MATHEMATICAL MODELLING OF HEPATITIS B VIRUS WITH INFECTIOUS LATENT

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILLMENT OF THE REQIUREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF TECHNOLOGY (MTech) IN MATHEMATICS

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ABSTRACT

In this research, a mathematical model for the transmission dynamics of Hepatitis B virus (HBV) incorporating treatment, using condom and vaccine as the control parameters, incorporating vaccinated compartment was formulated. It was assumed that a susceptible individual can get infected with HBV when there is an effective interaction with any of the three infectious classes: exposed, chronic or acute individuals. The basic reproduction number was obtained using the next generation matrix approach. The Jacobian stability technique and the Lyaponuv second method of stability were used to establish the local and global stabilities of the equilibrium states respectively. The stability analysis shows that HBV can be eradicated from the entire population when $R_0 \leq 1$but will continue to persevere within the population when $R_0 > 1$. The model was solved analytically using the homotopy perturbation method (HPM), and the stability analysis was verified with graphs using maple 18. The result shows that vaccination have a significant impact on all the compartments, but treatment only have effect on the infected compartments. It is therefore recommended that every susceptible individual to HBV should get vaccinated, and those who are acutely and chronically infected should receive early medical attention.

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Glossary of Symbols

Variable/Parameter	Desription
S	Susceptible
V	Vaccinated
E	Exposed
C	Chronic
A	Acute
R	Recovered
Λ	Recruitment rate
ω_1	Waning of vaccine
ω_2	Waning of immunity
ε	Efficcacy of vaccine
ψ	Compliance to usage of vaccine
ε_c	Efficcacy of condom
ψ_c	Compliance to usage of condom
μ	Natural mortality
c_1	Progression from exposed to chronic
c_2	Progression from exposed to acute
γ_1	Treatment rate of chronic
γ_2	Treatment rate of acute
δ	Disease induced death

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background of Study

Hepatitis B (HB) is an infectious disease initiated by the hepatitis B virus (HBV) that upsets the liver. It can result in both severe and prolonged infections. Many people have no symptoms during the early contagion. Some develop a speedy onset of sickness with vomiting, yellowish skin, tiredness, dark urine and abdominal pain. Frequently these indications last a few weeks and rarely does the primary infection result in death (Rapheal & David, 2008). It may take 30 to 180 days for symptoms to begin WHO (2014). In those who get infected around the time of birth 90% develop lingering hepatitis B while less than 10% of those infected after the age of five do. According to U.S. Centers for Disease Control and Prevention in 2011, most of those with chronic disease have no symptoms; however, cirrhosis and liver cancer may eventually develop. Cirrhosis or liver cancer occur in about 25% of those with chronic disease.

About a third of the world population has been infected at one point in their lives, including 343 million who have chronic infections (Schilsky, 2013). Another 129 million new infections occurred in 2013. Over 750,000 people die of hepatitis B each year. About 300,000 of these are due to liver cancer. The disease is now only common in East Asia and sub-Saharan Africa where between 5 and 10% of adults are chronically infected. Rates in Europe and North America are less than 1%. It was originally known as "serum hepatitis". Research is looking to create foods that contain HBV vaccine. The disease may affect other great apes as well.

The virus is transmitted by exposure to infectious blood or body fluids. Infection around the time of birth or from contact with other people's blood during childhood is the most

frequent method by which hepatitis B is acquired in areas where the disease is common. In areas where the disease is rare, intravenous drug use and sexual intercourse are the most frequent routes of infection. Other risk factors include working in healthcare, blood transfusions, dialysis, living with an infected person, travel in countries where the infection rate is high, and living in an institution (Centers for Disease Control and Prevention 2011). Tattooing and acupuncture led to a significant number of cases in the 1980s; however, this has become less common with improved sterilization (Boyce, 2016). The hepatitis B viruses cannot be spread by holding hands, sharing eating utensils, kissing, hugging, coughing, sneezing, or breastfeeding. The infection can be diagnosed 30 to 60 days after exposure. The diagnosis is usually confirmed by testing the blood for parts of the virus and for antibodies against the virus. It is one of five main hepatitis viruses.

In areas of high endemicity where at least 8% of the population are chronic HBV carriers, HBV is mainly contracted at birth and early childhood. Perinatal transmission from an infected mother to her baby is common. About 90% of those infected during the prenatal period, 30% of those infected in early childhood, and 6% of those infected after 5 years of age develop chronic infection. Transmission of HBV among adults occurs via contact with infected blood and body fluids such as semen, vaginal fluids, and saliva. Therefore transfusion of unscreened blood and its products, sexual activities, use of contaminated or inadequately sterilized instruments, sharing of sharp objects as could occur during some traditional or cultural practices, for example, local circumcision, are common means of spread. It could also occur by other means of iatrogenic or horizontal transmission such as long-term household contacts with no sexual involvements in regions of high endemicity. HBV infection is also recognized as an occupational health hazard for health-care practitioners (Bhattarai *et al*., 2014).

The infection has been preventable by vaccination since 1982. Pungpapong and Poterucha (2007) Vaccination is recommended by the World Health Organization in the first day of life if possible. Two or three more doses are required at a later time for full effect. This vaccine works about 95% of the time. About 180 countries gave the vaccine as part of national programs as of 2006 (Williams, 2006). It is also recommended that all blood be tested for hepatitis B before transfusion, and that condoms be used to prevent infection. During an initial infection, care is based on the symptoms that a person has. In those who develop chronic disease, antiviral medication such as tenofovir or interferon may be useful; however, these drugs are expensive. Liver transplantation is sometimes used for cirrhosis.

Acute hepatitis B infection does not usually require treatment and most adults clear the infection spontaneously (Hollinger & Lau, 2006). Early antiviral treatment may be required in fewer than 1% of people, whose infection takes a very aggressive course (fulminant hepatitis) or who are immunocompromised. On the other hand, treatment of chronic infection may be necessary to reduce the risk of cirrhosis and liver cancer. Chronically infected individuals with persistently elevated serum alanine aminotransferase, a marker of liver damage, and HBV DNA levels are candidates for therapy (Lai & Yuen 2007) Treatment lasts from six months to a year, depending on medication and genotype (Caporaso 2011). Treatment duration when medication is taken by mouth, however, is more variable and usually longer than one year.

Although none of the available medications can clear the infection, they can stop the virus from replicating, thus minimizing liver damage. As of 2018, there are eight medications licensed for the treatment of hepatitis B infection in the United States. These include antiviral medications lamivudine, adefovir, tenofovir disoproxil, tenofovira lafenamide, telbivudine, and entecavir, and the two immune

system modulators interferon alpha-2a and PEGylated interferon alpha-2a. In 2015 the World Health Organization recommended tenofovir or entecavir as first-line agents. Those with current cirrhosis are in most need of treatment.

Mathematical modelling of virus-related infections has resulted to superior indulgent of virus dynamics and helped in suggesting and curbing the spread of viral diseases such as HIV, Hepatitis B Virus (HBV), Hepatitis C Virus (HCV), and Dengue Fever to a minimal level. Amongst the earliest models of HIV infection branded as the basic model was used by Nowak (2000) and by (Perelson *et al.,* 1996) and was efficacious in mathematically imitating the dynamics of the early phases of HIV and its objective CD4+ cells resulting in an infection occurrence. Current works have concentrated on HIV viral and cellular infections integrating dynamics such as intracellular delays, latent infection and viral mutation, and spatial heterogeneity (Chen *et.al.*, 2016) For example, Pourbashash *et al.* (2015) investigated the global stability of within host virus models with cell-to-cell viral transmission and achieved a comprehensive analytic explanation of equilibria.

In this work, we suggest a transmission dynamics of HBV infection model, incorporating treatments of exposed class and vaccination of the susceptible class the two major control strategies. A mathematical analysis of the special effects of treatment on the infection dynamics of HBV incorporating effect of drugs sensitive strain is carried out.

1.2 Statement of the Research Problem

Even though there are various research and the handiness of effective HBV awareness programs and vaccines the mortality rate of the malady continue to increase drastically all around the world. HBV preventive measures has been made known to the society but due to ignorance or carelessness by some people the disease continues to spread.

HBV has persisted as a major peril to human race worldwide resulting in death of millions every year across the globe inspite of various research works and availability of

some safe and effective vaccines and therapies even at an affordable price as means of fighting the virus. The need to model a good solution to this life threating virus and control measure which will assist in the controlling of this menace has remained a key issue in this research.

Hence, this research formulated a mathematical model for Hepatitis B virus (HBV) transmission incorporating treatment as a means of combating the menace, we also put into consideration the individuals who are treatment sensitive.

1.3 Aim and Objectives of the Study

The aim of this research is to formulate a model for the spread and control of HBV incorporating condom, vaccine and treatment as control parameters.

The objectives of the study are to:

i. Obtain the disease free and endemic equilibrium points of the model
ii. compute the effective reproduction number of the model
iii. analyze the conditions for local and global stability of the equilibrium (DFE) state
iv. analyze the conditions for local and global stability of the equilibrium (EE) state
v. Obtain the analytical solution of the model using homotopy perturbation method
vi. Carryout the model simulation using maple and present graphical profiles of the system responses

1.4 Motivation of the study

The author is motivated owing to the distinctive transmission dynamics of HBV and HBV being one of the most devastating disease globally, which cannot be eradicated completely but can be curbed to bearable minimum.

1.5 Scope and Limitation of Study

The scope and limitations of the study include:

1. The model uses systems of ordinary differential equations
2. The study considered only sexual means of transmission
3. It also consider drugs sensitive individuals
4. The entire populations were splitted into six sub-populations.

1.6 Justification of study

HBV is on the rise though intervention from government and non-government agencies had been made. Although much work have been made, further studies to quantify and understand disease dynamics will help in the prevention and control of emerging infectious diseases.

1.7 Definition of Terms

Epidemiology: This deals with outlines, basis and the effects of healthy and unhealthy conditions in a well known population

Pandemic: This refers to a stage of any infectious disease (which is endemic) that has infected a large population of humans.

Mathematical modelling: Is a system of description of a system using mathematicl concepts and languages. They are used in natural sciences , physics, biology, earth sciences **Susceptibility:** likely or liable to be influenced or harmed by a particular thing , the dynamic state of being more likely or liable to be harmed by a health determinant.

Hepatitis B (HB) is an infectious disease initiated by the hepatitis B virus (HBV) that upsets the liver

Simulation is representation of the behaviour or characteristics of one system through the use of another system

Endemic is when an infection in a population is maintained in the population without the need for external inputs

Primary infection stage is an infection period that is sudden, onset, brief, intense, short term; sometimes used to mean severe.

Chronic infection is an infection lasting a long time, often of low intensity.

Epidemiology is the study of occurrence, spread or distribution and control of diseases, viruses, concepts etc. throughout populations or systems.

Equilibrium stability. An equilibrium is said to stable if the system always returns to it after small disturbances, otherwise the equilibrium is unstable.

Mathematical model is a representation of a system, process, or relationship in mathematical form, in which equations are used to simulate the behaviour of the system or process under study.

Mortality is the state of being alive, or susceptible to death. The measure of the number of deaths in a given population is mortality rate.

Recovery is an act or process of returning to a normal state of health mind or strength

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CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Overview of HBV

Hepatitis B (HB) is an infectious disease initiated by the hepatitis B virus (HBV) that upsets the liver. Countless individuals have no indications during the early infection. Some develop a speedy onset of sickness with vomiting, yellowish skin, tiredness, dark urine and abdominal pain. Frequently these indications last a few weeks and rarely does the primary infection result in death. These symptoms potency also result from the immune system combating other types of viruses. Nevertheless, people who noticed some of these symptoms and know of any reason they might have been at risk of contracting HBV over the last two months should get tested.

2.2 Mathematical Model of HBV and Control.

Mathematical models have been well-thought-out to judge the effect of public awareness programs, the use of anti-viral treatments and vaccination has provided long-time forecasts vis-à-vis HIV/AIDS prevalence and control in various regions. This destroyer virus still dawdles in developing countries and remains an important global health issue.

Birke and Purnachandra (2019) developed HBV model in which the infected population is classified into two categories viz: chronic and acute and thus developed a seven compartmental SEICIAR model. Also, both vaccination and treatments are included and studied their impact on the spread of hepatitis B virus. The present model is biologically meaningful and mathematically well posed since the solutions are proved to be positive as well as bounded. The basic reproduction number of the model is derived using the next generation matrix method. Further, the equilibrium points of the model are identified and

mathematical analysis pertaining to their stability is conducted using Routh – Hurtiz criteria. It is shown that the disease free equilibrium point is locally and globally stable If $R_0 < 1$. On the other hand, the endemic equilibrium point is proved to be stable if $R_0 > 1$. Also, the numerical simulation study of the model is carried out using ode45 of MATLAB: Rung – Kutta order four. It is observed that, if the vaccination and treatment rates are increased then the infective population size decreases and evenfall to zero over time. Hence, it is concluded that the use of vaccination and treatment at the highest possible rates is essential so as to control the spread hepatitis B virus.

Wiah *et al.,* (2011) simplified the mathematical model of immune responds to Hepatitis B Virus (HBV) infection. The model focuses on the control of the infection by the interferons, the innate and adaptive immunity. The model was compartmentalized as appropriate and the resulting model equations were solved numerically. A mathematical analysis of the model shows that both disease-free and endemic equilibrium point exist and we derive conditions for their stability. We perform sensitivity analysis on the model parameters, to account for the variability and speed of adaptation. Our results show that although each component of innate and adaptive immune response contributes to the recovery of HBV infection, the simulations suggest that, in the absence of one component of innate immunity, the remaining two defense mechanisms are sufficient for viral clearance.

Ruiqing *et al.,* (2019) developed a fractional-order model to describe the transmission of Hepatitis B Virus (HBV). Firstly, the existence and uniqueness of positive solutions are proved. Secondly, the basic reproduction number and the sufcient conditions for the existence of two equilibriums are obtained. Thirdly, the stability of equilibriums are analyzed. Afer that, some numerical simulations are performed to verify the theoretical

prediction. they found out that the value of cure rate is very important for the dynamics of the system. If the value is relatively small, the disease will persist; while the disease will eradicate if the value of is relatively big. In addition, they also found out that the initial values are not sensitive to the dynamical behaviors.

Avner and Nourridine (2018) considered the treatment of chronic HBV by a combination of IFN-α and adefovir, and raise the following question: What should be the optimal ratio between IFN-α and adefovir in order to achieve the best 'efficacy' under constraints on the total amount of the drugs; here the efficacy is measured by the reduction of the levels of inflammation and of fibrosis? We develop a mathematical model of HBV pathogenesis by a system of partial differential equations (PDEs) and use the model to simulate a'synergy map' which addresses the above question.

Kadelka and Ciupe (2019) proposed a mathematical models of within-host interactions which provide insight into hepatitis B e antibody formation, its influence on hepatitis B e antigen seroclearance, and reversion of anergic cytotoxic immune responses. They predict that antibody expansion causes immune activation and hepatitis B e antigen seroclearance. Quantification of the time between antibody expansion and hepatitis B e antigen seroclearance in the presence and absence of treatment shows that potent short-term treatment speeds up the time between antibody expansion and hepatitis B e antigen seroclearance. The monthly hepatocyte turnover during this time can be increased or decreased by treatment depending on the amount of core promoter or precore mutated virus produced.

According to Emerenini and Inyama (2017) Hepatitis B is a potentially life-threatening liver infection caused by the hepatitis B virus (HBV). In their research, the transmission

dynamics of hepatitis B is formulated with a mathematical model with considerations of different classes of individuals, namely immunized, susceptible, latent,infected and recovered class. The role of vaccination of new born babies against hepatitis B and the treatment of both latently and actively infected individuals in controlling the spread are factored into the model. The model in this study is based on the standard SEIR mo del. The disease-free equilibrium state of the model was established and its stability analyzed using the Routh-Hurwitz theorem. The result of the analysis of the stability of the disease-free equilibrium state shows that hepatitis B can totally be eradicated if effort is made to ensure that the sum of the rate of recovery of the latent class, the rate at which latently infected individuals become actively infected and the rate of natural death must have a lower bound.

Liang & Zhuang (2018) in their research on HBV stated that mathematical model of the transmission dynamics of infectious disease is an important theoretical epidemiology method, which has been used to simulate the prevalence of hepatitis B and evaluate different immunization strategies. However, differences lie in the mathematical processes of modeling HBV transmission in published studies, not only in the model structure, but also in the estimation of certain parameters. This review reveals that the dynamics model of HBV transmission only simulates the spread of HBV in the population from the macroscopic point of view and highlights several main shortcomings in the model structure and parameter estimation. First, age-dependence is the most important characteristic in the transmission of HBV, but an age-structure model and related age-dependent parameters were not adopted in some of the compartmental models describing HBV transmission. In addition, the numerical estimation of the force of HBV infection did not give sufficient weight to the age and time factors and is not suitable using the incidence data. Lastly, the current mathematical models did not well reflect the details of

the factors of HBV transmission, such as migration from high or intermediate HBV endemic areas to low endemic areas and the kind of HBV genotype. All of these shortcomings may lead to unreliable results. When the mathematical model closely reflects the fact of hepatitis B spread, the results of the model fit will provide valuable information for controlling the transmission of hepatitis B.

Meltem (2019) noted that Hepatitis B infection is one of the serious viral infections that is threathening the global health. Turkey has an intermediate endemicity for hepatitis B. In their study, a classical SIR model for hepatitis B virus (HBV) transmission is proposed and analyzed. Based on the available data of Republic of Turkey Ministry of Health, associated parameters are estimated and the Ötted model is shown by appropriate simulations. The basic reproductive number is obtained by using the estimated parameters. Finally, we discuss the sensitivity of parameters and the effect of changes of parameters in the spread of disease.

MacLachlan and Cowie (2015) in their research, analyzed Hepatitis B virus to find out the analytical solutions for reducing the HBV infection. They also found the analytical solutions using Homotopy Perturbation Method (HPM) to find out the solution of nonlinear ordinary differential equation systems. SEICR models have been used to control the viral infections. Thus, we concentrated on examining the dynamics of Hepatitis B viral infection and how it must be controlled by vaccination and treatment method. The exactness and effectiveness of two methods has been analyzed by solvable ordinary differential equation systems. We mainly concentrated on steady controls for both vaccination and treatment. Finally, their paper depicts the analytical results which show that optimal combination of vaccination and treatment that will be the most useful way to control Hepatitis B virus infection.

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Khan *et al.* (2013) presented characteristics of HBV virus transmission in the form of a mathematical model. They analyzed the effct of immigrants in the model to study the effct of immigrants for the host population. We added the following flw parameters: “the transmission between migrated and exposed class” and “the transmission between migrated and acute class.” With these new features, we obtained a compartment model of six differential equations. First, they found the basic threshold quantity Ro and then fid the local asymptotic stability of disease free equilibrium and endemic equilibrium. Furthermore, they found the global stability of the disease-free and endemic equilibria. Previous similar publications have not added the kind of information about the numerical results of the model. In our case, from numerical simulation, a detailed discussion of the parameters and their numerical results is presented. We claim that with these assumptions and by adding the migrated class, the model informs policy for governments, to be aware of the immigrants and subject them to tests about the disease status. Immigrants for short

visits and students should be subjected to tests to reduce the number of immigrants with disease.

Titus *et al.* (2018) in their model they studied the dynamics and control of hepatitis B virus (HBV) infection which is a major health problem worldwide by considering condom, vaccination and treatment as control measures. Initially they determined the basic reproduction number R0 for the model and observe that once $R_0 < 1$, the disease free equilibrium will be stable and HBV infection can be controlled using the three control measures and we also study the solution of the endemic equilibrium point of the model. Next they took the sensitivity analysis of the basic reproduction number of HBV infection and obtain that the endemicity of the infection will reduce with the controls. Finally, the numerical simulation result shows that combination of condom, vaccination and treatment is the most effective way to control hepatitis B infection.

McNaughton *et.al.*, (2019) evaluated the current and future role of HBV vaccination and prevention of mother to child transmission (PMTCT) as tools for eliminating the deadly disease. They first investigated the current impact of paediatric vaccination in a cohort of children with and without HIV infection in Kimberley, South Africa. Second, they used these data to inform a new parsimonious model to simulate the ongoing impact of preventive interventions. By applying these two approaches in parallel, they were able to determine both the current impact of interventions, and the future projected outcome of ongoing preventive strategies over time. Their model predicted that, if consistently deployed, combination efforts of vaccination and PMTCT can significantly reduce population prevalence (HBsAg) by 2030, such that a major public health impact is possible even without achieving elimination. However, the prevalence of HBV e-antigen (HBeAg)-positive carriers will decline more slowly, representing a persistent population

reservoir. They showed that HIV co-infection significantly reduces titres of vaccine-mediated antibody, but has a relatively minor role in influencing the projected time to elimination. Their model can also be applied to other settings in order to predict impact and time to elimination based on specific interventions.

Abdulrahman *et al.* (2015) developed mathematical model for the transmission dynamics and control of Human Papillomavirus (HPV) incorporating the impact of vaccination and condom usage. The effective reproduction number was obtained and used to find the best recipe for curbing transmission of the disease. Using Nigerian demographic data, numerical simulations revealed that 20% HPV vaccination coverage of sexually active individuals is better than 75% condom usage on limiting the spread of HPV. Furthermore, it revealed that vaccinating 30% of individuals who are sexually active is a b etter way of curbing the disease than vaccinating 75% of individuals that are not yet sexually active.

Birke and Purnachandra (2019) in their work, modified model and stability analysis of the spread of Hepatitis B Virus Disease classified the infected population into two categories viz: chronic and acute and thus developed a five compartmental SEICIAR model. Also, both vaccination and treatments are included and studied their impact on the spread of hepatitis B virus. The present model is biologically meaningful and mathematically well posed since the solutions are proved to be positive as well as bounded. The basic reproduction number RO of the model is derived using the next generation matrix method. Further, the equilibrium points of the model are identified and mathematical analysis pertaining to their stability is conducted using Routh – Hurtiz criteria. It is shown that the disease free equilibrium point is locally and globally stable If $R_0 < 1$. On the other hand, the endemic equilibrium point is proved to be stable if $R_0 > 1$. Also, the numerical simulation study of the model is carried out using ode45 of

MATLAB: Rung – Kutta order four. It is observed that, if the vaccination and treatment rates are increased then the infective population size decreases and evenfall to zero over time.

Hence, it is concluded that the use of vaccination and treatment at the highest possible rates is essential so as to control the spread hepatitis B virus.

Sarah and Stanca (2019) proposed mathematical models of within-host interactions; which provide insight into hepatitis B antibody formation, its influence on hepatitis B e antigen seroclearance, and reversion of anergic cytotoxic immune responses. They predict that antibody expansion causes immune activation and hepatitis B e antigen seroclearance. Quantification of the time between antibody expansion and hepatitis B e antigen seroclearance in the presence and absence of treatment shows that potent short-term treatment speeds up the time between antibody expansion and hepatitis B e antigen seroclearance. The monthly hepatocyte turnover during this time can be increased or decreased by treatment depending on the amount of core promoter or precore mutated virus produced. The results can inform human interventions.

CHAPTER THREE

3.0 MATERIAL AND METHODS

3.1 Development of Model

In this chapter, the mathematical model was formulated and analyzed to account for the transmission dynamics of HBV infection incorporating treatment and condom usage.

Following Aniji *et al.* (2019), we divide the total human population into six compartments; Susceptible S, Vaccinated V, Exposed E, Chronic C, Acute A, and Recovered R. The schematic diagram of the model is as shown in the figure 3.1:

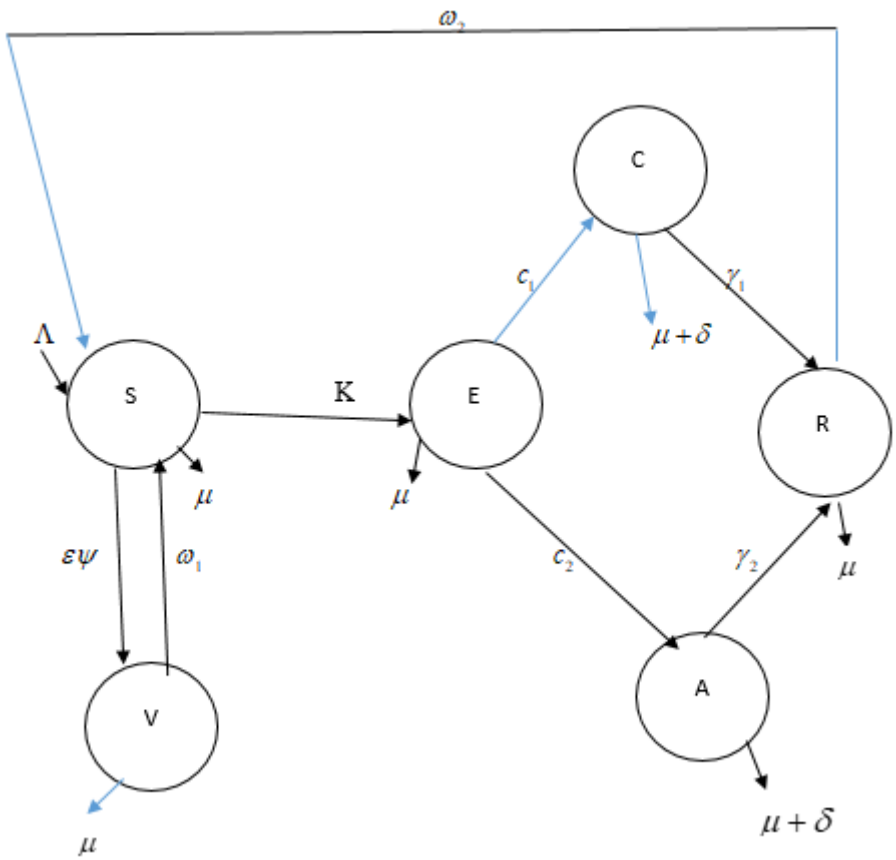

Figure 3.1: The schematic diagram of spread dynamics of HBV with treatment and condom usage as control measures.

The susceptible individuals (S) are generated through constant recruitment of individuals by immigration at the rate Λ, waning of both the vaccines and the immunity developed from drugs treatment at the rate ω_1 and ω_2 respectively . They decreases due to effective contact with E, C and A given by the force of infection:

$$K = \beta(E + C + A)[(1 - \varepsilon_c\psi_c) + (1 - \varepsilon\psi)] \quad (3.1)$$

It further decreases due to usage of effective vaccine at the rate $\varepsilon\psi$ where ε is the efficacy level of the vaccine and ψ is the compliance level to the usage of the vaccine, and usage of high protection condom at the rate $\varepsilon_c\psi_c$ where ε_c is the efficacy level of the condom and ψ_c is the compliance level to the usage of the condom. Finally, there is a further reduction in the compartment due to natural death at the rate μ.

The vaccinated class (V) are generated when the susceptible class are given vaccine at the rate $\varepsilon\psi$. They decreases due to the waning of the vaccine at the rate ω_1 and further reduction occurs in the vaccinated compartment due to natural motarlity at the rate μ.

The exposed class (E) are generated due to effective contact with E, C and A given by the force of infection:

$$K = \beta(E + C + A)[(1 - \varepsilon_c\psi_c) + (1 - \varepsilon\psi)]$$

It decreases due to the progression to either chronic or acute stage at the rate c_1 and c_2 respectively. It further decreases due to natural mortality at rate μ.

The chronic class (C) are generated due to the progression of the exposed class at the rate c_1. It decreases due to treatment at the rate γ_1, the class experiences a further decline due to natural death or disease induced death at the rate μ and δ respectively.

The acute class (A) are generated due to the progression of the exposed class at the rate c_2. It decreases due to treatment at the rate γ_2, the class experiences a further decline due to natural death or disease induced death at the rate μ and δ respectively.

The recovered class (R) are generated from the treatment of the chronic and acute classes at the γ_1 and γ_2 respectively. It decreases due to waning of the administered treatments at the rate ω_2 further reduction occurs in the class due to natural death at the rate μ.

The model equations are as shown below:

$$\frac{dS}{dt} = \Lambda - \beta(E+C+A)[(1-\varepsilon_c\psi_c)+(1-\varepsilon\psi)]S - \mu S + \omega_1 V + \omega_2 R \tag{3.2}$$

$$\frac{dV}{dt} = \varepsilon\psi S - (\omega_1+\mu)V \tag{3.3}$$

$$\frac{dE}{dt} = \beta(E+C+A)[(1-\varepsilon_c\psi_c)+(1-\varepsilon\psi)]S - (c_1+c_2+\mu)E \tag{3.4}$$

$$\frac{dC}{dt} = c_1 E - (\gamma_1+\mu+\delta)C \tag{3.5}$$

$$\frac{dA}{dt} = c_2 E - (\gamma_2+\mu+\delta)\,\mathrm{A} \tag{3.6}$$

$$\frac{dR}{dt} = \gamma_1 C + \gamma_2 A - (\omega_2+\mu)\,\mathrm{R} \tag{3.7}$$

$$N = S+V+E+C+A+R \tag{3.8}$$

3.2 Invariant Region of the Model

The entire population size N may be determined by (3.8),

Adding equations (3.2) to (3.7) gives

$$\frac{dN}{dt} = \Lambda - \mu N - \delta C - \delta A \tag{3.9}$$

In the absence of disease i.e $\delta = 0$, (3.9) gives,

$$\frac{dN}{dt} = \Lambda - \mu N \tag{3.10}$$

Birkhorff and Rota theorem.

Theorem 3.1.

The system (3.2) to (3.7) has solutions which are contained in the feasible region Ω for all $t > 0$.

Proof:

Let $\Omega = (S, V, E, C, A, R) \in \mathbb{R}^6$ be any solution to the system (3.2) to (3.7) having non-negative initial conditions.

Using theorem on differential inequality, Birkoff and Rota (1982) on (3.10)

$$\frac{dN}{dt} \leq \Lambda - \mu N \tag{3.11}$$

$$0 \leq N \leq \frac{\Lambda}{\mu} \tag{3.12}$$

Hence

$$\Lambda - \mu N \geq Be^{-\mu t} \tag{3.13}$$

B the constant

Hence, all feasible solutions of the entire population of the model system are in the region:

$$\Omega = [(S,V,E,C,A,R) \in \mathbb{R}^6 : S,V,E,C,A,R \geq 0, N \leq \frac{\Lambda}{\mu}] \tag{3.14}$$

Which is a positively invariant (i.e solutions remain positive for all time, t) and the model is epidemiologically meaningful and well pose mathematically.

3.3 Positivity of Solutions

Lemma 3.1

Let the initials be

$$\{(S(0),V(0),E(0),C(0),A(0),R(0)) \geq 0\} \in \Omega \tag{3.15}$$

Then the solution set

$$(S(t),V(t),E(t),C(t),A(t),R(t)) \tag{3.16}$$

of the system (3.2) to (3.7) is positive for all $t > 0$.

Proof

From (3.2)

$$\frac{dS}{dt} = \Lambda - \beta(E+C+A)[(1-\varepsilon_c\psi_c)+(1-\varepsilon\psi)]S - \mu S + \omega_1 V + \omega_2 R \tag{3.17}$$

$$\frac{dS}{dt} \geq -\mu S \tag{3.18}$$

$$\frac{dS}{S} \geq -\mu dt \tag{3.19}$$

Integrating

$$\int \frac{dS}{S} \geq -\int \mu dt \tag{3.20}$$

$$\ln S \geq -\mu t + k \tag{3.21}$$

$$S \geq Ke^{-\mu t} \tag{3.22}$$

$$S(0) \geq K = S_0 \tag{3.23}$$

$$S(t) \geq S_0 e^{-\mu t} \tag{3.24}$$

From (3.3)

$$\frac{dV}{dt} = \varepsilon \psi S - (\omega_1 + \mu)V \tag{3.25}$$

$$\frac{dV}{dt} \geq -(\omega_1 + \mu)V \tag{3.26}$$

$$\frac{dV}{V} \geq -(\omega_1 + \mu)dt \tag{3.27}$$

Integrating

$$\int \frac{dV}{V} \geq -\int (\omega_1 + \mu)dt \tag{3.28}$$

$$\ln V \geq -(\omega_1 + \mu)t + k \tag{3.29}$$

$$V \geq Ke^{-(\omega_1 + \mu)t} \tag{3.30}$$

$$V(0) \geq K = V_0 \tag{3.31}$$

$$V(t) \geq V_0 e^{-(\omega_1 + \mu)t} \tag{3.32}$$

From (3.4)

$$\frac{dE}{dt} = \beta(E + C + A)[(1 - \varepsilon_c \psi_c) + (1 - \varepsilon\psi)]S - (c_1 + c_2 + \mu)E \tag{3.33}$$

$$\frac{dE}{dt} \geq -(c_1 + c_2 + \mu)E \tag{3.34}$$

$$\frac{dE}{E} \geq -(c_1 + c_2 + \mu)dt \tag{3.35}$$

Integrating

$$\int \frac{dE}{E} \geq -\int (c_1 + c_2 + \mu)dt \tag{3.36}$$

$$\ln E \geq -(c_1 + c_2 + \mu)t + k \tag{3.37}$$

$$E \geq Ke^{-(c_1 + c_2 + \mu)t} \tag{3.38}$$

$$E(0) \geq K = E_0 \tag{3.39}$$

$$E(t) \geq E_0 e^{-(c_1 + c_2 + \mu)t} \tag{3.40}$$

From (3.5)

$$\frac{dC}{dt} = c_1 E - (\gamma_1 + \mu + \delta)C \tag{3.41}$$

$$\frac{dC}{dt} \geq -(\gamma_1 + \mu + \delta)C \tag{3.42}$$

$$\frac{dC}{C} \geq -(\gamma_1 + \mu + \delta)dt \tag{3.43}$$

integrating

$$\int \frac{dC}{C} \geq -\int (\gamma_1 + \mu + \delta)dt \tag{3.44}$$

$$\ln C \geq -(\gamma_1 + \mu + \delta)t + k \tag{3.45}$$

$$C \geq Ke^{-(\gamma_1+\mu+\delta)t} \tag{3.46}$$

$$C(0) \geq K = C_0 \tag{3.47}$$

$$C \geq C_0 e^{-(\gamma_1+\mu+\delta)t} \tag{3.48}$$

From (3.6)

$$\frac{dA}{dt} = c_2 E - (\gamma_2 + \mu + \delta)\,\mathrm{A} \tag{3.49}$$

$$\frac{dA}{dt} \geq -(\gamma_2 + \mu + \delta)\,\mathrm{A} \tag{3.50}$$

$$\frac{dA}{\mathrm{A}} \geq -(\gamma_2 + \mu + \delta)dt \tag{3.51}$$

Integrating

$$\int \frac{dA}{\mathrm{A}} \geq -\int (\gamma_2 + \mu + \delta)dt \tag{3.52}$$

$$\ln\mathrm{A} \geq -(\gamma_2 + \mu + \delta)\,\mathrm{t} + \mathrm{k} \tag{3.53}$$

$$\mathrm{A} \geq Ke^{-(\gamma_2+\mu+\delta)\mathrm{t}} \tag{3.54}$$

$$\mathrm{A}(0) \geq K = A_0 \tag{3.55}$$

$$\mathrm{A(t)} \geq A_0 e^{-(\gamma_2+\mu+\delta)\mathrm{t}} \tag{3.56}$$

From (3.7)

$$\frac{dR}{dt} = \gamma_1 C + \gamma_2 A - (\omega_2 + \mu)\,\mathrm{R} \tag{3.57}$$

$$\frac{dR}{dt} \geq -(\omega_2 + \mu)\,\mathrm{R} \tag{3.58}$$

$$\frac{dR}{\mathrm{R}} \geq -(\omega_2 + \mu)dt \tag{3.59}$$

Integrate

$$\int \frac{dR}{\mathrm{R}} \geq -\int (\omega_2 + \mu)dt \tag{3.60}$$

$$\ln\mathrm{R} \geq -(\omega_2 + \mu)t + k \tag{3.61}$$

$$\mathrm{R} \geq Ke^{-(\omega_2+\mu)t} \tag{3.62}$$

$$\mathrm{R}(0) \geq K = R_0 \tag{3.63}$$

$$\mathrm{R}(t) \geq R_0 e^{-(\omega_2+\mu)t} \tag{3.64}$$

Hence, all the solution of the system (3.2) to (3.7) are positive for all $t > 0$.

3.4 Equilibrium Points of the Model

At equilibrium

$$\frac{dS}{dt} = \frac{dV}{dt} = \frac{dE}{dt} = \frac{dC}{dt} = \frac{dA}{dt} = \frac{dR}{dt} = 0 \tag{3.65}$$

Let

$$(S, V, E, C, A, R) = (S^*, V^*, E^*, C^*, A^*, R^*) \tag{3.66}$$

Be arbitrary equilibrium point

Then (3.2) to (3.7) becomes;

$$\Lambda - \beta(E^* + C^* + A^*)\,\mathrm{F}_1\, S^* - \mu S^* + \omega_1 V^* + \omega_2 R^* = 0 \tag{3.67}$$

$$\varepsilon\psi S^* - F_2 V^* = 0 \tag{3.68}$$

$$\beta(E^* + C^* + A^*)\,\mathrm{F}_1\, S^* - F_3 E^* = 0 \tag{3.69}$$

$$c_1 E^* - F_4 C^* = 0 \tag{3.70}$$

$$c_2 E^* - F_5 A^* = 0 \tag{3.71}$$

$$\gamma_1 C^* + \gamma_2 A^* - F_6\,\mathrm{R}^* = 0 \tag{3.72}$$

Where,

$$\begin{aligned}&\mathrm{F}_1 = (1-\varepsilon_c\psi_c) + (1-\varepsilon\psi), \mathrm{F}_2 = (\omega_1 + \mu), \mathrm{F}_3 = (c_1 + c_2 + \mu),\\ &\mathrm{F}_4 = (\gamma_1 + \mu + \delta), \mathrm{F}_5 = (\gamma_2 + \mu + \delta), \mathrm{F}_6 = (\omega_2 + \mu)\end{aligned} \tag{3.73}$$

From (3.71)

$$A^* = \frac{c_2 E^*}{F_5} \tag{3.74}$$

From (3.70)

$$C^* = \frac{c_1 E^*}{F_4} \tag{3.75}$$

Substituting (3.74) and (3.75) in (3.69)

$$\beta\left(E^* + \frac{c_1 E^*}{F_4} + \frac{c_2 E^*}{F_5}\right)\mathrm{F}_1\, S^* - F_3 E^* = 0 \tag{3.76}$$

$$\left(\beta\left(1 + \frac{c_1}{F_4} + \frac{c_2}{F_5}\right)\mathrm{F}_1\, E^* S^* - F_3 E^*\right) = 0 \tag{3.77}$$

$$\left(\beta\left(1 + \frac{c_1}{F_4} + \frac{c_2}{F_5}\right)\mathrm{F}_1\, S^* - F_3\right) E^* = 0 \tag{3.78}$$

$$E^* = 0 \tag{3.79}$$

Or

$$\beta(1+\frac{c_1}{F_4}+\frac{c_2}{F_5})\mathrm{F}_1 S^* - F_3 = 0 \tag{3.80}$$

Thus equation (3.78) gives the existence of two different equilibria; one satisfying (3.79) and the other satisfying (3.80)

3.5 Disease Free Equilibrium (DFE) Point of the Model

Let

$$\mathrm{E}^0 = (S,V,E,C,A,R) = (S^0,V^0,E^0,C^0,A^0,R^0) \tag{3.81}$$

Substituting (3.79) in (3.75)

$$C^0 = 0 \tag{3.82}$$

Substituting (3.79) in (3.74)

$$A^0 = 0 \tag{3.83}$$

Substituting (3.87) and (3.88) in (3.72)

$$R^0 = 0 \tag{3.84}$$

Substituting (3.79), (3.82), (3.83) and (3.84) in (3.67)

$$\Lambda - \mu S^0 + \omega_1 V^0 = 0 \tag{3.85}$$

$$\mu S^0 + \omega_1 V^0 = \Lambda \tag{3.86}$$

$$S^0 = \frac{\Lambda + \omega_1 V^0}{\mu} \tag{3.87}$$

From (3.68)

$$\varepsilon\psi S^0 - F_2 V^0 = 0 \tag{3.88}$$

Substituting (3.87) in (3.88)

$$\varepsilon\psi\left(\frac{\Lambda + \omega_1 V^0}{\mu}\right) - F_2 V^0 = 0 \tag{3.89}$$

$$\varepsilon\psi\Lambda + \varepsilon\psi\omega_1 V^0 - F_2\mu V^0 = 0 \tag{3.90}$$

$$F_2\mu V^0 - \varepsilon\psi\omega_1 V^0 = \varepsilon\psi\Lambda \tag{3.91}$$

$$(F_2\mu - \varepsilon\psi\omega_1)V^0 = \varepsilon\psi\Lambda \tag{3.92}$$

$$V^0 = \frac{\varepsilon\psi\Lambda}{F_2\mu - \varepsilon\psi\omega_1} \tag{3.93}$$

Substituting (3.93) in (3.87)

$$S^0 = \frac{\Lambda + \omega_1\left(\frac{\varepsilon\psi\Lambda}{F_2\mu - \varepsilon\psi\omega_1}\right)}{\mu} \tag{3.94}$$

$$S^0 = \frac{\Lambda(F_2\mu - \varepsilon\psi\omega_1) + \omega_1\varepsilon\psi\Lambda}{\mu(F_2\mu - \varepsilon\psi\omega_1)} \tag{3.95}$$

$$S^0 = \frac{\Lambda F_2\mu - \varepsilon\psi\omega_1\Lambda + \omega_1\varepsilon\psi\Lambda}{\mu(F_2\mu - \varepsilon\psi\omega_1)} \tag{3.96}$$

$$S^0 = \frac{\Lambda F_2}{F_7} \tag{3.97}$$

where

$$\mathrm{F}_7 = (F_2\mu - \varepsilon\psi\omega_1) \tag{3.98}$$

Thus DFE exist at the points

$$(S^0, V^0, E^0, C^0, A^0, R^0) = \left(\frac{\Lambda F_2}{F_7}, \frac{\varepsilon\psi\Lambda}{F_7}, 0, 0, 0, 0 \right) \tag{3.99}$$

3.6 Basic Reproduction Number, R_0

The basic reproduction number is the average number of secondary infections caused by a single infectious individual during his/her entire infectious life time. Applying next generation matrix operator to compute the Basic Reproduction Number of the model as used by (Diekmann *et al.* 1990) and improved by (van den Driessche & Watmough, 2002). The basic reproduction number is obtained by dividing the whole population into n compartments in which there are $m < n$ infected compartments. Let $x_i, i = 1, 2, 3, ..., m$ be the number of infected individuals in the i^{th} infected compertment at time t. The largest eigenvalue or spectra radius of FV^{-1} is the basic reproduction number of the model.

$$FV^{-1} = \left[\frac{\partial F_i(E^0)}{\partial x_i} \right] \left[\frac{\partial V_i(E^0)}{\partial x_i} \right]^{-1} \tag{3.100}$$

Where F_i is the rate of appearance of new infection in compartment i to another and E^0 is the disease-Free Equilibrium.

$$f_i = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \beta(E + C + A)F_1 S \\ 0 \\ 0 \end{pmatrix} \tag{3.101}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial C} & \frac{\partial f_1}{\partial A} \\ \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial C} & \frac{\partial f_2}{\partial A} \\ \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial C} & \frac{\partial f_3}{\partial A} \end{pmatrix} \tag{3.102}$$

$$F(E_0) = \begin{pmatrix} \beta F_1 S^0 & \beta F_1 S^0 & \beta F_1 S^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.103}$$

Recall,

$$S^0 = \frac{\Lambda F_2}{F_7} \tag{3.104}$$

$$F(E_0) = \begin{pmatrix} \beta F_1 F_8 & \beta F_1 F_8 & \beta F_1 F_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.105}$$

$$F_8 = \frac{\Lambda F_2}{F_7} \tag{3.106}$$

$$v_i = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} F_3 E \\ F_4 C \\ F_5 A \end{pmatrix} \tag{3.107}$$

$$V = \begin{pmatrix} \frac{\partial v_1}{\partial E} & \frac{\partial v_1}{\partial C} & \frac{\partial v_1}{\partial A} \\ \frac{\partial v_2}{\partial E} & \frac{\partial v_2}{\partial C} & \frac{\partial v_2}{\partial A} \\ \frac{\partial v_3}{\partial E} & \frac{\partial v_3}{\partial C} & \frac{\partial v_3}{\partial A} \end{pmatrix} \tag{3.108}$$

$$V = \begin{pmatrix} F_3 & 0 & 0 \\ 0 & F_4 & 0 \\ 0 & 0 & F_5 \end{pmatrix} \tag{3.109}$$

$$v^{-1} = \frac{adjv}{\det v} \tag{3.110}$$

$$\det v = F_3 F_4 F_5 \tag{3.111}$$

$$adjv = \begin{pmatrix} F_4 F_5 & 0 & 0 \\ 0 & F_3 F_5 & 0 \\ 0 & 0 & F_3 F_4 \end{pmatrix} \tag{3.112}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{F_3} & 0 & 0 \\ 0 & \frac{1}{F_4} & 0 \\ 0 & 0 & \frac{1}{F_5} \end{pmatrix} \tag{3.113}$$

$$fv^{-1} = \begin{pmatrix} \beta F_1 F_8 & \beta F_1 F_8 & \beta F_1 F_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{F_3} & 0 & 0 \\ 0 & \frac{1}{F_4} & 0 \\ 0 & 0 & \frac{1}{F_5} \end{pmatrix} \tag{3.114}$$

$$fv^{-1} = \begin{pmatrix} \frac{\beta F_1 F_8}{F_3} & \frac{\beta F_1 F_8}{F_4} & \frac{\beta F_1 F_8}{F_5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.115}$$

$$(fv^{-1} - \lambda I) = 0 \tag{3.116}$$

$$\begin{pmatrix} \frac{\beta F_1 F_8}{F_3} - \lambda & \frac{\beta F_1 F_8}{F_4} & \frac{\beta F_1 F_8}{F_5} \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = 0 \tag{3.116}$$

$$(\frac{\beta F_1 F_8}{F_3} - \lambda)\lambda^2 = 0 \tag{3.118}$$

then

$$\lambda^2 = 0 \tag{3.119}$$

or

$$\frac{\beta F_1 F_8}{F_3} - \lambda = 0 \tag{3.120}$$

$$\lambda = \frac{\beta F_1 F_8}{F_3} \tag{3.121}$$

$$R_0 = \frac{\beta F_1 F_8}{F_3} \tag{3.122}$$

Hence the basic reproduction number of our model is given by (3.122) which is the average number of secondary infections caused by a single infectious individual during his/her entire infectious life time.

3.7 Local Stability of Disease Free Equilibrium (DFE), E^0

Following Deikmann and Heesterbeek (2000). theorem, the DFE is LAS if R_0 exist, and the $R_0 < 1$. We want to further establish the theorem using Jacobian methods for stability.

Lemma 3.2: The Disease Free Equilibrium of the model is locally asymptotically stable (LAS) if $R_0 < 1$.

Proof:

$$J_{E_0} = \begin{pmatrix} -\mu & \omega_1 & -\beta F_1 S & -\beta F_1 S & -\beta F_1 S & \omega_2 \\ \varepsilon\psi & -F_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta F_1 S - F_3 & \beta F_1 S & \beta F_1 S & 0 \\ 0 & 0 & c_1 & -F_4 & 0 & 0 \\ 0 & 0 & c_2 & 0 & -F_5 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -F_6 \end{pmatrix} \quad (3.123)$$

$$R_2 = R_2 + \frac{\varepsilon\psi}{\mu} R_1$$

(3.124)

$$J_{E_0} = \begin{pmatrix} -\mu & \omega_1 & -\beta F_1 F_8 & -\beta F_1 F_8 & -\beta F_1 F_8 & \omega_2 \\ 0 & \dfrac{\mu F_2 - \varepsilon\psi\omega_1}{\mu} & \dfrac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \dfrac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \dfrac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \dfrac{\varepsilon\psi\omega_2}{\mu} \\ 0 & 0 & \beta F_1 F_8 - F_3 & \beta F_1 F_8 & \beta F_1 F_8 & 0 \\ 0 & 0 & c_1 & -F_4 & 0 & 0 \\ 0 & 0 & c_2 & 0 & -F_5 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -F_6 \end{pmatrix}$$

(3.125)

$$R_4 = R_4 - \frac{c_1}{\beta F_1 F_8 - F_3} R_3$$

$$R_5 = R_5 - \frac{c_2}{\beta F_1 F_8 - F_3} R_3$$

(3.126)

$$J_{E_0} = \begin{pmatrix} -\mu & \omega_1 & -\beta F_1 F_8 & -\beta F_1 F_8 & -\beta F_1 F_8 & \omega_2 \\ 0 & \dfrac{\mu F_2 - \varepsilon\psi\omega_1}{\mu} & \dfrac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \dfrac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \dfrac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \dfrac{\varepsilon\psi\omega_2}{\mu} \\ 0 & 0 & \beta F_1 F_8 - F_3 & \beta F_1 F_8 & \beta F_1 F_8 & 0 \\ 0 & 0 & 0 & \dfrac{-(\beta F_1 F_4 F_8 + \beta c_1 F_1 F_8 + F_3 F_4)}{\beta F_1 F_8 - F_3} & \dfrac{-(\beta c_1 F_1 F_8)}{\beta F_1 F_8 - F_3} & 0 \\ 0 & 0 & 0 & \dfrac{-(\beta c_2 F_1 F_8)}{\beta F_1 F_8 - F_3} & \dfrac{-(\beta F_1 F_5 F_8 + \beta c_2 F_1 F_8 + F_3 F_5)}{\beta F_1 F_8 - F_3} & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -F_6 \end{pmatrix} \quad (3.127)$$

$$R_5 = R_5 - \frac{\beta c_2 F_1 F_8}{\beta F_1 F_4 F_5 + \beta c_1 F_1 F_3 F_4 F_8} R_4$$

$$R_6 = R_6 + \frac{\sigma_1}{\frac{\beta F_1 F_4 F_8 + \beta c_1 F_1 F_8 - F_3 F_4}{\beta F_1 F_8 - F_3}} R_3$$

(3.128)

$$J_{E_0} = \begin{pmatrix} -\mu & \omega_1 & -\beta F_1 F_8 & -\beta F_1 F_8 & -\beta F_1 F_8 & \omega_2 \\ 0 & \frac{\mu F_2 - \varepsilon\psi\omega_1}{\mu} & \frac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \frac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \frac{-\varepsilon\psi\beta F_1 F_8}{\mu} & \frac{\varepsilon\psi\omega_2}{\mu} \\ 0 & 0 & \beta F_1 F_8 - F_3 & \beta F_1 F_8 & \beta F_1 F_8 & 0 \\ 0 & 0 & 0 & \frac{-(\beta F_1 F_4 F_8 + \beta c_1 F_1 F_8 + F_3 F_4)}{\beta F_1 F_8 - F_3} & \frac{-(\beta c_1 F_1 F_8)}{\beta F_1 F_8 - F_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{-(\beta F_1 F_5 F_8 + \beta F_1 F_4 F_8 + \beta c_1 F_1 F_5 F_8 + F_1 F_4 F_5)}{\beta F_1 F_4 F_8 - \beta c_1 F_1 F_8 - F_1 F_4} & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 - \frac{\beta\sigma_1 c_1 F_1 F_8}{\beta F_1 F_4 F_8 - \beta c_1 F_1 F_8 - F_1 F_4} & -F_6 \end{pmatrix} \quad (3.129)$$

$$R_6 = R_6 - \frac{\sigma_2 \beta F_1 F_4 + \beta c_1 F_1 F_8 - F_3 F_4 - \sigma_1 \beta c_1 F_1 F_8}{\frac{\beta F_1 F_4 F_8 + \beta c_1 F_1 F_8 - F_3 F_4}{\beta F_1 F_4 F_5 + \beta c_1 F_1 F_8 - F_1 F_4}} R_5$$

(3.129)

$$J_{E_0} = \begin{pmatrix} -\mu & \omega_1 & -\beta F_1 F_8 & -\beta F_1 F_8 & -\beta F_1 F_8 & \omega_2 \\ 0 & K_2 & -K_3 & -K_4 & -K_5 & -K_6 \\ 0 & 0 & -K_7 & K_8 & K_9 & 0 \\ 0 & 0 & 0 & -K_{10} & -K_{11} & 0 \\ 0 & 0 & 0 & 0 & -K_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_6 \end{pmatrix}$$

(3.130)

Where

$$K_2 = \frac{F_2\mu - \varepsilon\psi\omega_1}{\mu}, K_3 = \frac{\varepsilon\psi(\beta F_1 \mathrm{F}_8)}{\mu}, K_4 = \frac{\varepsilon\psi(\beta F_1 \mathrm{F}_8)}{\mu},$$
$$K_5 = \frac{\varepsilon\psi(\beta F_1 \mathrm{F}_8)}{\mu}, K_6 = \frac{\varepsilon\psi\omega_2}{\mu}, K_7 = (F_3 - \beta F_1 \mathrm{F}_8),$$
$$K_8 = K_9 = \beta F_1 F_8, K_{10} = \frac{F_4(\beta F_1 \mathrm{F}_8) + c_1(\beta F_1 \mathrm{F}_8) - F_4F_3}{\beta F_1 F_{\ 8} - F_3}, \quad (3.131)$$
$$K_{11} = \frac{c_1(\beta F_1 \mathrm{F}_8)}{\beta F_1 F_8 - F_3}, K_{12} = \frac{\beta F_1F_4F_5F_8 + \beta F_1F_4c_2F_8 + \beta F_1F_5c_1F_8 - F_3F_4F_5}{F_4F_1\beta F_8 + \beta F_1c_1F_8 - F_3F_4}$$

$$[J_{E_0} - \lambda I] = J_{E_0} = \begin{pmatrix} -\mu-\lambda & \omega_1 & -\beta F_1F_8 & -\beta F_1F_8 & -\beta F_1F_8 & \omega_2 \\ 0 & -K_2-\lambda & -K_3 & -K_4 & -K_5 & K_6 \\ 0 & 0 & -K_7-\lambda & K_8 & K_9 & 0 \\ 0 & 0 & 0 & -K_{10}-\lambda & -K_{11} & 0 \\ 0 & 0 & 0 & 0 & -K_{12}-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_6-\lambda \end{pmatrix} \quad (3.132)$$

$$\lambda_1 = -\mu \quad (3.133)$$

$$\lambda_2 = -K_2 \quad (3.134)$$

$$\lambda_3 = -K_7 \quad (3.135)$$

$$\lambda_4 = -K_{10} \quad (3.136)$$

$$-\lambda_5 - \frac{\beta F_1F_4F_5F_8 + \beta F_1F_4c_2F_8 + \beta F_1F_5c_1F_8 - F_3F_4F_5}{F_4F_1\beta F_8 + \beta F_1c_1F_8 - F_3F_4} = 0 \quad (3.137)$$

$$\lambda_5 = -\frac{(F_4F_5 + F_4c_2 + F_5c_1)\beta F_1F_8 - F_3F_4F_5}{F_4F_1\beta F_8 + \beta F_1c_1F_8 - F_3F_4} \quad (3.138)$$

$$\lambda_5 = -\frac{(F_4F_5 + F_4c_2 + F_5c_1)\beta F_1F_8 - 1}{F_4F_5(F_4F_1\beta F_8 + \beta F_1c_1F_8 - F_3F_4)F_3} \quad (3.139)$$

$$\frac{(F_4F_5 + \beta F_4c_2 + F_5c_1)}{F_4F_5(F_4F_1\beta F_8 + \beta F_1c_1F_8 - F_3F_4)}(\mathrm{R}_0 - 1) < 0 \quad (3.140)$$

$$R_0 < 1 \tag{3.141}$$

This implies

$$\lambda_5 < 0 \text{ if } R_C < 1. \tag{3.142}$$

Hence, DFE LAS.

3.8 Global Stability of Disease Free Equilibrium (DFE), E^0

The restraint on the first lemma was gotten rid of by the Global stability of equilibrium. For all initial conditions in global asymptotic stability, solutions approach the equilibrium. Lyaponuv theorem and (Castilo-Chavez and Song 2004). global stability theorem are examples of ways in which we can test for the global stability of disease-free equilibrium; but this research uses the Lyapunov method.

Lemma 3.3: The DFE E^0, is globally asymptotically stable if $R_c \leq 1$

Proof: To establish the global stability of the disease free equilibrium, we consider the Lyaponuv function.

$$L = F_4F_5E + \beta F_1F_5CF_8 + \beta F_1F_4AF_8 \tag{3.143}$$

$$\frac{dL}{dt} = F_4F_5\frac{dE}{dt} + \beta F_1F_5F_8\frac{dC}{dt} + \beta F_1F_4F_8\frac{dA}{dt} \tag{3.144}$$

$$\begin{aligned}\frac{dL}{dt} &= F_4F_5[\beta(E+C+A)F_1F_8 - F_3E] + \beta F_1F_5\,\mathrm{F}_8(c_1E - F_4C) \\ &+\beta F_1F_4\,\mathrm{F}_8(c_2E - F_5A)\end{aligned} \tag{3.145}$$

$$\begin{aligned}\frac{dL}{dt} &= F_4F_5\beta F_1F_8E + F_4F_5\beta F_1F_8C + F_4F_5\beta F_1F_8A - F_3F_4F_5E \\ &+\beta F_1F_5c_1F_8E - \beta F_1F_4F_5F_8C + \beta F_1F_4c_2F_8E - \beta F_1F_4F_5SA\end{aligned} \tag{3.146}$$

$$\frac{dL}{dt} = F_4F_5\beta F_1F_8E + \beta F_1F_5c_1F_8E + \beta F_1F_4c_2F_8E - F_3F_4F_5E \quad (3.147)$$

$$\frac{dL}{dt} = [(F_4F_5\beta F_1 + \beta F_1F_5c_1 + \beta F_1F_4c_2)\mathrm{F}_8 - F_3F_4F_5]E \quad (3.148)$$

$$N^0 \le N \text{ and } S^0 \le S \quad (3.149)$$

$$\frac{dL}{dt} \le \frac{(F_4F_{51} + F_5c_1 + F_4c_2)\beta F_1F_8}{F_4F_5F_3} - 1 \quad (3.150)$$

$$\frac{dL}{dt} \le \{R_c - 1\} \quad (3.151)$$

$$\frac{dL}{dt} \le 0 \text{ if } R_c \le 1 \quad (3.152)$$

Thus, there is an asymptotical stability in the disease-free equilibrium point and thus, this completes the proof.

3.9 Endemic Equilibrium (EE) Points of the Model

The endemic equilibrium point (EEP) in terms of forces of infection are computed in this section

Let

$$\mathrm{E}^1 = (S,V,E,C,A,R) = (S^{**},V^{**},E^{**},C^{**},A^{**},R^{**}) \quad (3.153)$$

From (3.80)

$$(1 + \frac{c_1}{F_4} + \frac{c_2}{F_5})\frac{\beta \mathrm{F}_1 S^{**}}{F_3} - 1 = 0 \quad (3.154)$$

$$S^{**} = \frac{F_3}{\beta(1 + \frac{c_1}{F_4} + \frac{c_2}{F_5})\mathrm{F}_1} \quad (3.155)$$

$$S^{**} = \frac{F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \tag{3.156}$$

Using (3.156) in (3.68)

$$V^{**} = \frac{\varepsilon \psi S^{**}}{F_2} \tag{3.157}$$

$$V^{**} = \frac{\varepsilon \psi \left(\dfrac{F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \right)}{F_2} \tag{3.158}$$

$$V^{**} = \frac{\varepsilon \psi F_3 F_4 F_5}{\beta \mathrm{F}_1 F_2 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \tag{3.159}$$

Using (3.74) and (3.75) in (3.72)

$$R^{**} = \left(\frac{F_5 \gamma_1 c_1 - F_4 \gamma_2 c_2}{F_4 F_5 F_6} \right) E^{**} \tag{3.160}$$

Using (3.74), (3.75), (3.1), (3.159) and (3.1) in (3.67)

$$\begin{aligned} &\Lambda - \left(\beta (1 + \frac{c_1}{F_4} + \frac{c_2}{F_5}) \mathrm{F}_1 \frac{F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \right) E^{**} - \frac{\mu F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \\ &+ \frac{\omega_1 \varepsilon \psi F_3 F_4 F_5}{\beta \mathrm{F}_1 F_2 (F_4 F_5 + c_1 F_5 + c_2 F_4)} + \omega_2 \left(\frac{F_5 \gamma_1 c_1 - F_4 \gamma_2 c_2}{F_4 F_5 F_6} \right) E^{**} = 0 \end{aligned} \tag{3.161}$$

$$\begin{aligned} &\left(\beta (1 + \frac{c_1}{F_4} + \frac{c_2}{F_5}) \mathrm{F}_1 \frac{F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} - \omega_2 \left(\frac{F_5 \gamma_1 c_1 - F_4 \gamma_2 c_2}{F_4 F_5 F_6} \right) \right) E^{**} \\ &- \frac{\mu F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} = \Lambda + \frac{\omega_1 \varepsilon \psi F_3 F_4 F_5}{\beta \mathrm{F}_1 F_2 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \end{aligned} \tag{3.162}$$

$$\begin{aligned} &\left(\beta (\frac{\mathrm{F}_4 F_5 + c_1 F_5 + c_2 F_4}{F_4 F_5}) \mathrm{F}_1 \frac{F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} - \omega_2 \left(\frac{F_5 \gamma_1 c_1 - F_4 \gamma_2 c_2}{F_4 F_5 F_6} \right) \right) E^{**} \\ &= \Lambda + \frac{\omega_1 \varepsilon \psi F_3 F_4 F_5}{\beta \mathrm{F}_1 F_2 (F_4 F_5 + c_1 F_5 + c_2 F_4)} + \frac{\mu F_3 F_4 F_5}{\beta \mathrm{F}_1 (F_4 F_5 + c_1 F_5 + c_2 F_4)} \end{aligned} \tag{3.163}$$

$$\left(F_3-\omega_2\left(\frac{F_5\gamma_1c_1-F_4\gamma_2c_2}{F_4F_5F_6}\right)\right)E^{**}=\Lambda+\frac{\omega_1\varepsilon\psi F_3F_4F_5}{\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)}+\frac{\mu F_3F_4F_5}{\beta\mathrm{F}_1(F_4F_5+c_1F_5+c_2F_4)} \quad (3.164)$$

$$\left(\frac{F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2)}{F_4F_5F_6}\right)E^{**}=\frac{\Lambda\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5}{\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)} \quad (3.165)$$

$$E^{**}=\frac{F_4F_5F_6(\Lambda\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))} \quad (3.166)$$

From (3.74)

$$A^{**}=\frac{c_2E^{**}}{F_5} \quad (3.167)$$

$$A^{**}=\frac{c_2F_4F_5F_6(\Lambda\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{F_5\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))} \quad (3.168)$$

From (3.75)

$$C^{**}=\frac{c_1E^{**}}{F_4} \quad (3.169)$$

$$C^{**}=\frac{c_1F_4F_5F_6(\Lambda\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{F_4\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))} \quad (3.170)$$

From (3.160)

$$R^{**}=\left(\frac{F_5\gamma_1c_1-F_4\gamma_2c_2}{F_4F_5F_6}\right)E^{**} \quad (3.171)$$

$$R^{**}=\left(\frac{F_5\gamma_1c_1-F_4\gamma_2c_2}{F_4F_5F_6}\right)\frac{F_4F_5F_6(\Lambda\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{\beta\mathrm{F}_1F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))} \quad (3.172)$$

Hence EEP is given by;

$$\mathrm{E}^1 = (S,V,E,C,A,R) = (S^{**},V^{**},E^{**},C^{**},A^{**},R^{**}) = \{\frac{F_3F_4F_5}{\beta \mathrm{F}_1(F_4F_5+c_1F_5+c_2F_4)},$$
$$\frac{\varepsilon\psi F_3F_4F_5}{\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)}, \frac{F_4F_5F_6(\Lambda\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))},$$
$$\frac{c_1F_4F_5F_6(\Lambda\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{F_4\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))}, \quad (3.173)$$
$$\frac{c_2F_4F_5F_6(\Lambda\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{F_5\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))},$$
$$\left(\frac{F_5\gamma_1c_1-F_4\gamma_2c_2}{F_4F_5F_6}\right)\frac{F_4F_5F_6(\Lambda\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)+\omega_1\varepsilon\psi F_3F_4F_5+\mu F_2F_3F_4F_5))}{\beta \mathrm{F}_1 F_2(F_4F_5+c_1F_5+c_2F_4)(F_3F_4F_5F_6-\omega_2(F_5\gamma_1c_1-F_4\gamma_2c_2))}\}$$

3.10 Analysis of Local Stability of Endemic Equilibrium Point

An important criterion by Routh-Hurwitz gives the necessary and sufficient conditions for the all roots of the characteristics (with real coefficients) to lie in the left half of the complex plane. In other words, all the roots of the polynomial are negative or have real roots if and only if the determinant of all Hurwitz matrices is positive.

Theorem 3.2 (Routh-Hurwitz Conditions)

Let $J = \begin{pmatrix} f_x(x_*,y_*) & f_y(x_*,y_*) \\ g_x(x_*,y_*) & g_y(x_*,y_*) \end{pmatrix}$ be the Jacobian matrix of the non-linear system

$$\frac{dx}{dt} = f(x,y) \quad (3.174)$$

$$\frac{dy}{dt} = g(x,y) \quad (3.175)$$

Evaluated at the critical point (x_*,y_*), then the critical point (x_*,y_*);

1. Is locally asymptotically stable if trace (J)<0 and determinant>0
2. Is stable but not asymptotically stable if trace (J)=0 and determinant (J)>0
3. Is unstable if either, trace (J)>0 or determinant (J)<0

Jacobian matrix of the system of equations at endemic equilibrium state is:

$$J\left(x^*_1, x^*_2, x^*_3, x^*_4, x^*_5,\right) = \begin{pmatrix} -\beta(E^*+C^*+A^*)\mathrm{F}_1-\mu & \omega_1 & -\beta\mathrm{F}_1 S^* & -\beta\mathrm{F}_1 S^* & -\beta\mathrm{F}_1 S^* & \omega_2 \\ \varepsilon\psi & -F_2 & 0 & 0 & 0 & 0 \\ \beta(E^*+C^*+A^*)\mathrm{F}_1 & 0 & \beta\mathrm{F}_1 S^* - F_3 & \beta\mathrm{F}_1 S^* & \beta\mathrm{F}_1 S^* & 0 \\ 0 & 0 & c_1 & -F_4 & 0 & 0 \\ 0 & 0 & c_2 & 0 & F_5 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -F_6 \end{pmatrix} \quad (3.176)$$

The Trace is

$$-\beta(E+C+A)F_1 - \mu - F_2 + \beta F_1 S - F_3 - F_4 - F_5 - F_6 \quad (3.177)$$

Gives

$$-\left\{\beta(E+C+A)F_1 + \mu + F_2 + F_3 + F_4 + F_5 + F_6\right\} + \beta F_1 S \quad (3.178)$$

Negative if

$$\left\{\beta(E+C+A)F_1 + \mu + F_2 + F_3 + F_4 + F_5 + F_6\right\} > \beta F_1 S \quad (3.179)$$

And the determinant is given by

$$\begin{aligned} &\varepsilon\psi(\beta F_1 S)F_4F_5F_6\omega_1 + \varepsilon\psi(\beta F_1 S)F_4F_6c_2\omega_1 + \varepsilon\psi(\beta F_1 S)F_4F_6c_1\omega_1 \\ &-\varepsilon\psi F_3F_4F_5F_6\omega_1 - 2(\beta F_1 S)(\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1)\mathrm{F}_2\,F_4F_6c_2 - \\ &2(\beta F_1 S)(\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1)\mathrm{F}_2\,F_4F_6c_1 - \beta F_1 S\mu F_2\,\mathrm{F}_2\,F_4F_6 - \\ &\beta F_1 S\mu F_2\,\mathrm{F}_2\,F_4F_6c_2 - \beta F_1 S\mu F_2\,\mathrm{F}_2\,F_4F_6c_1 + \beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_3F_4F_5F_6 \\ &-\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_4c_2\gamma_2\omega_2 - \beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_5c_1\gamma_1\omega_2 + \mu\,\mathrm{F}_2\,F_3F_4F_5F_6 \end{aligned} \quad (3.180)$$

This gives

$$\begin{aligned} &\varepsilon\psi(\beta F_1 S)F_4F_6\omega_1(F_5 + c_2 + c_1) - \varepsilon\psi F_3F_4F_5F_6\omega_1 \\ &-2(\beta F_1 S)(\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1)\mathrm{F}_2\,F_4F_6(c_2 + c_1) \\ &-\beta F_1 S\mu F_2{}^2F_4F_6(1 + c_2 + c_1) + \beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_3F_4F_5F_6 \\ &-\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,\omega_2(F_4c_2\gamma_2 + F_5c_1\gamma_1) + \mu\,\mathrm{F}_2\,F_3F_4F_5F_6 \end{aligned} \quad (3.181)$$

$$\begin{aligned} &\varepsilon\psi(\beta F_1 S)F_4F_6\omega_1(F_5 + c_2 + c_1) + \beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_3F_4F_5F_6 \\ &+\mu\,\mathrm{F}_2\,F_3F_4F_5F_6 - \varepsilon\psi F_3F_4F_5F_6\omega_1 - 2(\beta F_1 S)(\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1)\mathrm{F}_2\,F_4F_6(c_2 + c_1) \\ &-\beta F_1 S\mu F_2{}^2F_4F_6(1 + c_2 + c_1) - \beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,\omega_2(F_4c_2\gamma_2 + F_5c_1\gamma_1) \end{aligned} \quad (3.182)$$

$$\begin{Bmatrix} \varepsilon\psi(\beta F_1 S)F_4F_6\omega_1(F_5+c_2+c_1)+\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_3F_4F_5F_6 \\ +\mu\,\mathrm{F}_2\,F_3F_4F_5F_6 \end{Bmatrix}$$
$$-\begin{Bmatrix} \varepsilon\psi F_3F_4F_5F_6\omega_1+2(\beta F_1 S)(\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1)\mathrm{F}_2\,F_4F_6(c_2+c_1) \\ +\beta F_1 S\mu F_2{}^2F_4F_6(1+c_2+c_1)+\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,\omega_2(F_4c_2\gamma_2+F_5c_1\gamma_1) \end{Bmatrix} \quad (3.183)$$

The determinant is negative if

$$M > H \quad (3.184)$$

Where

$$M=\begin{Bmatrix} \varepsilon\psi(\beta F_1 S)F_4F_6\omega_1(F_5+c_2+c_1)+\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,F_3F_4F_5F_6 \\ +\mu\,\mathrm{F}_2\,F_3F_4F_5F_6 \end{Bmatrix} \quad (3.185)$$

$$H=\begin{Bmatrix} \varepsilon\psi F_3F_4F_5F_6\omega_1+2(\beta F_1 S)(\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1)\mathrm{F}_2\,F_4F_6(c_2+c_1) \\ +\beta F_1 S\mu F_2{}^2F_4F_6(1+c_2+c_1)+\beta(\mathrm{E}+\mathrm{C}+\mathrm{A})\mathrm{F}_1\mathrm{F}_2\,\omega_2(F_4c_2\gamma_2+F_5c_1\gamma_1) \end{Bmatrix} \quad (3.186)$$

The endemic equilibrium is locally asymptotically stable according to the first stability criteria as stated in the conditions for stability earlier, since the determinant is positive if (3.184) holds and the trace of the Jacobian is negative.

3.11 Analytical Solution of the Model

Analytical solution of the Model using Homotopy Perturbation Method (HPM)

Ji-Haun (2000) discovered Homotopy Perturbation Method (HPM) of solution to systems of differential equations. The Homotopy Perturbation Method (HPM), which gives analytical estimated solution, is applied to various linear and non-linear equations. The homotopy perturbation method (HPM) is a series expansion method used in the solution of nonlinear partial differential equations (Jiya 2010).

To show the simple concepts of this method, he considered the following non-linear differential equation:

$$B_3(U)-G(r)=0,\ r\in\Omega \quad (3.185)$$

Subject to the boundary condition

$$D_3\left(U, \frac{\partial U}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{3.186}$$

Where B_3 is a general differential operator, D_3 a boundary operator, $G(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A_3 can be divided into two parts L and N, where L is the linear part, and N is the nonlinear part. Equation (3.202) can be written as:

$$L(U) + N(U) - G(r) = 0, \text{r } \epsilon\Omega \tag{3.187}$$

The Homotopy Perturbation structure is shown as follows

$$H(V, \text{p}) = (1-p)\left[L(V) - L(U_0)\right] + p\left[A(V) - G(r)\right] = 0 \tag{3.188}$$

Where

$$V(r, P): \Omega \in [0,1] \to R \tag{3.189}$$

In equation (3.188) $P \in [0,1]$ is an embedding parameter and U_0 is the approximation that satisfies the boundary condition. It can be assumed that the solution of the equation can be written as power series in h as follows:

$$V = V_0 + pV_1 + p^2V_2 + ... \tag{3.190}$$

And the best approximation for the solution is:

$$U = \lim_{h \to 1} v = v_0 + pv_1 + p^2v_2 + ... \tag{3.191}$$

The series is convergent for most cases. However, the convergent rate depends on the nonlinear operator A (V)

Solution of the Model Equations

$$\frac{dS}{dt} = \Lambda - \beta(E + C + A)[(1 - \varepsilon_c \psi_c) + (1 - \varepsilon\psi)]S - \mu S + \omega_1 V + \omega_2 R \quad (3.192)$$

$$\frac{dV}{dt} = \varepsilon\psi S - (\omega_1 + \mu)V \quad (3.193)$$

$$\frac{dE}{dt} = \beta(E + C + A)[(1 - \varepsilon_c \psi_c) + (1 - \varepsilon\psi)]S - (c_1 + c_2 + \mu)E \quad (3.194)$$

$$\frac{dC}{dt} = c_1 E - (\gamma_1 + \mu + \delta)C \quad (3.195)$$

$$\frac{dA}{dt} = c_2 E - (\gamma_2 + \mu + \delta)\,\mathrm{A} \quad (3.196)$$

$$\frac{dR}{dt} = \gamma_1 C + \gamma_2 A - (\omega_2 + \mu)\,\mathrm{R} \quad (3.197)$$

With the following initial condition

$$S(0) = S_0, \mathrm{V}(0) = V_0, \mathrm{E}(0) = E_0, \mathrm{C}(0) = C_0, \mathrm{A}(0) = A_0, R(0) = R_0$$

(3.198)

Let

$$S = a_0 + pa_1 + p^2 a_2 + \ldots \quad (3.199)$$

$$V = b_0 + pb_1 + p^2 b_2 + \ldots \quad (3.200)$$

$$E = d_0 + pd_1 + p^2 d_2 + \ldots \quad (3.201)$$

$$C = e_0 + pe_1 + p^2 e_2 + \ldots \quad (3.202)$$

$$A = f_0 + pf_1 + p^2 f_2 + \ldots \quad (3.203)$$

$$R = g_0 + pg_1 + p^2 g_2 + ... \tag{3.204}$$

Applying HPM to equation (3.192)

$$(1-p)\frac{dS}{dt} + p\left[\frac{dS}{dt} + G\left(E + C + A\right)S + \mu S - w_1 V - w_2 R - \Lambda\right] = 0 \tag{3.205}$$

$$G = (1 - \varepsilon_c \psi_c) + (1 - \varepsilon\psi) \tag{3.206}$$

$$(1-p)(a_0^1 + pa_1^1 + p^2 a_2^1 + ...) + p\begin{bmatrix} (a_0^1 + pa_1^1 + p^2 a_2^1 + ...) \\ +G\begin{pmatrix} (\mathrm{d}_0 + pd_1 + p^2 d_2 + ...) + \\ (\mathrm{e}_0 + pe_1 + h^2 e_2 + ...) + \\ (\mathrm{f}_0 + pf_1 + p^2 f_2 + ...) \end{pmatrix}(a_0 + pa_1 + p^2 a_2 + ...) \\ +\mu(a_0 + pa_1 + p^2 a_2 + ...) - \\ w_1(\mathrm{b}_0 + pb_1 + p^2 b_2 + ...) - \mathrm{w}_2(\mathrm{g}_0 + pg_1 + p^2 g_2 + ...) - \Lambda \end{bmatrix} = 0 \tag{3.207}$$

Expanding and Collecting the coefficient power of p

$$p^0 : a_0^1 = 0 \tag{3.208}$$

$$p^1 : a_1^1 - G(\mathrm{d}_0 a_0 + e_0 a_0 + f_0 \mathrm{a}_0) + \mu a_0 - w_1 b_0 - w_2 g_0 - \Lambda = 0 \tag{3.209}$$

$$p^2 : a_2^1 - G(\mathrm{d}_1 a_0 + e_1 a_0 + f_1 \mathrm{a}_0 + \mathrm{e}_0 \mathrm{a}_1 + d_0 a_1 + f_0 a_1) + \mu a_1 - w_1 b_1 - w_2 g_1 = 0 \tag{3.210}$$

Also applying homotopy perturbation method to (3.193)

$$(1-p)\frac{dV}{dt} + p\left[\frac{dV}{dt} + F_2 V - \varepsilon\psi S\right] = 0 \tag{3.211}$$

$$(1-p)(\mathrm{b}_0^1 + pb_1^1 + p^2 b_2^1 + ...) + p\begin{bmatrix} (\mathrm{b}_0^1 + pb_1^1 + p^2 b_2^1 + ...) \\ +F_2(\mathrm{b}_0 + pb_1 + p^2 b_2 + ...) \\ -\varepsilon\psi(a_0 + pa_1 + p^2 a_2 + ...) \end{bmatrix} = 0 \tag{3.212}$$

$$(\mathrm{b}_0^1 + pb_1^1 + p^2 b_2^1 + ...) + p\begin{bmatrix} F_2(\mathrm{b}_0 + pb_1 + p^2 b_2 + ...) \\ -\varepsilon\psi(a_0 + pa_1 + p^2 a_2 + ...) \end{bmatrix} = 0 \tag{3.213}$$

Expanding and Collecting the coefficient power of p

$$p^0 : \mathrm{b}_0^1 = 0 \tag{3.214}$$

$$p^1 : \mathrm{b}_1^1 + F_2 b_0 - \varepsilon\psi a_0 = 0 \tag{3.215}$$

$$p^2 : \mathrm{b}_2^1 + F_2 b_1 - \varepsilon\psi a_1 = 0 \tag{3.216}$$

Also applying homotopy perturbation method to (3.194)

$$(1-p)\frac{dE}{dt} + p\left[\frac{dE}{dt} + F_3 E - G(E + C + A)S\right] = 0 \tag{3.217}$$

$$(1-p)(\mathrm{d}_0^1 + p d_1^1 + p^2 d_2^1 + ...) + p\left[\begin{array}{l}(\mathrm{d}_0^1 + p d_1^1 + p^2 d_2^1 + ...) + \\ F_3(\mathrm{d}_0 + p d_1 + p^2 d_2 + ...) - \\ +G\left(\begin{array}{l}(\mathrm{d}_0 + p d_1 + p^2 d_2 + ...) + \\ (\mathrm{e}_0 + p e_1 + p^2 e_2 + ...) + \\ (\mathrm{f}_0 + p f_1 + p^2 f_2 + ...)\end{array}\right)(a_0 + p a_1 + p^2 a_2 + ...)\end{array}\right] = 0 \tag{3.218}$$

$$(\mathrm{d}_0^1 + p d_1^1 + p^2 d_2^1 + ...) + p\left[\begin{array}{l}F_3(\mathrm{d}_0 + p d_1 + p^2 d_2 + ...) - \\ G\left(\begin{array}{l}(\mathrm{d}_0 + p d_1 + p^2 d_2 + ...) + \\ (\mathrm{e}_0 + h e_1 + h^2 e_2 + ...) + \\ (\mathrm{f}_0 + p f_1 + p^2 f_2 + ...)\end{array}\right)(a_0 + p a_1 + p^2 a_2 + ...)\end{array}\right] = 0 \tag{3.219}$$

Expanding and Collecting the coefficient power of p

$$p^0 : \mathrm{d}_0^1 = 0 \tag{3.220}$$

$$p^1 : \mathrm{d}_1^1 + F_3 d_0 - G(d_0 a_0 + e_0 a_0 + f_0 a_0) = 0 \tag{3.221}$$

$$p^2 : \mathrm{d}_2^1 + F_3 d_1 - G(\mathrm{d}_0 + e_0 + f_0)a_1 - G(d_1 + e_1 + f_1)a_0 = 0 \tag{3.222}$$

Also applying homotopy perturbation method to (3.195)

$$(1-p)\frac{dC}{dt} + p\left[\frac{dC}{dt} + F_4 C - c_1 E\right] = 0 \tag{3.223}$$

$$(1-p)(e_0^1+pe_1^1+p^2e_2^1+....)+p\begin{bmatrix}(e_0^1+pe_1^1+p^2e_2^1+....)\\+F_4(e_0+pe_1+p^2e_2+....)\\-c_1(d_0+pd_1+p^2d_2+....)\end{bmatrix}=0 \tag{3.224}$$

$$(e_0^1+pe_1^1+p^2e_2^1+....)+p\begin{bmatrix}F_4(e_0+pe_1+p^2e_2+....)\\-c_1(d_0+pd_1+p^2d_2+....)\end{bmatrix}=0 \tag{3.225}$$

Expanding and Collecting the coefficient power of p

$$p^0 : e_0^1 = 0 \tag{3.226}$$

$$p^1 : e_1^1+F_4e_0-c_1d_0=0 \tag{3.227}$$

$$p^2 : e_2^1+F_4e_1-c_1d_1=0 \tag{3.228}$$

Also applying homotopy perturbation method to (3.196)

$$(1-p)\frac{dA}{dt}+p\left[\frac{dA}{dt}+F_5A-c_2E\right]=0 \tag{3.229}$$

$$(1-p)(f_0^1+pf_1^1+p^2f_2^1+....)+p\begin{bmatrix}(f_0^1+pf_1^1+p^2f_2^1+....)\\+F_5(f_0+pf_1+p^2f_2+....)\\-c_2(e_0+pe_1+p^2e_2+....)\end{bmatrix}=0 \tag{3.230}$$

$$(f_0^1+pf_1^1+p^2f_2^1+....)+p\begin{bmatrix}F_5(f_0+pf_1+p^2f_2+....)\\-c_2(e_0+pe_1+p^2e_2+....)\end{bmatrix}=0 \tag{3.231}$$

Expanding and Collecting the coefficient power of p

$$p^0 : f_0^1 = 0 \tag{3.232}$$

$$p^1 : f_1^1+F_5f_0-c_2e_0=0 \tag{3.233}$$

$$p^2 : f_2^1+F_5f_1-c_2e_1=0 \tag{3.234}$$

Also applying homotopy perturbation method to (3.197)

$$(1-p)\frac{dR}{dt}+p\left[\frac{dR}{dt}+F_6R-\gamma_1C-\gamma_2A\right]=0 \tag{3.235}$$

$$(1-\mathrm{p})(\mathrm{g}_0^1+pg_1^1+p^2g_2^1+....)+p\begin{bmatrix}(\mathrm{g}_0^1+pg_1^1+p^2g_2^1+....)+\\F_6(\mathrm{g}_0+pg_1+p^2g_2+....)\\-\gamma_1(\mathrm{e}_0+pe_1+p^2e_2+....)-\\\gamma_2(\mathrm{f}_0+pf_1+p^2f_2+....)\end{bmatrix}=0 \tag{3.236}$$

$$(\mathrm{g}_0^1+pg_1^1+p^2g_2^1+....)+p\begin{bmatrix}F_6(\mathrm{g}_0+pg_1+p^2g_2+....)\\-\gamma_1(\mathrm{e}_0+pe_1+p^2e_2+....)-\\\gamma_2(\mathrm{f}_0+pf_1+p^2f_2+....)\end{bmatrix}=0 \tag{3.237}$$

$$p^0:\mathrm{g}_0^1=0 \tag{3.238}$$

$$p^1:\mathrm{g}_1^1+F_6g_0-\gamma_1e_0-\gamma_2f_0=0 \tag{3.239}$$

$$p^2:\mathrm{g}_2^1+F_6g_1-\gamma_1e_1-\gamma_2f_1=0 \tag{3.240}$$

From (3.208)

$$p^0:a_0^1=0 \tag{3.241}$$

Integrating (3.241)

$$a_0=C_1 \tag{3.242}$$

Applying initial condition

$$a_0(0)=S(0)=S_0 \tag{3.243}$$

$$a_0=S_0 \tag{3.244}$$

From (3.209)

$$p^0 : \mathrm{b}_0^1 = 0 \tag{3.245}$$

Integrating (3.245)

$$b_0 = C_2 \tag{3.246}$$

Applying initial condition to (3.246)

$$b_0(0) = V(0) = V_0 \tag{3.247}$$

$$b_0 = V_0 \tag{3.248}$$

From (3.220)

$$p^0 : \mathrm{d}_0^1 = 0 \tag{3.249}$$

Integrating (3.249)

$$d_0 = C_3 \tag{3.250}$$

Applying initial condition to (3.250)

$$d_0(0) = E(0) = E_0 \tag{3.251}$$

$$d_0 = E_0 \tag{3.252}$$

From (3.242)

$$p^0 : \mathrm{e}_0^1 = 0 \tag{3.253}$$

Integrating (3.253)

$$e_0 = C_4 \tag{3.254}$$

Applying initial condition to (3.254)

$$e_0(0) = C(0) = C_0 \tag{3.255}$$

$$e_0 = C_0 \tag{3.256}$$

From (3.232)

$$p^0 : \mathrm{f}_0^1 = 0 \tag{3.257}$$

Integrating (3.257)

$$f_0 = C_5 \tag{3.258}$$

Applying initial condition to (3.258)

$$f_0(0) = A(0) = A_0 \tag{3.259}$$

$$f_0 = A_0 \tag{3.260}$$

From (3.238)

$$p^0 : \mathrm{g}_0^1 = 0 \tag{3.261}$$

Integrating (3.261)

$$g_0 = C_6 \tag{3.262}$$

Applying initial condition (3.262)

$$g_0(0) = R(0) = R_0 \tag{3.263}$$

$$g_0 = R_0 \tag{3.264}$$

From (3.209)

$$a_1^1 = \Lambda + w_1 b_0 + w_2 g_0 - G(\mathrm{d}_0\, a_0 + e_0 a_0 + f_0\, \mathrm{a}_0) - \mu a_0 \tag{3.265}$$

Integrating (3.265)

$$a_1 = [\Lambda + w_1 b_0 + w_2 g_0 - G(\mathrm{d}_0\, a_0 + e_0 a_0 + f_0\, \mathrm{a}_0) - \mu a_0]t \tag{3.266}$$

$$a_1 = [\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\,\mathrm{S}_0 - \mu S_0]t \tag{3.267}$$

From (3.215)

$$\mathrm{b}_1^1 = \varepsilon\psi a_0 - F_2 b_0 \tag{3.268}$$

Integrating (3.268)

$$\mathrm{b}_1 = (\varepsilon\psi a_0 - F_2 b_0)t \tag{3.269}$$

$$\mathrm{b}_1 = (\varepsilon\psi S_0 - F_2 V_0)t \tag{3.270}$$

From (3.221)

$$\mathrm{d}_1^1 = G(d_0 a_0 + e_0 a_0 + f_0 a_0) - F_3 d_0 \tag{3.271}$$

Integrating (3.271)

$$\mathrm{d}_1 = [G(d_0 a_0 + e_0 a_0 + f_0 a_0) - F_3 d_0]t \tag{3.272}$$

$$\mathrm{d}_1 = [G(\mathrm{E}_0 + C_0 + A_0)\,\mathrm{S}_0 - F_3 E_0]t \tag{3.273}$$

From (3.227)

$$\mathrm{e}_1^1 = c_1 d_0 - F_4 e_0 \tag{3.274}$$

Integrating (3.274)

$$\mathrm{e}_1 = [c_1 d_0 - F_4 e_0]\,\mathrm{t} \tag{3.275}$$

$$\mathrm{e}_1 = [c_1 E_0 - F_4 C_0]\,\mathrm{t} \tag{3.276}$$

From (3.233)

$$\mathrm{f}_1^1 = c_2 e_0 - F_5 f_0 \tag{3.277}$$

Integrating (3.277)

$$\mathrm{f}_1 = [c_2 e_0 - F_5 f_0]t \tag{3.278}$$

$$\mathrm{f}_1 = [c_2 C_0 - F_5 A_0]t \tag{3.279}$$

From (3.239)

$$\mathrm{g}_1^1 = \gamma_1 e_0 + \gamma_2 f_0 - F_6 g_0 \tag{3.280}$$

Integrating (3.280)

$$\mathrm{g}_1 = [\gamma_1 e_0 + \gamma_2 f_0 - F_6 g_0]t \tag{3.281}$$

$$\mathrm{g}_1 = [\gamma_1 C_0 + \gamma_2 A_0 - F_6 R_0]t \tag{3.282}$$

From (3.210)

$$\begin{aligned} a_2^1 &= -G(\mathrm{d}_1 + e_1 + f_1)\mathrm{a}_0 - \mathrm{G}(\mathrm{e}_0 + d_0 + f_0)a_1 + \\ &-\mu a_1 + w_1 b_1 + w_2 g_1 = 0 \end{aligned} \tag{3.283}$$

Substituting (3.267), (3.270), (3.273) and (3.276), (3.279), and (3.282) into (3.283)

$$\begin{aligned} a_2^1 &= -G([G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0]t + \\ &[c_1 E_0 - F_4 C_0]\mathrm{t} + [c_2 C_0 - F_5 A_0]t)\mathrm{S}_0 - \\ &\mathrm{G}(\mathrm{C}_0 + E_0 + A_0)[\Lambda + w_1 V_0 + w_2 R_0 - \\ &G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0]t_1 + \\ &-\mu[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 \\ &-\mu S_0]t + w_1(\varepsilon\psi S_0 - F_2 V_0)t + \\ &w_2[\gamma_1 C_0 + \gamma_2 A_0 - F_6 R_0]t \end{aligned} \tag{3.284}$$

Integrating (3.284)

$$a_2 = \left\{ \begin{array}{l} -G([G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] + \\ [c_1 E_0 - F_4 C_0] + [c_2 C_0 - F_5 A_0])\mathrm{S}_0 - \\ \mathrm{G}(\mathrm{C}_0 + E_0 + A_0)[\Lambda + w_1 V_0 + w_2 R_0 - \\ G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0] + \\ -\mu[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 \\ -\mu S_0] + w_1(\varepsilon\psi S_0 - F_2 V_0) + \\ w_2[\gamma_1 C_0 + \gamma_2 A_0 - F_6 R_0] \end{array} \right\} \frac{t^2}{2} \qquad (3.285)$$

From (3.216)

$$\mathrm{b}_2^1 = \varepsilon\psi a_1 - F_2 b_1 \qquad (3.286)$$

Substituting (3.267), (3.270) into (3.286),

$$\begin{array}{l} b_2^1 = \varepsilon\psi[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0]t \\ -F_2(\varepsilon\psi S_0 - F_2 V_0)t \end{array} \qquad (3.287)$$

Integrating (3.287)

$$b_2 = \left\{ \begin{array}{l} \varepsilon\psi[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0] \\ -F_2(\varepsilon\psi S_0 - F_2 V_0) \end{array} \right\} \frac{t^2}{2} \qquad (3.288)$$

From (3.289)

$$\mathrm{d}_2^1 = G(d_1 + e_1 + f_1)a_0 + G(\mathrm{d}_0 + e_0 + f_0)a_1 - F_3 d_1 \qquad (3.290)$$

Substituting (3.267),(3.273),(3.276) and (3.279) into (3.290)

$$\begin{array}{l} \mathrm{d}_2^1 = G([G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0]t + \\ [c_1 E_0 - F_4 C_0]\mathrm{t} + f_1)a_0 + G(\mathrm{d}_0 + e_0 + f_0)[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0]t \\ -F_3[G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0]t \end{array} \qquad (3.291)$$

Integrating (3.291)

$$\mathrm{d}_2 = \left\{ \begin{array}{l} G([G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] + \\ [c_1 E_0 - F_4 C_0] + f_1)\mathrm{S}_0 + G(\mathrm{C}_0 + E_0 + A_0)[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0] \\ -F_3[G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] \end{array} \right\} \frac{t^2}{2} \qquad (3.292)$$

From (3.228)

$$\mathrm{e}_2^1 = c_1 d_1 - F_4 e_1 \tag{3.293}$$

Substituting (3.273) and (3.276) into (3.293)

$$\mathrm{e}_2^1 = c_1[G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0]t - F_4[c_1 E_0 - F_4 C_0]\mathrm{t} \tag{3.294}$$

Integrating (3.294)

$$\mathrm{e}_2 = \{c_1[G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] - F_4[c_1 E_0 - F_4 C_0]\}\frac{\mathrm{t}^2}{2} \tag{3.295}$$

From (3.279)

$$\mathrm{f}_2^1 = c_2 e_1 - F_5 f_1 \tag{3.296}$$

Substituting (3.276) and (3.279) into (3.296)

$$\mathrm{f}_2^1 = \{c_2[c_1 E_0 - F_4 C_0] - F_5[c_2 C_0 - F_5 A_0]\}t \tag{3.297}$$

Integrating (3.297)

$$\mathrm{f}_2 = \{c_2[c_1 E_0 - F_4 C_0] - F_5[c_2 C_0 - F_5 A_0]\}\frac{t^2}{2} \tag{3.298}$$

From (3.282)

$$\mathrm{g}_2^1 = \gamma_1 e_1 + \gamma_2 f_1 - F_6 g_1 \tag{3.299}$$

Substituting (3.276), (3.279) and (3.282) into (3.299)

$$\begin{aligned}\mathrm{g}_2^1 &= \gamma_1[c_1 E_0 - F_4 C_0]\mathrm{t} + \gamma_2[c_2 C_0 - F_5 A_0]t \\ &-F_6[\gamma_1 C_0 + \gamma_2 A_0 - F_6 R_0]t\end{aligned} \tag{3.300}$$

Integrating (3.300)

$$g_2 = \begin{Bmatrix} \gamma_1[c_1E_0 - F_4C_0] + \gamma_2[c_2C_0 - F_5A_0] \\ -F_6[\gamma_1C_0 + \gamma_2A_0 - F_6R_0] \end{Bmatrix} \frac{t^2}{2} \tag{3.301}$$

But

$$S(t) = a_0 + pa_1 + p^2a_2 + ... \tag{3.302}$$

$$S(t) = \lim_{p \to 1}\left(a_0 + pa_1 + p^2a_2 + ...\right) \tag{3.303}$$

$$S(t) = a_0 + a_1 + a_2 + ... \tag{3.304}$$

$$S(t) = S_0 + [\Lambda + w_1V_0 + w_2R_0 - G(E_0 + C_0 + A_0)S_0 - \mu S_0]t + \begin{Bmatrix} -G([G(E_0 + C_0 + A_0)S_0 - F_3E_0] + \\ [c_1E_0 - F_4C_0] + [c_2C_0 - F_5A_0])S_0 - \\ G(C_0 + E_0 + A_0)[\Lambda + w_1V_0 + w_2R_0 - \\ G(E_0 + C_0 + A_0)S_0 - \mu S_0] + \\ -\mu[\Lambda + w_1V_0 + w_2R_0 - G(E_0 + C_0 + A_0)S_0 \\ -\mu S_0] + w_1(\varepsilon\psi S_0 - F_2V_0) + \\ w_2[\gamma_1C_0 + \gamma_2A_0 - F_6R_0] \end{Bmatrix} \frac{t^2}{2} \tag{3.305}$$

Also

$$V = b_0 + pb_1 + p^2b_2 + ... \tag{3.306}$$

$$V(t) = \lim_{p \to 1}\left(b_0 + pb_1 + p^2b_2 + ...\right) \tag{3.307}$$

$$V(t) = b_0 + b_1 + b_2 + ... \tag{3.308}$$

$$V(t) = V_0 + (\varepsilon\psi S_0 - F_2V_0)t + \begin{Bmatrix} \varepsilon\psi[\Lambda + w_1V_0 + w_2R_0 - G(E_0 + C_0 + A_0)S_0 - \mu S_0] \\ -F_2(\varepsilon\psi S_0 - F_2V_0) \end{Bmatrix} \frac{t^2}{2} \tag{3.309}$$

also

$$E = d_0 + pd_1 + p^2d_2 + ... \tag{3.310}$$

$$E(t) = \lim_{p \to 1} \left(c_0 + pc_1 + p^2 c_2 + ... \right) \tag{3.311}$$

$$E = d_0 + d_1 + d_2 + ... \tag{3.312}$$

$$\begin{aligned} &\mathrm{E}(t) = E_0 + [G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0]t + \\ &\left\{ \begin{aligned} &G([G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] + \\ &[c_1 E_0 - F_4 C_0] + f_1)\mathrm{S}_0 + G(\mathrm{C}_0 + E_0 + A_0)[\Lambda + w_1 V_0 + w_2 R_0 - G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - \mu S_0] \\ &-F_3[G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] \end{aligned} \right\} \frac{t^2}{2} \end{aligned} \tag{3.313}$$

Also

$$C = e_0 + pe_1 + p^2 e_2 + ... \tag{3.314}$$

$$C(t) = \lim_{p \to 1} \left(e_0 + pe_1 + p^2 e_2 + ... \right) \tag{3.315}$$

$$C = e_0 + e_1 + e_2 + ... \tag{3.316}$$

$$\begin{aligned} &\mathrm{C}(t) = C_0 + [c_1 E_0 - F_4 C_0]\mathrm{t} + \\ &\left\{ c_1 [G(\mathrm{E}_0 + C_0 + A_0)\mathrm{S}_0 - F_3 E_0] - F_4 [c_1 E_0 - F_4 C_0] \right\} \frac{\mathrm{t}^2}{2} \end{aligned} \tag{3.317}$$

Also

$$A = f_0 + pf_1 + p^2 f_2 + ... \tag{3.318}$$

$$A(t) = \lim_{p \to 1} \left(f_0 + pf_1 + p^2 f_2 + ... \right) \tag{3.319}$$

$$A(t) = f_0 + f_1 + f_2 + ... \tag{3.320}$$

$$\begin{aligned} &A(t) = A_0 + [c_2 C_0 - F_5 A_0]t + \\ &\left\{ c_2 [c_1 E_0 - F_4 C_0] - F_5 [c_2 C_0 - F_5 A_0] \right\} \frac{t^2}{2} \end{aligned} \tag{3.321}$$

Also

$$R = g_0 + pg_1 + p^2 g_2 + ... \tag{3.322}$$

$$R(t) = \lim_{p \to 1} \left(g_0 + p g_1 + p^2 g_2 + ... \right) \tag{3.323}$$

$$\mathrm{R}(t) = g_0 + g_1 + g_2 + ... \tag{3.324}$$

$$\begin{aligned} \mathrm{R}(t) = {} & R_0 + [\gamma_1 C_0 + \gamma_2 A_0 - F_6 R_0] + \\ & \left\{ \begin{aligned} & \gamma_1 [c_1 E_0 - F_4 C_0] + \gamma_2 [c_2 C_0 - F_5 A_0] \\ & -F_6 [\gamma_1 C_0 + \gamma_2 A_0 - F_6 R_0] \end{aligned} \right\} \frac{t^2}{2} \end{aligned} \tag{3.325}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Numerical Simulations

In this section, we plot the graph of analytical solution of our model equations using maple software in order to view the effects of our control parameters on different compartment of the model population.

4.2 Estimation of Variables and Parameters Value

The values of the variables of the model were approximated based on the Nigeria demographic sketch and also on HBV epidemiology. Thus Table 4.1 is a referenced and hypothetical values for the variables of the model and the corresponding values of each of the model parameters are shown in table 4.2.

Table 4.1 Baseline Values for Variables of the HBV in Nigeria

Variable	Value	Source
S	135008399	Calculated
V	105131190	WHO (2020)
E	15000000	WHO (2020)
C	3000000	WHO (2020)
A	1500000	WHO (2020)
R	500000	Assumed

Table 4.2: Baseline Values for Parameters of HBV.

Parameters and State Variables	Value	Source
Λ	2430151	Calculated
β	0.2	Assumed
ω_1	0.002	Assumed
ω_2	0.002	Assumed
ε	(0-1)	Varies
ψ	(0-1)	Varies
ε_c	0.8	Assumed
ψ_c	(0-1)	Varies
μ	0.018	Assumed
c_1	0.03	Assumed
c_2	0.01	Assumed
γ_1	(0-1)	Varies
γ_2	(0-1)	Varies
δ	0.014	Assumed

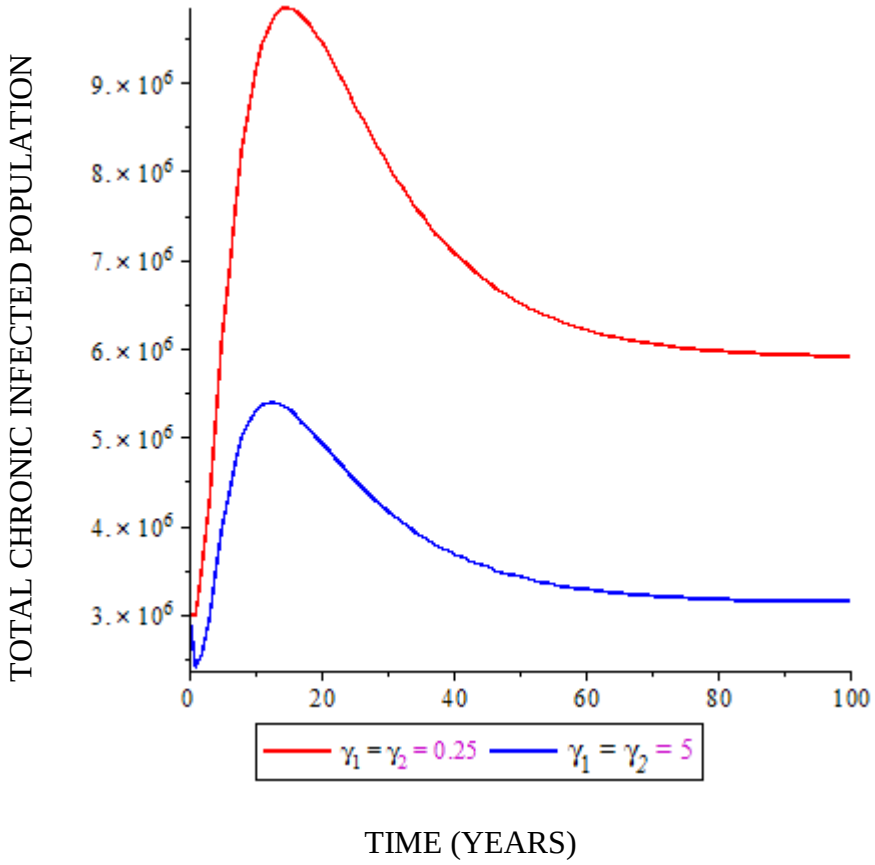

Figure 4.1: Effect of Treatment on Chronic Infected Population

From figure 4.1 the graph shows that the higher the treatment rate, the lower the chronically infected population. This shows that the treatment rate have a significant effect on the chronic infectious population. Although there is an initial increase in the population with treatment, but within 10years the population started declining.

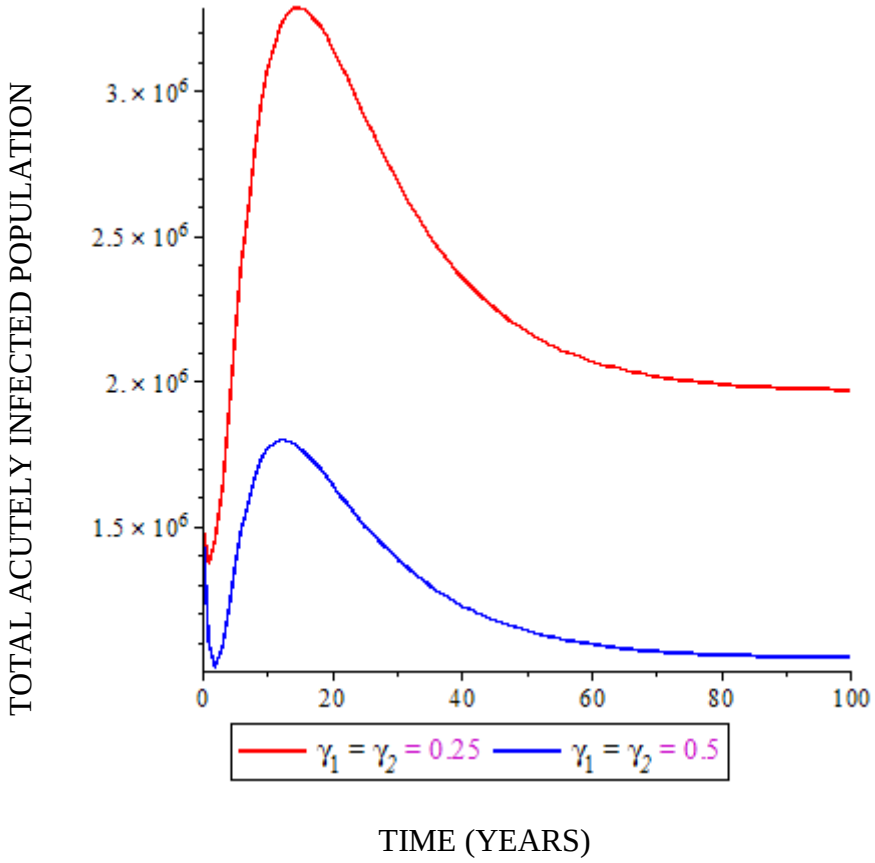

Figure 4.2: Effect of Treatment on Acutely Infected Population

From figure 4.2 the graph shows that the higher the treatment rate, the lower the acutely infected population. This shows that the treatment rate have a significant effect on the acutely infectious population. Although there is an initial increase in the population with treatment, but took between 10 to 20 years for the population to start declining.

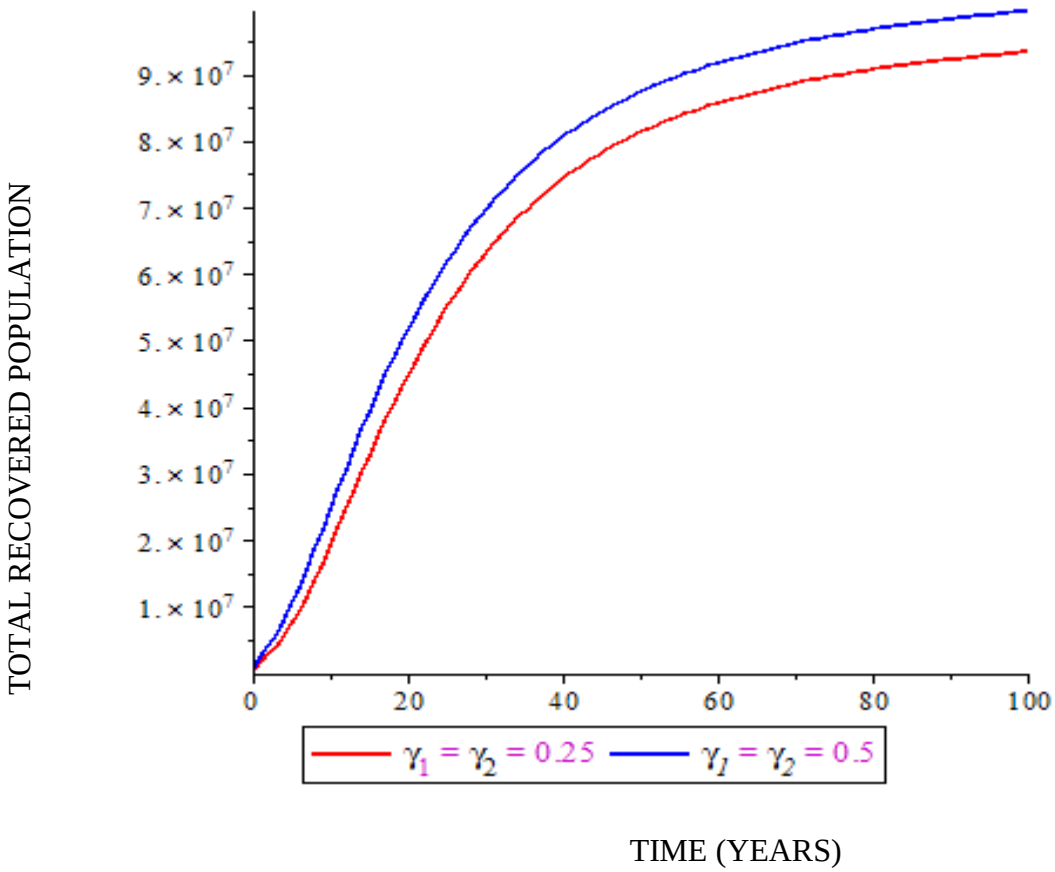

Figure 4.3: Effect of Treatment on Recovered Population

From figure 4.3 the graph shows that the higher the treatment rate, the higher the recovery population. This shows that the treatment rate have a significant effect on the recovery population.

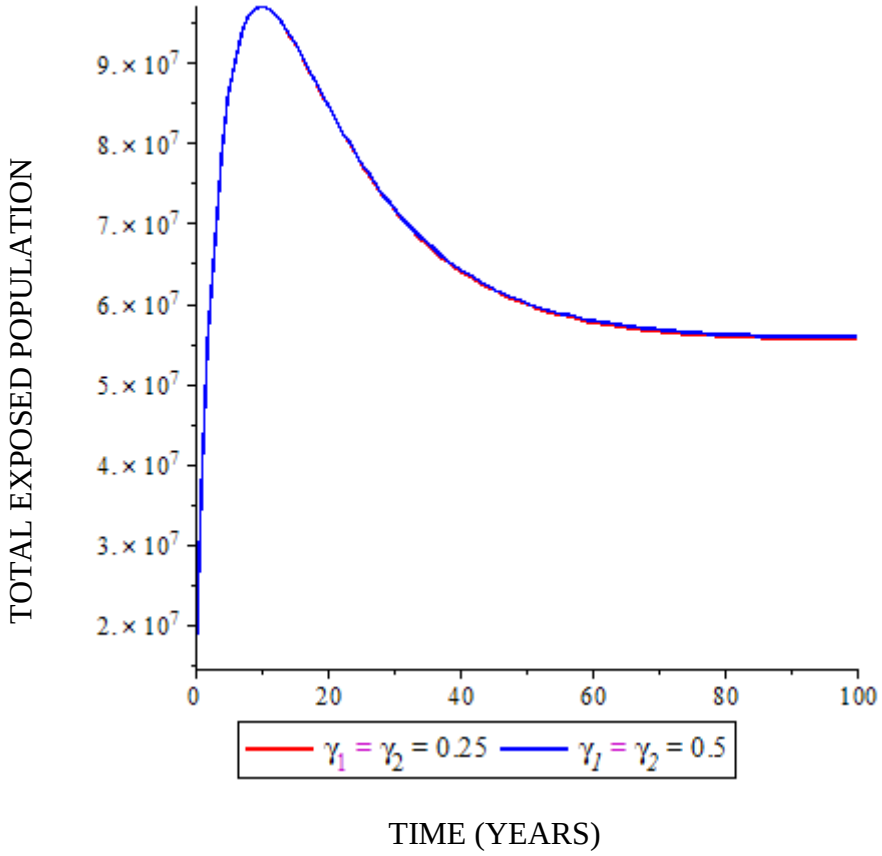

Figure 4.4: Effect of Treatment on Exposed Population

From figure 4.4 the graph shows that the treatment rate doesnot have a significant effect on the exposed population. For different level of treatment, the exposed population increased exponentially, before declining.

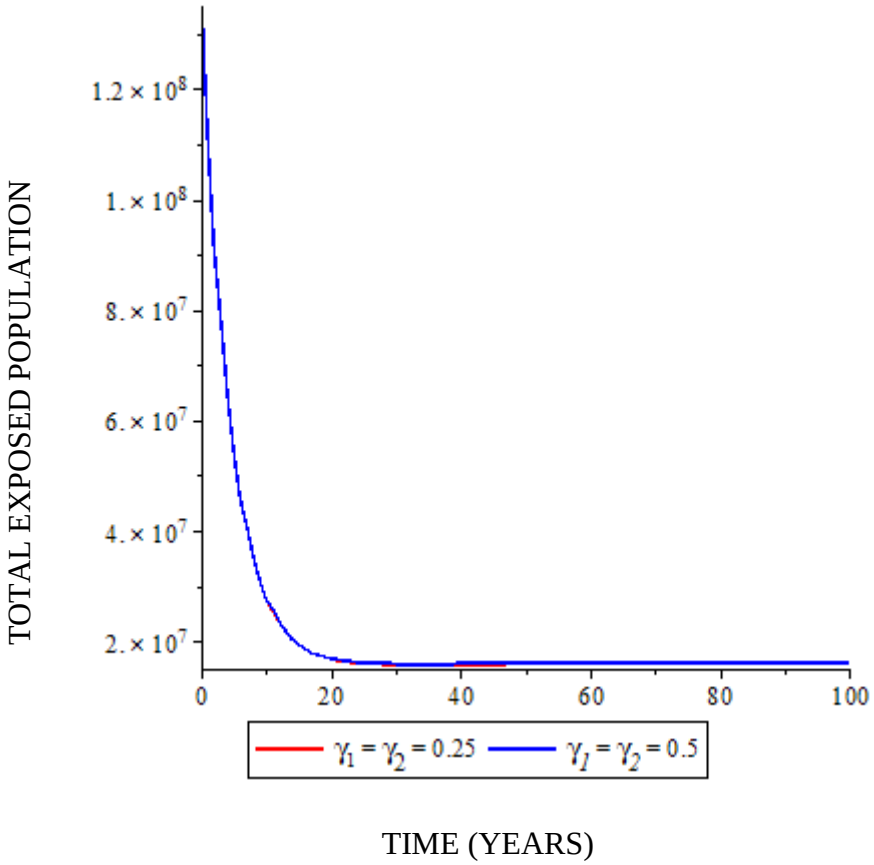

Figure 4.5: Effect of Treatment on Exposed Population

From figure 4.5 the graph shows that the treatment rate doesnot have a significant effect on the exposed population. For different level of treatment, the exposed population declined.

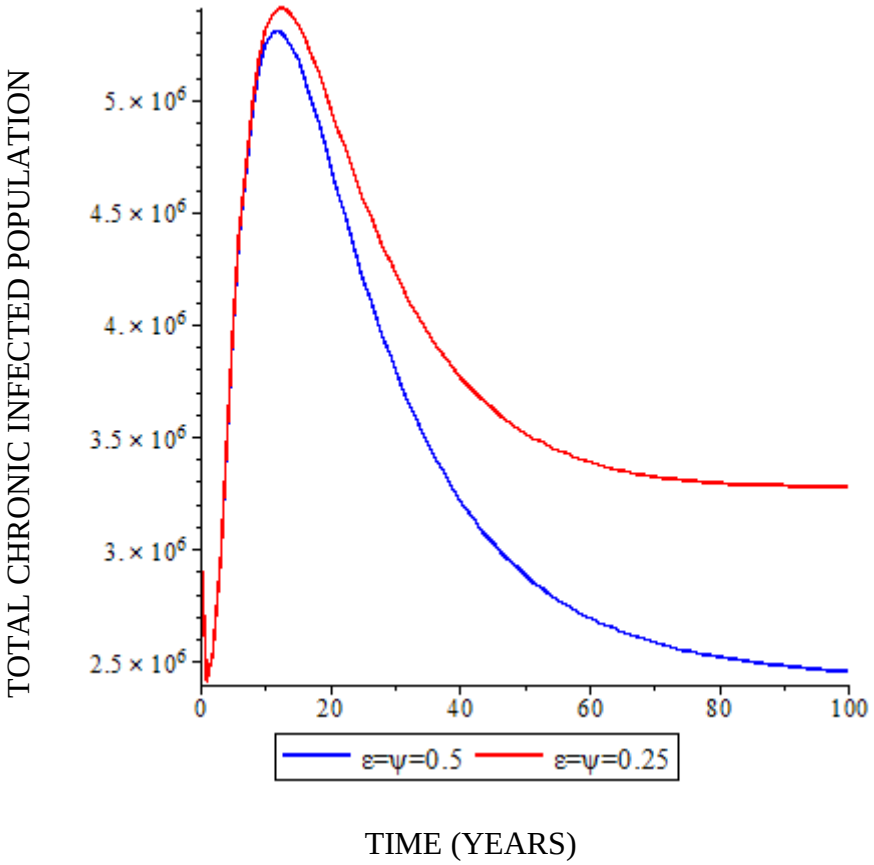

Figure 4.6: Effect of Vaccination on Chronic Infected Population

From figure 4.6 the graph shows that the higher the vaccination rate, the lower the chronically infected population. This shows that the vaccination rate have a significant effect on the chronic infectious population. Although there is an initial increase in the population with vaccination, but got to the peak within 20years, before the population started declining.

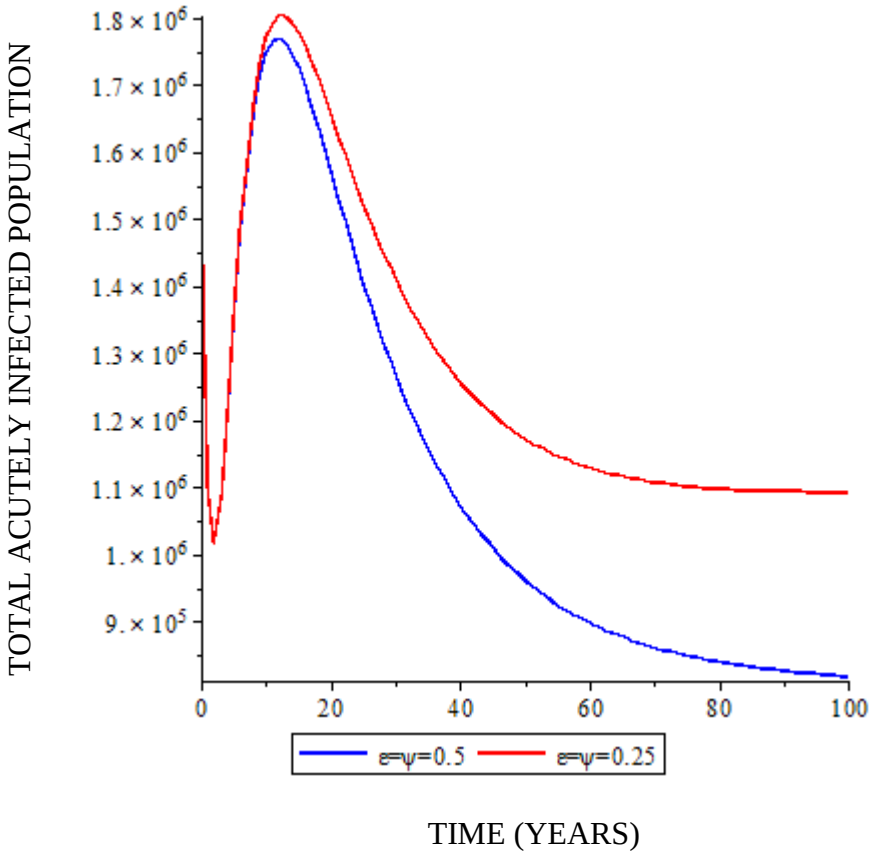

Figure 4.7: Effect of Vaccination on Acutely Infected Population

From figure 4.7 the graph shows that the higher the vaccination rate, the lower the acutely infected population. This shows that the vaccination rate have a significant effect on the acute infectious population. Although there is an initial increase in the population with vaccination, but got to the peak within 20years, before the population started declining.

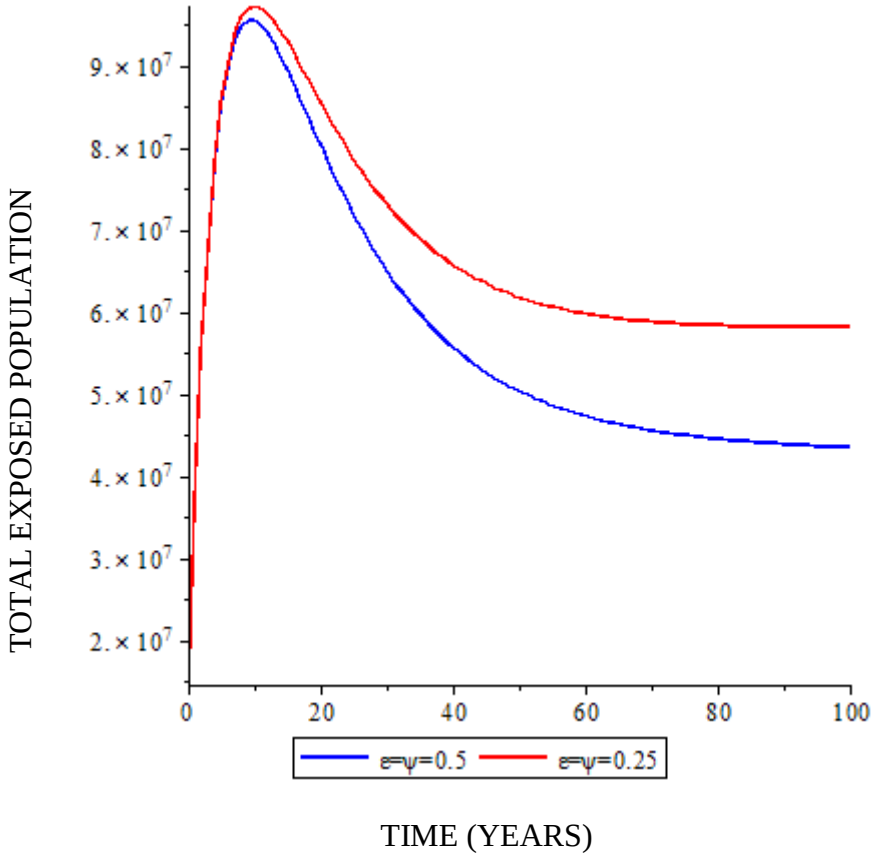

Figure 4.8: Effect of Vaccination on Exposed Population

From figure 4.8 the graph shows that the higher the vaccination rate, the lower the exposed population. This shows that the vaccination rate have a significant effect on the exposed population. Although the is an initial increase in the population with vaccination was the same, but got to the peak within 20years, before the population started declining and the exponential drop reveals clearly the effect of the vaccination at different level.

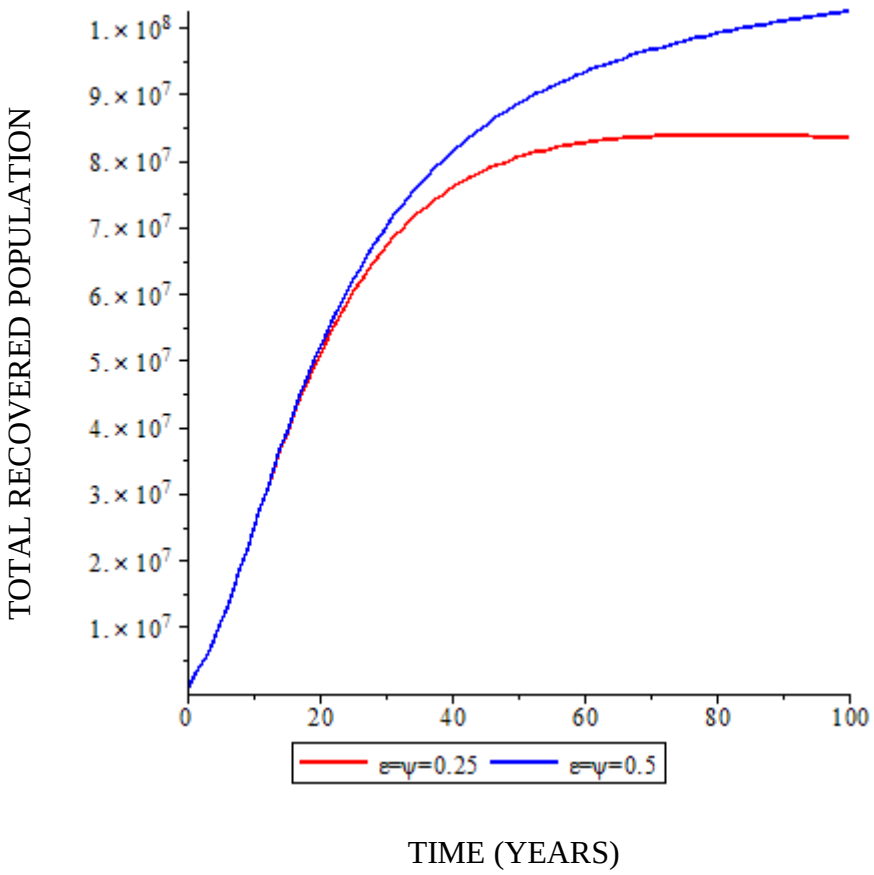

Figure 4.9: Effect of Vaccination on Recovered Population

From figure 4.9 the graph shows that the higher the vaccination rate, the higher the recovery population. This shows that the vaccination rate have a significant effect on the recovery population.

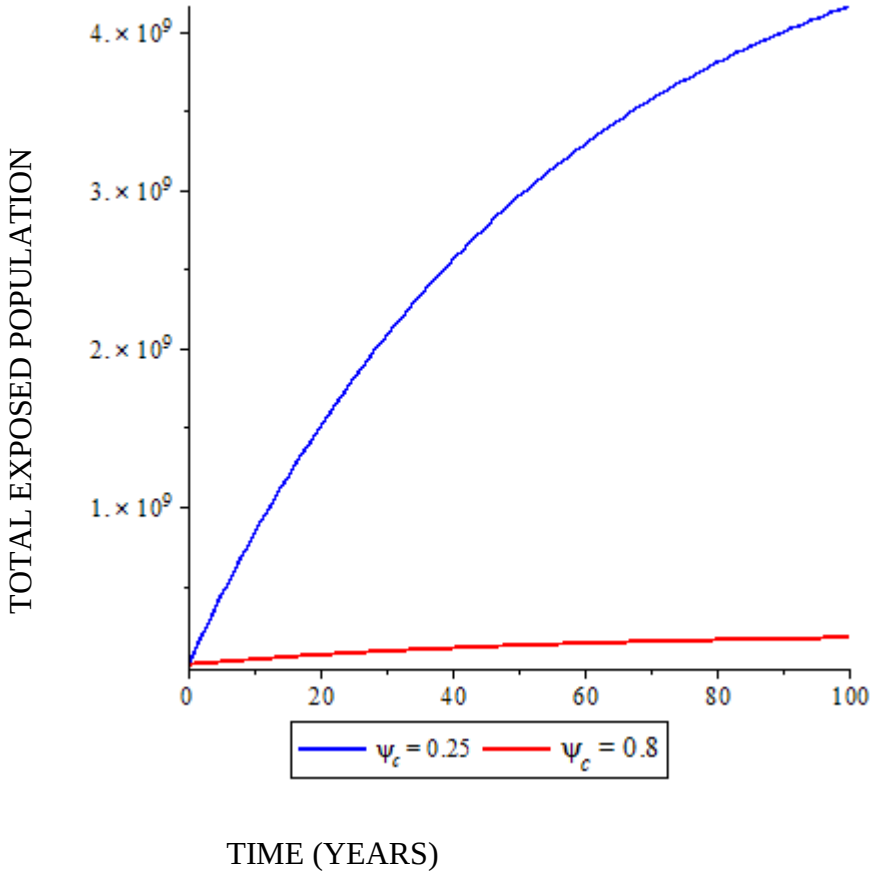

Figure 4.10: Effect of Condom Usage on Exposed Population

From figure 4.10 the graph shows that the higher the condom usage rate, the lower the exposed population. This shows that the condom usgae rate have a significant effect on the exposed population.

4.3 Discussion of Results

The result of the research shows that higher treatment rate, results in the decline of the chronically and acutely infected population, also the higher the treatment rate, the lower the chronically and acutely infected population, this in turn resulted in increase in the recovered population. This shows that the treatment rate have a significant effect on the infected and recovered populations, although the treatment rate does not have a significant effect on the exposed population. Also for different levels of treatment, the vaccination rate have a significant effect on the chronically infectious population. Finally the result also shows that the higher the condom usage rate, the lower the exposed population which means the risk of contacting HBV with usage of highly effective condom is very minimal.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

This work, developed a mathematical model for the transmission dynamics and control of hepatitis B virus incorporating vaccinated class using ordinary differential equations. It can be concluded that the local stability analysis of the Disease Free Equilibrium State (DFE) of the model will be stable if $R_C < 1$ and globally stable if $R_C \leq 1$. Also the analysis of the Endemic Equilibrium State (EE) shows that it is stable.

5.2 Contribution to Knowledge

(i) This research developed model for the transmission dynamics of HBV incorporating vaccinated class and treatment was used as the control parameter using ordinary differential equations.

(ii) The work has shown the criteria for the stability of the model equilibrium points

(iii) The work has shown the effect of treatment and vaccination on all the compartments of the model

5.3 Recommendations

i. Early treatment of acute and chronic infected individuals is highly recommended.

ii. Every individuals who are susceptible to HBV are stongly advised to get vaccinated.

iii. We also want to recommend to World Health Organization, CDC and NAFDAC that the efficacy level of the produced condoms should be at 95% and above in order to reduced the risk of getting infected through sex.

iii. One of the limitations of this study is the unavailability of valid and sufficient records of HBV cases; therefore health workers should make data available for researchers.

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