} and collocating (3.21) at x_n, x_{n+1} and x_{n+2} and representing in matrix form:

$$\begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2(x_{n+1}) & 3(x_{n+1})^2 \\ 0 & 1 & 2(x_{n+2}) & 3(x_{n+2})^2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}=\begin{pmatrix} y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \tag{3.22}$$

Using matrix inversion the values of a_j's are given as

$$\left.\begin{aligned} a_0 &= y_{n+1}-\tfrac{5}{12}hf_n-\tfrac{2}{3}hf_{n+1}+\tfrac{1}{12}hf_{n+2} \\ a_1 &= f_n \\ a_2 &= \tfrac{-3}{4h}f_n+\tfrac{1}{h}f_{n+1}-\tfrac{1}{4h}f_{n+2} \\ a_3 &= \tfrac{1}{6h^2}f_n-\tfrac{1}{3h^2}f_{n+1}+\tfrac{1}{6h^2}f_{n+2} \end{aligned}\right\} \tag{3.23}$$

Substituting (3.23) into (3.20) and collecting like terms gives the continuous scheme

$$y_{n+1}+\left(\tfrac{1}{6}\tfrac{x^3}{h^2}-\tfrac{3}{4}\tfrac{x^2}{h}+x-\tfrac{5}{12}h\right)f_n+\left(-\tfrac{1}{3}\tfrac{x^3}{h^2}+\tfrac{x^2}{h}-\tfrac{2}{3}h\right)f_{n+1}+\left(\tfrac{1}{6}\tfrac{x^3}{h^2}-\tfrac{1}{4}\tfrac{x^2}{h}+\tfrac{1}{12}h\right)f_{n+2} \tag{3.23}$$

Evaluating (3.23) at $x=x_n, x_{n+1}$ and x_{n+2} gives the block members of the 2 step Adams Moulton

$$\left.\begin{aligned} y_{n+1} &= y_n + \tfrac{5}{12}hf_n + \tfrac{2}{3}hf_{n+1} - \tfrac{1}{12}hf_{n+2} \\ y_{n+2} &= y_{n+1} - \tfrac{1}{12}hf_n + \tfrac{2}{3}hf_{n+1} + \tfrac{5}{12}hf_{n+2} \end{aligned}\right\} \tag{3.24}$$

3.3.2 Three-step Block Adams Moulton Method

For this method one interpolation point and four collocation points are considered, $r=1$ and $s=4$ and a polynomial of degree 4 is obtained as follows;

$$y(x) = \sum_{j=0}^{4} a_j x^j \tag{3.25}$$

The derivative is given as

$$y'(x) = \sum_{j=1}^{4} j a_j x^{j-1} \tag{3.26}$$

Interpolating (3.25) at x_{n+2} and collocating (3.26) at x_n, x_{n+1}, x_{n+2} and x_{n+3} and representing in matrix form:

$$\begin{pmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2(x_{n+1}) & 3(x_{n+1})^2 & 4(x_{n+1})^3 \\ 0 & 1 & 2(x_{n+2}) & 3(x_{n+2})^2 & 4(x_{n+2})^3 \\ 0 & 1 & 2(x_{n+2}) & 3(x_{n+3})^2 & 4(x_{n+3})^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{3.27}$$

Using matrix inversion the values of a_j's are given as

$$\left.\begin{aligned} a_0 &= y_{n+2} - \tfrac{1}{3}hf_n - \tfrac{4}{3}hf_{n+1} - \tfrac{1}{3}hf_{n+2} \\ a_1 &= f_n \\ a_2 &= -\tfrac{11}{12}\tfrac{f_n}{h} + \tfrac{3}{2}\tfrac{f_{n+1}}{h} - \tfrac{3}{4}\tfrac{f_{n+2}}{h} + \tfrac{1}{6}\tfrac{f_{n+3}}{h} \\ a_3 &= \tfrac{1}{3}\tfrac{f_n}{h^2} - \tfrac{5}{6}\tfrac{f_{n+1}}{h^2} + \tfrac{2}{3}\tfrac{f_{n+2}}{h^2} - \tfrac{1}{6}\tfrac{f_{n+3}}{h^2} \\ a_4 &= -\tfrac{1}{24}\tfrac{f_n}{h^3} + \tfrac{1}{8}\tfrac{f_{n+1}}{h^3} - \tfrac{1}{8}\tfrac{f_{n+2}}{h^3} + \tfrac{1}{24}\tfrac{f_{n+3}}{h^3} \end{aligned}\right\} \tag{3.28}$$

Substituting (3.28) into (3.25) and collecting like terms gives the continuous scheme

$$y(x) = y_{n+2} + \left(-\tfrac{1}{24}\tfrac{x^4}{h^3} + \tfrac{1}{3}\tfrac{x^3}{h^2} - \tfrac{11}{12}\tfrac{x^2}{h} + x - \tfrac{1}{3}h\right) f_n + \left(\tfrac{1}{8}\tfrac{x^4}{h^3} - \tfrac{5}{6}\tfrac{x^3}{h^2} + \tfrac{3}{2}\tfrac{x^2}{h} - \tfrac{4}{3}h\right) f_{n+1}$$
$$+\left(-\tfrac{1}{8}\tfrac{x^4}{h^3} + \tfrac{2}{3}\tfrac{x^3}{h^2} - \tfrac{3}{4}\tfrac{x^2}{h^2} - \tfrac{1}{3}h\right) f_{n+2} + \left(\tfrac{1}{24}\tfrac{x^4}{h^3} - \tfrac{1}{6}\tfrac{x^3}{h^2} + \tfrac{1}{6}\tfrac{x^2}{h}\right) f_{n+3} \qquad (3.29)$$

Evaluating (3.29) at $x = x_n, x_{n+1}, x_{n+2}$ and x_{n+3} gives the block members of the 3 step Adams Moulton

$$\left.\begin{aligned}
y_{n+1} &= y_{n+2} + \tfrac{1}{24}hf_n - \tfrac{13}{24}hf_{n+1} - \tfrac{13}{24}hf_{n+2} + \tfrac{1}{24}hf_{n+3} \\
y_{n+2} &= y_n + \tfrac{1}{3}hf_n + \tfrac{4}{3}hf_{n+1} + \tfrac{1}{3}hf_{n+2} \\
y_{n+3} &= y_{n+2} + \tfrac{1}{24}hf_n - \tfrac{5}{24}hf_{n+1} + \tfrac{19}{24}hf_{n+2} + \tfrac{3}{8}hf_{n+3}
\end{aligned}\right\} \qquad (3.30)$$

3.3.3 Four-step block Adams Moulton method

Following similar procedure in the above two methods derived, the four-step block Adams Moulton method is presented as follows:

$$\left.\begin{aligned}
y_{n+1} &= y_{n+3} + \tfrac{1}{90}hf_n - \tfrac{17}{45}hf_{n+1} - \tfrac{19}{15}hf_{n+2} - \tfrac{17}{45}hf_{n+3} + \tfrac{1}{90}hf_{n+4} \\
y_{n+2} &= y_{n+3} - \tfrac{11}{720}hf_n + \tfrac{37}{360}hf_{n+1} - \tfrac{19}{30}hf_{n+2} - \tfrac{173}{360}hf_{n+3} + \tfrac{19}{720}hf_{n+4} \\
y_{n+3} &= y_n + \tfrac{27}{80}hf_n + \tfrac{51}{40}hf_{n+1} + \tfrac{9}{10}hf_{n+2} + \tfrac{21}{40}hf_{n+3} - \tfrac{3}{80}hf_{n+4} \\
y_{n+4} &= y_{n+3} - \tfrac{19}{720}hf_n + \tfrac{53}{360}hf_{n+1} - \tfrac{11}{30}hf_{n+2} + \tfrac{323}{360}hf_{n+3} + \tfrac{251}{720}hf_{n+4}
\end{aligned}\right\} \qquad (3.31)$$

3.3.4 Five-step block Adams Moulton method

Similarly, the five-step block Adams Moulton method is presented as follows:

$$\left.\begin{aligned}
y_{n+1} &= y_{n+4} + \tfrac{3}{160}hf_n - \tfrac{69}{160}hf_{n+1} - \tfrac{87}{80}hf_{n+2} - \tfrac{87}{80}hf_{n+3} - \tfrac{69}{160}hf_{n+4} + \tfrac{3}{160}hf_{n+5} \\
y_{n+2} &= y_{n+4} + \tfrac{1}{90}hf_{n+1} - \tfrac{17}{45}hf_{n+2} - \tfrac{19}{15}hf_{n+3} - \tfrac{17}{45}hf_{n+4} + \tfrac{1}{90}hf_{n+5} \\
y_{n+3} &= y_{n+4} + \tfrac{11}{1440}hf_n - \tfrac{77}{1440}hf_{n+1} + \tfrac{43}{240}hf_{n+2} - \tfrac{511}{720}hf_{n+3} - \tfrac{637}{1440}hf_{n+4} + \tfrac{3}{160}hf_{n+5} \\
y_{n+4} &= y_n + \tfrac{14}{45}hf_n + \tfrac{64}{45}hf_{n+1} + \tfrac{8}{15}hf_{n+2} + \tfrac{64}{45}hf_{n+3} + \tfrac{14}{45}hf_{n+4} \\
y_{n+5} &= y_{n+4} + \tfrac{3}{160}hf_n - \tfrac{173}{1440}hf_{n+1} + \tfrac{241}{720}hf_{n+2} - \tfrac{133}{240}hf_{n+3} + \tfrac{1427}{1440}hf_{n+4} + \tfrac{95}{288}hf_{n+5}
\end{aligned}\right\} \qquad (3.32)$$

3.4 Block Error Analysis of Adams Methods

Following Nwachukwu and Okor (2018), the individual scheme of a linear multistep method can be written as:

$$L\left[y(x);h\right]=\sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right)+h\left(\sum_{j=0}^{k}\beta_j y'(x+jh)\right) \tag{3.33}$$

Expanding (3.33) in Taylor series, the local truncation error associated with (3.5) is the linear difference operator

$$L\left[y(x);h\right]=\sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right)-h\left(\sum_{j=0}^{k}\beta_j y'(x+jh)\right) \tag{3.34}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (3.34) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right]=c_0 y(x)+c_1 hy'(x)+\ldots+c_q h^q y^{(q)}(x)+\ldots \tag{3.35}$$

where the constant $c_q, q=0,1,\ldots$ are given as follows

$$\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k}\alpha_j \\
c_1 &= \sum_{j=1}^{k} j\alpha_j+\sum_{j=0}^{k}\beta_j \\
c_2 &= \frac{1}{2!}\sum_{j=1}^{k}(j)^2\alpha_j+\sum_{j=1}^{k} j\beta_j \\
&\vdots \\
c_q &= \frac{1}{\text{q}!}\sum_{j=1}^{k}(j)^q\alpha_j+\frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j
\end{aligned}\right\} \tag{3.36}$$

A linear multistep method is said to be of order of accuracy p if $c_0=c_1=\ldots\text{c}_p=0, c_{p+1}\neq 0$. c_{p+1} is called the error constant (Akinfenwa *et al.*, 2015). However, this approach is normally used to determine the order of the individual members of the block. This approach is extended to determine the order of the entire block. To achieve this, the block linear multi-step method is expressed in the form:

$$\sum_{i=0}^{k} \alpha_{ij} y_{n+j} = h \sum_{j=0}^{k} \beta_{ij} f_{n+j} \tag{3.37}$$

Equation (3.37) is expanded to give the following system of equation.

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{21} & \dots & \alpha_{1k} \\ \alpha_{02} & \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2k} \\ \alpha_{03} & \alpha_{12} & \alpha_{23} & \dots & \alpha_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0k} & \alpha_{1k} & \alpha_{2k} & \cdots & \alpha_{kk} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+k} \end{pmatrix} = h \begin{pmatrix} \beta_{01} & \beta_{11} & \beta_{21} & \dots & \beta_{1k} \\ \beta_{02} & \beta_{12} & \beta_{22} & \cdots & \beta_{2k} \\ \beta_{03} & \beta_{12} & \beta_{23} & \dots & \beta_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{0k} & \beta_{1k} & \beta_{2k} & \cdots & \beta_{kk} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+k} \end{pmatrix} \tag{3.38}$$

where

$$\vec{\alpha}_0 = \begin{pmatrix} 01 \\ 02 \\ 03 \\ \vdots \\ 0k \end{pmatrix}, \vec{\alpha}_1 = \begin{pmatrix} 11 \\ 12 \\ 13 \\ \vdots \\ 1k \end{pmatrix} \ \vec{\alpha}_k = \begin{pmatrix} k1 \\ k2 \\ k3 \\ \vdots \\ kk \end{pmatrix} \text{ and } \vec{\beta}_0 = \begin{pmatrix} 01 \\ 02 \\ 03 \\ \vdots \\ 0k \end{pmatrix}, \vec{\beta}_1 = \begin{pmatrix} 11 \\ 12 \\ 13 \\ \vdots \\ 1k \end{pmatrix} \ \vec{\beta}_k = \begin{pmatrix} k1 \\ k2 \\ k3 \\ \vdots \\ kk \end{pmatrix}$$

Extending (3.34) to the vector form in (3.38),

$$L\left[y(x);h\right] = \sum_{j=0}^{k} \left[\vec{\alpha}_j y(x+jh) - h\vec{\beta}_j y'\left(x+jh, y(y+jh)\right) \right] \tag{3.39}$$

where, $y(x)$ is the exact solution satisfying $y'(x) = f\left(x, y(x)\right)$. Taking the Taylor's series expansion of (3.39), about x yields

$$L\left[y(x);h\right] = \vec{C}_0 y(x) + \vec{C}_1 h y'(x) + \vec{C}_2 h^2 y''(x) + \ldots + \vec{C}_q h^q y^{(q)}(x) \tag{3.40}$$

where

$$\vec{C}_0 = \begin{pmatrix} c_{01} \\ c_{02} \\ c_{03} \\ \vdots \\ c_{0p} \end{pmatrix}, \vec{C}_1 = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ \vdots \\ c_{1p} \end{pmatrix} \quad \vec{C}_p = \begin{pmatrix} c_{p1} \\ c_{p2} \\ c_{p3} \\ \vdots \\ c_{pp} \end{pmatrix} \tag{3.41}$$

The block linear multistep method is said to be of order p if $\vec{C}_0 = \vec{C}_1 = \vec{C}_2 = \ldots = \vec{C}_p = 0$, $\vec{C}_{p+1} \neq 0$

and the local truncation error is expressed as $T_n = \vec{C}_{p+1} h^{p+1} y^{p+1}(x_n)$ (Chollom *et al.*, 2007).

3.4.1 Local truncation error of two-step Adams Bashforth block method

Applying the procedure described above, the order and error constant of the two-step Adams Bashforth block method is presented as thus:

$$\vec{C}_0 = \vec{\alpha}_0 + \vec{\alpha}_1 + \vec{\alpha}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.42}$$

$$\vec{C}_1 = \vec{\alpha}_1 + 2\vec{\alpha}_2 - \left(\beta_0 + \beta_1\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.43}$$

$$\vec{C}_2 = \frac{1}{2!}\left(\vec{\alpha}_1 + 2^2\vec{\alpha}_2\right) - \beta_1 = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.44}$$

$$\vec{C}_3 = \frac{1}{3!}\left(\vec{\alpha}_1 + 2^3\vec{\alpha}_2\right) - \frac{1}{2!}\beta_1 = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2^3\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \frac{1}{2!}\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \tag{3.45}$$

Hence the two-step Adams Bashforth block method is of order $\left(2,2\right)^T$ and the error constant is

$\left(-\frac{1}{12}, \frac{5}{12}\right)^T$

3.4.2 Local truncation error of Three-step Adams Bashforth block method

The order and error constant of the three-step Adams Bashforth block method in equation (3.17) is presented as thus:

$$\vec{C}_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.46}$$

$$\vec{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} \frac{1}{12} \\ \frac{1}{3} \\ \frac{5}{12} \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{pmatrix} + \begin{pmatrix} -\frac{5}{12} \\ \frac{1}{3} \\ \frac{23}{12} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.47}$$

$$\vec{C}_2 = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \left(\begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{pmatrix} + 2\begin{pmatrix} -\frac{5}{12} \\ \frac{1}{3} \\ \frac{23}{12} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.48}$$

$$\vec{C}_3 = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \frac{1}{2!}\left[\begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{5}{12} \\ \frac{1}{3} \\ \frac{23}{12} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.49}$$

$$\vec{C}_4 = \frac{1}{4!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^4\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] - \frac{1}{3!}\left[\begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{pmatrix} + 2^3\begin{pmatrix} -\frac{5}{12} \\ \frac{1}{3} \\ \frac{23}{12} \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix} \tag{3.50}$$

$$\vec{C}_5 = \frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + 2^5\begin{pmatrix}-1\\1\\-1\end{pmatrix} + 3^5\begin{pmatrix}0\\0\\1\end{pmatrix}\right] - \frac{1}{4!}\left[\begin{pmatrix}-\frac{2}{3}\\\frac{4}{3}\\-\frac{4}{3}\end{pmatrix} + 2^4\begin{pmatrix}-\frac{5}{12}\\\frac{1}{3}\\\frac{23}{12}\end{pmatrix}\right] = \begin{pmatrix}\frac{17}{360}\\-\frac{1}{90}\\\frac{193}{360}\end{pmatrix} \tag{3.51}$$

Hence the three-step Adams Bashforth block method is of order $(3,4,3)^T$ and the error constant is

$\left(\frac{1}{24}, -\frac{1}{90}, \frac{3}{8}\right)^T$

3.4.3 Local truncation error of four-step Adams Bashforth block method

The order and error constant of the four-step Adams Bashforth block method in equation (3.18) is presented as thus:

$$\vec{C}_0 = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.52}$$

$$\vec{C}_1 = \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right] - \left[\begin{pmatrix}0\\-\frac{1}{24}\\\frac{3}{8}\\-\frac{3}{8}\end{pmatrix} + \begin{pmatrix}-\frac{1}{3}\\\frac{5}{24}\\\frac{9}{8}\\\frac{37}{24}\end{pmatrix} + \begin{pmatrix}-\frac{4}{3}\\-\frac{19}{24}\\\frac{9}{8}\\-\frac{59}{24}\end{pmatrix} + \begin{pmatrix}-\frac{1}{3}\\-\frac{3}{8}\\\frac{3}{8}\\\frac{55}{24}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.53}$$

$$\vec{C}_2 = \frac{1}{2!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2^2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3^2\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right] - \left[\begin{pmatrix}-\frac{1}{3}\\\frac{5}{24}\\\frac{9}{8}\\\frac{37}{24}\end{pmatrix} + 2\begin{pmatrix}-\frac{4}{3}\\-\frac{19}{24}\\\frac{9}{8}\\-\frac{59}{24}\end{pmatrix} + 3\begin{pmatrix}-\frac{1}{3}\\-\frac{3}{8}\\\frac{3}{8}\\\frac{55}{24}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.54}$$

$$\vec{C}_3 = \frac{1}{3!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2^3\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3^3\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right] - \frac{1}{2!}\left[\begin{pmatrix}-\frac{1}{3}\\ \frac{5}{24}\\ \frac{9}{8}\\ \frac{37}{24}\end{pmatrix} + 2^2\begin{pmatrix}-\frac{4}{3}\\ -\frac{19}{24}\\ \frac{9}{8}\\ -\frac{59}{24}\end{pmatrix} + 3^2\begin{pmatrix}-\frac{1}{3}\\ -\frac{3}{8}\\ \frac{3}{8}\\ \frac{55}{24}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.55}$$

$$\vec{C}_4 = \frac{1}{4!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2^4\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3^4\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right] - \frac{1}{3!}\left[\begin{pmatrix}-\frac{1}{3}\\ \frac{5}{24}\\ \frac{9}{8}\\ \frac{37}{24}\end{pmatrix} + 2^3\begin{pmatrix}-\frac{4}{3}\\ -\frac{19}{24}\\ \frac{9}{8}\\ -\frac{59}{24}\end{pmatrix} + 3^3\begin{pmatrix}-\frac{1}{3}\\ -\frac{3}{8}\\ \frac{3}{8}\\ \frac{55}{24}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.56}$$

$$\vec{C}_5 = \frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2^5\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3^5\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right] - \frac{1}{4!}\left[\begin{pmatrix}-\frac{1}{3}\\ \frac{5}{24}\\ \frac{9}{8}\\ \frac{37}{24}\end{pmatrix} + 2^4\begin{pmatrix}-\frac{4}{3}\\ -\frac{19}{24}\\ \frac{9}{8}\\ -\frac{59}{24}\end{pmatrix} + 3^4\begin{pmatrix}-\frac{1}{3}\\ -\frac{3}{8}\\ \frac{3}{8}\\ \frac{55}{24}\end{pmatrix}\right] = \begin{pmatrix}\frac{1}{90}\\ \frac{19}{720}\\ -\frac{3}{80}\\ \frac{251}{720}\end{pmatrix} \tag{3.57}$$

Hence the four-step Adams Bashforth block method is of order $(4,4,4)^T$ and the error constant is

$$\left(\frac{1}{90}, \frac{19}{720}, -\frac{3}{80}, \frac{251}{720}\right)^T$$

3.4.4 Local truncation error of five-step Adams Bashforth block method

The order and error constant of the four-step Adams Bashforth block method in equation (3.19) is presented as thus:

$$\vec{C}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.58}$$

$$\vec{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{3}{80} \\ \frac{1}{90} \\ \frac{19}{720} \\ \frac{14}{45} \\ \frac{251}{720} \end{pmatrix} + \begin{pmatrix} -\frac{21}{40} \\ -\frac{2}{45} \\ -\frac{53}{360} \\ \frac{64}{45} \\ -\frac{637}{360} \end{pmatrix} + \begin{pmatrix} -\frac{9}{10} \\ -\frac{4}{15} \\ \frac{11}{30} \\ \frac{8}{15} \\ \frac{109}{30} \end{pmatrix} + \begin{pmatrix} -\frac{51}{40} \\ -\frac{62}{45} \\ -\frac{323}{360} \\ \frac{64}{45} \\ -\frac{1387}{360} \end{pmatrix} + \begin{pmatrix} -\frac{27}{80} \\ -\frac{29}{90} \\ -\frac{251}{720} \\ \frac{14}{45} \\ \frac{1901}{720} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.59}$$

$$\vec{C}_2 = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3^2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4^2\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 5^2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] - \left[\begin{pmatrix} -\frac{21}{40} \\ -\frac{2}{45} \\ -\frac{53}{360} \\ \frac{64}{45} \\ -\frac{637}{360} \end{pmatrix} + 2\begin{pmatrix} -\frac{9}{10} \\ -\frac{4}{15} \\ \frac{11}{30} \\ \frac{8}{15} \\ \frac{109}{30} \end{pmatrix} + 3\begin{pmatrix} -\frac{51}{40} \\ -\frac{62}{45} \\ -\frac{323}{360} \\ \frac{64}{45} \\ -\frac{1387}{360} \end{pmatrix} + 4\begin{pmatrix} -\frac{27}{80} \\ -\frac{29}{90} \\ -\frac{251}{720} \\ \frac{14}{45} \\ \frac{1901}{720} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.60}$$

$$\vec{C}_3 = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^3\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3^3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4^3\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 5^3\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] - \frac{1}{2!}\left[\begin{pmatrix} -\frac{21}{40} \\ -\frac{2}{45} \\ -\frac{53}{360} \\ \frac{64}{45} \\ -\frac{637}{360} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{9}{10} \\ -\frac{4}{15} \\ \frac{11}{30} \\ \frac{8}{15} \\ \frac{109}{30} \end{pmatrix} + 3^2\begin{pmatrix} -\frac{51}{40} \\ -\frac{62}{45} \\ -\frac{323}{360} \\ \frac{64}{45} \\ -\frac{1387}{360} \end{pmatrix} + 4^2\begin{pmatrix} -\frac{27}{80} \\ -\frac{29}{90} \\ -\frac{251}{720} \\ \frac{14}{45} \\ \frac{1901}{720} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.61}$$

$$\vec{C}_4=\frac{1}{4!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^4\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^4\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^4\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^4\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{3!}\left[\begin{pmatrix}-\frac{21}{40}\\-\frac{2}{45}\\-\frac{53}{360}\\\frac{64}{45}\\-\frac{637}{360}\end{pmatrix}+2^3\begin{pmatrix}-\frac{9}{10}\\-\frac{4}{15}\\\frac{11}{30}\\\frac{8}{15}\\\frac{109}{30}\end{pmatrix}+3^3\begin{pmatrix}-\frac{51}{40}\\-\frac{62}{45}\\-\frac{323}{360}\\\frac{64}{45}\\-\frac{1387}{360}\end{pmatrix}+4^3\begin{pmatrix}-\frac{27}{80}\\-\frac{29}{90}\\-\frac{251}{720}\\\frac{14}{45}\\\frac{1901}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\quad(3.62)$$

$$\vec{C}_5=\frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^5\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^5\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^5\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^5\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{4!}\left[\begin{pmatrix}-\frac{21}{40}\\-\frac{2}{45}\\-\frac{53}{360}\\\frac{64}{45}\\-\frac{637}{360}\end{pmatrix}+2^4\begin{pmatrix}-\frac{9}{10}\\-\frac{4}{15}\\\frac{11}{30}\\\frac{8}{15}\\\frac{109}{30}\end{pmatrix}+3^4\begin{pmatrix}-\frac{51}{40}\\-\frac{62}{45}\\-\frac{323}{360}\\\frac{64}{45}\\-\frac{1387}{360}\end{pmatrix}+4^4\begin{pmatrix}-\frac{27}{80}\\-\frac{29}{90}\\-\frac{251}{720}\\\frac{14}{45}\\\frac{1901}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\quad(3.62)$$

$$\vec{C}_6=\frac{1}{6!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^6\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^6\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^6\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^6\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{5!}\left[\begin{pmatrix}-\frac{21}{40}\\-\frac{2}{45}\\-\frac{53}{360}\\\frac{64}{45}\\-\frac{637}{360}\end{pmatrix}+2^5\begin{pmatrix}-\frac{9}{10}\\-\frac{4}{15}\\\frac{11}{30}\\\frac{8}{15}\\\frac{109}{30}\end{pmatrix}+3^5\begin{pmatrix}-\frac{51}{40}\\-\frac{62}{45}\\-\frac{323}{360}\\\frac{64}{45}\\-\frac{1387}{360}\end{pmatrix}+4^5\begin{pmatrix}-\frac{27}{80}\\-\frac{29}{90}\\-\frac{251}{720}\\\frac{14}{45}\\\frac{1901}{720}\end{pmatrix}\right]=\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\quad(3.63)$$

$$\vec{C}_7 = \frac{1}{7!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + 2^7\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 3^7\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + 4^7\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix} + 5^7\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right] - \frac{1}{6!}\left[\begin{pmatrix}-\frac{21}{40}\\-\frac{2}{45}\\-\frac{53}{360}\\\frac{64}{45}\\-\frac{637}{360}\end{pmatrix} + 2^6\begin{pmatrix}-\frac{9}{10}\\-\frac{4}{15}\\\frac{11}{30}\\\frac{8}{15}\\\frac{109}{30}\end{pmatrix} + 3^6\begin{pmatrix}-\frac{51}{40}\\-\frac{62}{45}\\-\frac{323}{360}\\\frac{64}{45}\\-\frac{1387}{360}\end{pmatrix} + 4^6\begin{pmatrix}-\frac{27}{80}\\-\frac{29}{90}\\-\frac{251}{720}\\\frac{14}{45}\\\frac{1901}{720}\end{pmatrix}\right] = \begin{pmatrix}\frac{23}{560}\\\frac{5}{189}\\\frac{641}{15120}\\-\frac{8}{945}\\\frac{12253}{15120}\end{pmatrix} \quad (3.64)$$

Hence the five-step Adams Bashforth block method is of order $(5,5,5,6,5)^T$ and the error constant is $\left(\frac{3}{160},\frac{1}{900},\frac{3}{160},-\frac{8}{945},\frac{95}{288}\right)^T$. Table 3.1 present the summary of Order and Error Constants of Block Adams Bashforth Methods.

Table 3.1: Summary of Order and Error Constants of Block Adams Bashforth Methods

Step Number	Order	Error constants
K=2	$(2,2)^T$	$\left(-\frac{1}{12},\frac{5}{12}\right)^T$
K=3	$(3,4,3)^T$	$\left(\frac{1}{24},-\frac{1}{90},\frac{3}{8}\right)^T$
K=4	$(4,4,4)^T$	$\left(\frac{1}{90},\frac{19}{720},-\frac{3}{80},\frac{251}{720}\right)^T$
K=5	$(5,5,5,6,5)^T$	$\left(\frac{3}{160},\frac{1}{900},\frac{3}{160},-\frac{8}{945},\frac{95}{288}\right)^T$

3.4.5 Local truncation error of two-step Adams Moulton block method

Applying the procedure described above, the order and error constant of the two-step Adams Moulton block method in (3.24) is presented as thus:

$$\vec{C}_0 = \vec{\alpha}_0 + \vec{\alpha}_1 + \vec{\alpha}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.65}$$

$$\vec{C}_1 = \vec{\alpha}_1 + 2\vec{\alpha}_2 - \left(\beta_0 + \beta_1 + \beta_2\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} \frac{5}{12} \\ -\frac{1}{12} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -\frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.66}$$

$$\vec{C}_2 = \frac{1}{2!}\left[\vec{\alpha}_1 + 2^2\vec{\alpha}_2\right] - \left(\beta_1 + 2\beta_2\right) = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \left(\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} + 2\begin{pmatrix} -\frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.67}$$

$$\vec{C}_3 = \frac{1}{3!}\left[\vec{\alpha}_1 + 2^3\vec{\alpha}_2\right] - \frac{1}{2!}\left(\beta_1 + 2\beta_2\right) = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2^3\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \frac{1}{2!}\left(\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.68}$$

$$\vec{C}_4 = \frac{1}{4!}\left[\vec{\alpha}_1 + 2^4\vec{\alpha}_2\right] - \frac{1}{3!}\left(\beta_1 + 2^3\beta_2\right) = \frac{1}{4!}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2^4\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \frac{1}{3!}\left(\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} + 2^3\begin{pmatrix} -\frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{24} \\ -\frac{1}{24} \end{pmatrix} \tag{3.69}$$

Hence the two-step Adams Moulton block method is of order $(3,3)^T$ and the error constant is

$$\left(\frac{1}{24}, -\frac{1}{24}\right)^T$$

3.4.6 Local truncation error of three-step Adams Moulton block method

The order and error constant of the three-step Adams Moulton block method in (3.30) is presented as thus:

$$\vec{C}_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.70}$$

$$\vec{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{1}{24} \\ \frac{1}{3} \\ \frac{1}{24} \end{pmatrix} + \begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + \begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + \begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.71}$$

$$\vec{C}_2 = \frac{1}{2!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] - \left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.72}$$

$$\vec{C}_3 = \frac{1}{3!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] - \frac{1}{2!}\left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3^2\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.73}$$

$$\vec{C}_4 = \frac{1}{4!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^4\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] - \frac{1}{3!}\left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2^3\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3^3\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.74}$$

$$\vec{C}_5 = \frac{1}{5!}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2^5\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + 3^5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] - \frac{1}{4!}\left[\begin{pmatrix} -\frac{13}{24} \\ \frac{4}{3} \\ -\frac{5}{24} \end{pmatrix} + 2^4\begin{pmatrix} -\frac{13}{24} \\ \frac{1}{3} \\ \frac{19}{24} \end{pmatrix} + 3^4\begin{pmatrix} \frac{1}{24} \\ 0 \\ \frac{3}{8} \end{pmatrix}\right] = \begin{pmatrix} -\frac{11}{720} \\ -\frac{1}{90} \\ -\frac{19}{720} \end{pmatrix} \tag{3.75}$$

Hence the three-step Adams Moulton block method is of order $(4,4,4)^T$ and the error constant is

$\left(-\frac{11}{720},-\frac{1}{90},-\frac{10}{720}\right)^T$

3.4.7 Local truncation error of four-step Adams Moulton block method

The order and error constant of the four-step Adams Moulton block method in (3.31) is presented as thus:

$$\vec{C}_0=\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.76}$$

$$\vec{C}_1=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\left[\begin{pmatrix}\frac{1}{90}\\-\frac{11}{720}\\\frac{27}{80}\\-\frac{19}{720}\end{pmatrix}+\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.77}$$

$$\vec{C}_2=\frac{1}{2!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^2\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{3.78}$$

$$\vec{C}_3=\frac{1}{3!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^3\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^3\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{2!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^2\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^2\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^2\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad(3.79)$$

$$\vec{C}_4=\frac{1}{4!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^4\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^4\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{3!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^3\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^3\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^3\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad(3.80)$$

$$\vec{C}_5=\frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^5\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^5\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{4!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^4\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^4\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^4\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad(3.81)$$

$$\vec{C}_6=\frac{1}{6!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+2^6\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+3^6\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right]-\frac{1}{5!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix}+2^5\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix}+3^5\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix}+4^5\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right]=\begin{pmatrix}0\\-\frac{11}{1440}\\\frac{3}{160}\\-\frac{3}{160}\end{pmatrix}\quad(3.82)$$

$$\vec{C}_7 = \frac{1}{7!}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2^7\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3^7\begin{pmatrix}-1\\-1\\1\\-1\end{pmatrix}\right] - \frac{1}{6!}\left[\begin{pmatrix}-\frac{17}{45}\\\frac{37}{360}\\\frac{51}{40}\\\frac{53}{360}\end{pmatrix} + 2^6\begin{pmatrix}-\frac{19}{15}\\-\frac{19}{30}\\\frac{9}{10}\\-\frac{11}{30}\end{pmatrix} + 3^6\begin{pmatrix}-\frac{17}{45}\\-\frac{173}{360}\\\frac{21}{40}\\\frac{323}{360}\end{pmatrix} + 4^6\begin{pmatrix}\frac{1}{90}\\\frac{19}{720}\\-\frac{3}{80}\\\frac{251}{720}\end{pmatrix}\right] = \begin{pmatrix}-\frac{1}{756}\\-\frac{241}{15120}\\\frac{19}{560}\\-\frac{641}{15120}\end{pmatrix} \quad (3.83)$$

Hence the four-step Adams Moulton block method is of order $(6,5,5,5)^T$ and the error constant is $\left(-\frac{1}{756},-\frac{11}{1440},\frac{3}{160},-\frac{3}{160}\right)^T$

3.4.8 Local truncation error of five-step Adams Moulton block method

The order and error constant of the five-step Adams Moulton block method in (3.32) is presented as thus:

$$\vec{C}_0 = \begin{pmatrix}0\\0\\0\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (3.84)$$

$$\vec{C}_1 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + 4\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix} + 5\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix} - \left[\begin{pmatrix}\frac{3}{360}\\\frac{11}{1440}\\\frac{14}{45}\\\frac{3}{160}\\\frac{251}{720}\end{pmatrix} + \begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix} + \begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix} + \begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix} + \begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix} + \begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (3.85)$$

$$\vec{C}_2=\frac{1}{2!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^2\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^2\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^2\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\quad(3.86)$$

$$\vec{C}_3=\frac{1}{3!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^3\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^3\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^3\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{2!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^2+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^2\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^2\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^2\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\quad(3.87)$$

$$\vec{C}_4=\frac{1}{4!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+2^4\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+3^4\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+4^4\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix}+5^4\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right]-\frac{1}{3!}\left[\begin{pmatrix}-\frac{69}{160}\\\frac{1}{90}\\-\frac{77}{1440}\\\frac{64}{45}\\-\frac{173}{1440}\end{pmatrix}2^3+\begin{pmatrix}-\frac{87}{80}\\-\frac{17}{45}\\-\frac{43}{240}\\\frac{8}{15}\\\frac{241}{720}\end{pmatrix}+3^3\begin{pmatrix}-\frac{87}{80}\\-\frac{19}{15}\\-\frac{511}{720}\\\frac{64}{45}\\-\frac{133}{240}\end{pmatrix}+4^3\begin{pmatrix}-\frac{69}{160}\\-\frac{17}{45}\\-\frac{637}{1440}\\\frac{14}{45}\\\frac{1427}{1440}\end{pmatrix}+5^3\begin{pmatrix}\frac{3}{160}\\\frac{1}{90}\\\frac{3}{160}\\0\\\frac{95}{288}\end{pmatrix}\right]=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\quad(3.88)$$

$$\vec{C}_5 = \frac{1}{5!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + 2^5\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 3^5\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + 4^5\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix} + 5^5\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right] - \frac{1}{4!}\left[\begin{pmatrix}-\frac{69}{160}\\ \frac{1}{90}\\ -\frac{77}{1440}\\ \frac{64}{45}\\ -\frac{173}{1440}\end{pmatrix}2^4 + \begin{pmatrix}-\frac{87}{80}\\ -\frac{17}{45}\\ -\frac{43}{240}\\ \frac{8}{15}\\ \frac{241}{720}\end{pmatrix}3^4 + \begin{pmatrix}-\frac{87}{80}\\ -\frac{19}{15}\\ -\frac{511}{720}\\ \frac{64}{45}\\ -\frac{133}{240}\end{pmatrix} + 4^4\begin{pmatrix}-\frac{69}{160}\\ -\frac{17}{45}\\ -\frac{637}{1440}\\ \frac{14}{45}\\ \frac{1427}{1440}\end{pmatrix} + 5^4\begin{pmatrix}\frac{3}{160}\\ \frac{1}{90}\\ \frac{3}{160}\\ 0\\ \frac{95}{288}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (3.89)$$

$$\vec{C}_6 = \frac{1}{6!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + 2^6\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 3^6\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + 4^6\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix} + 5^6\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right] - \frac{1}{5!}\left[\begin{pmatrix}-\frac{69}{160}\\ \frac{1}{90}\\ -\frac{77}{1440}\\ \frac{64}{45}\\ -\frac{173}{1440}\end{pmatrix}2^5 + \begin{pmatrix}-\frac{87}{80}\\ -\frac{17}{45}\\ -\frac{43}{240}\\ \frac{8}{15}\\ \frac{241}{720}\end{pmatrix}3^5 + \begin{pmatrix}-\frac{87}{80}\\ -\frac{19}{15}\\ -\frac{511}{720}\\ \frac{64}{45}\\ -\frac{133}{240}\end{pmatrix} + 4^5\begin{pmatrix}-\frac{69}{160}\\ -\frac{17}{45}\\ -\frac{637}{1440}\\ \frac{14}{45}\\ \frac{1427}{1440}\end{pmatrix} + 5^5\begin{pmatrix}\frac{3}{160}\\ \frac{1}{90}\\ \frac{3}{160}\\ 0\\ \frac{95}{288}\end{pmatrix}\right] = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \quad (3.90)$$

$$\vec{C}_7 = \frac{1}{7!}\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + 2^7\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 3^7\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + 4^7\begin{pmatrix}-1\\-1\\-1\\1\\-1\end{pmatrix} + 5^7\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right] - \frac{1}{6!}\left[\begin{pmatrix}-\frac{69}{160}\\ \frac{1}{90}\\ -\frac{77}{1440}\\ \frac{64}{45}\\ -\frac{173}{1440}\end{pmatrix}2^6 + \begin{pmatrix}-\frac{87}{80}\\ -\frac{17}{45}\\ -\frac{43}{240}\\ \frac{8}{15}\\ \frac{241}{720}\end{pmatrix}3^6 + \begin{pmatrix}-\frac{87}{80}\\ -\frac{19}{15}\\ -\frac{511}{720}\\ \frac{64}{45}\\ -\frac{133}{240}\end{pmatrix} + 4^6\begin{pmatrix}-\frac{69}{160}\\ -\frac{17}{45}\\ -\frac{637}{1440}\\ \frac{14}{45}\\ \frac{1427}{1440}\end{pmatrix} + 5^6\begin{pmatrix}\frac{3}{160}\\ \frac{1}{90}\\ \frac{3}{160}\\ 0\\ \frac{95}{288}\end{pmatrix}\right] = \begin{pmatrix}-\frac{13}{2240}\\ -\frac{1}{756}\\ -\frac{271}{60480}\\ -\frac{8}{945}\\ -\frac{863}{60480}\end{pmatrix} \quad (3.91)$$

Hence the five-step Adams Moulton block method is of order $(6,6,6,6,6)^T$ and the error constant is $\left(-\frac{13}{2240}, -\frac{1}{756}, -\frac{271}{60480}, -\frac{8}{945}, -\frac{863}{60480}\right)^T$. Table 3.2 present the summary of order and error constants of block Adams Moulton methods.

Table 3.2: Summary of Order and Error Constants of Block Adams Moulton Methods

Step Number	Order	Error constants
K=2	$(3,3)^T$	$\left(\frac{1}{24},-\frac{1}{24}\right)^T$
K=3	$(4,4,4)^T$	$\left(-\frac{11}{720},-\frac{1}{90},-\frac{10}{720}\right)^T$
K=4	$(6,5,5,5)^T$	$\left(-\frac{1}{756},-\frac{11}{1440},\frac{3}{160},-\frac{3}{160}\right)^T$
K=5	$(6,6,6,6,6)^T$	$\left(-\frac{13}{2240},-\frac{1}{756},-\frac{271}{60480},-\frac{8}{945},-\frac{863}{60480}\right)^T$

3.5 Zero Stability of the Block Adams Methods

In what follows, the block Adams Bashforth and Adams Moulton methods can generally be written as a matrix difference equation as follows

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + h\left[B^{(0)}F_{w-1} + B^{(1)}F_w\right] \tag{3.92}$$

where

$$\left.\begin{aligned} Y_w &= \left(y_{n+1}, y_{n+2}, \ldots, y_{n+k}\right)^T \\ Y_{w-1} &= \left(y_{n-k+1}, y_{n-k+2}, \ldots, y_{n-1}, y_n\right)^T \\ F_w &= \left(f_{n+1}, f_{n+2}, \ldots, f_{n+k}\right)^T \\ F_{w-1} &= \left(f_{n-k+1}, f_{n-k+2}, \ldots, f_{n-1}, f_n\right)^T \end{aligned}\right\} \tag{3.93}$$

and the matrices $A^{(1)}, A^{(0)}, \mathrm{B}^{(1)}$ and $B^{(0)}$ are matrices whose entries are given by the coefficients of the methods. In order to find the zero stability of the derived methods, we only consider the first characteristic polynomial of the method,

$$\rho(R) = \left| RA^{(1)} - A^0 \right| \tag{3.93}$$

Definition 3.1 A linear multistep method is said to be zero-stable if the first characteristic polynomial $\rho(r)$ satisfies $|r| \leq 1$ and if every root satisfying $|r| = 1$ have multiplicity not be greater than one (Skwame *et al.*, 2018).

It is however, worth noting that the first characteristic polynomials of both Adams Bashforth and Adams Moulton k-step methods are the same. Therefore, we generally analyze the zero-stability of the Block Adams methods for the k-steps considered in this project.

3.5.1 Zero stability of 2-step Adams method

Writing the two-step Adams method in the form (3.93),

$$P(R) = \left| R\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right| = 0 \tag{3.94}$$

$$\begin{aligned} P(R) &= R(R-1) = 0 \\ R &= \{0,1\} \end{aligned} \tag{3.95}$$

Therefore, the method is zero stable since it satisfies $|R_j| \leq 1$.

3.5.2 Zero stability of 3-step Adams method

Writing the three-step Adams method in the form (3.93),

$$P(R) = \left| R\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right| = 0 \tag{3.96}$$

$$\begin{aligned} P(R) &= R^2(R-1) = 0 \\ R &= \{0,0,1\} \end{aligned} \tag{3.97}$$

Therefore, the method is zero stable since it satisfies $|R_j| \leq 1$.

3.5.3 Zero stability of 4-step Adams method

Writing the four-step Adams method in the form (3.93),

$$P(R)=\left|R\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}-\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\right| \tag{3.98}$$

$$\begin{aligned}&P\left(R\right)=R^3(R-1)=0\\&R=\left\{0,0,0,1\right\}\end{aligned} \tag{3.99}$$

Therefore, the method is zero stable since it satisfies $\left|R_j\right|\leq 1$.

3.5.4 Zero stability of 5-step Adams method

Writing the five-step Adams method in the form (3.93),

$$P(R)=\left|R\begin{bmatrix}1&0&0&-1&0\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&0&1&0\\0&0&0&-1&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\right| \tag{3.100}$$

$$\begin{aligned}&P\left(R\right)=R^4(R-1)=0\\&R=\left\{0,0,0,0,1\right\}\end{aligned} \tag{3.101}$$

Therefore, the method is zero stable since it satisfies $\left|R_j\right|\leq 1$.

3.6 Convergence

The necessary and sufficient condition for linear multistep method to be convergent is for it to be consistent and zero stable (Ngwane and Jator, 2012). Following this theorem, each of the block Adams methods developed are convergent.

3.7 Absolute Stability Properties of the Adams Methods

The stability characteristic of the block Adams methods is analyzed using the linear general method. Applying the scaler test equation $y' = \lambda y,\ \lambda < 0$ to the methods, and arranging the block members of the Adams methods in matrix form as given in (3.92), the stability polynomial of the methods can be compute following Akinfenwa *et al.* (2014) as:

$$\sigma(z) = \left(A^{(1)} - zB^{(1)}\right)^{-1}\left(A^{(0)} + zB^{(0)}\right) \tag{3.102}$$

where $z = \lambda h$

The matrix $\sigma(z)$ has eigenvalues $\{0, 0, 0, ..., \lambda_k\}$, and the dominant eigenvalue $\lambda_k : \mathbb{C} \to \mathbb{C}$ is a rational function (called the stability function) with real coefficients.

Definition 3.2: A numerical method is said to be A-Stable if the region of absolute stability contains the left half plane (Akinfenwa *et al.*, 2014).

3.7.1 Absolute Stability Properties of the Two-step Adams Bashforth Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z) = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - z\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & 0 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}\right) \tag{3.103}$$

and the stability function is given as:

$$\sigma(z) = -\frac{2z^2 + 3z + 2}{z - 2} \tag{3.104}$$

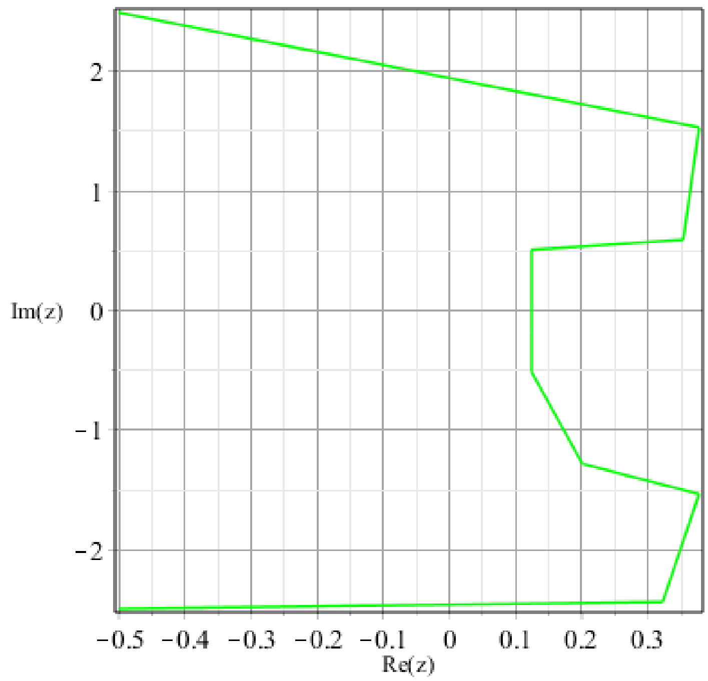

Figure 3.1: Stability Region of Two-step Block Adams Bashforth Method

Figure 3.1 shows the region of absolute stability of the two-step block Adams Bashforth method. The method is not A-stable as the unstable region covers the entire plane as shown in the figure

3.7.2 Absolute Stability Properties of the Three-step Adams Bashforth Method

Computing the characteristic polynomial as given in (3.102).

$$\sigma(z)=\left(\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}-z\begin{bmatrix}-\frac{2}{3} & -\frac{5}{12} & 0\\ \frac{4}{3} & \frac{1}{3} & 0\\ -\frac{4}{3} & \frac{23}{12} & 0\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}+z\begin{bmatrix}0 & 0 & \frac{1}{12}\\ 0 & 0 & \frac{1}{3}\\ 0 & 0 & \frac{5}{12}\end{bmatrix}\right) \tag{3.105}$$

and the stability function is given as

$$\sigma(z)=\frac{1}{2}\left[\frac{6z^3+11z^2+12z+6}{z^2-3z+3}\right] \tag{3.106}$$

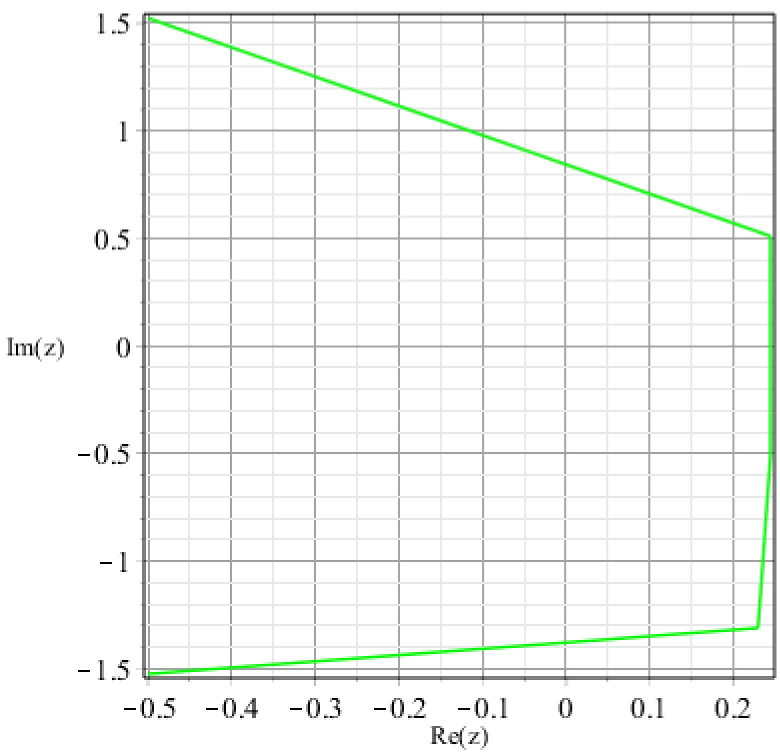

Figure 3.2: Stability Region of Three-step Block Adams Bashforth Method

Figure 3.2 shows the region of absolute stability of the three-step block Adams Bashforth method.

The method is not A-stable as the unstable region covers the entire plane as shown in the figure

3.7.3 Absolute Stability Properties of the Four-step Adams Bashforth Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z)=\left(\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}-z\begin{bmatrix}-\frac{1}{3}&-\frac{1}{4}&-\frac{1}{3}&0\\\frac{5}{24}&-\frac{19}{24}&-\frac{3}{8}&0\\\frac{9}{8}&\frac{9}{8}&\frac{3}{8}&0\\\frac{37}{24}&-\frac{59}{24}&\frac{55}{24}&0\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}+z\begin{bmatrix}0&0&0&0\\0&0&0&-\frac{1}{24}\\0&0&0&\frac{3}{8}\\0&0&0&-\frac{3}{8}\end{bmatrix}\right) \quad (3.107)$$

and the stability function is given as

$$\sigma(z)=-\frac{12z^4+25z^3+35z^2+30z+12}{3z^3-11z^2+18z-12} \quad (3.108)$$

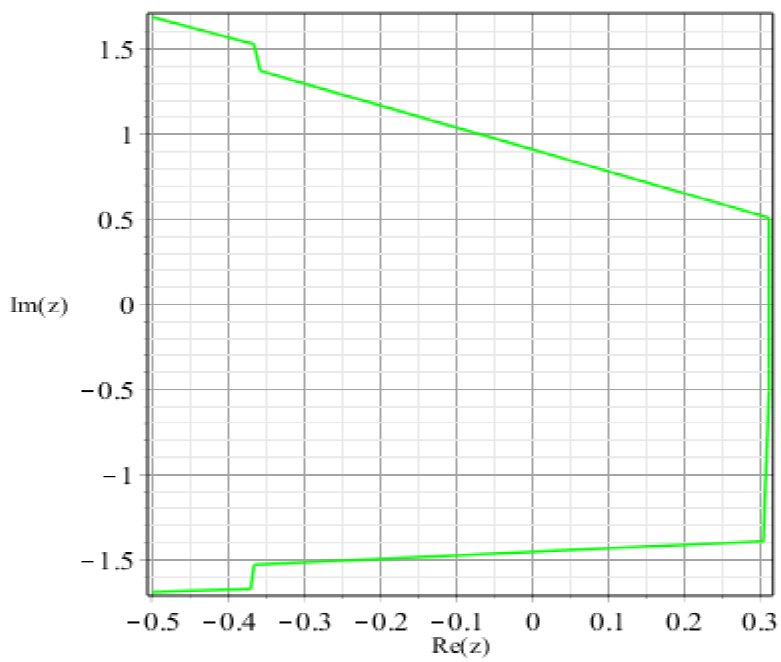

Figure 3.3: Stability Region of Four-step Block Adams Bashforth Method

Figure 3.3 shows the region of absolute stability of the four-step block Adams Bashforth method. The method is not A-stable as the unstable region covers the entire plane as shown in the figure

3.7.4 Absolute Stability Properties of the Five-step Adams Bashforth Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z)=\left(\begin{bmatrix}1&0&0&-1&0\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&0&1&0\\0&0&0&-1&1\end{bmatrix}-z\begin{bmatrix}-\frac{21}{40}&-\frac{9}{10}&-\frac{51}{40}&-\frac{27}{80}&0\\-\frac{2}{45}&-\frac{4}{15}&-\frac{62}{45}&-\frac{29}{90}&0\\-\frac{53}{360}&\frac{11}{30}&-\frac{62}{45}&-\frac{251}{720}&0\\\frac{64}{45}&\frac{8}{15}&\frac{64}{45}&\frac{14}{45}&0\\-\frac{637}{360}&\frac{109}{30}&-\frac{1387}{360}&-\frac{1901}{720}&0\end{bmatrix}\right)^{-1}\times$$
$$\left(\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}+z\begin{bmatrix}0&0&0&0&\frac{3}{80}\\0&0&0&0&\frac{1}{90}\\0&0&0&0&\frac{19}{720}\\0&0&0&0&\frac{14}{45}\\0&0&0&0&\frac{251}{720}\end{bmatrix}\right) \tag{3.109}$$

and the stability function is given as

$$\sigma(z)=\frac{2}{3}\left[\frac{54420z^5+194324z^4+418941z^3+460856z^2+225540z+64800}{3796z^4-12126z^3+44979z^2-65640z+43200}\right] \tag{3.110}$$

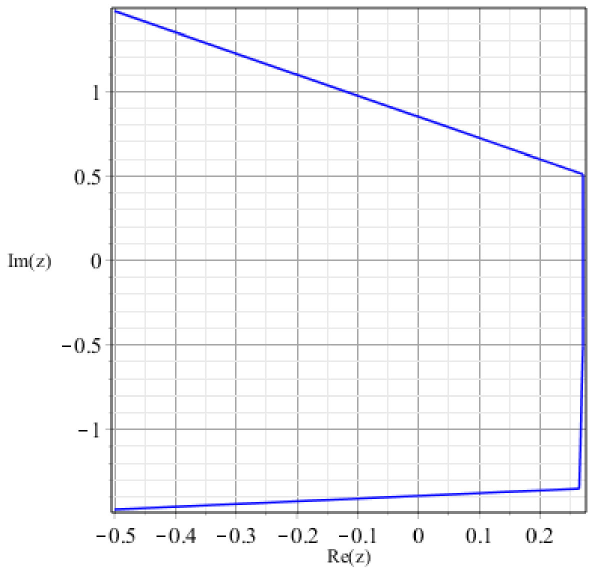

Figure 3.4: Stability Region of Four-step Block Adams Bashforth Method

Figure 3.4 shows the region of absolute stability of the five-step block Adams Bashforth method.

The method is not A-stable as the unstable region covers the entire plane as shown in the figure

3.7.5 Absolute Stability Properties of the Two-step Adams Moulton Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z)=\left(\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}-z\begin{bmatrix}\frac{3}{2} & -\frac{1}{12}\\ \frac{3}{2} & \frac{5}{12}\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+z\begin{bmatrix}0 & \frac{5}{12}\\ 0 & -\frac{1}{12}\end{bmatrix}\right) \tag{3.111}$$

and the stability function is given as:

$$\sigma(z)=\frac{9z^2+22z+12}{9z^2-22z+12} \tag{3.112}$$

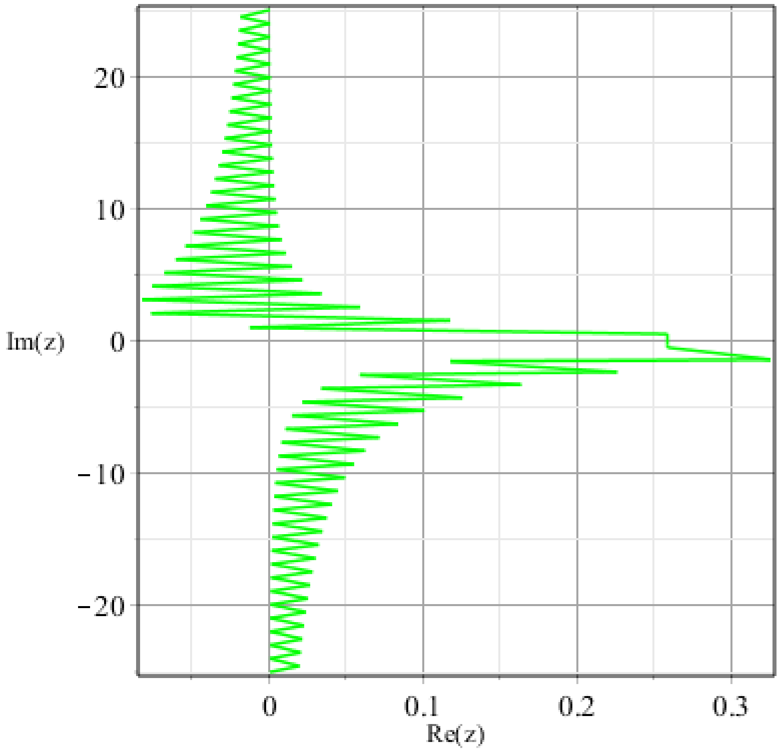

Figure 3.5: Stability Region of Two-step Block Adams Moulton Method

Figure 3.5 shows the region of absolute stability of the two-step block Adams Moulton method. However, the method is A-stable as the region of its absolute stability covers the entire half left plane as shown in the figure.

3.7.6 Absolute Stability Properties of the Three-step Adams Moulton Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z)=\left(\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}-z\begin{bmatrix}-\frac{13}{24} & -\frac{13}{24} & \frac{1}{24}\\ \frac{4}{3} & \frac{1}{3} & 0\\ -\frac{5}{24} & \frac{19}{24} & \frac{3}{8}\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}+z\begin{bmatrix}0 & 0 & \frac{1}{24}\\ 0 & 0 & \frac{1}{3}\\ 0 & 0 & \frac{1}{24}\end{bmatrix}\right) \qquad (3.113)$$

and the stability function is given as

$$\sigma(z) = -\frac{3z^3 + 11z^2 + 18z + 12}{3z^3 - 11z^2 + 18z - 12} \tag{3.114}$$

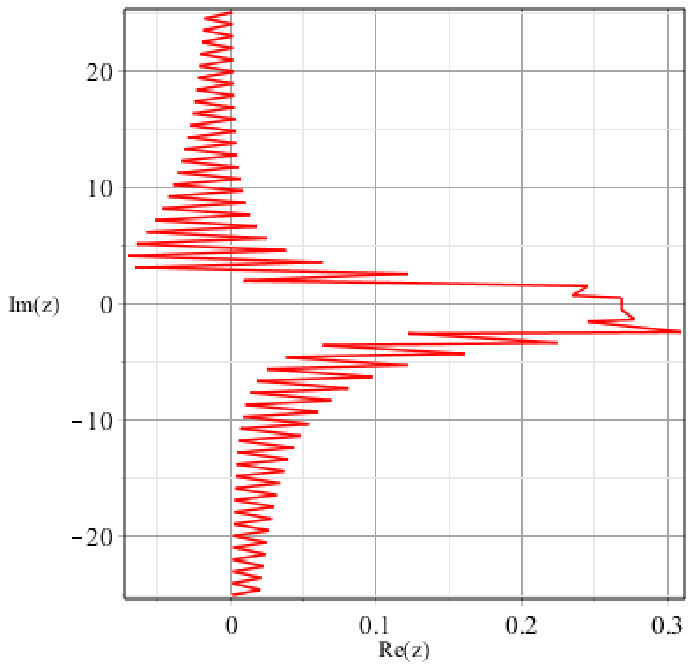

Figure 3.6: Stability Region of Three-step Block Adams Moulton Method

Figure 3.6 shows the region of absolute stability of the three-step block Adams Moulton method. The method is A-stable as the region of its absolute stability covers the entire half left plane as shown in the figure.

3.7.7 Absolute Stability Properties of the Four-step Adams Bashforth Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z) = \left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} - z \begin{bmatrix} -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{37}{360} & -\frac{19}{30} & -\frac{173}{360} & \frac{19}{720} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & -\frac{3}{80} \\ \frac{53}{360} & -\frac{11}{30} & \frac{323}{360} & \frac{251}{720} \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & -\frac{11}{720} \\ 0 & 0 & 0 & \frac{27}{80} \\ 0 & 0 & 0 & -\frac{19}{720} \end{bmatrix} \right) \tag{3.115}$$

and the stability function is given as

$$\sigma(z)=\frac{12z^4+50z^3+105z^2+120z+60}{12z^4-50z^3+105z^2-120z+60} \tag{3.116}$$

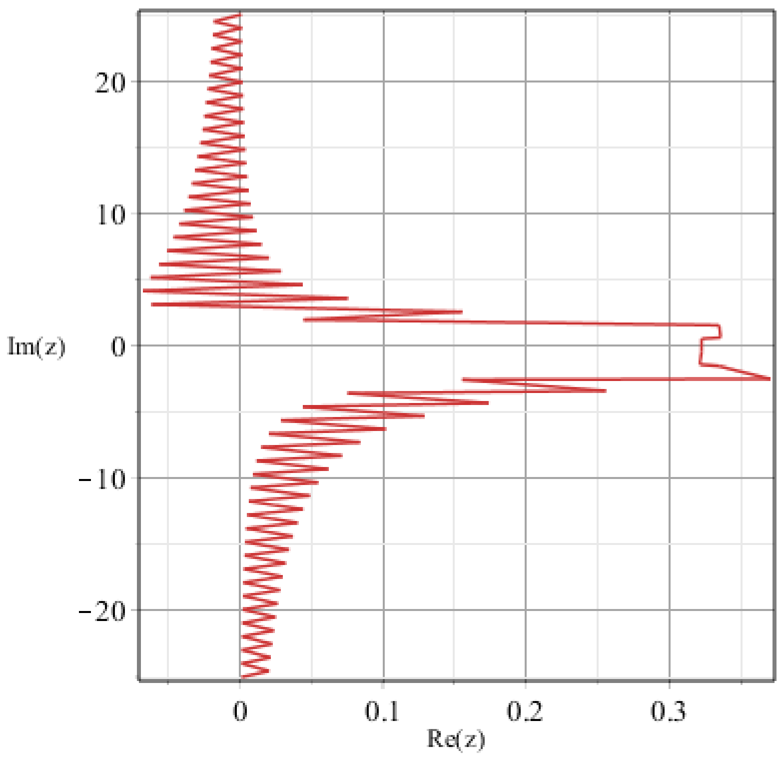

Figure 3.7: Stability Region of Four-step Block Adams Moulton Method

Figure 3.7 shows the region of absolute stability of the four-step block Adams Moulton method. The method is A-stable as the region of its absolute stability covers the entire half left plane as shown in the figure.

3.7.8 Absolute Stability Properties of the Five-step Adams Bashforth Method

Computing the characteristic polynomial as given in (3.102)

$$\sigma(z)=\left(\begin{bmatrix}1&0&0&-1&0\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&0&1&0\\0&0&0&-1&1\end{bmatrix}-z\begin{bmatrix}-\frac{69}{160}&-\frac{87}{80}&-\frac{87}{80}&-\frac{69}{160}&\frac{3}{160}\\\frac{1}{90}&-\frac{17}{45}&-\frac{19}{15}&-\frac{17}{45}&\frac{1}{90}\\-\frac{77}{1440}&\frac{43}{240}&-\frac{511}{720}&-\frac{637}{1440}&\frac{3}{160}\\\frac{64}{45}&\frac{8}{15}&\frac{64}{45}&\frac{14}{45}&0\\-\frac{173}{1440}&\frac{241}{720}&-\frac{133}{240}&\frac{1427}{1440}&\frac{95}{288}\end{bmatrix}\right)^{-1}\times$$

$$\left(\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}+z\begin{bmatrix}0&0&0&0&\frac{3}{160}\\0&0&0&0&0\\0&0&0&0&\frac{11}{1440}\\0&0&0&0&\frac{14}{45}\\0&0&0&0&\frac{3}{360}\end{bmatrix}\right) \tag{3.117}$$

and the stability function is given as

$$\sigma(z)=-\frac{60z^5+274z^4+675z^3+1020z^2+900z+360}{60z^5-274z^4+675z^3-1020z^2+900z-360} \tag{3.110}$$

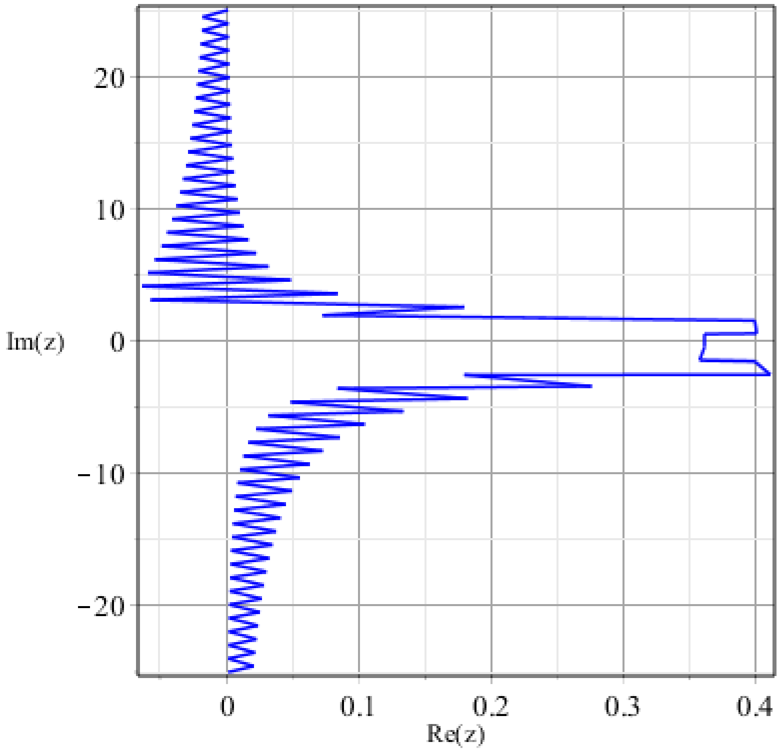

Figure 3.8: Stability Region of Five-step Block Adams Moulton Method

Figure 3.8 shows the region of absolute stability of the three-step block Adams Moulton method. The method is A-stable as the region of its absolute stability covers the entire half left plane as shown in the figure.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSIONS

4.1 Numerical Experiments

In this section, some numerical examples are given to illustrate the accuracy of the Adams Bashforth and Adams Moulton methods considered in this project. We find the absolute errors of the approximate solution on the partition π_N as $\left|y(x)-y(x_n)\right|$.

Problem 1: consider a linear stiff problem

$$y' = -y$$

$$y(0) = 1, 0 \le x \le 1$$

Exact solution is given as

$$y(x) = e^{-x}$$

Problem 2: we consider the nonlinear problem: (Musa *et al.*, 2012)

$$y' = \frac{y(1-y)}{2y-1} \quad y(0) = \frac{5}{6}, \ 0 \le x \le 1$$

Exact solution is given as

$$y(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{6}e^{-x}}$$

Problem 3: (Sunday *et al.* 2014).

$$y' = -10(y-1)^2 \qquad y(0) = 2$$

Exact solution: $1+\dfrac{1}{1+10x}$

Problem 4: Consider the oscillatory problem (Mohammad *et al.*, 2018).

$$y' = -20y + 20\sin x + \cos x \qquad\qquad y(0) = 1,$$

With exact solution

$$y(x) = \sin x + e^{-20x}$$

Problem 5: Consider the linear stiff problem (Ehiemua and Agbeboh, 2019).

$$y' = -8y + 8x + 1 \quad y(0) = 2$$

Exact solution: $y(x) = x + 2e^{-8x}$

Table 4.1: Comparison of Adams Bashforth Schemes for Problem 1 (*h=0.1*)

X	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.1	0.90483741803595957316	7.55×10^{-5}	3.59×10^{-6}	2.15×10^{-7}	1.45×10^{-8}
0.2	0.81873075307798185867	3.17×10^{-4}	3.65×10^{-7}	6.95×10^{-8}	7.21×10^{-9}
0.3	0.74081822068171786607	2.25×10^{-4}	3.27×10^{-5}	2.81×10^{-7}	1.25×10^{-8}
0.4	0.67032004603563930074	5.19×10^{-4}	3.23×10^{-5}	2.67×10^{-6}	3.43×10^{-9}
0.5	0.60653065971263342360	4.19×10^{-4}	2.65×10^{-5}	2.27×10^{-6}	2.61×10^{-7}
0.6	0.54881163609402643263	6.37×10^{-4}	4.85×10^{-5}	2.14×10^{-6}	2.45×10^{-7}
0.7	0.49658530379140951470	5.35×10^{-4}	4.58×10^{-5}	1.79×10^{-6}	2.18×10^{-7}
0.8	0.44932896411722159143	6.96×10^{-4}	3.95×10^{-5}	3.58×10^{-6}	2.01×10^{-7}
0.9	0.40656965974059911188	5.96×10^{-4}	5.39×10^{-5}	3.15×10^{-6}	1.73×10^{-7}
1.0	0.36787944117144232160	7.12×10^{-4}	5.02×10^{-5}	2.90×10^{-6}	3.15×10^{-7}

The numerical results for problem 1 obtained from the Adams Bashforth methods are displayed in Table 4.1. The absolute errors are compared between the step numbers considered for the Adams Bashforth, it is however observed that as the step number k increases the accuracy of the methods increases.

Table 4.2: Comparison of Adams Moulton Schemes for Problem 1 (*h=0.1*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.1	0.90483741803595957316	3.58×10^{-6}	2.15×10^{-7}	1.45×10^{-8}	1.05×10^{-9}
0.2	0.81873075307798185867	3.65×10^{-7}	6.95×10^{-8}	7.21×10^{-9}	6.20×10^{-10}
0.3	0.74081822068171786607	2.60×10^{-6}	2.81×10^{-7}	1.25×10^{-8}	7.96×10^{-10}
0.4	0.67032004603563930074	6.00×10^{-7}	2.14×10^{-7}	3.43×10^{-9}	3.86×10^{-10}
0.5	0.60653065971263342360	1.86×10^{-6}	2.81×10^{-7}	6.65×10^{-9}	1.42×10^{-9}
0.6	0.54881163609402643263	7.33×10^{-7}	4.16×10^{-7}	2.02×10^{-9}	1.92×10^{-9}
0.7	0.49658530379140951470	1.30×10^{-6}	4.94×10^{-7}	5.84×10^{-9}	1.54×10^{-9}
0.8	0.44932896411722159143	8.01×10^{-7}	3.79×10^{-7}	4.60×10^{-9}	1.53×10^{-9}
0.9	0.40656965974059911188	8.85×10^{-7}	4.62×10^{-7}	2.38×10^{-9}	1.18×10^{-9}
1.0	0.36787944117144232160	8.19×10^{-7}	5.06×10^{-7}	5.26×10^{-10}	1.72×10^{-9}

The numerical results for problem 1 obtained from the Adams Moulton methods are displayed in Table 4.2. The absolute errors are compared between the step numbers considered for the Adams Moulton, it is observed that as the step number k increases the accuracy of the methods increases. More so, the Adams Moulton methods perform better than the Adams Basforth methods when compared with the corresponding step numbers.

Table 4.3: Comparison of Adams Bashforth Schemes for Problem 2 (*h=0.1*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.1	0.85260195175848715618	5.42×10^{-5}	8.12×10^{-6}	1.99×10^{-6}	6.43×10^{-7}
0.2	0.86917122776001362287	2.13×10^{-4}	5.11×10^{-7}	6.65×10^{-7}	3.18×10^{-7}
0.3	0.88354736403848810521	1.56×10^{-4}	6.35×10^{-5}	2.26×10^{-6}	4.95×10^{-7}
0.4	0.89610603833589965639	2.79×10^{-4}	5.81×10^{-5}	1.89×10^{-6}	6.21×10^{-8}
0.5	0.90713588713778054775	2.28×10^{-4}	4.88×10^{-5}	1.63×10^{-5}	8.15×10^{-6}
0.6	0.91686468026639944627	2.89×10^{-4}	6.32×10^{-5}	1.46×10^{-5}	7.25×10^{-6}
0.7	0.92547599100050262368	2.45×10^{-4}	5.70×10^{-5}	1.27×10^{-5}	6.41×10^{-6}
0.8	0.93312030595224194344	2.77×10^{-4}	4.97×10^{-5}	1.46×10^{-5}	5.71×10^{-6}
0.9	0.93992271105581099311	2.38×10^{-4}	5.30×10^{-5}	1.29×10^{-5}	5.05×10^{-6}
1.0	0.94598837784255433542	2.54×10^{-4}	4.78×10^{-5}	1.16×10^{-5}	5.12×10^{-6}

The numerical results for problem 2 obtained from the Adams Bashforth methods are displayed in Table 4.3.The absolute errors are compared between the step numbers considered for the Adams Bashforth, it is however observed that as the step number k increases the accuracy of the methods increases.

Table 4.4: Comparison of Adams Moulton Schemes for Problem 2 (*h=0.1*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.1	0.85260195175848715618	8.12×10^{-6}	1.99×10^{-6}	6.43×10^{-7}	2.50×10^{-7}
0.2	0.86917122776001362287	5.11×10^{-7}	6.65×10^{-7}	3.18×10^{-7}	1.45×10^{-7}
0.3	0.88354736403848810521	2.98×10^{-6}	2.26×10^{-6}	4.95×10^{-7}	1.72×10^{-7}
0.4	0.89610603833589965639	6.42×10^{-7}	2.40×10^{-6}	6.21×10^{-8}	9.10×10^{-8}
0.5	0.90713588713778054775	1.18×10^{-6}	1.88×10^{-6}	9.13×10^{-9}	2.56×10^{-7}
0.6	0.91686468026639944627	6.40×10^{-7}	2.05×10^{-6}	1.65×10^{-8}	2.37×10^{-7}
0.7	0.92547599100050262368	4.37×10^{-7}	1.96×10^{-6}	8.72×10^{-9}	2.07×10^{-7}
0.8	0.93312030595224194344	5.90×10^{-7}	1.66×10^{-6}	4.70×10^{-8}	1.86×10^{-7}
0.9	0.93992271105581099311	1.03×10^{-7}	1.61×10^{-6}	2.88×10^{-8}	1.63×10^{-7}
1.0	0.94598837784255433542	5.25×10^{-7}	1.50×10^{-6}	3.09×10^{-8}	1.54×10^{-7}

The numerical results for problem 2 obtained from the Adams Moulton methods are displayed in Table 4.4. The absolute errors are compared between the step numbers considered for the Adams Moulton, it is observed that as the step number k increases the accuracy of the methods increases. More so, the Adams Moulton methods perform better than the Adams Basforth methods when compared with the corresponding step numbers.

Table 4.5: Comparison of Adams Bashforth Schemes for Problem 3 (*h=0.01*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.01	1.9090909090909090909	3.79×10^{-4}	5.96×10^{-5}	1.38×10^{-5}	4.04×10^{-6}
0.02	1.8333333333333333333	1.52×10^{-3}	6.45×10^{-6}	4.07×10^{-6}	1.85×10^{-6}
0.03	1.7692307692307692308	1.10×10^{-3}	4.83×10^{-4}	1.58×10^{-5}	3.03×10^{-6}
0.04	1.7142857142857142857	1.88×10^{-3}	4.35×10^{-4}	1.43×10^{-4}	7.71×10^{-7}
0.05	1.6666666666666666667	1.53×10^{-3}	3.61×10^{-4}	1.22×10^{-4}	5.63×10^{-5}
0.06	1.6250000000000000000	1.87×10^{-3}	4.67×10^{-4}	1.08×10^{-4}	4.99×10^{-5}
0.07	1.5882352941176470588	1.59×10^{-3}	4.20×10^{-4}	9.39×10^{-4}	4.40×10^{-5}
0.08	1.5555555555555555556	1.73×10^{-3}	3.61×10^{-4}	1.11×10^{-4}	3.94×10^{-5}
0.09	1.5263157894736842105	1.52×10^{-3}	3.88×10^{-4}	9.87×10^{-5}	3.51×10^{-5}
0.1	1.5000000000000000000	1.57×10^{-3}	3.53×10^{-4}	8.94×10^{-5}	3.68×10^{-5}

The numerical results for problem 3 obtained from the Adams Bashforth methods are displayed in Table 4.5. The absolute errors are compared between the step numbers considered for the Adams Bashforth, it is however observed that as the step number k increases the accuracy of the methods increases.

Table 4.6: Comparison of Adams Moulton Schemes for Problem 3 (*h=0.01*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.01	1.9090909090909090909	5.96×10^{-6}	5.96×10^{-5}	4.04×10^{-6}	1.38×10^{-6}
0.02	1.8333333333333333333	6.45×10^{-6}	6.45×10^{-6}	1.85×10^{-6}	7.52×10^{-7}
0.03	1.7692307692307692308	2.05×10^{-5}	4.83×10^{-4}	3.03×10^{-6}	9.11×10^{-7}
0.04	1.7142857142857142857	7.15×10^{-6}	4.35×10^{-4}	7.71×10^{-7}	4.24×10^{-7}
0.05	1.6666666666666666667	6.51×10^{-6}	3.61×10^{-4}	1.48×10^{-7}	1.46×10^{-6}
0.06	1.6250000000000000000	6.51×10^{-6}	4.67×10^{-4}	3.34×10^{-7}	1.37×10^{-6}
0.07	1.5882352941176470588	1.06×10^{-6}	4.20×10^{-4}	1.01×10^{-7}	1.19×10^{-6}
0.08	1.5555555555555555556	5.64×10^{-6}	3.61×10^{-4}	5.48×10^{-7}	1.08×10^{-6}
0.09	1.5263157894736842105	1.14×10^{-6}	3.88×10^{-4}	3.83×10^{-7}	9.42×10^{-7}
0.1	1.5000000000000000000	4.83×10^{-6}	3.53×10^{-4}	3.88×10^{-7}	9.24×10^{-7}

The numerical results for problem 3 obtained from the Adams Moulton methods are displayed in Table 4.6. The absolute errors are compared between the step numbers considered for the Adams Moulton, it is observed that as the step number k increases the accuracy of the methods increases. More so, the Adams Moulton methods perform better than the Adams Basforth methods when compared with the corresponding step numbers.

Table 4.7: Comparison of Adams Bashforth Schemes for Problem 4 (h=0.01)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.2	0.216984969683795	6.70×10^{-4}	9.17×10^{-5}	1.50×10^{-5}	2.66×10^{-6}
0.4	0.389753804936553	2.60×10^{-5}	3.66×10^{-6}	5.49×10^{-7}	9.73×10^{-8}
0.6	0.564648617607388	1.64×10^{-6}	1.07×10^{-7}	1.50×10^{-8}	2.67×10^{-9}
0.8	0.717356203434698	8.30×10^{-7}	6.10×10^{-9}	3.23×10^{-10}	6.49×10^{-11}
1.0	0.841470986869051	6.47×10^{-7}	5.90×10^{-9}	2.73×10^{-11}	1.07×10^{-12}
1.2	0.932039086004977	4.55×10^{-7}	7.65×10^{-9}	2.50×10^{-11}	4.52×10^{-13}
1.4	0.985449729989151	2.45×10^{-7}	5.36×10^{-9}	1.36×10^{-11}	5.38×10^{-13}
1.6	0.999573603041518	2.47×10^{-7}	7.23×10^{-9}	1.47×10^{-12}	5.25×10^{-13}
1.8	0.973847630878195	1.96×10^{-7}	8.25×10^{-9}	1.07×10^{-11}	5.25×10^{-13}
2.0	0.909297426825682	4.09×10^{-7}	5.18×10^{-9}	2.25×10^{-11}	4.93×10^{-13}

The numerical results for problem 4 obtained from the Adams Bashforth methods are displayed in Table 4.7. The absolute errors are compared between the step numbers considered for the Adams Bashforth, it is however observed that as the step number k increases the accuracy of the methods increases.

Table 4.8: Comparison of Adams Moulton Schemes for Problem 4 (*h=0.01*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.2	0.216984969683795	2.63×10^{-6}	1.40×10^{-6}	6.13×10^{-8}	2.38×10^{-8}
0.4	0.389753804936553	9.64×10^{-8}	5.68×10^{-8}	2.25×10^{-9}	8.71×10^{-10}
0.6	0.564648617607388	2.79×10^{-9}	1.53×10^{-9}	6.17×10^{-11}	2.39×10^{-11}
0.8	0.717356203434698	2.56×10^{-10}	3.32×10^{-11}	1.50×10^{-12}	5.86×10^{-13}
1.0	0.841470986869051	2.29×10^{-10}	3.47×10^{-12}	2.90×10^{-14}	5.00×10^{-15}
1.2	0.932039086004977	2.54×10^{-10}	2.62×10^{-12}	2.00×10^{-15}	3.00×10^{-15}
1.4	0.985449729989151	2.71×10^{-10}	1.13×10^{-12}	2.00×10^{-15}	1.30×10^{-14}
1.6	0.999573603041518	2.77×10^{-10}	8.60×10^{-14}	4.00×10^{-15}	3.00×10^{-15}
1.8	0.973847630878195	2.73×10^{-10}	1.14×10^{-12}	5.00×10^{-15}	1.00×10^{-15}
2.0	0.909297426825682	2.57×10^{-10}	1.71×10^{-12}	1.20×10^{-14}	1.00×10^{-14}

The numerical results for problem 4 obtained from the Adams Moulton methods are displayed in Table 4.8. The absolute errors are compared between the step numbers considered for the Adams Moulton, it is observed that as the step number k increases the accuracy of the methods increases. More so, the Adams Moulton methods perform better than the Adams Basforth methods when compared with the corresponding step numbers.

Table 4.9: Comparison of Adams Bashforth Schemes for Problem 5 (h=0.1)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.1	0.998657928234444	4.14×10^{-2}	1.12×10^{-2}	3.87×10^{-3}	1.50×10^{-3}
0.2	0.603793035989310	2.25×10^{-1}	6.80×10^{-3}	5.99×10^{-4}	3.95×10^{-5}
0.3	0.481435906578825	8.80×10^{-2}	1.21×10^{-1}	3.94×10^{-3}	6.73×10^{-4}
0.4	0.481524407956732	1.16×10^{-1}	5.45×10^{-2}	6.23×10^{-2}	1.37×10^{-3}
0.5	0.536631277777468	4.80×10^{-2}	2.41×10^{-2}	2.77×10^{-2}	3.30×10^{-2}
0.6	0.616459494098040	4.56×10^{-2}	1.46×10^{-2}	1.26×10^{-2}	1.49×10^{-2}
0.7	0.707395727432966	1.92×10^{-2}	6.57×10^{-3}	5.37×10^{-3}	6.67×10^{-3}
0.8	0.803323114546348	1.62×10^{-2}	2.94×10^{-3}	7.02×10^{-3}	3.00×10^{-3}
0.9	0.901493171616753	6.87×10^{-3}	1.44×10^{-3}	3.13×10^{-3}	1.34×10^{-3}
1.0	1.00067092525581	5.46×10^{-3}	6.46×10^{-4}	1.42×10^{-3}	6.64×10^{-4}

The numerical results for problem 5 obtained from the Adams Bashforth methods are displayed in Table 4.9. The absolute errors are compared between the step numbers considered for the Adams Bashforth, it is however observed that as the step number k increases the accuracy of the methods also increases.

Table 4.10: Comparison of Adams Moulton Schemes for Problem 5 (*h=0.1*)

x	Exact Solution	Error in k=2	Error in k=3	Error in k=4	Error in k=5
0.1	0.998657928234444	1.12×10^{-2}	3.87×10^{-3}	1.50×10^{-3}	6.24×10^{-4}
0.2	0.603793035989310	6.80×10^{-3}	5.99×10^{-4}	3.95×10^{-5}	7.74×10^{-5}
0.3	0.481435906578825	7.49×10^{-4}	3.94×10^{-3}	6.73×10^{-4}	1.82×10^{-4}
0.4	0.481524407956732	2.77×10^{-3}	2.12×10^{-3}	1.37×10^{-3}	1.32×10^{-4}
0.5	0.536631277777468	7.71×10^{-4}	7.43×10^{-4}	5.51×10^{-4}	6.45×10^{-4}
0.6	0.616459494098040	8.46×10^{-4}	7.08×10^{-4}	2.74×10^{-4}	3.01×10^{-4}
0.7	0.707395727432966	2.83×10^{-4}	3.48×10^{-4}	9.60×10^{-5}	1.32×10^{-4}
0.8	0.803323114546348	2.30×10^{-4}	1.38×10^{-4}	1.12×10^{-4}	6.18×10^{-5}
0.9	0.901493171616753	8.32×10^{-5}	9.52×10^{-5}	4.79×10^{-5}	2.39×10^{-5}
1.0	1.00067092525581	5.85×10^{-5}	4.55×10^{-5}	2.26×10^{-5}	2.34×10^{-5}

The numerical results for problem 5 obtained from the Adams Moulton methods are displayed in Table 4.10. The absolute errors are compared between the step numbers considered for the Adams Moulton, it is observed that as the step number k increases the accuracy of the methods increases. More so, the Adams Moulton methods perform better than the Adams Basforth methods when compared with the corresponding step numbers.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this project, an efficient approach of carrying out the analysis of order and error constants of linear multistep methods is proposed. To demonstrate this, some class of Adams Bashforth and Adams Moulton schemes were considered. The derivation of the continuous formulation of both Adams Bashforth and Adams Moulton schemes for cases when $k = 2,3,4$ and 5 are done through the collocation technique using power series as the basis function. The continuous schemes of each step number of the Adams class derived enable us to generate the sufficient number of discrete schemes which are combined for implementation in each method as a block form thus addressing the setback associated with the predictor-corrector methods of linear multistep methods. The convergence analysis of each method is carried out on the entire block which reveal the order and error constants of each block method and the zero-stability. The new approach is faster than the conventional approach of analyzing the individual member of a block method. The stability analysis also reveals that all the explicit Adams class are not A-stable but the implicit Adams class are A-stable. Furthermore, numerical experiments on some initial value problems of first order ODEs were carried on each block method; results reveal that as the step number of the Adams class increases, the accuracy also increases. Also, the Adams Moulton methods have relatively higher accuracy than the corresponding step number of the Adams Bashforth methods.

5.2 Recommendations

As research is continuum of which numerical analysis will not be an exception, further research is therefore recommended to be carried out in the following areas.

1. Adams class can also be extended to higher order differential equations
2. The step length of Adams method can be extended
3. The new approach of carrying out the analysis of the order and error constants of a block linear multistep method is highly recommended for general use.
4. The other classes of existing linear multistep methods (including the hybrid methods) can be considered to further ascertain the efficiency of the new approach of block analysis.

5.3 Contributions to Knowledge

1. The block members of the Adams Bashforth and Adams Moulton when $k = 2, 3, 4, 5$ were derived.
2. Rather than using the conventional approach of obtaining order and error constants of individual members in the block method, this work proposed block analysis that yielded the error constants of all the members at once, thereby saving computing time.
3. The work further established that all Adams Bashforth classes of methods considered are not A-stable, while the Adams Moulton methods are A-stable
4. All the cases considered also validate the notion that as the step number increases, the order also increases and thus produce better and more accurate results.

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APPENDIX A

```
> restart;
> WithLinearAlgebra :
> Digits := 20 :
> N := 8 :
>
  for n from 0 by 2 to N do
   h := 0.1; y[0] := 1;
   f[n + 0] := -y[n + 0];
   f[n + 1] := -y[n + 1];
   f[n + 2] := -y[n + 2];
```

$$A := \left(y_{n+1} = y_n + \frac{1}{2}hf_n + \frac{1}{2}hf_{n+1}, y_{n+2} = y_{n+1} - \frac{1}{2}hf_n + \frac{3}{2}hf_{n+1}\right); P$$

```
      := fsolve({A}); Q := eval([y[n + 1], y[n + 2]], P); y[n + 1] := Q[1]; y[n + 2]
      := Q[2]; y end do:

> N := 10 :
> for n from 0  to N do y_n := y_n end do:
> for n from 0 to N do x[n] := 0.1·n; Y[n] := exp(-x_n) end do:
> for n from 0 to N do E[n] := abs(Y[n] - y[n]) end do;
```

$$E_0 := 0.$$

$$E_1 := 0.00007551327405481126$$

$$E_2 := 0.00031686596963718895$$

$$E_3 := 0.00022486321850889130$$

$$E_4 := 0.00051895623193439540$$

$$E_5 := 0.00041891376755230147$$

$$E_6 := 0.00063745147751011849$$

$$E_7 := 0.00053529924950450774$$

$$E_8 := 0.00069600284613215520$$

$$E_9 := 0.00059578655957808745$$

$$E_{10} := 0.00071243653187598516$$

```
> K = 3 :
> restart;
> WithLinearAlgebra :
> Digits := 20 :
> N := 9 :
>
  for n from 0 by 3 to N do
   h := 0.1; y[0] := 1;
   f[n + 0] := -y[n + 0];
   f[n + 1] := -y[n + 1];
   f[n + 2] := -y[n + 2];
   f[n + 3] := -y[n + 3];
```

$$A := \left(y_{n+1} = y_{n+2} + \frac{1}{12} h f_n - \frac{2}{3} h f_{n+1} - \frac{5}{12} h f_{n+2}, y_{n+2} = y_n + \frac{1}{3} h f_n + \frac{4}{3} h f_{n+1} + \frac{1}{3} h f_{n+2}, y_{n+3} = y_{n+2} + \frac{5}{12} h f_n - \frac{4}{3} h f_{n+1} + \frac{23}{12} h f_{n+2} \right);$$

```
P := fsolve({A}); Q := eval([y[n + 1], y[n + 2], y[n + 3]], P); y[n + 1] := Q[1]; y[n + 2] := Q[2]; y[n + 3] := Q[3]; end do:

> N := 10 :
> for n from 0  to N do y_n := y_n end do:
> for n from 0 to N do x[n] := 0.1·n; Y[n] := exp(-x_n) end do:
> for n from 0 to N do E[n] := abs(Y[n] - y[n]) end do;
```

$$E_0 := 0.$$

$$E_1 := 0.0000035811779535307 8$$

$$E_2 := 3.6474679155528\ 10^{-7}$$

$$E_3 := 0.000032722192291884 15$$

$$E_4 := 0.000032261148681350 27$$

$$E_5 := 0.000026520466003692 14$$

$$E_6 := 0.000048481321799088 89$$

$$E_7 := 0.000045832932551238 37$$

$$E_8 := 0.000039492989506757 10$$

$$E_9 := 0.000053872580024656 61$$

$$E_{10} := 0.000050201731587187 57$$