DETERMINATION OF OPTIMAL NUMBER OF SERVERS IN FIRST CITY MONUMENT BANK, MINNA, NIGER STATE, NIGERIA

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MTech/SPS/2018/8867

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A THESIS SUBMITED TO THE POSTGRADUATE SCHOOL, FEDERAL UNIVERSITY OF TECHNOLOGY MINNA, NIGERIA IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF TECHNOLOGY (MTech) IN APPLIED MATHEMATICS

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ABSTRACT

It is a common knowledge that most customers in Nigeria spent lot of their useful times in commercial banks queues before being served. Hence, the need to use scientific techniques to determine optimal number of banking personnel (Servers) at different units of First City Monument Bank (FCMB) Minna branch become imperative to reduce waiting time. In this thesis, a network queuing model that determines optimal numbers of servers at the nodes of the Bank network queuing system is presented. The relevant data were collected for a period of four (4) weeks, through direct observations and personal interview. The number of arrivals and departures were also obtained. The total expected waiting time of customers in the current system before modification was 52 minutes with total number of 11 servers in the system while the total new expected waiting time of the customers in the system after modification was reduced to 11 minutes with optimal number of 17 servers (personnel) in all the nodes. The study has determined optimal number of servers (personnel) at the nodes of the bank network system. Result from this study is important information to the management of the First City Monument Bank, Minna branch for efficient and better service delivery.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

A Common situation that occurs in everyday life is that of queuing or waiting in line, when the demand for a service exceeds the capacity of the service, waiting is unsurprising and inevitable (Kembe *et al.*, 2012). Queues or waiting lines are usually seen at hotels, hospitals, bus stops, supermarkets, traffic, airports, gas stations, bank counters and so on. Service delay is unavoidable as a system gets blocked (Kandenmir & Cavas 2007).

Queuing systems theories have been used to study waiting time and predict the efficiency of services to be provided (Narayanamoorthy & Ramya, 2017). In queuing theory, there are three basic components of a queuing process which are:- Arrivals patterns, the actual waiting line and service facilities. Customers arrive to the facility from an infinite calling population, with a random arrival pattern following poison process. Once customers arrive, they are served immediately if the server(s) is empty, or otherwise the customers wait in the queue for the next empty server. Mostly, the service is on a first come first serve (FCFS) basis although other methods like service at random order (SARO) can be used. Preference service depending on the level of risk, urgency or the social, economic or political standing of the customers and Hold on line (HL) discipline, where important arriving customer takes the lead of the queue is rampant in many facilities. Customers who may feel to have waited for long in queue can balk or renege and seek alternative equivalent services elsewhere, however, the queue length and waiting time depends on the traffic intensity, which is the ratio of arrival and service rates. The service discipline follows an exponential pattern, with individual service time variation due to different nature of the problems to be handled (Rotich, 2016).

In an open queuing network, jobs enter and depart from the network. In a closed queuing network, jobs neither enter nor depart from the network. Open queuing networks can be further divided into two categories; open feed forward queuing networks and open feedback queuing networks. In an open feed forward queuing network, a job cannot appear in the same queue for more than one time. In an open feedback queuing network, after a job is served by a queue, it may reenter the same queue (Tin & Talthi, 2014). In a Mixed Networks, Network has multiple job classes and is open with respect to some classes but closed with repeat to the others (Vasilios *et al*., 2014).

1.2 Statement of the Research Problem

It is well known that in Nigeria most customers spent a lot of their useful time in commercial banking halls before been served. First City Monument Bank (FCMB) Minna is not left out from these protracted waiting times. These times wasting is mostly as a result of improper allocation of banking personnel (Servers) at the different units of the bank, which result in redundancies or deficiencies in some of the units of the bank. Hence, the need to use scientific techniques to allocate optimal number of banking personnel (Servers) to different units in First City Monument Bank (FCMB) Minna branch.

1.3 Aim and Objectives of the study

1.3.1 Aim of the Study

The aim of this study is to formulate a mathematical model that captures the network queuing system of First City Monument Bank (FCMB) Minna branch and determine optimal number of banking personnel (servers) at different units of the bank.

1.3.2 Objectives of the Study

The objectives of this present study are to:-

i. capture the operations of the units.

ii. determine the arrival rate of customers at each unit of the bank

iii. determine the departure rate of customers at each unit of the bank

iv. determine the waiting time of customers in the queuing network of the bank

v. determine the optimal number of banking personnel at each unit of the bank

1.4 Significance of the Study

The research will serve as a useful tool to First City Monument Bank (FCMB) Minna branch, in order to evaluate and determine the required optimal number of banking personnel (servers) required in the bank and also to take into consideration the system parameters such as arrival rate of the customers and departure rate of the customers respectively.

1.5 Scope and Limitations

The scope and area covered by this study is the First City Monument Bank (FCMB) Minna branch. It is limited to the various operation unit(s) of the bank where customers do wait in queue to receive service.

1.6 Definition of Terms

Queue: Is a line or sequence of people or vehicle waiting to be served, or an aggregation of items waiting for a service.

Customer: Is a Person that arrives and occupies a server for some period of time and departs after been served.

Server: A server is one who provides services to customers.

Arrival rate: Number of arrivals per unit of time, denoted by the Greek letter lamda (λ).

Departure rate: The number of departure per unit time. It is always representing by the Greek letter Mu (μ).

Service discipline: The order by which customers on queue are to be served or attended to, such as first come first served (FCFS), last come first served (LCFS) and Priority selection rule etc

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Related Literature

Queuing theory is the mathematical study of waiting lines or queues (Adaji *et al.*, 2021). Danish Engineer, Erlang (1913), first analyzed queue or queuing theory or waiting lines in the context of telephone facilities (Gupta *et al*., 1978). He started with the problem of the congestion of telephone traffic and later on, extended to business application and waiting lines. The ideas have since seen applications including telecommunication, traffic engineering, computing and the design of factories, shops, offices and hospitals (Thomas, 2014). In queuing theory, a model is constructed so that queue lengths and waiting times can be predicted (Adaji *et al.*, 2021). Queuing theory is generally considered as branch of operations research because the results are often used when making business decisions about the resources needed to provide a service. Queuing theory has its origin in research by Agner Krarup Erlang when he created models to describe the Copenhagen telephone exchange (Adaji *et al.*, 2021)

Johnson (2007), there are many attempts made to apply the queuing theory and its powerful application. But only recently, professionals discovered the benefits of applying queuing theory techniques in the reduction of waiting time to their organizations. He vigorously advised that queuing theory has a lot benefits if properly applied in the management of organization in areas of waiting line, service capacities and the operating cost (service cost and the waiting cost). There have been studies on outpatient clinics with the most common objectives of these studies on the reduction of patients waiting time in the system, improvement on customer service, better resource utilization and reduction of operating cost (service cost and the waiting cost).

Thomas (2014) conducted a research work on queue theory and the research was to minimize the waiting time in the queues. The objective was to reduce the average time a customer spent in the system, focusing on customer waiting time as well as other areas that can be improved. The data collected was on time studies and inputting the various values into simulation software, which was ran to represent the current system as well as various other possible scenarios encountered. In conclusion, it was observed that the company should increase the flit size by one trailer.

Owoloko *et al.* (2015) in their paper entitled "on the application of the open Jackson queuing network" said for an efficient hospital planning, a good patient flow means that the patient queuing time is minimised, while poor patient flow means the patient suffer considerable queuing delays. The data collected was on time studies and substituting the various values into simulation software, which was ran to represent the current system. In conclusion, it was observed that one can easily deduce that the more the number of service channels, the less the waiting time in the queue and thereby making the amount of waiting time in the service less.

Sitzia and Wood (1997), described patients' perception of health care has gained increasing attention over the years. Patients' evaluation of service quality is now affected not only by actual waiting time but also by the perceived waiting time. The amount of time patients spend waiting can significantly influence their satisfaction (Davis & Vollmann, 1990).

In Taylor (1994), it was clearly stated that recent research has demonstrated that customer satisfaction is affected not just by waiting time but also by customers' expectations or attribution of the causes for the waiting. Reported in Filipowicz and Kwiecien (2008) is an article that describes queuing systems and queuing networks which are successfully used for performance analyses of different system such as computer, communications, transportation networks and manufacturing. The paper was titled "Queuing systems and networks models and applications" and it incorporates classical markovian systems with exponential service times and a poison arrival process and queuing systems with individual service. The model of fork join system applied to parallel processing analysis.

Bakari and Chamalwa (1963) gave an extensive study of queuing system and its application to customer service delivery in Fidelity Bank. In their studies, they observed that, standing in line can cause extreme boredom, annoyance and even rage to customers. Customers are often forced to waiting on line whenever the service facility is busy. Although, Automated Teller Machine (ATM) have been designed to provide efficient and improve services to customers at the shortest time possible, yet customers wait too long before they are finally serviced by the facility.

John (2010) explained that, if patients are attended to almost as soon as they join the queue, queuing is minimized. If not, then patients could suffer considerable queuing delays. Olaniyi (2004) warned that the danger of keeping patients in a queue is that their waiting time could become a cost to them because the time wasted on the queue would have been judiciously utilized elsewhere. Failing to take account of the patients waiting time leads to an exaggeration of productivity of the organization. On the other hand Mital (2010) asserted that if so many service channels are employed the movement of patients would be very fast and customer delay would be minimized but warned that with such a system, some servers might be underutilized. If this happens, it means the organization would have incurred expenses in getting the servers which were not fully utilized.

Green *et al.* (2006) carried out a research using queuing theory to increase the effectiveness of emergency department provider staffing. He used the lag SIPP queuing analysis to gain insights on how to change mechanics of the system so as to provide appropriate staffing patterns to reduce the fraction of patients who leave without being seen by the doctor. Their conclusion was that queuing models can be extremely useful in most effective allocation of staff. Similarly, Johanna (2007) carried out a research on the effect of waiting time on health service outcomes and service utilisation. He came up with the conclusion that people who wait longer than necessary end up spending more simply because they use the services for longer period. Ozden (1990) in his paper, entitled “understanding the efficiency of multi-server service system” said server utilization increases as the number of servers (and the arrival rate) increases. He also showed how increasing variability in the arrival and service processes tends to reduce server utilization with a given grade of service.

Kembe *et al.* (2012) in their paper entitled “A study of waiting and service costs of A Multi-Server Queuing Model in A Specialist Hospital” said the operation managers can recognize the trade-off that must take place between the cost of providing good service and the cost of patients waiting time. Service cost increases as a firm attempts to raise it’s level of service. As service improves, the cost of time spent waiting on the line decreases. This could be done by expanding the service facilities or using models that consider cost optimization.

Because of the increasing demand of quality and efficacy of service delivery from highly aware and educated patients due to increase in knowledge of technological advancement and telecommunications, have started putting more pressure on the healthcare managers to respond to these concerns (Singh, 2011). He also stated that any system in which arrivals place demand upon finite capacity, resources may be termed as a queuing system. In the case of the hospital system, it is observed that patients arrive randomly. In the study, they looked at a typical hospital system that consists of about five different departments, of which they assumed in this study that patients who come into the hospital for services will start by going first to record department to register and then proceed to the nursing stations; this procedure are followed until the patient depart from the system (Hospital).

The crowded phenomenon in the health care of most countries in the world happens more seriously and prevalently, deserving more immediate attention. In recent years, the queuing theory is introduced into the field of medical service and combined with the information system, which become the practical solution to tackle the queuing problem and relieve the crowd in health care. The queuing theory has its advantage in producing the simple models especially for the application of less random data. It is easy to construct a queuing model for a large general hospital (Lakshmi & Sivakumar, 2013).

The queuing theory also can be applied for the rapid evaluation and the comparison among various alternatives. Consequently, the queuing model becomes a powerful tool for the outpatient research. Queuing theory can solve the basic problem of maximum capacity to associate with a specific service standard in the health care, allocate rationally and effectively various service resources like facility, staff, space *etc.*, and choose the priority queuing discipline to determine service order among patients (Mital, 2010). Therefore, queuing theory is widely used in the academic research and practical application on the service system. The service process of foreign outpatients has a significant difference with Chinese outpatient process. In the foreign outpatients, there is almost no queuing phenomenon because most foreign outpatients provide the reservation system for the patients. For this reason, the queuing theory in foreign researches is mostly applied in the emergency, inpatient, operating room and other departments.

Mehandiratta (2011) studied the application of queuing theory in the medical system and presented it's efficiency in the inpatient, office, public health, emergency preparedness, long-term care and so on. Filipowicz and Kwiecien (2008) analyzed the medical resource allocation based on the queuing theory and summarized the contribution of queuing theory in the health care. Cochran (2006) derived an open queuing network model of an emergency department design intended to increase the capacity of emergency department to treat patients according to the patient acuity, arrival patterns and volumes.

Davis (2003) assert that providing ever-faster service, with the ultimate goal of having zero customer waiting time, has recently received managerial attention for several reasons. First, in the more highly developed countries, where standards of living are high, time becomes more valuable as a commodity and consequently, patients are less willing to wait for service. Secondly, this is a growing realization by organizations that the way they treat their patients today significantly impact on whether or not they will remain loyal patients tomorrow. Finally, advances in technology such as computers, internet etc., have provided firms with the ability to provide faster services. For these reasons hospital managers and health providers are always finding way to deliver more rapidly services, believing that the waiting will negatively affect the organization performance evaluation. Cochran (2006) also argue that higher operational efficiency of the hospital is likely to help to control the cost of medical services and consequently to provide more affordable care and improve access to the public. Researchers have argued that service waits can be controlled by two techniques: operations management or perceptions management (Hall, 2006).

The operation management feature deals with the organization of how patients (patients), queues and servers can be coordinated towards the goal of rendering efficient service at the minimum cost. The act of waiting has significant impact on patients' satisfaction. The amount of time patients must spend waiting can significantly influence their satisfaction. Additionally, research has demonstrated that customer satisfaction is affected not just by waiting time but also by customer expectations or attribution of the causes for the waiting.

Yankovic and Green (2011) represented the nursing system as a variable finite-source queuing model and developed a reliable, tractable, easily parameterized two-dimensional model to approximate the actual interdependent dynamics of bed occupancy levels and demands for nursing. Chiang *et al.* (2013) presented two scheduling policies to stay away from the risk of over-provisioning and under-provision in both quality of servers and service rate. They modeled a cloud server farm as an M/M/R queuing model, such that system congestion cost and balking event loss can be estimated analytically. A cost function is developed by taking system operating cost, working mode cost, system congestion cost etc. into consideration. Simulation result showed that the optimal quality of service rate can be obtained to minimum cost.

Adaji (2018) presented a research work on how to reduce customer waiting time in order to improve service processes. In his research work, a study of waiting and service costs of a multi-servers queuing model in general hospital, minna was conducted. This was done using queuing model to determine the waiting line performance such as: average arrival rate of patients, average service rate of patients, system utilisation factors, service cost, waiting cost, the probability of finding a specific number of patient in the system as well as the probability of finding a specific number of patient in the queue. He added that the management of hospitals services such as outpatient clinic is very complex and demanding to manage different policies to address long waiting time been tried some of them are more efficient than others.

Narayanamoorthy and Ramya (2017), explained vigorously their paper title" Multi Server Fuzzy Queuing Model Using DSW Algorithm with Hexagonal Fuzzy Number" said The queuing theory or waiting line theory is development by Erlang in 1903 based on congestion of telephone traffic. He directed his first efforts at finding the delay or one operator and later on the results were extended to find the delay for many operators. Moline & Thornton D-Fry who was further developed this telephone traffic in 1927 and 1928 respectively. Queuing theory is studying about such waiting line through performance measures. The queuing model comprises one or more queue and one or more service facilities under a set of rules. The parameters arrival rate and service rate μ follow poison and exponential distribution. They added further to described waiting line (queue) is one of the un avoid situation in our daily life. Queuing theory is studying about such waiting line through performance measures. (Fuzzy queuing model was first introduced by Lie and Lee (1989) further developed this model by many authors (Gupta *et al.*, 1978; Taylor, 1994; Sundarapandian, 2009, Jeeva & Rathnakumari, 2015). Here the parameters fuzzy arrival rate and fuzzy service rate are best described by linguistic terms very high, low, very low and moderate. The aim of their paper discussed about multi server queuing model and first come first served discipline using hexagonal fuzzy numbers and Octagonal fuzzy number undercut representation through DSW algorithm here fuzzy set decompose into distinct leaven points through - cut method. The approximate method DSW algorithm is used to define a membership function of the performance measures in multi-server fuzzy queuing model (Dong & Wong, 2017).

Jeeva and Rathnakumari (2015), analysed their work title "Fuzzy Cost Computations of M/M/1 and M/G/1 Queueing Models" that the two models of planning queuing system and its effect on the cost of each system by using two fuzzy queuing models of M/M/1 and M/G/1 are studied. These two fuzzy queuing models based on the cost of each model are compared and fuzzy ranking methods are used to select the optimal model due to the resulted complexity. Fuzzy queuing is more practical and realistic than deterministic queuing models. The basic idea is to transform a fuzzy queuing cost problem to a family of conventional crisp queue cost problem by applying the α-cut approach and Zadeh's extension principle. A set of parametric nonlinear programs are developed to calculate the lower and upper bound of the minimal expected total cost per unit time at α, through which the membership function of the total cost is constructed. Numerical example is illustrated to check the validity of the proposed method.

Sahu and Sahu (2014) focused on the bank lines system. They proposed an M/M/1: FCFS/inf/inf for queuing management system in a bank with a single channel. The model illustrated in the bank for patients on a level with services the single-channel queuing model with poison arrival and exponential service time (M/M/1). Data was obtained from a bank in the city. Four-week average Customer arrival was taken as the input data for both the queuing model and the service rate was obtained by the average service rate for patients they have given.

Ismaila and Lawal (2020) conducted a research titled "Determination of optimal number of servers at network queuing node to reduce waiting time" using the open Jackson queuing network consisting of several service stations. In their work, patients arrive from outside the network to get service and then leave the network, also added that the open Jackson theory is used to determine the queue network model and the performance measures of each service station in the network. The method used in collecting the data is through interview and direct observation through the number of arrival and departure at each service station. The open Jackson queue network model is built to capture the queuing system of the Federal polytechnics, School clinic as network of queue. Furthermore, the average arrival rate and the average departure rate are substituted into the linear equation system of the formed queue network model. The concept of the Moore-Penrose Generalized Inverse of Matrices has been comprehensively studied and reported, the linear equation system is solved using Moore-Penrose pseudo-inverse to obtain the value of the transition probability between stations. The analysis of result at all service nodes has shown.

The notion of the generalized inverse of a (Square or rectangular) Matrix was first introduced by Moore in 1970 and again by Penrose in 1955, who was apparently unaware of Moore’s work. These two definitions are equivalent (as it was pointed by Zuhair in 1976) and since then, the generalized inverse of a matrix is called the Moore-Penrose inverse (Vasilios *et al*., 2014). We use the Moore-Penrose Generalized Inverse (MPGI) of matrices to solve linear systems of algebraic equations when the coefficients matrix is singular or rectangular (Asmaa *et al.*, 2019). The solution of the linear system using the Moore-Penrose Generalized Inverse (MPGI) is often an approximate unique solution, but for some cases we can get an exact unique solution (Asmaa *et al.*, 2019). The concept of the Moore-Penrose Generalized Inverse of Matrices has been comprehensively studied and reported in the following works (Campbell & Meyer, 1979; Bahadur-Thapa *et al*,, 2018; Zuhair, 1976; Penrose, 1955; Ozden, 1990; Ben- Isreal, 1980; Rao & Mitra, 1971; Kanan, 2017; Hearon, 1968; Greville, 1960; Golub & Kanan , 1965). These systems are studied utilizing the theory of the Moore-Penrose generalized inverse or shortly (MPGI) of matrices. Some important algorithms and theorems for computation of the Moore-Penrose generalized inverse (MPGI) of matrices are given. The singular value decomposition (SVD) of a matrix has a very important role in computation, the Moore-Penrose generalized inverse (MPGI), and hence it is useful to study the solutions of over- and under-determined linear systems. A closed form for solution of linear system of algebraic equations is given when the coefficients matrix is of full rank or is not of full rank, singular square matrix or non-square matrix. The results are taken from the works mentioned in the references. A few examples including linear systems with coefficients matrix of full rank and not of full rank were presented in their research papers.

$$Let\ Ax = b \tag{2.1}$$

be a linear system of equations.

CASE 1: Let A be a non-square (rectangular) matrix, if A is of full column rank then the Pseudo inverse of A is defined as: $A^{+} = (A^{T}A)^{-1}A^{T}$ (2.2)

$now\ x = A^{+}b$ Is the unique solution to equation (2.1).

Where $A^{+}A \approx I\ \ and\ \ A^{+}A \neq I$

CASE 2: Let A be a non-square (rectangular) matrix, if A is of full row rank then the Pseudo inverse of A is defined as:

$$A^{+} = A^{T}(AA^{T})^{-1} \tag{2.3}$$

$$now\ x = A^{+}b \tag{2.4}$$

Equation (2.4) is the unique solution to equation (2.1)

Where $AA^{+} \approx I\ \ and\ \ A^{+}A \neq I$

CASE 3: Let A be a non-square (rectangular) matrix, if A is not of full rank that is A has rank deficiency, then the Pseudo Inverse of A is defined as:

$$A^{+} = V\sum{}^{+}U^{T} = V\sum{}^{-1}U^{T}. \tag{2.5}$$

Where V and U are unitary matrices and $\sum$ is a diagonal matrix.

Then $x = A^{+}A$ (2.6)

Is a solution to equation (2.1). This method is called Singular Value Decomposition.

Numerical Examples

Example (1)

Consider the system

$$\left.\begin{aligned} x_1 + 3x_2 &= 17 \\ 5x_1 + 7x_2 &= 19 \\ 11x_1 + 13x_2 &= 23 \end{aligned}\right\} \tag{2.7}$$

This system of equation can be written in the form of equation as (Asmaa *et al*., 2019)

$$\left.\begin{aligned} \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 19 \\ 23 \end{pmatrix} \\ \textit{that is } Ax = b \end{aligned}\right\} \tag{2.8}$$

The equation (2.8) becomes equation (2.9)

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, \quad b = \begin{pmatrix} 17 \\ 19 \\ 23 \end{pmatrix} \tag{2.9}$$

$$\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 7 - 15 = -8 \neq 0$$

Since rank (A) = 2 = n, then A is of full column rank, hence the pseudo inverse is:

$$A^{+} = (A^T A)^{-1} A^T = \begin{pmatrix} -0.317 & -0.2169 & 0.2373 \\ 0.4277 & 0,2041 & -0.1313 \end{pmatrix} \tag{2.10}$$

Maple Result of equation (2.10)

$> With(LinearA\lg ebra):$

$A := << 1,5,11 > | < 3,7,13 >>:$

$$MatrixInverse(A, method = Pseudo)$$

$$\begin{bmatrix} -\frac{79}{152} & -\frac{33}{152} & \frac{9}{39} \\ \frac{65}{152} & \frac{31}{152} & -\frac{5}{38} \end{bmatrix} \tag{2.11}$$

Recall from equation (2.8), hence, equation (2.8) becomes equation (2.12)

$$x = A^{+}b \tag{2.12}$$

Substituting equation (2.10) into equation (2.12), we have

$$x = \begin{pmatrix} -0.317 & -0.2169 & 0.2373 \\ 0.4277 & 0,2041 & -0.1313 \end{pmatrix} \begin{pmatrix} 17 \\ 19 \\ 23 \end{pmatrix} \tag{2.13}$$

Equation (2.13) gave an approximate solution represented in equation (2.13).

$$x = \begin{pmatrix} -7.5 \\ 8.13 \end{pmatrix} \tag{2.14}$$

Example (2): Find the Pseudo Inverse of

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{2.15}$$

$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5 \neq 0.$ Rank (A) = 2, since for the given matrix the rank is equal to the number of rows, therefore the matrix has row rank.

Then, the Pseudo Inverse of A is represented in equation (2.16)

$$A^{+} = A^{T}(AA^{T})^{-1} \tag{2.16}$$

Taking the A Transpose of equation (2.4), then we have

$$now\ the\ A^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \tag{2.17}$$

Multiplying equation (2.14) with (2.16), we have equation (2.18)

$$AA^T = \begin{pmatrix} 15 & 15 \\ 15 & 30 \end{pmatrix} \tag{2.18}$$

Finding the inverse of equation (2.18), then we have

$$(AA^T)^{-1} = \begin{pmatrix} \frac{2}{15} & -\frac{1}{15} \\ -\frac{1}{15} & \frac{1}{15} \end{pmatrix} \tag{2.19}$$

Multiplying equation (2.17) with equation (2.19), then Pseudo inverse is represented in (2.20).

$$A^+ = A^T (AA^T)^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{15} & -\frac{1}{15} \\ -\frac{1}{15} & \frac{1}{15} \end{pmatrix} \tag{2.20}$$

Equation (2.20) becomes equation (2.21)

$$\left. \begin{aligned} A^+ &= \begin{pmatrix} -\frac{2}{15} & \frac{3}{15} \\ \frac{1}{15} & \frac{1}{15} \\ 0 & \frac{1}{15} \\ \frac{5}{15} & -\frac{2}{15} \end{pmatrix} \\ A^+ &= \frac{1}{15} \begin{pmatrix} -2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & -2 \end{pmatrix} \end{aligned} \right\} \tag{2.21}$$

Maple Result of pseudo inverse in equation (2.21) represented as in equation (2.22)

$with(LinearA\lg ebra):$
$B := << 1, -3 > | < -2, 6 >>:$
$MatrixInverse(B, method = Pseudo)$

$$\begin{bmatrix} \frac{-2}{15} & \frac{1}{5} \\ \frac{1}{15} & \frac{1}{15} \\ 0 & \frac{1}{15} \\ \frac{1}{3} & \frac{-2}{15} \end{bmatrix} \tag{2.22}$$

Example (3) Find the Pseudo inverse of:

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix} \tag{2.23}$$

Solution:

$$Given\ \ A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}$$

Here all minors of order two are zero, Therefore, rank (A) = 1,

Since for the given matrix the rank is not equal to the number of rows or columns, therefore it has to be solved by singular value decomposition.

Then the Pseudo inverse of A using SVD is represented in equation (2.24).

$$A^{+}=V\sum{}^{+}U^{T} = V\sum{}^{-1}U^{T}. \tag{2.24}$$

$$Compute\ A^{T}A=\begin{bmatrix}5 & 10 & 15\\ 10 & 20 & 30\\ 15 & 30 & 45\end{bmatrix} \tag{2.25}$$

$$\left.\begin{array}{l} It\ has\ eigen\ values\ \lambda_1=70,\ \ \lambda_2=0,\ \lambda_3=0. \\ \text{And With the Corresponding Eigen vectors are: } \begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-2\\1\\-5\end{bmatrix},\begin{vmatrix}3\\6\\-5\end{vmatrix}. \end{array}\right\} \tag{2.26}$$

The Singular Values of A are:

$$\sigma_1=\sqrt{\lambda_1}=\sqrt{70},\ \sigma_2=\sqrt{\lambda_2}=\sqrt{0},\ \sigma_3=\sqrt{\lambda_3}=\sqrt{0}, \tag{2.27}$$

$$Normalized\ vectors\ are\ V_1=\begin{bmatrix}1/\sqrt{14}\\ 2/\sqrt{14}\\ 3/\sqrt{14}\end{bmatrix},\ V_2=\begin{bmatrix}-2/\sqrt{5}\\ 1/\sqrt{5}\\ 0\end{bmatrix},\ V_3=\begin{bmatrix}3/\sqrt{70}\\ 6/\sqrt{70}\\ -5/\sqrt{70}\end{bmatrix}, \tag{2.28}$$

$$Thus, V = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & 3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & 6/\sqrt{70} \\ 3/\sqrt{14} & 0 & -6/\sqrt{70} \end{bmatrix}, \ and \sum = \begin{bmatrix} \sqrt{70} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.29}$$

Also, we can find $u_1 = \frac{1}{\sigma_1} Av_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ $and\ u_2 = \frac{1}{\sigma_2} Av_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ (2.30)

Equation (2.30) becomes equation (2.31)

$$u = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \tag{2.31}$$

Multiplying equation (2.29) with equation (2.31), then we have equation (2.32)

$$\therefore A^{+} = V \sum {}^{+} U^{T} = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & 3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & 6/\sqrt{70} \\ 3/\sqrt{14} & 0 & -6/\sqrt{70} \end{bmatrix} \begin{bmatrix} \sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \tag{2.32}$$

Equation (32) gave pseudo inverse using SVD represented in equation (33)

$$\therefore A^{+} = V \sum {}^{+} U^{T} = \frac{1}{70} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \tag{2.33}$$

Maple Result of pseudo inverse in equation (2.33) can be represented in equation (2.34)

$with(LinearA\lg ebra):$

$C := << 1, 2 > | < 2, 4 > | < 3, 6 >>:$

$MatrixInverse(C, method = Pseudo)$ (2.34)

$$\begin{bmatrix} \frac{1}{70} & \frac{1}{35} \\ \frac{1}{35} & \frac{2}{35} \\ \frac{3}{70} & \frac{3}{35} \end{bmatrix}$$

Arum and Irwan (2019) conducted research in their paper entitled “On the Application of the Open Jackson Queuing Network in bank” Said, The open Jackson queuing network is the open queuing network consisting of several service stations. On the open Jackson queuing networks, customers arrive from outside the network to get service and then leave the network. Also added that the open Jackson theory is used to determine the queue network model and the performance measures of each service station in the network, the data collecting technique used in their research is by interview and observation. The interview obtained a diagram of a queue network at the bank, and the observation results consist of the number of arrival and departure data at each service station. The open Jackson queue network model is built based on a diagram of the service stages in bank. Furthermore, the average arrival rate and the average departure rate are substituted into the linear equation system of the formed queue network model. The linear equation system is solved using Moore-Penrose pseudo-inverse, to obtain the value of the transition probability between stations. The next steps were investigating steady-state conditions and counting performance measures. The analysis of result at all service stations has shown that the queuing network is a steady-state condition, such that queue performance measures can be calculated. The performance measures are the average of customers and the waiting time for customers in the queuing network.

CHAPTER THREE

3.0 MATERIALS AND METHODS

The type of queuing system being adopted by any organization solely depends on the type of service being provided. The First City Monument Bank practice network type of queuing system, queuing network is composed of several random queue systems, mostly limited and single queue systems. Diverse types of customers go through the network in many ways and are served by the service nodes within the network system. A queuing network system has a set of nodes (i). Each node has a number of servers and a single node can be regarded as a queuing system. Customers can have access to the queuing network from any node. The arrival rate from the outside is λ and the arrival rate of Node i is λ_i. After the customers queue and gets the service at a node (the service rate of Node i is μ_i), he/she can leave the network system or go to another node, or even return to the former node. (Ismaila & Lawal, 2020).

3.1 Model Formulation

The First City Monument Bank Minna branch consists of five main units, which are the Meter Greeter Unit, Customers Service Unit, Marketing Unit, Tellers Unit, and Customers Service Manager Unit. In this study, each department is regarded as node of the network system. The data used in this research were collected from the five different departments of the Bank and they were collected based on the arrival and departure rate as well as time spent at each node. The method adopted for the data collection was direct observation and personal interview. It was done for complete one month, started from Monday to Friday. The collection of the data was for a total of six (6) hours at different time of the day, for each node. In a day, the number of arrivals and departures together with service time were taken at intervals of 5 minutes arrivals of customers into a node

(λ), while the departure rate was obtained also by the average number of five (5) minutes departures of customers at that particular node.

3.2 Model Assumptions

The following are the model assumptions made for Network Queuing System of the First City Monument Bank (FCMB), Minna.

1. The First City Monument Bank in the network queuing system is considered as an independent queuing system.
2. Queuing discipline is usually first come first served in the bank.
3. The arrival of the customers in the bank follows a Poisson arrival process.
4. The service follows exponential distribution.
5. The way customers enter the bank is not restricted.

We consider a banking network queuing system based on Jackson open network queuing model, the First City Monument Bank, Minna constitute of five units. In this study, we assumed that customers who come in to bank for services will start by going first to the meter greeter unit and then move to the customer's service unit, then some customers proceed to tellers unit or customers service manager until all customers are served and depart from the bank as in Figure 3.1.

A Schematic Diagram of FCMB Queuing Network

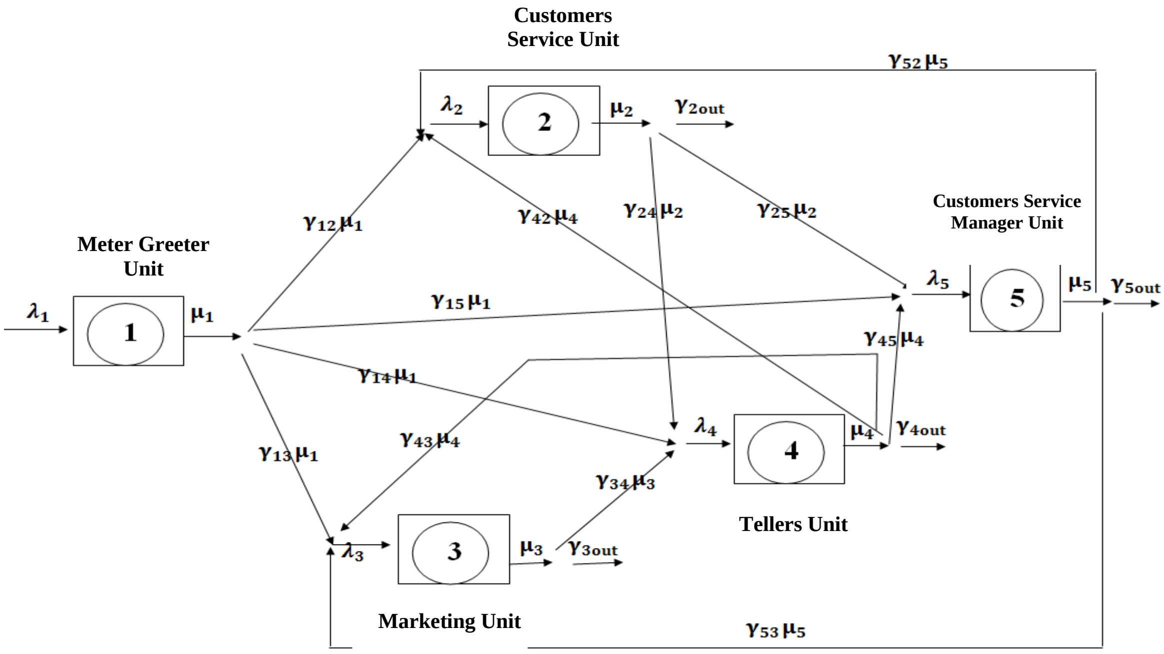

Figure 3.1 A Schematic diagram of the Bank (FCMB) Queuing Network

Where: λ_i Is the arrival rate of the customer, for i = 1, 2,...,5

μ_i Is the departure rate out of the system, for i = 1, 2,…,5

γ_{ij} Are the weights of moving from server i to server j.

m_i for i = 1,…,5, is the number of servers at the various node points in the system.

The following are the nodes in the network queuing system of the bank and Node1, Node2, Node3, Node4, and Node5 are defined as follow: Meter Greeter Unit denoted by node1; Customer Service Unit denoted by node2; Marketing Unit denoted by node3; Tellers Unit denoted by node4; Customer Service Manager Unit denoted by node5.

3.3 Model Equations

The following are model equations, obtained from the schematic diagram in figure 3.1. The system of linear equations which captures the probabilities at which a customer enters a particular node and also leaves a node to any other node in the system or out of the system is given below:

$$\lambda_2 = \gamma_{12}\mu_1 + \gamma_{42}\mu_4 + \gamma_{52}\mu_5 \quad (3.1)$$

$$\lambda_3 = \gamma_{13}\mu_1 + \gamma_{43}\mu_4 + \gamma_{53}\mu_5 \quad (3.2)$$

$$\lambda_4 = \gamma_{14}\mu_1 + \gamma_{24}\mu_2 + \gamma_{34}\mu_3 \quad (3.3)$$

$$\lambda_5 = \gamma_{15}\mu_1 + \gamma_{25}\mu_2 + \gamma_{45}\mu_4 \quad (3.4)$$

Also; on the application of the open Jackson Queuing Network

$$\mu_1 = \gamma_{12}\mu_1 + \gamma_{13}\mu_1 + \gamma_{14}\mu_1 + \gamma_{15}\mu_1 \quad (3.5)$$

$$\mu_2 = \gamma_{24}\mu_2 + \gamma_{25}\mu_2 + \gamma_{2out}\mu_2 \quad (3.6)$$

$$\mu_3 = \gamma_{34}\mu_3 + \gamma_{3out}\mu_3 \quad (3.7)$$

$$\mu_4 = \gamma_{42}\mu_4 + \gamma_{43}\mu_4 + \gamma_{45}\mu_4 + \gamma_{4out}\mu_4 \quad (3.8)$$

$$\mu_5 = \gamma_{52}\mu_5 + \gamma_{53}\mu_5 + \gamma_{5out}\mu_5 \quad (3.9)$$

Where:

$\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{24}, \gamma_{25}, \gamma_{2out}, \gamma_{34}, \gamma_{3out}, \gamma_{42}, \gamma_{43}, \gamma_{45}, \gamma_{4out}, \gamma_{52}, \gamma_{53}, \gamma_{5out}$

Are to be determined (Owoloko *et al.*, 2015)

Equation (3.1 – 3.9) can also be represented in the following forms, thus:

$$\lambda_2 = \mu_1\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + \mu_4\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + \mu_5\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.10)$$

$$\lambda_3 = 0\gamma_{12} + \mu_1\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + 0\gamma_{42} + \mu_4\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + \mu_5\gamma_{53} + 0\gamma_{5out} \quad (3.11)$$

$$\lambda_4 = 0\gamma_{12} + 0\gamma_{13} + \mu_1\gamma_{14} + 0\gamma_{15} + \mu_2\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + \mu_3\gamma_{34} + 0\gamma_{3out} + 0\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.12)$$

$$\lambda_5 = 0\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + \mu_1\gamma_{15} + 0\gamma_{24} + \mu_2\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + 0\gamma_{42} + 0\gamma_{43} + \mu_4\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.13)$$

$$\mu_1 = \mu_1\gamma_{12} + \mu_1\gamma_{13} + \mu_1\gamma_{14} + \mu_1\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + 0\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.14)$$

$$\mu_2 = 0\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + \mu_2\gamma_{24} + \mu_2\gamma_{25} + \mu_2\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + 0\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.15)$$

$$\mu_3 = 0\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + \mu_3\gamma_{34} + \mu_3\gamma_{3out} + 0\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.16)$$

$$\mu_4 = 0\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + \mu_4\gamma_{42} + \mu_4\gamma_{43} + \mu_4\gamma_{45} + \mu_4\gamma_{4out} + 0\gamma_{52} + 0\gamma_{53} + 0\gamma_{5out} \quad (3.17)$$

$$\mu_5 = 0\gamma_{12} + 0\gamma_{13} + 0\gamma_{14} + 0\gamma_{15} + 0\gamma_{24} + 0\gamma_{25} + 0\gamma_{2out} + 0\gamma_{34} + 0\gamma_{3out} + 0\gamma_{42} + 0\gamma_{43} + 0\gamma_{45} + 0\gamma_{4out} + \mu_5\gamma_{52} + \mu_5\gamma_{53} + \mu_5\gamma_{5out} \quad (3.18)$$

From model equation (3.10 - 3.18) can be represented in the matrix form as:

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 \\ 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & \mu_2 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & \mu_4 & \mu_4 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & \mu_5 & \mu_5 \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} \tag{3.19}$$

The matrix above is obviously rectangular and hence its determinant is zero. Therefore, in other to solve the above matrix, we shall introduce a pseudo inverse matrix called the moore-penrose pseudo inverse.

Therefore, equation (3.19) can be represented in the form (Asmaa *et al*, 2019)

$$\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & \mu_5 & 0 \\ 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & \mu_2 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & \mu_4 & \mu_4 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & \mu_5 & \mu_5 \end{bmatrix}^{+} \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} \tag{3.20}$$

3.4 Mathematical Formulation for new Departure Rate

Reducing waiting time of the customers in the banking hall and increasing the efficiency of the bank is thrust of this research, hence we formulate new departure rate of each of the nodes in our network system. This is done using equation (3.1 – 3.4), thus, we have the following equations.

$$\lambda_2 = \gamma_{12}\mu_1 + 0\mu_2 + 0\mu_3 + \gamma_{42}\mu_4 + \gamma_{52}\mu_5 \tag{3.21}$$

$$\lambda_3 = \gamma_{13}\mu_1 + 0\mu_2 + 0\mu_3 + \gamma_{43}\mu_4 + \gamma_{53}\mu_5 \tag{3.22}$$

$$\lambda_4 = \gamma_{14}\mu_1 + \gamma_{24}\mu_2 + \gamma_{34}\mu_3 + 0\mu_4 + 0\mu_5 \tag{3.23}$$

$$\lambda_5 = \gamma_{15}\mu_1 + \gamma_{25}\mu_2 + 0\mu_3 + \gamma_{45}\mu_4 + 0\mu_5 \tag{3.24}$$

Model equation (3.21-3.24) can be transform to matrix as in equation (3.25)

$$\begin{bmatrix} \gamma_{12} & 0 & 0 & \gamma_{42} & \gamma_{52} \\ \gamma_{13} & 0 & 0 & \gamma_{43} & \gamma_{53} \\ \gamma_{14} & \gamma_{24} & \gamma_{34} & 0 & 0 \\ \gamma_{15} & \gamma_{25} & 0 & \gamma_{45} & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \tag{3.25}$$

Where:

The arrival rate $\lambda_i = \frac{1}{mean\ number\ of\ arrival}$, for i = 1,2,...5 (3.26)

The departure rate $\mu_i = \frac{1}{mean\ number\ of\ departure}$, for i= 1,2,…5 (3.27)

$$\rho = \frac{\lambda_i}{\mu_i}, \quad for\ i = 1, 2, \cdots 5. \tag{3.28}$$

The expected number in the queue is given as

$$l_q = \frac{\rho}{m-\rho}, \tag{3.29}$$

Where m stands for the number of servers at the node. The expected waiting time in the queue is given as:

$$w_{qi} = \frac{l_q}{\lambda_i}, \tag{3.30}$$

(Owoloko *et al.*, 2015)

The expected number of customers in the system is given as

$$l_s = l_q + \rho \tag{3.31}$$

Finally, the expected waiting time in the system for node 1-5 is given as

$$w_i = \frac{l_s}{\lambda_i}, \text{ For } i = 1,2,\ldots 5. \tag{3.32}$$

(Owoloko *et al.*, 2015)

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Analysis of Results

We begin this section with computation of data collected for all the nodes in the network system, so as to minimize the waiting time, the results of the computation is presented in Tables 4.1 to 4.7

Table 4.1: Data for Node1 (Meter Greeter Unit)

Days	Week 1		Week 2		Week 3		Week 4		Total	
	No of Arrivall	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure
Monday	104	114	96	96	76	66	82	75	**358**	**351**
Tuesday	111	80	95	90	168	98	98	62	**472**	**330**
Wednesday	120	120	114	114	112	106	95	74	**441**	**414**
Thursday	125	90	90	86	158	95	100	78	**473**	**349**
Friday	96	96	90	90	260	138	90	90	**536**	**414**

Table 4.2: Data for Node2 (Customer Service Unit)

Days	Week 1		Week 2		Week 3		Week 4		Total	
	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure
Monday	98	75	76	74	64	60	92	84	**330**	**293**
Tuesday	112	96	89	76	120	100	89	80	**410**	**352**
Wednesday	100	96	92	89	100	87	70	60	**362**	**332**
Thursday	72	89	68	60	268	144	80	70	**488**	**363**
Friday	108	90	92	76	180	176	84	76	**464**	**418**

Table 4.3: Data for Node3 (Marketing Unit)

Days	Week 1		Week 2		Week 3		Week 4		Total	
	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure
Monday	135	40	95	96	89	78	60	67	**379**	**281**
Tuesday	94	80	70	45	85	65	70	56	**319**	**246**
Wednesday	95	89	87	85	80	68	75	55	**337**	**297**
Thursday	90	78	60	40	95	55	55	55	**300**	**228**
Friday	85	75	70	70	69	67	80	70	**304**	**282**

Table 4.4: Data for Node4 (Tellers Unit)

Days	Week 1		Week 2		Week 3		Week 4		Total	
	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure
Monday	125	70	95	95	95	75	75	70	**390**	**310**
Tuesday	94	60	85	70	125	95	80	75	**384**	**300**
Wednesday	100	65	95	85	115	90	95	85	**405**	**325**
Thursday	99	40	110	75	112	116	130	85	**451**	**316**
Friday	85	89	90	65	180	90	93	75	**448**	**319**

Table 4.5: Data for Node5 (Customers Service Manager Unit)

Days	Week 1		Week 2		Week 3		Week 4		Total	
	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure
Monday	75	60	60	70	75	65	80	76	**290**	**271**
Tuesday	95	86	80	70	70	80	75	60	**320**	**296**
Wednesday	90	70	75	70	80	70	65	80	**310**	**290**
Thursday	85	70	85	80	78	70	95	85	**343**	**305**
Friday	95	75	89	75	74	69	90	75	**348**	**294**

Table 4.6: Data analysis for the accumulation of the nodes

Days	Node 1		Node 2		Node 3		Node 4		Node 5	
	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure	No of Arrival	No of Departure
Monday	358	351	330	293	379	281	390	310	290	271
Tuesday	472	330	410	352	319	246	384	300	320	296
Wednesday	441	414	362	332	337	297	405	325	310	290
Thursday	473	349	488	363	300	228	451	316	343	305
Friday	536	414	464	418	304	282	448	319	348	294
Total	**2280**	**1858**	**2054**	**1758**	**1639**	**1334**	**2078**	**1570**	**1611**	**1456**

Table 4.7: Data analysis for the Mean arrival as well as Mean departure

	Node 1	Node 2	Node 3	Node 4	Node 5
Mean arrival	1.920	1.712	1.366	1.732	1.343
Mean departure	1.548	1.465	1.112	1.308	1.213

From Table 4.7 above, we have obtained the Mean arrival and Mean departure for each node, the expected number of customer in the system, the expected number of customer in the queue and the expected waiting time in the system.

For node1, the arrival rate: $\lambda_1 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.920} = 0.52$ person per minute.

For node2, the arrival rate: $\lambda_2 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.712} = 0.58$ person per minute.

For node3, the arrival rate: $\lambda_3 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.366} = 0.73$ person per minute.

For node4, the arrival rate: $\lambda_4 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.732} = 0.58$ person per minute.

For node5, the arrival rate: $\lambda_2 = \frac{1}{mean\ number\ of\ arrival} = \frac{1}{1.343} = 0.74$ person per minute.

The departure rate for nodes is given as:

$$\mu_1 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.548} = 0.646$$

$$\mu_2 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.465} = 0.683$$

$$\mu_3 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.112} = 0.899$$

$$\mu_4 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.308} = 0.765$$

$$\mu_5 = \frac{1}{mean\ number\ of\ departure} = \frac{1}{1.213} = 0.824$$

Node 1:

$$\rho = \frac{\lambda_1}{\mu_1} = \frac{0.52}{0.646} = 0.8$$

The expected number of customer in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{1-0.8} = 4.0$$

Where m stands for the number of servers at the node1 (Meter Greeter Unit)

The expected waiting time in the queue is given as

$$w_{q1} = \frac{l_q}{\lambda_1} = \frac{4.0}{0.52} = 7.69 \text{ Minutes}$$

The expected number of customers in the system is given as

$l_s = l_q + \rho = 4.0 + 0.8 = 4.8$

The expected waiting time in the system for node1 is given as

$$w_1 = \frac{l_s}{\lambda_1} = \frac{4.8}{0.52} = 9.2 \text{ Minutes}$$

Node 2:

$$\rho = \frac{\lambda_2}{\mu_2} = \frac{0.58}{0.683} = 0.8$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{2-0.8} = 0.7$$

Where m stands for the number of servers at the node2 (Customer Service Unit)

The expected waiting time in the queue is given as

$$w_{q2} = \frac{l_q}{\lambda_2} = \frac{0.7}{0.58} = 1.2 \text{ Minutes}$$

The expected number of customers in the system is given as

$l_s = l_q + \rho = 0.7 + 0.8 = 1.5$

The expected waiting time in the system for node 2 is given as

$$w_2 = \frac{l_s}{\lambda_2} = \frac{1.5}{0.58} = 26 \text{ Minutes}$$

Node 3:

$$\rho = \frac{\lambda_3}{\mu_3} = \frac{0.73}{0.899} = 0.8$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.8}{3-0.8} = 0.4$$

Where m stands for the number of servers at the node3 (Marketing Unit)

The expected waiting time in the queue is given as

$$w_{q3} = \frac{l_q}{\lambda_3} = \frac{0.4}{0.73} = 0.54 \text{ Minute}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 0.4 + 0.8 = 1.2$$

The expected waiting time in the system for node3 is given as

$$w_3 = \frac{l_s}{\lambda_3} = \frac{1.2}{0.73} = 1.6 Minutes$$

Node 4:

$$\rho = \frac{\lambda_4}{\mu_4} = \frac{0.58}{0.765} = 0.8$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m - \rho} = \frac{0.8}{4 - 0.8} = 0.3$$

Where m stands for the number of servers at the node4 (Tellers Unit)

The expected waiting time in the queue is given as

$$w_{q4} = \frac{l_q}{\lambda_4} = \frac{0.3}{0.58} = 0.52 \text{ Minutes}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 0.3 + 0.8 = 1.1$$

The expected waiting time in the system for node 4 is given as

$$w_4 = \frac{l_s}{\lambda_4} = \frac{1.1}{0.58} = 1.9 \text{ Minutes}$$

Node 5:

$$\rho = \frac{\lambda_5}{\mu_5} = \frac{0.74}{0.824} = 0.9$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.9}{1-0.9} = 9.0$$

Where m stands for the number of servers at the node5 (Customers Service Manager Unit).

The expected waiting time in the queue is given as

$$w_{q5} = \frac{l_q}{\lambda_5} = \frac{0.3}{0.74} = 0.41 \text{ Minutes}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 9.0 + 0.9 = 9.9$$

The expected waiting time in the system for node5 is given as

$$w_5 = \frac{l_s}{\lambda_5} = \frac{9.9}{0.74} = 13.4 \, Minutes$$

However, the total expected waiting time in the system before modification is:

$$W_t = w_1 + w_2 + w_3 + w_4 + w_5 = 9.2 + 26 + 1.6 + 1.9 + 13.4 = 52.1 \cong 52 \min utes$$

Recall that from table 4:9**,** all the departure rates (μ's) and arrival rates (λ's) for each node in the system have been calculated, therefore by putting these values in equation (3.20) we have;

$$
\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} 0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0 & 0 & 0 & 0.82 & 0 & 0 \\ 0 & 0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0 & 0 & 0 & 0.82 & 0 \\ 0 & 0 & 0.65 & 0 & 0.68 & 0 & 0 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0 & 0.68 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0 & 0 & 0 & 0 \\ 0.65 & 0.65 & 0.65 & 0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.68 & 0.68 & 0.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0.77 & 0.77 & 0.77 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.82 & 0.82 & 0.82 \end{bmatrix}^{+} \begin{bmatrix} 1.00 \\ 0.56 \\ 0.58 \\ 0.62 \\ 0.55 \\ 1.00 \\ 0.79 \\ 0.67 \\ 0.65 \end{bmatrix} \quad (4.1)
$$

The coefficient of the unknown in equation (4.1) is a rectangular matrix, to find the inverse of this Matrix; we use Moore-Penrose Generalized Inverse also called Moore-Penrose Inverse built-in function in maple 17 software. Hence, we obtain equation (4.2)

$$
\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} 0.3529 & -0.0322 & -0.1548 & -0.1134 & 0.3715 & 0.0894 & -0.0774 & -0.0518 & -0.1069 \\ -0.0322 & 0.3529 & -0.1548 & -0.1134 & 0.3715 & 0.0894 & 0.0774 & -0.0518 & -0.1069 \\ -0.2222 & -0.2222 & 0.3901 & -0.1626 & 0.4388 & -0.0758 & -0.195 & 0.1517 & 0.1481 \\ -0.0986 & -0.0986 & -0.0805 & 0.3893 & 0.3567 & -0.103 & 0.0402 & -0.048 & 0.0657 \\ 0.0285 & 0.0285 & 0.4256 & -0.095 & -0.0969 & 0.38 & -0.2128 & 0.0095 & -0.019 \\ 0.1577 & 0.1577 & -0.0666 & 0.4824 & -0.1828 & 0.3516 & 0.0333 & -0.1995 & -0.1051 \\ -0.1862 & -0.1862 & -0.359 & -0.3874 & 0.2797 & 0.739 & 0.1795 & 0.19 & 0.1241 \\ 0.1405 & 0.1405 & 0.5135 & 0.1914 & -0.2465 & -0.235 & 0.305 & -0.1181 & -0.0937 \\ -0.1405 & -0.1405 & -0.5135 & -0.1914 & 0.2465 & 0.235 & 0.8185 & 0.1181 & 0.0937 \\ 0.4789 & 0.0227 & 0.0388 & -0.0515 & -0.1222 & 0.0043 & -0.0194 & 0.2122 & -0.1672 \\ 0.0227 & 0.4789 & 0.0388 & -0.0515 & -0.1222 & 0.0043 & -0.0194 & 0.2122 & -0.1672 \\ -0.056 & -0.056 & 0.1268 & 0.544 & -0.1397 & -0.2236 & -0.0634 & 0.2167 & 0.0374 \\ -0.4455 & -0.4455 & -0.2043 & -0.441 & 0.3841 & 0.2151 & 0.1022 & 0.6577 & 0.297 \\ 0.4901 & 0.0042 & 0.0863 & 0.1382 & -0.1797 & -0.0748 & -0.0431 & -0.1581 & 0.2417 \\ 0.0042 & 0.4901 & 0.0863 & 0.1382 & -0.1797 & -o.o748 & -0.0431 & -0.1581 & 0.2417 \\ -0.4943 & -0.4943 & -0.1726 & -0.2765 & 0.3594 & 0.1497 & 0.0863 & 0.3163 & 0.736 \end{bmatrix} \begin{bmatrix} 0.58 \\ 0.73 \\ 0.58 \\ 0.74 \\ 0.65 \\ 0.68 \\ 0.89 \\ 0.77 \\ 0.82 \end{bmatrix} \quad (4.2)
$$

Also solving equation (4.2) using maple 17 software, we obtain equation (4.3) below

$$\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} 0.251085 \\ 0.308850 \\ 0.113229 \\ 0.326733 \\ -0.211641 \\ 0.435043 \\ 0.353324 \\ 0.407175 \\ 0.592740 \\ 0.211245 \\ 0.279675 \\ 0.300992 \\ 0.208421 \\ 0.310075 \\ 0.382960 \\ 0.307033 \end{bmatrix} \tag{4.3}$$

To normalize the values of the probabilities in equation (4.3), since we do not have negative probabilities, we take the absolute value of the estimates and rescale them so that the sum of each node sums up to 1. Thus we have

$$\begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{2out} \\ \gamma_{34} \\ \gamma_{3out} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{45} \\ \gamma_{4out} \\ \gamma_{52} \\ \gamma_{53} \\ \gamma_{5out} \end{bmatrix} = \begin{bmatrix} 0.251111 \\ 0.308882 \\ 0.113241 \\ 0.326767 \\ 0.211639 \\ 0.435040 \\ 0.353321 \\ 0.407210 \\ 0.592790 \\ 0.211175 \\ 0.279582 \\ 0.300892 \\ 0.208352 \\ 0.310054 \\ 0.382934 \\ 0.307012 \end{bmatrix} \tag{4.4}$$

From equation (4.4), the following deductions are made:

- At node1 (Meter Greeter Unit), the weights are $\gamma_{12}, \gamma_{13}, \gamma_{14}$ and γ_{15} with the values 0.251111, 0.308882, 0.113241 and 0.326767 respectively, shows that there is a high probability of a customer leaving node1 (Meter Greeter Unit) to join the queue for service at node5 (Customer Service Manager Unit) than any other node. The least probability is that a customer leaves node1 to node4 (Tellers Unit).

- At node2 (Customer Service Unit), the weights are γ_{24}, γ_{25} *and* γ_{2out} with values 0.211639, 0.435040 and 0.353321 respectively, shows that there is a high probability that a customer leaves node2 (Customer Service Unit) and goes directly to node5 (Customer Service Manager Unit).

- At node3 (Marketing Unit), the weights are γ_{34} *and* γ_{3out} with the values 0.407210 and 0.592790 respectively, which shows that there is a high probability that a customer leaves node3 (Marketing Unit) moves out of the system. The least probability is that a customer leaves node3 (Marketing Unit) to join the service at the node4 (Tellers Unit).

- At node4 (Tellers Unit), the weights are $\gamma_{42}, \gamma_{43}, \gamma_{45}$, *and* γ_{4out} with the values 0.211175, 0.279582, 0.300892 and 0.208352 respectively, which shows that, there is a high probability that a customer leaves node4 (Tellers Unit) to join the service at the node5 (Customers Service Manager Unit).

- At node5 (Customers Service Manager Unit) the weights are γ_{52}, γ_{53} *and* γ_{5out} with the values 0.310054, 0.382934 and 0.307012 respectively, which shows that there is a high probability that a customer leaves node5 (Customer Service Manager Unit) to join the queue for service either at node3 (Marketing Unit). The least probability here is that a customer leaves node5 (Customer Service Manager Unit) and goes out of the system.

4.2 Solution for New Departure Rates

We Recall equation (3.25), as presented below

$$\begin{bmatrix} \gamma_{12} & 0 & 0 & \gamma_{42} & \gamma_{52} \\ \gamma_{13} & 0 & 0 & \gamma_{43} & \gamma_{53} \\ \gamma_{14} & \gamma_{24} & \gamma_{34} & 0 & 0 \\ \gamma_{15} & \gamma_{25} & 0 & \gamma_{45} & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \tag{4.5}$$

Substituting $\lambda_i, (i = 2,3,4,5)$, We obtain equation (4.6)

$$\begin{bmatrix} \gamma_{12} & 0 & 0 & \gamma_{42} & \gamma_{52} \\ \gamma_{13} & 0 & 0 & \gamma_{43} & \gamma_{53} \\ \gamma_{14} & \gamma_{24} & \gamma_{34} & 0 & 0 \\ \gamma_{15} & \gamma_{25} & 0 & \gamma_{45} & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.73 \\ 0.58 \\ 0.74 \end{bmatrix} \tag{4.6}$$

After the necessary substitution, the solution to the equation (4.6) becomes equation (4.7).

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 54.732 & 43.8693 & -0.1849 & 0.7306 \\ 6.2475 & -5.9549 & 0.2033 & 1.4693 \\ 17.8841 & 14.8199 & 2.3845 & -0.9486 \\ 69.3682 & 56.9901 & -0.0948 & 0.3747 \\ 6.0785 & -3.2277 & 0.2133 & -0.8430 \end{bmatrix} \begin{bmatrix} 0.58 \\ 0.73 \\ 0.58 \\ 0.74 \end{bmatrix} \tag{4.7}$$

Equation (4.7) becomes equation (4.8).

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 16.023 \\ 18.014 \\ 18.022 \\ 22.812 \\ 7.4242 \end{bmatrix} \tag{4.8}$$

Therefore, to serve Customer with the space interval of 5 minutes, we divide the values in equation (4.8) by 5. The recommended number of servers therefore for each node is given in equation (4.9)

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 3.2046 \\ 3.6028 \\ 3.6044 \\ 4.5624 \\ 1.4848 \end{bmatrix} \cong \begin{bmatrix} 3 \\ 4 \\ 4 \\ 5 \\ 1 \end{bmatrix} \tag{4.9}$$

The result in equation (4.9) shows that for the Bank to work optimally and to serve customers within the average time frame of 5 minutes, node1 will require a total of three servers, node2 will require a total of four servers, node3 will require a total of four servers, node4 will require a total of five servers, node five will require a total of one server. With the new estimated number of servers at each node given by equation (4.9). The arrival rates for each node are assumed to remain the same since we do not have control over it. A new departure rates and the expected waiting time for each node is estimated as follow.

The new departure rates for nodes are as follows:

$$\mu_1 = recommended\ departure\ rate\ per\ 5\min. = \tfrac{3}{5} = 0.6$$

$$\mu_2 = recommended\ departure\ rate\ per\ 5\min. = \tfrac{4}{5} = 0.8$$

$\mu_3 = recommended\ departure\ rate\ per\ 5\min. = \frac{4}{5} = 0.8$

$\mu_4 = recommended\ departure\ rate\ per\ 5\min. = \frac{5}{5} = 1.0$

$\mu_5 = recommended\ departure\ rate\ per\ 5\min. = \frac{1}{5} = 0.2$

4.3 Finding the New Expected Waiting Time in the System

Node 1:

$\rho = \frac{\lambda_1}{\mu_1} = \frac{0.52}{0.6} = 0.9$

The expected number of customers in the queue is given as

$l_q = \frac{\rho}{m-\rho} = \frac{0.9}{3-0.9} = 0.4$

Where m stands for the number of servers at the node1 (Meter Greeter Unit)

The expected waiting time in the queue is given as

$w_{q1} = \frac{l_q}{\lambda_1} = \frac{0.4}{0.52} = 0.8$ Minutes

The expected number of customers in the system is given as

$l_s = l_q + \rho = 0.4 + 0.9 = 1.3$

The expected waiting time in the system for node1 is given as

$w_1 = \frac{l_s}{\lambda_1} = \frac{1.3}{0.52} = 2.5$ Minutes

Node 2:

$\rho = \frac{\lambda_2}{\mu_2} = \frac{0.58}{0.8} = 0.7$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.7}{4-0.7} = 0.2$$

Where m stands for the number of servers at the node2 (Customer Service Unit)

The expected waiting time in the queue is given as

$$w_{q2} = \frac{l_q}{\lambda_2} = \frac{0.2}{0.58} = 0.34 \text{ Minutes}$$

The expected number of customers in the system is given as

The expected waiting time in the system for node2 is given as

$$w_2 = \frac{l_s}{\lambda_2} = \frac{0.9}{0.58} = 1.6 \text{ Minutes}$$

Node 3:

$$\rho = \frac{\lambda_3}{\mu_3} = \frac{0.73}{0.8} = 0.9$$

The expected number of in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.9}{4-0.9} = 0.3$$

Where m stands for the number of servers at the node3 (Marketing Unit)

The expected waiting time in the queue is given as

$$w_{q3} = \frac{l_q}{\lambda_3} = \frac{0.3}{0.73} = 0.41 \text{Minutes}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 0.2 + 0.9 = 1.1$$

The expected waiting time in the system for node3 is given as

$$w_3 = \frac{l_s}{\lambda_3} = \frac{1.1}{0.73} = 1.5 \text{ Minutes}$$

Node 4:

$$\rho = \frac{\lambda_4}{\mu_4} = \frac{0.58}{1.0} = 0.6$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.6}{5-0.6} = 0.1$$

Where m stands for the number of servers at the node4 (Tellers Unit)

The expected waiting time in the queue is given as

$$w_{q4} = \frac{l_q}{\lambda_4} = \frac{0.1}{0.58} = 0.17 \text{ Minutes}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 0.1 + 0.6 = 0.7$$

The expected waiting time in the system for node1 is given as

$$w_4 = \frac{l_s}{\lambda_4} = \frac{0.7}{0.58} = 1.2 \text{ Minutes}$$

Node 5:

$$\rho = \frac{\lambda_5}{\mu_5} = \frac{0.74}{1.0} = 0.7$$

The expected number of customers in the queue is given as

$$l_q = \frac{\rho}{m-\rho} = \frac{0.7}{1-0.7} = 2.3$$

Where m stands for the number of servers at the node5 (Customer Service Manager Unit)

The expected waiting time in the queue is given as

$$w_{q5} = \frac{l_q}{\lambda_5} = \frac{2.3}{0.74} = 3.10 \text{ Minutes}$$

The expected number of customers in the system is given as

$$l_s = l_q + \rho = 2.3 + 0.7 = 3.0$$

The expected waiting time in the system for node1 is given as

$$w_5 = \frac{l_s}{\lambda_5} = \frac{3.0}{0.74} = 4.1 \text{Minutes}$$

However, the total expected waiting time in the system after modification is

$$W_t = w_1 + w_2 + w_3 + w_4 + w_5 = 2.5 + 1.6 + 1.5 + 1.2 + 4.1 = 10.9 \cong 11 \, minutes$$

4.4 Discussion of Results

The summary of the computed performance measure for determination of optimal number of servers at network queuing nodes to reduce waiting time at the First City Monument Bank, Minna (**(FCMB)**, is given in Table 4.8

Table 4.8: Showing all the results obtained before modification.

Nodes i	Number of Servers (m_i)	Probabilities (α_{ij})	ρ_i	Lq	Ls	W_q	W_S
1	1	0.2511111. 0.308882 0.113241 0.326767	0.8	4.0	4.8	7.69	9.2
2	2	0.211639. 0.435040. 0.353321	0.8	0.7	1.5	1.20	26
3	3	0.407210. 0.592790.	0.8	0.4	1.6	0.54	1.6
4	4	0.211175. 0.279582. 0.300892 0.208352	0.8	0.3	1.1	0.52	1.9
5	1	0.310054. 0.382934. 0.307012	0.9	9.0	9.9	0.41	13.4
Total	**11**		**4.1**	**14.4**	**18.9**	**10.36**	**52.1**

Table 4.9: Showing all the results obtained after modification

Nodes i	Number of Servers (m_i)	ρ_i	Lq	Ls	W_q	W_S
1	3	0.9	0.4	1.3	0.8	2.5
2	4	0.7	0.2	0.9	0.34	1.6
3	4	0.9	0.3	1.1	0.41	1.5
4	5	0.6	0.1	0.7	0.17	1.2
5	1	0.7	2.3	3.0	3.10	4.1
Total	**17**	**3.8**	**3.3**	**7**	**4.8**	**10.9**

Table 4.10: Showing the comparison between current number of servers and optimal number of servers obtained

Nodes1	Current number of servers	Optimal number of servers obtained
1	1	3
2	2	4
3	3	4
4	4	5
5	1	1
Total	**11**	**17**

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

The FCMB network queuing system has been investigated and studied effectively. The study has determined optimal number of servers at the nodes of the network queuing system to reduce waiting time of the customers. The result from this study is important information to the management of FCMB for proper planning and efficient service delivery. The analysis has shown that the arrival rate which results are: Node1=0.52 person per unit, Node2=0.58 person per unit, Node3=0.73 person per unit, Node4=0.58 person per unit, Node5=0.74 person per unit and departure rate which results are: Node1=0.646, Node2=0.683, Node3=0.899, Node4=0.765, Node5=0.824, as well as probabilities at each node (unit) were obtained. The current number of servers at the Node1, Node2, Node3, Node4 and Node5 are 1, 2, 3, 4 and 1 which sum up to 11 servers and the Optimal number of servers obtained after modification at the Node1, Node2, Node3, Node4 and Node5 are 3, 4, 4, 5 and 1 which sum up to 17 servers. The total expected waiting time of the customers in the system before modification is about 52 minutes while the total expected waiting time of the customers in the system after modification is about 11 minutes. This demonstrated that the optimal number of servers at the nodes of FCMB network queuing system is achieved.

5.2 Recommendations

To empower the FCMB consistently in order to meet up with its high standard of giving satisfactory services to her customers, the management of FCMB is advised to implement the recommendations given below:

(1) The number of servers at the node1 (Meter Greeter Unit) ought to be Three (3) servers.

(2) The number of servers at the node2 (Customers Service Unit) ought to be four (4) servers.

(3) The number of servers at the node3 (Marketing Unit) ought to be Four (4) servers at each time for legitimate Proficiency.

(4) The number of servers at the node4 (Tellers Unit) also, ought to be five (5) servers for legitimate Service delivery.

(5) The number of servers at the node5 (Customers Service Manager Unit) similarly, ought to be One (1) server.

5.3 Contribution to Knowledge

The research is able to determine the current number of servers before modification to be 11 servers while the optimal number of servers obtained after modification to be 17 servers. The total expected waiting time of the customers in the system before modification is about 52 minutes while the total expected waiting time of the customers in the system after modification is about 11 minutes. This shows that the total expected waiting time of the customers before and after modification, the optimal number of servers at each node of the First City Monument Bank (FCMB) Minna network queuing system is achieved to guarantee optimal service delivery.

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